MATH. SCAND. 22 (1968), 130–142

NOTE ON COFIBRATIONS II

ARNE STRØM

Introduction.

The present paper is a continuation of [7] and contains some results of a general topological nature concerning fibrations and cofibrations. Section 1 is devoted to the proof of a dual of theorem 1 of [7], while the second section contains a characterization of cofibrations and some immediate consequences of this result. Theorem 3 of [7] is strengthened and dualized in section 3, and in the last section we prove that the pull-back of a closed cofibration over a fibration is a cofibration and we prove a conjecture of Per Holm (see [2]), who has also made a number of valuable suggestions. After the work described here was completed Puppe has published his article [5], which slightly overlaps this one.

A few words about notation. The set Y^X of all continuous functions from X to Y is given the compact-open topology. Continuous maps $i\colon T \to X$ and $p\colon Y \to Z$ induce continuous maps $i^\#\colon Y^X \to Y^T$ and $p_\#\colon Y^X \to Z^X$ such that $i^\#(f)=fi$ and $p_\#(f)=pf$. We denote by I the closed unit interval $[0,1]$ with the usual topology and boundary $\dot{I}=\{0,1\}$. For any space X continuous maps $i_0\colon X \to X\times I$, $\pi_0,\pi_1\colon X^I \to X$ are defined by $i_0(x)=(x,0)$, $\pi_0(f)=f(0)$, $\pi_1(f)=f(1)$. By $pr_1\colon X\times Y \to X$ and $pr_2\colon X\times Y \to Y$ we denote the projections. Further $a\wedge b$ denotes the smaller of two real numbers a and b. All maps considered will be continuous.

We shall have occasion to use the following theorem of "exponential correspondence".

EXP. *For arbitrary spaces X, Y, and Z there is an injection (not necessarily continuous)*

$$\vartheta\colon\ Y^{X\times Z} \to (Y^X)^Z$$

such that $[\vartheta(f)(z)](x)=f(x,z)$. If X is locally compact and regular, ϑ is a bijection.

See [3, V. 3] for a proof. The maps f and $f'=\vartheta(f)$ are called *associate* maps. An immediate consequence of EXP is that $i\colon A \to X$ is a cofibration if and only if every commutative diagram

Received April 14, 1967.

$$\begin{array}{ccc} A & \xrightarrow{F'} & Y^I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ X & \xrightarrow[f]{} & Y \end{array}$$

can be filled in with a commutativity preserving map $\bar{F}': X \to Y^I$.

1.

In [7] it was proved that all cofibrations are imbeddings. In the case of a fibration $p: E \to B$ it will not always be true that $p(E)$ is a quotient space of E (see 2.4.8 of [6] for a counterexample), but we do have the following result.

THEOREM 1. *If $p: E \to B$ is a surjective fibration with a locally path connected base space B, then p is a quotient map.*

PROOF. The proof is modelled on the proof of theorem 1 of [7].

Consider the subspace

$$\bar{B} = \{(e,\omega) \in E \times B^I \mid \omega(0) = p(e)\}$$

of $E \times B^I$ and define $\bar{p}: E^I \to \bar{B}$ by $\bar{p}(\omega) = (\omega(0), p\omega)$. It is well known that there exists a section λ of $\bar{p}$ (cf. [4]). The map $\pi_1: E^I \to E$ also has a section $s: E \to E^I$ sending each point of E to the constant path at that point. Consequently $\bar{p}$ and π_1 are quotient maps. We define a map $\pi: \bar{B} \to B$ by $\pi(e,\omega) = \omega(1)$ and so obtain a commutative diagram

$$\begin{array}{ccc} E^I & \xrightarrow{\pi_1} & E \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \bar{B} & \xrightarrow[\pi]{} & B \end{array}$$

Because π_1 and $\bar{p}$ are quotient maps, p is a quotient map if and only if π is a quotient map. We shall prove that π is a quotient map.

Let A be a subset of B such that $\pi^{-1}(A)$ is open in $\bar{B}$ and suppose that $b \in A$. If ω_b is the constant path at b and e is a point of $p^{-1}(b)$, then $(e, \omega_b) \in \pi^{-1}(A)$ and there exists an open set $W \subset B^I$ such that

$$(e,\omega_b) \in (e \times W) \cap \bar{B} \subset \pi^{-1}(A)\,.$$

Because ω_b is a constant path it is easily seen that there exists an open set $U \subset B$ such that

$$\omega_b \in U^I \subset W ,$$

U^I being regarded as a subspace of B^I. Now, b belongs to U, and because B is locally path connected the path component V of U containing b is open. If b' is an arbitrary point of V there exists a path ω in U such that $\omega(0)=b$ and $\omega(1)=b'$. Then

$$(e,\omega) \in (e \times W) \cap \bar{B} \subset \pi^{-1}(A)$$

and $b'=\pi(e,\omega) \in A$. Therefore $V \subset A$, and so b is an interior point of A. But b was an arbitrary point of A and consequently A is open.

2.

Theorem 2. *The pair (X,A) is cofibered if and only if $X \times 0 \cup A \times I$ is a retract of $X \times I$.*

Proof. If (X,A) is a cofibered pair the identity map

$$X \times 0 \cup A \times I \to X \times 0 \cup A \times I$$

extends to a retraction

$$r: X \times I \to X \times 0 \cup A \times I .$$

Conversely, if such a retraction exists, then every continuous map

$$f: X \times 0 \cup A \times I \to Y$$

has a continuous extension

$$fr: X \times I \to Y .$$

It remains to show that every function $f: X \times 0 \cup A \times I \to Y$ whose restrictions $f|X \times 0$ and $f|A \times I$ are continuous is itself continuous. This is an immediate consequence of the following lemma (which is trivial if A is closed).

Lemma 3. *If (X,A) is a pair such that $X \times 0 \cup A \times I$ is a retract of $X \times I$, then a subset C of $X \times 0 \cup A \times I$ is open in $X \times 0 \cup A \times I$ if and only if $C \cap X \times 0$ and $C \cap A \times I$ are open in $X \times 0$ and $A \times I$ respectively.*

Proof. The "only if"-part is obvious. To prove the "if"-part let $C \subset X \times 0 \cup A \times I$ be such that $C \cap X \times 0$ and $C \cap A \times I$ are open in $X \times 0$

and $A \times I$ respectively. It is then easily seen that C is the union of $C \cap (A \times \langle 0,1])$ (which is open in $X \times 0 \cup A \times I$) and the set

$$B = U \times 0 \cup \bigcup_{n=1}^{\infty} ((A \cap U_n) \times [0, 1/n\rangle),$$

where $U, U_1, U_2, \ldots$ are open subsets of X given by

$$U = \{x \in X \mid (x,0) \in C\},$$
$$U_n = \bigcup \{V \mid V \text{ open in } X \text{ and } (V \cap A) \times [0, 1/n\rangle \subset C\}.$$

Then $A \cap U = A \cap \bigcup_{n=1}^{\infty} U_n$ and if V is an open subset of X such that $V \cap A \subset U_n$, then $V \subset U_n$.

We prove $U \subset \bigcup_{n=1}^{\infty} U_n$. Suppose $x \in X - \bigcup_{n=1}^{\infty} U_n$. Then $x \in \bar{A}$. Let $t \in \langle 0,1]$. We then have

$$r(x,t) \in r(\bar{A} \times t) = A \times t.$$

If $r(x,t)$ belongs to some $U_n \times I$ there must exist open neighborhoods V and W of x and t respectively such that

$$r(V \times W) \subset U_n \times I.$$

We should then have

$$(V \cap A) \times t = r((V \cap A) \times t) \subset U_n \times I,$$

that is, $V \cap A \subset U_n$. But this, in turn, would imply $V \subset U_n$, and so

$$x \in U_n \subset \bigcup_{n=1}^{\infty} U_n,$$

contrary to hypothesis.

Consequently

$$r(x,t) \in \left(A - \bigcup_{n=1}^{\infty} U_n\right) \times I = (A - U) \times I \subset (X - U) \times I$$

for each $t \in \langle 0,1]$, and, since r is continuous and $X - U$ is closed,

$$(x,0) = r(x,0) \in (X - U) \times I, \qquad x \in X - U,$$

which shows that $X - \bigcup_{n=1}^{\infty} U_n \subset X - U$, that is, $U \subset \bigcup_{n=1}^{\infty} U_n$.

Let $V_n = U \cap U_n$, $n = 1, 2, \ldots$. Then each V_n is open in X, $U = \bigcup_{n=1}^{\infty} V_n$, $A \cap U_n = A \cap V_n$, and

$$B = (X \times 0 \cup A \times I) \cap \bigcup_{n=1}^{\infty} (V_n \times [0, 1/n\rangle)$$

is open in $X \times 0 \cup A \times I$. But then

$$C = B \cup (C \cap (A \times \langle 0,1]))$$

is also open in $X \times 0 \cup A \times I$, and the lemma is proved.

If A is a subspace of a space X the mapping cylinder of the inclusion map $A \subset X$ may be identified with the subset $X \times 0 \cup A \times I$ of $X \times I$. Lemma 3 shows that if $X \times 0 \cup A \times I$ is a retract of $X \times I$, then the subspace topology inherited from $X \times I$ is identical with the mapping cylinder topology. These topologies are also identical if A is closed, even if no retraction of $X \times I$ to $X \times 0 \cup A \times I$ exists, but examples are easily constructed to show that they need not be identical for arbitrary pairs (X,A).

We can now prove

Lemma 4. *The pair (X,A) is cofibered if and only if there exist a continuous function $\varphi: X \to I$ such that $A \subset \varphi^{-1}(0)$ and a homotopy $H: X \times I \to X$ such that*

$$H(x,0) = x, \quad x \in X,$$
$$H(a,t) = a, \quad a \in A,\ t \in I,$$

and such that $H(x,t) \in A$ whenever $t > \varphi(x)$.

If, in addition, A is a strong deformation retract of X we may assume that φ is everywhere less than 1.

Proof. If there exists a retraction $r: X \times I \to X \times 0 \cup A \times I$ we may define φ and H as follows:

$$\varphi(x) = \sup_{t \in I} |t - pr_2 r(x,t)|, \quad x \in X,$$
$$H(x,t) = pr_1 r(x,t), \quad x \in X,\ t \in I.$$

Conversely, given φ and H a retraction $r: X \times I \to X \times 0 \cup A \times I$ is defined by

$$r(x,t) = \begin{cases} (H(x,t),0), & t \leqq \varphi(x), \\ (H(x,t),t-\varphi(x)), & t \geqq \varphi(x). \end{cases}$$

Finally, given φ and H and a strong deformation retraction $D: X \times I \to X$ of X to A we may replace $\varphi(x)$ and $H(x,t)$ by $\varphi'(x) = \frac{1}{2} \wedge \varphi(x)$ and $H'(x,t) = D(H(x,t), 2t \wedge 1)$.

Note that if $\varphi(x) < 1$, then $H(x,\varphi(x)) \in \overline{H(x \times \langle \varphi(x), 1])} \subset \bar{A}$. Thus, replacing $H(x,t)$ by $\bar{H}(x,t) = H(x, t \wedge \varphi(x))$ we obtain

Corollary 5. *If (X,A) is a cofibered pair, so is $(X,\bar{A})$.*

We use lemma 4 to prove

Theorem 6. *If (X,A) and (Y,B) are cofibered pairs with A closed in X, then the product pair*

$$(X,A)\times(Y,B) = (X\times Y, X\times B\cup A\times Y)$$

is also cofibered. If, in addition, A (or B) is a strong deformation retract of X (or Y), then $X\times B\cup A\times Y$ is a strong deformation retract of $X\times Y$.

Proof. Let $\varphi\colon X\to I$ and $H\colon X\times I\to X$ be as described in lemma 4 and let ψ and G be the corresponding maps for (Y,B). Define $\eta\colon X\times Y\to I$ and $F\colon X\times Y\times I\to X\times Y$ by

$$\eta(x,y) = \varphi(x)\wedge\psi(y)\,,$$
$$F(x,y,t) = \big(H(x,t\wedge\psi(y)),\, G(y,t\wedge\varphi(x))\big)\,.$$

Then $X\times B\cup A\times Y\subset\eta^{-1}(0)$ and $F(x,y,t)=(x,y)$ if $t=0$ or $(x,y)\in X\times B\cup A\times Y$.

Because A is closed $H(x,\varphi(x))\in A$ whenever $\varphi(x)<1$. Now suppose that $t\in I$ and $t>\eta(x,y)$. Then either $\varphi(x)\leqq\psi(y)$ and $\varphi(x)<t$, in which case $t\wedge\psi(y)\geqq\varphi(x)$ and $F(x,y,t)\in A\times Y$, or $\psi(y)<\varphi(x)$ and $\psi(y)<t$, so that $t\wedge\varphi(x)>\psi(y)$ and $F(x,y,t)\in X\times B$. This shows that $F(x,y,t)\in X\times B\cup A\times Y$ whenever $t>\eta(x,y)$, and it follows from lemma 4 that $(X\times Y, X\times B\cup A\times Y)$ is cofibered.

If A (or B) is a strong deformation retract of X (or Y), then we may assume that φ (or ψ) is everywhere less than 1. But then $\eta(x,y)<1$ for all $(x,y)\in X\times Y$, and so $F(x,y,1)\in X\times B\cup A\times Y$, which shows that F is a strong deformation retraction of $X\times Y$ to $X\times B\cup A\times Y$.

See [5] for an example showing that $(X\times Y, X\times B\cup A\times Y)$ need not be cofibered if neither A nor B is closed.

In the way of a converse of theorem 6 we have

Theorem 7. *Suppose that $A\subset X$, that there exists a continuous function $\varphi\colon X\to I$ with $A\subset\varphi^{-1}(0)$, and that there exists a point $x_0\in X-A$ such that $\varphi(x_0)\neq 0$. Then if (Y,B) is a pair such that $(X\times Y, X\times B\cup A\times Y)$ is cofibered, (Y,B) itself is cofibered.*

Proof. Let $\eta\colon X\times Y\to I$ and $F\colon X\times Y\times I\to X\times Y$ be functions for $(X\times Y, X\times B\cup A\times Y)$ as described in lemma 4. We may obviously assume that $\varphi(x_0)=1$, and the functions $G\colon Y\times I\to Y$ and $\psi\colon Y\to I$ defined by

$$G(y,t) = pr_2F(x_0,y,t)\,,$$
$$\psi(y) = \max\big(\eta(x_0,y),\, 1-\inf_{t\in I}\varphi\, pr_1F(x_0,y,t)\big)\,,$$

will then satisfy the conditions of lemma 4.

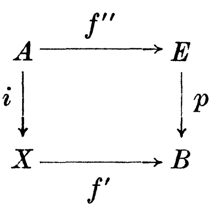
Note, in particular, that (X,A) is cofibered if and only if $(X \times I, X \times 0 \cup A \times I)$ is cofibered, and then $X \times 0 \cup A \times I$ is a strong deformation retract of $X \times I$.

3.

According to 1.4.10 and 1.4.11 of [6] a cofibration $i\colon A \subset X$ is a homotopy equivalence if and only if A is a strong deformation retract of X (the closedness restriction on A in [6] is unnecessary in our case in view of lemma 3). Correspondingly, a fibration $p\colon E \to B$ is a homotopy equivalence if and only if there exists a section $s\colon B \to E$ of p such that $sp \underset{p}{\simeq} 1_E$ (see 6.2 of [1]).

We shall strenghten theorem 3 of [7] and also prove its dual. But first a definition.

Definition. If $i\colon A \to X$ and $p\colon E \to B$ are maps, a *map pair* $f = (f'',f')\colon i \to p$ is a pair of maps $f''\colon A \to E$ and $f'\colon X \to B$ such that $pf'' = f'i$, that is, the diagram



commutes. A map $\bar{f}\colon X \to E$ defines a map pair

$$\varrho(\bar{f}) = (\bar{f}i, p\bar{f}) : i \to p .$$

$\bar{f}$ is called a *lifting* of the pair $\varrho(\bar{f})$.

Theorem 8. *Let $i\colon A \to X$ be a map such that $i(A)$ is closed in X. The following are then equivalent.*

(i) *Every map pair $f\colon i \to p$ with $p\colon E \to B$ a fibration has a lifting.*

(ii) *i is a cofibration and a homotopy equivalence.*

When (i) *and* (ii) *hold the lifting $\bar{f}$ of f is unique up to homotopy relative to $i(A)$.*

Proof. (ii) $\Rightarrow$ (i) and the uniqueness property are just theorem 3 of [7]. To prove that (i) $\Rightarrow$ (ii) note that $\pi_0\colon Y^I \to Y$ is a fibration for any space Y (2.8.2 of [6]), and so $i\colon A \to X$ must be a cofibration, and we may assume that i is an inclusion map. Because $A \to *$ is a fibration ($*$ denotes a one-point space), a retraction $r\colon X \to A$ is obtained as a lifting of the map pair

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ i\cap & & \downarrow \\ X & \longrightarrow & * \end{array} .$$

The map $p\colon X^I \to X \times X$ defined by $p(\omega) = (\omega(0), \omega(1))$ is also a fibration ([6], 2.8.3), and the map pair

$$\begin{array}{ccc} A & \xrightarrow{f''} & X^I \\ i\cap & & \downarrow p \\ X & \xrightarrow[f']{} & X \times X \end{array}$$

with $f''(a)(t) = a$, $f'(x) = (x, r(x))$ has a lifting $\bar{f}\colon X \to X^I$ associate to a strong deformation retraction of X to A.

In a similar fashion we prove

THEOREM 9. *For a map $p\colon E \to B$ the following are equivalent.*

(i) *Every map pair $f\colon i \to p$ with $i\colon A \subset X$ a closed cofibration has a lifting.*

(ii) *p is a fibration and a homotopy equivalence.*

When (i) *and* (ii) *hold the lifting $\bar{f}$ of f is unique up to fiber homotopy over p.*

PROOF. (i) $\Rightarrow$ (ii): Applying (i) to map pairs

$$\begin{array}{ccc} X \times 0 & \longrightarrow & E \\ \cap & & \downarrow p \\ X \times I & \longrightarrow & B \end{array}$$

we see that p must be fibration. The pair $(B, \emptyset)$ is a cofibered pair, and a section $s\colon B \to E$ of p is obtained as a lifting of

$$\begin{array}{ccc} \emptyset & \subset & E \\ \cap & & \downarrow p \\ B & \xrightarrow[1_B]{} & B \end{array}$$

Finally, let $F\colon E\times I\to E$ be a lifting of the map pair

$$\begin{array}{ccc} E\times \dot{I} & \xrightarrow{f''} & E \\ \cap & & \downarrow p \\ E\times I & \xrightarrow[f']{} & B \end{array}$$

with $f''(e,0)=sp(e), f''(e,1)=e$, $f'(e,t)=p(e)$. Then $F\colon sp\underset{p}{\simeq} 1_E$.

(ii) $\Rightarrow$ (i): We know that there exists a section s of p and a fiber homotopy $F\colon sp\underset{p}{\simeq}1_E$. Let $A\subset X$ be a closed cofibration and consider the map pair

$$\begin{array}{ccc} A & \xrightarrow{f''} & E \\ \cap & & \downarrow p \\ X & \xrightarrow[f']{} & B \end{array}\ .$$

Define $F''\colon X\times 0\cup A\times I\to E$ and $F'\colon X\times I\to B$ by $F''(x,0)=sf'(x)$, $F''(a,t)=F(f''(a),t)$, and $F'(x,t)=f'(x)$. The diagram

$$\begin{array}{ccc} X\times 0\cup A\times I & \xrightarrow{F''} & E \\ \cap & & \downarrow p \\ X\times I & \xrightarrow[F']{} & B \end{array}$$

is then commutative and has a lifting $\bar{F}\colon X\times I\to E$ (theorem 4 of [7]). A lifting $\bar{f}$ of (f'',f') is given by $\bar{f}(x)=\bar{F}(x,1)$. Finally, any lifting $\bar{f}$ of (f'',f') is fiber homotopic to $sp\bar{f}=sf'$.

For maps $i\colon A\to X$ and $p\colon E\to B$ the set of map pairs $i\to p$ may be identified with the fibered product $E^A\times{}'B^X$ of the maps $i^{\#}\colon B^X\to B^A$ and $p_{\#}\colon E^A\to B^A$, and the function ϱ mentioned above is then a continuous map from E^X to $E^A\times{}'B^X$. We have the following analogue of 7.8.10 of [6].

Theorem 10. *If $i\colon A\subset X$ is a closed cofibration with X locally compact and regular and $p\colon E\to B$ is a fibration, then $\varrho\colon E^X\to E^A\times{}'B^X$ is a fibration, and if i or p is a homotopy equivalence, so is ϱ.*

PROOF. Given a map pair

$$\begin{array}{ccc} Y & \xrightarrow{f'} & E^X \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle \varrho} \\ Y \times I & \xrightarrow[F']{} & E^A \times {}' B^X \end{array}$$

we shall prove the existence of a lifting $\overline{F}'\colon Y \times I \to E^X$. By EXP there exist maps $f\colon Y \times X \times 0 \cup Y \times A \times I \to E$ and $F\colon Y \times X \times I \to B$ such that

$$\begin{aligned} f(y,x,0) &= f'(y)(x)\,, \\ f(y,a,t) &= [pr_1 F'(y,t)](a)\,, \\ F(y,x,t) &= [pr_2 F'(y,t)](x)\,. \end{aligned}$$

The diagram

$$\begin{array}{ccc} Y \times X \times 0 \cup Y \times A \times I & \xrightarrow{f} & E \\ \cap & & \downarrow{\scriptstyle p} \\ Y \times X \times I & \xrightarrow[F]{} & B \end{array}$$

is then commutative. By theorem 6 $(Y \times X, Y \times A) = (Y,\emptyset) \times (X,A)$ is a cofibered pair, and since $Y \times A$ is closed in $Y \times X$ theorem 4 of [7] gives us a lifting $\overline{F}\colon Y \times X \times I \to E$ of (f,F). The associate map $\overline{F}'\colon Y \times I \to E^X$ is then a lifting of (f',F').

Now, suppose that i or p is a homotopy equivalence and let $C \subset Z$ be a closed cofibration. Every map pair

(1)
$$\begin{array}{ccc} C & \longrightarrow & E^X \\ \cap & & \downarrow{\scriptstyle \varrho} \\ Z & \longrightarrow & E^A \times {}' B^X \end{array}$$

corresponds to a map pair

(2)
$$\begin{array}{ccc} Z \times A \cup C \times X & \longrightarrow & E \\ \cap & & \downarrow{\scriptstyle p} \\ Z \times X & \longrightarrow & B \end{array}$$

(EXP again), and theorem 6 together with theorem 8 or 9 gives a lifting $Z \times X \to E$ of (2). The associate map $Z \to E^X$ is then a lifting of (1). Consequently ϱ is a homotopy equivalence.

The following theorem is related to theorem 10 in very much the same way as theorem 7 is to theorem 6.

Theorem 11. *Let* (X,A) *be a topological pair and* $p\colon E \to B$ *a map. Suppose that* $\varrho\colon E^X \to E^A \times' B^X$ *is a fibration and that there exist a continuous function* $\varphi\colon X \to I$ *and a point* $x_0 \in X$ *such that* $A \subset \varphi^{-1}(0)$ *and* $\varphi(x_0) \neq 0$. *Then* $p\colon E \to B$ *is a fibration.*

Proof. We may assume $\varphi(x_0)=1$. In order to establish the existence of a lifting of the map pair

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & E \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle p} \\ Y\times I & \xrightarrow[\ F\ ]{} & B \end{array}$$

we define $g\colon Y \to E^X$ and $G\colon Y\times I \to E^A \times' B^X$ by

$$\begin{aligned} g(y)(x) &= f(y)\,, \\ [pr_1 G(y,t)](a) &= f(y)\,, \\ [pr_2 G(y,t)](x) &= F(y, t \wedge \varphi(x))\,. \end{aligned}$$

We thus obtain a map pair $(g,G)\colon i_0 \to \varrho$, and since ϱ is a fibration (g,G) has a lifting $\bar{G}\colon Y\times I \to E^X$. The map $\bar{F}\colon Y\times I \to E$ defined by $\bar{F}(y,t)=\bar{G}(y,t)(x_0)$ is then a lifting of (f,F).

If we put $(X,A)=(I,0)$ it follows that, in the notation used in the proof of theorem 1, $p\colon E\to B$ is a fibration if and only if $\bar{p}\colon E^I \to \bar{B} \approx E^0 \times' B^I$ is a fibration, and then $\bar{p}$ is a homotopy equivalence, which implies that the lifting function $\lambda\colon \bar{B} \to E^I$ for p is unique up to fiber homotopy over $\bar{p}$ (cf. [4]), corresponding to the fact that the retraction $X\times I \to X\times 0 \cup A\times I$ for a cofibration $A\subset X$ is unique up to homotopy relative to $X\times 0\cup A\times I$.

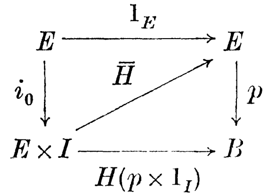
It is well known (and an easy consequence of theorem 10) that if X is locally compact and regular and $p\colon E\to B$ is a fibration, then $p_\#\colon E^X \to B^X$ is also a fibration. Conversely, it follows from theorem 11 (with $A=\emptyset$) that, if X is non-empty and $p_\#\colon E^X\to B^X$ is a fibration, then $p\colon E\to B$ is also a fibration.

4.

Consider the following situation. The pair (B,A) is cofibered, and $p\colon E\to B$ is a map. We denote $p^{-1}(A)$ by $E|A$. In general it need not be true that $(E,E|A)$ is cofibered, but we do have

Theorem 12. *If* (B,A) *is a cofibered pair with* A *closed and* $p\colon E \to B$ *is a fibration, then* $(E,E|A)$ *is a cofibered pair.*

Proof. Let $\varphi\colon B \to I$ and $H\colon B\times I \to B$ be as given by lemma 4. Since p is a fibration there exists a homotopy $\bar{H}\colon E\times I \to E$ making commutative the diagram



Define $\tilde{H}\colon E\times I \to E$ by $\tilde{H}(e,t)=\bar{H}(e,t\wedge\varphi p(e))$. $\tilde{H}$ and φp then satisfy the requirements of lemma 4, which completes the proof.

Finally we prove

Theorem 13. *Suppose that* (B,A) *is a cofibered pair with* A *closed, that* $p\colon E \to B$ *is a fibration, and that there exists a section* s *of* p. *Suppose further that there exist a continuous function* $\psi\colon E \to I$ *such that* $E'=s(B)=\psi^{-1}(0)$ *and a fiber deformation* $D\colon E\times I \to E$ *relative to* E' *such that* $D(\psi^{-1}([0,1\rangle)\times 1)\subset E'$.

Then $(E,E'\cup E|A)$ *is a cofibered pair.*

Proof. As before, let $\varphi\colon B \to I$ and $H\colon B\times I \to B$ be as described in lemma 4. Replacing $D(e,t)$, if necessary, by

$$D'(e,t) = \begin{cases} D(e,t/\psi(e)), & t<\psi(e) \\ D(e,1), & t\geqq\psi(e) \end{cases},$$

it follows that we may assume $D(e,t)\in E'$ whenever $t>\psi(e)$. (E,E') is obviously a cofibered pair, and by theorem 4 of [7] there exists a homotopy $\bar{H}\colon E\times I \to E$ such that $\bar{H}(e,0)=e$, $p\bar{H}(e,t)=H(p(e),t)$, and $\bar{H}(s(b),t)=sH(b,t)$ for $e\in E$, $b\in B$, and $t\in I$. Define $\eta\colon E \to I$ and $G\colon E\times I \to E$ by

$$\begin{aligned} \eta(e) &= \psi(e)\wedge\varphi p(e)\,, \\ G(e,t) &= \bar{H}\big(D(e,t\wedge\varphi p(e)),\, t\wedge\eta(e)\big)\,. \end{aligned}$$

Then $\eta^{-1}(0)=E'\cup E|A$ and $G(e,t)=e$ if $t=0$ or $e\in E'\cup E|A$. If $t>\eta(e)$, then either $\psi(e)\geqq\varphi p(e)$ and

$$\begin{aligned} pG(e,t) &= p\bar{H}\big(D(e,\varphi p(e)),\, \varphi p(e)\big) \\ &= H\big(pD(e,\varphi p(e)),\, \varphi p(e)\big) \\ &= H(p(e),\, \varphi p(e)) \in A\ , \end{aligned}$$

so that $G(e,t) \in E\,|\,A$, or $\varphi p(e) > \psi(e)$, in which case

$$G(e,t) = \bar{H}\big(D(e,\, t \wedge \varphi p(e)),\, \psi(e)\big) \in \bar{H}(E' \times I) = E'\ .$$

Thus, $G(e,t) \in E' \cup E\,|\,A$ whenever $t > \eta(e)$, and by lemma 4 $(E, E' \cup E\,|\,A)$ is cofibered.

REFERENCES

1. A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math. 78 (1963), 223–255.
2. P. Holm, *Microbundles and S-duality*, Acta Math. 118 (1967), 271–296.
3. S.-T. Hu, *Elements of general topology*, San Francisco, 1964.
4. W. Hurewicz, *On the concept of fiber space*, Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 956–961.
5. D. Puppe, *Bemerkungen über die Erweiterung von Homotopien*, Arch. Math. 18 (1967), 81–88.
6. E. H. Spanier, *Algebraic topology*, New York, 1966.
7. A. Strøm, *Note on cofibrations*, Math. Scand. 19 (1966), 11–14.

UNIVERSITY OF OSLO, NORWAY