LICHTENBAUM–QUILLEN FOR TRUNCATED BROWN–PETERSON SPECTRA

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1. Statements of results

Let p be any prime and $n \ge 0$ an integer. Recall from [BM13] that BP is a retract of $MU_{(p)}$ in $\mathbb{E}_4$-rings. Following [HW22, Thm. A], let

$$R := BP\langle n \rangle$$

be an $\mathbb{E}_3$-BP-algebra such that the composite ring homomorphism

$$\mathbb{Z}_{(p)}[v_1, \dots, v_n] \subset BP_* \to R_*$$

is an isomorphism. Its mod p homology is

$$H_*R = \Lambda(\bar\tau_k \mid k \ge n+1) \otimes \mathbb{F}_p[\bar\xi_k \mid k \ge 1] \subset \mathscr{A}_*$$

(with the usual adjustments when $p = 2$). Let C_{p^k} denote the subgroup of $\mathbb{T}$ of order p^k when $0 \le k < \infty$, and $\mathbb{T}$ itself when $k = \infty$.

The topological Hochschild homology spectrum $THH(R)$ is a cyclotomic $\mathbb{E}_2$-$THH(BP)$-algebra, with (p-)cyclotomic structure map

$$\varphi \colon THH(R) \longrightarrow THH(R)^{tC_p},$$

and canonical maps

$$\mathrm{can} \colon THH(R)^{hC_{p^k}} \longrightarrow THH(R)^{tC_{p^k}}$$

for $0 \le k \le \infty$, all compatible with the (residual) $\mathbb{T}$-actions. A Bökstedt spectral sequence argument [AR05, Prop. 5.7] gives an isomorphism

$$H_*THH(R) \cong H_*R \otimes \mathbb{F}_p[\sigma\bar\tau_{n+1}] \otimes \Lambda(\sigma\bar\xi_1, \dots, \sigma\bar\xi_{n+1})$$

of $\mathscr{A}_*$-comodule algebras. Hence

Lemma 1.1.

$$\pi_*(\mathbb{F}_p \otimes_R THH(R)) \cong \mathbb{F}_p[\mu_{n+1}] \otimes \Lambda(\lambda_1, \dots, \lambda_{n+1})$$

with μ_{n+1} in degree $2p^n$ and λ_k in degree $2p^k - 1$ detected by $\sigma\bar\tau_{n+1} - \bar\tau_0 \cdot \sigma\bar\xi_{n+1}$ and $\sigma\bar\xi_k$, respectively.

Theorem 1.2 (Segal conjecture, [HW22, Thm. C, Thm. 4.0.1])**.** *Let U be any type $\ge n+1$ finite p-local spectrum. The cyclotomic structure map $U \otimes \varphi$ is truncated, i.e., induces an isomorphism*

$$U_*\varphi \colon U_*THH(R) \xrightarrow{\cong} U_*THH(R)^{tC_p}$$

in all sufficiently large degrees $* \gg 0$.

Proposition 1.3 ([HW22, Prop. 6.2.1])**.** *There is a finite p-local $\mathbb{E}_1$-ring U with a non-nilpotent central v_{n+1}-element $v \in U_*$ of degree $|v| = (2p^{n+1} - 2)e$, such that*

1. *v has Adams filtration e;*
2. *$U \otimes R$ splits as an R-module as a finite sum of suspensions of $\mathbb{F}_p$;*
3. *the homomorphism $U_*BP \to U_*R$ is surjective.*

Part (1) asks that v has maximal Adams filtration.

In Part (2) we may arrange that one of the summands of $R \to U \otimes R \simeq \bigvee^{?} \Sigma^{?}\mathbb{F}_p$ is the ring map $R \to \mathbb{F}_p$. (Proof: Let $U = F(X,X)$ with $X \otimes R \simeq \bigvee_\alpha \Sigma^{d_\alpha}\mathbb{F}_p$. Unit map from R to $F(X,X)\otimes R \simeq F_R(X\otimes R, X\otimes R) \simeq \bigvee_{\alpha,\beta} \Sigma^{-d_\alpha+d_\beta} F_R(\mathbb{F}_p,\mathbb{F}_p)$ factors through $\tau_{\ge 0}F_R(\mathbb{F}_p,\mathbb{F}_p) = \mathbb{F}_p$ of summands with $\alpha = \beta$.)

Part (3) ensures that the Landweber filtration of $U_*BP \cong BP_*U$ only has suspensions of $BP_*/(p,\dots,v_n)$ in its associated graded.

The following consequence of the Hopkins–Smith nilpotence theorem was explained to me by Jeremy Hahn.

Lemma 1.4. *We may assume that the images of $v \in U_*$ and $v_{n+1}^e \in BP_*$ in U_*BP are equal.*

The cofiber U/v is a type $n+2$ finite p-local spectrum.

Theorem 1.5 (Canonical vanishing, [HW22, Thm. D, Thm. 6.3.1])**.** *There are U and v as above, and an integer d, such that for each $0 \le k \le \infty$ the canonical homomorphism*

$$(U/v)_*\mathrm{can}\colon (U/v)_*THH(R)^{hC_{p^k}} \xrightarrow{0} (U/v)_*THH(R)^{tC_{p^k}}$$

is zero whenever $* \ge d$.

The Segal conjecture and canonical vanishing together imply cyclotomic boundedness.

Corollary 1.6 (Bounded TR, [HW22, Thm. G, Thm. 3.3.2(f)])**.** *For each type $n+2$ finite p-local spectrum V, the graded abelian group $V_*TR(R)$ is bounded.*

This conclusion is equivalent to saying that $V \otimes THH(R)$ is bounded in the cyclotomic t-structure, by [AN21, Thm. 9].

The relative topological Hochschild homology

$$THH(R/BP) = R \otimes_{R\otimes_{BP} R^{op}} R$$

is an $\mathbb{E}_2$-BP-algebra with $\mathbb{T}$-action, with homotopy fixed points $TC^-(R/BP) = THH(R/BP)^{h\mathbb{T}}$. Letting v_{n+1} be the lowest-degree generator of

$$(v_{n+1}, v_{n+2}, \dots) = \ker(BP_* \to R_*)\,,$$

its suspension $\sigma v_{n+1} \in [v_{n+1}]$ is the lowest-degree generator of

$$\ker(\pi_*(R \otimes_{BP} R^{op}) \to R_*)\,,$$

and its double suspension $\sigma^2 v_{n+1} \in [\sigma v_{n+1}]$ is the lowest-degree generator of

$$\ker(\pi_* THH(R/BP) \to R_*)\,.$$

Theorem 1.7 (Polynomial THH, [HW22, Thm. E, Thm. 2.5.4])**.** *There is an isomorphism of even R_*-algebras*

$$\pi_*THH(R/BP) \cong R_*[\gamma_{p^i}\sigma^2 v_{n+1} \mid i \ge 0]\,,$$

with lowest-degree generator $\sigma^2 v_{n+1}$ in degree $2p^{n+1}$.

Theorem 1.8 (Detection, [HW22, Thm. F, Thm. 5.0.1])**.** *There is an isomorphism of even R_*-algebras*

$$\pi_*TC^-(R/BP) \cong \pi_*THH(R/BP)\,[[t]]$$

with $|t| = -2$. The unit map $\iota\colon BP \to TC^-(R/BP)$ takes v_{n+1} to $t \cdot \sigma^2 v_{n+1}$.

The $\mathbb{E}_2$-ring maps

$$TC(R) \xrightarrow{\pi} TC^-(R) \longrightarrow TC^-(R/BP)$$

lead to the following variant of [HW22, Thm. B], where we may assume $T(n+1) = v^{-1}U$.

Corollary 1.9. *Multiplication by v acts non-nilpotently on $U_*TC^-(R/BP)$, so $T(n+1)_*TC^-(R/BP) \neq 0$ and $T(n+1)_*TC(R) \neq 0$.*

2. Proof of Segal conjecture for $THH(R)$

Proof of Theorem 1.2 (= Thm. C). By Proposition 1.3(2) it suffices to prove that

$$\mathbb{F}_p \otimes_R \varphi \colon THH(R)/(p, v_1, \dots, v_n) \longrightarrow THH(R)^{tC_p}/(p, v_1, \dots, v_n)$$

is truncated. We exhaustively filter $R_p^\wedge$ by the sequence $\mathrm{fil}^\star R$ of spectra

$$\mathrm{fil}^w R = \lim_{[q]\in\Delta} \tau_{\geq w}(\overbrace{\mathbb{F}_p \otimes \cdots \otimes \mathbb{F}_p}^{1+q} \otimes R)$$

for (double-)weights $w \geq 0$, with associated graded $\mathrm{gr}^* R$ given by the cofiber sequences

$$\mathrm{fil}^{w+1} R \longrightarrow \mathrm{fil}^w R \longrightarrow \mathrm{gr}^w R\,.$$

(In the words of [Pst23], we form the $\mathbb{F}_p$-synthetic analogue.) The filtration is conditionally convergent, in the sense that $\lim_w \mathrm{fil}^w R = 0$. The associated spectral sequence

$$\pi_* \mathrm{gr}^* R \Longrightarrow \pi_* R_p^\wedge$$

has starting page equal to the classical Adams E_2-page

$${}^{Ad}E_2^{*,*} = \mathrm{Ext}_{\mathscr{A}_*}(\mathbb{F}_p, H_*R) \cong \mathbb{F}_p[v_0, v_1, \dots, v_n]\,,$$

with v_k in (even) stem $2p^k - 2$ and weight $w = 2p^k - 1$, and collapses at this stage.

Looping the inclusion $i_1 \colon BU(1) \to BU$ twice gives an $\mathbb{E}_2$-map $\Omega^2 i_1 \colon \mathbb{Z} \to \mathbb{Z} \times BU$. For $m \in \mathbb{Z}$ consider the composite

$$\eta_m \colon \mathbb{Z}_{\geq 0} \xrightarrow{m\cdot} \mathbb{Z} \xrightarrow{\Omega^2 i_1} \mathbb{Z} \times BU\,.$$

Here $\mathbb{Z}_{\geq 0}$ admits a CW $\mathbb{E}_2$-space structure, with one $\mathbb{E}_2$-cell in each non-negative even dimension. This can be deduced along the lines of [GKRW], as shown to me by Oscar Randal-Williams. The associated Thom $\mathbb{E}_2$-ring

$$\mathbb{S}[y_{2m}] := \mathrm{Th}(\eta_m) \simeq \bigvee_{j\geq 0} S^{2mj}$$

inherits a CW $\mathbb{E}_2$-ring structure of the same kind, with bottom $\mathbb{E}_2$-cell the free $\mathbb{E}_2$-ring on S^{2m}. We can view this as an $\mathbb{E}_2$-algebra $\mathbb{S}[y_{2m}^w]$ in graded spectra, placing the summand S^{2mj} in weight wj.

Proposition 2.1 ([HW22, Prop. 4.2.1])**.**

$$\mathbb{F}_p \otimes \mathbb{S}[a_0] \otimes \mathbb{S}[a_1] \otimes \cdots \otimes \mathbb{S}[a_n] \xrightarrow{\simeq} \mathrm{gr}^* R$$

as graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebras, with $a_k = y_{2p^k-2}^{2p^k-1}$ for $0 \leq k \leq n$.

Proof. The two sides have bigraded homotopy rings that are isomorphic and of finite type. The left-hand side is a CW graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebra, containing the free algebra on $S^0 \vee S^{2p-2} \vee \cdots \vee S^{2p^n-2}$ as a subcomplex, with remaining $\mathbb{E}_2$-cells only in even dimensions. We first send S^{2p^k-2} in weight $2p^k - 1$ within the bottom $\mathbb{E}_2$-cell of $\mathbb{S}[a_k]$ to v_k, for each $0 \leq k \leq n$. Since $\pi_* \mathrm{gr}^* R$ is even, there is no obstruction to extending this over the remaining $\mathbb{E}_2$-cells. The resulting $\mathbb{E}_2$-map is surjective on π_*, hence is an equivalence. $\square$

Proposition 2.2 ([HW22, Prop. 4.2.2]). *The graded cyclotomic structure map*

$$\varphi\colon \mathrm{gr}^* THH(R) \longrightarrow \mathrm{gr}^{p*} THH(R)^{tC_p}$$

induces the localization homomorphism

$$\mathbb{F}_p[\mu_0, v_0, v_1, \dots, v_n] \otimes \Lambda(\sigma v_0, \sigma v_1, \dots, \sigma v_n) \\ \longrightarrow \mathbb{F}_p[\mu_0^{\pm 1}, v_0, v_1, \dots, v_n] \otimes \Lambda(\sigma v_0, \sigma v_1, \dots, \sigma v_n)$$

in homotopy. Here $|\mu_0| = 2$ *in weight* 0*, while* σv_k *has degree and weight* $2p^k - 1$*.*

Corollary 2.3.

$$\mathbb{F}_p \otimes_{\mathrm{gr}^* R} \varphi\colon \mathrm{gr}^* THH(R)/(v_0, v_1, \dots, v_n) \longrightarrow \mathrm{gr}^{p*} THH(R)^{tC_p}/(v_0, v_1, \dots, v_n)$$

induces

$$\mathbb{F}_p[\mu_0] \otimes \Lambda(\sigma v_0, \sigma v_1, \dots, \sigma v_n) \longrightarrow \mathbb{F}_p[\mu_0^{\pm 1}] \otimes \Lambda(\sigma v_0, \sigma v_1, \dots, \sigma v_n)\,,$$

which is truncated.

It follows that $\mathbb{F}_p \otimes_R \varphi$ is also truncated, proving Theorem 1.2 (= Thm. C). □

Proof of Proposition 2.2. The convolution product of filtrations gives a conditionally convergent filtration $\mathrm{fil}^\star THH(R)$, with associated graded cyclotomic $\mathbb{E}_1$-ring

$$\begin{aligned} \mathrm{gr}^* THH(R) &\simeq THH(\mathrm{gr}^* R) \\ &\simeq THH(\mathbb{F}_p \otimes \mathbb{S}[a_0] \otimes \cdots \otimes \mathbb{S}[a_n]) \\ &\simeq THH(\mathbb{F}_p) \otimes THH(\mathbb{S}[a_0]) \otimes \cdots \otimes THH(\mathbb{S}[a_n])\,. \end{aligned}$$

By [HM97], φ for $\mathbb{F}_p$ induces the localization homomorphism

$$\pi_* THH(\mathbb{F}_p) \cong \mathbb{F}_p[\mu_0] \longrightarrow \mathbb{F}_p[\mu_0^{\pm 1}] \cong \pi_* THH(\mathbb{F}_p)^{tC_p}\,,$$

with $|\mu_0| = 2$, all in weight 0.

We claim that φ for each $\mathbb{S}[a_k]$ is a p-equivalence. A collapsing Bökstedt spectral sequence shows that

$$H_* THH(\mathbb{S}[y_{2m}^w]) \cong HH_*(\mathbb{F}_p[y_{2m}^w]) \cong \mathbb{F}_p[y_{2m}^w] \otimes \Lambda(\sigma y_{2m}^w)$$

as a bigraded $\mathbb{F}_p$-algebra, with σy_{2m}^w in degree $2m+1$ and weight w. Moreover, as in [Rog09], the cyclic bar construction on $\mathbb{Z}_{\ge 0}$ decomposes as

$$B^{cy}(\mathbb{Z}_{\ge 0}) \simeq \{0\} \sqcup \coprod_{j>0} \mathbb{T}/C_j\,,$$

which Thomifies to a splitting

$$THH(\mathbb{S}[y_{2m}^w]) \simeq \mathbb{S} \vee \bigvee_{j>0} \mathbb{T}_+ \wedge_{C_j} (S^{2m})^{\otimes j}$$

with the j-th summand in weight wj. Here C_j acts by cyclic permutations on $(S^{2m})^{\otimes j}$, and $\mathbb{T}_+ \wedge_{C_j} (S^{2m})^{\otimes j}$ is a finite C_p-spectrum. The graded cyclotomic structure map

$$\varphi\colon THH(\mathbb{S}[y_{2m}^w]) \longrightarrow THH(\mathbb{S}[y_{2m}^w])^{tC_p}$$

multiplies weights by p. It is the sum of $\varphi^0\colon \mathbb{S} \to \mathbb{S}^{tC_p}$ and

$$\varphi^{wj}\colon \mathbb{T}_+ \wedge_{C_j} (S^{2m})^{\otimes j} \longrightarrow (\mathbb{T}_+ \wedge_{C_{pj}} (S^{2m})^{\otimes pj})^{tC_p}$$

for $j > 0$, all of which are p-equivalences by the classical Segal conjecture (proved by Lin and Gunawardena in these cases). The remaining target terms

$$(\mathbb{T}_+ \wedge_{C_k} (S^{2m})^{\otimes k})^{tC_p}\,,$$

with $p \nmid k$, are all trivial, since C_p acts freely.

It follows that

$$\begin{aligned} &THH(\mathbb{F}_p) \otimes THH(\mathbb{S}[a_0]) \otimes \cdots \otimes THH(\mathbb{S}[a_n]) \\ &\qquad \xrightarrow{\varphi\otimes\varphi\otimes\cdots\otimes\varphi} THH(\mathbb{F}_p)^{tC_p} \otimes THH(\mathbb{S}[a_0])^{tC_p} \otimes \cdots \otimes THH(\mathbb{S}[a_n])^{tC_p} \\ &\qquad\qquad \xrightarrow{\lambda} (THH(\mathbb{F}_p) \otimes THH(\mathbb{S}[a_0]) \otimes \cdots \otimes THH(\mathbb{S}[a_n]))^{tC_p} \end{aligned}$$

induces the asserted localization homomorphism in homotopy. (The C_p-equivariant finiteness of each $\mathbb{T}_+ \wedge_{C_j} (S^{2m})^{\otimes j}$ ensures that the C_p-Tate lax structure map λ is an equivalence.) □

3. Proof of canonical vanishing for $THH(R)$

Proof of Theorem 1.5 (= Thm. D). Contemplate

$$\begin{array}{ccccc} U & \longrightarrow & U \otimes BP & \longleftarrow & BP \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\ U \otimes TC^-(R) & \longrightarrow & U \otimes TC^-(R/BP) & \longleftarrow & TC^-(R/BP) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ U \otimes THH(R) & \xrightarrow{\alpha} & U \otimes THH(R/BP) & \longleftarrow & THH(R/BP) \\ \uparrow & & \uparrow & \swarrow & \\ \mathbb{F}_p \otimes_R THH(R) & \longrightarrow & \mathbb{F}_p \otimes_R THH(R/BP) & & \end{array}$$

in homotopy. Maps to the right are induced by $\mathbb{S} \to BP$. Maps to the left are induced by $\mathbb{S} \to U$. The maps $F\colon TC^- \to THH$ forget $\mathbb{T}$-invariance. The lower maps are induced by the R-algebra maps $R \to \mathbb{F}_p \to U \otimes R$.

The classes $v \in U_*$ and $v_{n+1}^e \in BP_*$ have the same image in U_*BP, by Lemma 1.4.

The $\mathbb{T}$-homotopy fixed point spectral sequence for $\pi_*TC^-(R/BP)$ collapses at the E_2-page, by the evenness in Theorem 1.7 (= Thm. E), and $\iota(v_{n+1})$ is detected by $t \cdot \sigma^2 v_{n+1}$, in filtration 2. (Proof: By Adams spectral sequence for $F(S^3_+, THH(R/BP))^{\mathbb{T}}$ as in [AR02, Prop. 4.8], or by [HW22, Lem. A.4.1].)

The $\mathbb{T}$-homotopy fixed point spectral sequence for $U_*TC^-(R/BP)$ collapses at the E_2-page, by [HW22, Lem. 6.3.4], using Proposition 1.3(3), so $\iota(v_{n+1}^e)$ and $\iota(v) \in U_*TC^-(R)$ map to a class detected by $t^e \cdot (\sigma^2 v_{n+1})^e$, in filtration $2e$.

Since $U \otimes THH(R)$ is a $U \otimes R$-module, it is also an $\mathbb{F}_p$-module. Hence the associated graded of the $\mathbb{T}$-homotopy fixed point filtration of $U \otimes TC^-(R)$ consists of $\mathbb{F}_p$-modules, and is trivial in odd gradings. The Adams filtration of v is e, so $\iota(v)$ must be detected in filtration $\geq 2e$ in the $\mathbb{T}$-homotopy fixed point spectral sequence for $U_*TC^-(R)$.

Combining the last two paragraphs, we see that $\iota(v)$ must be detected by the E_∞-class of an infinite cycle $t^e \cdot z$, for some $z \in U_*THH(R)$ that maps by the homomorphism labeled α to $(\sigma^2 v_{n+1})^e \in U_*THH(R/BP)$.

Recalling Lemma 1.1, the bottom horizontal arrow induces

$$\mathbb{F}_p[\mu_{n+1}] \otimes \Lambda(\lambda_1, \dots, \lambda_{n+1}) \longrightarrow \mathbb{F}_p[\gamma_{p^i}\sigma^2 v_{n+1} \mid i \geq 0]$$

with $\mu_{n+1} \mapsto \sigma^2 v_{n+1}$.

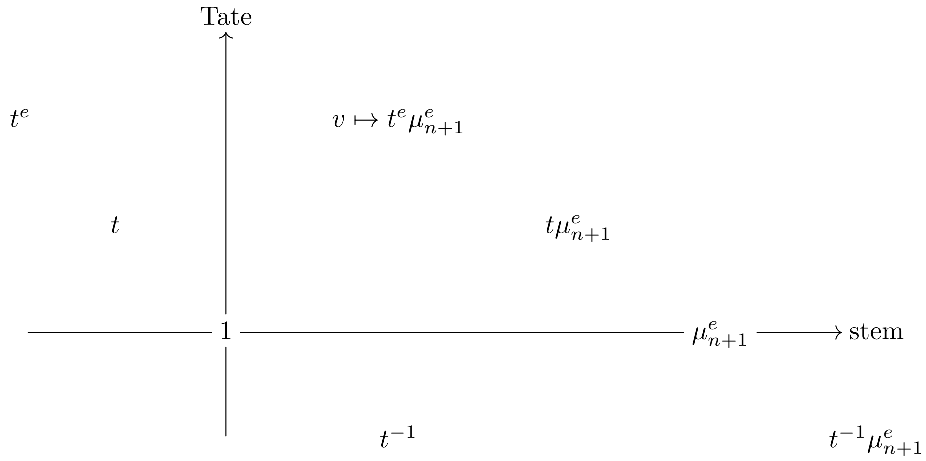
Hence $\mu_{n+1}^e \in U_*THH(R)$ is also a class that maps by α to $(\sigma^2 v_{n+1})^e$. By [HW22, Prop. 6.1.1] (see below) the kernel of α is nilpotent. It follows that by replacing v with some power of itself we may arrange that $z = \mu_{n+1}^e$. Then $\iota(v)$ is detected by $t^e \cdot \mu_{n+1}^e$.

Since $U \otimes THH(R) \simeq (U \otimes R) \otimes_R THH(R)$ is a finite sum of suspensions of $\mathbb{F}_p \otimes_R THH(R)$, it follows that $U_*THH(R)$ is finitely generated and free as a

$\mathbb{F}_p[\mu_{n+1}]$-module, hence also as a $\mathbb{F}_p[\mu_{n+1}^e]$-module. Thus the C_{p^k}-Tate spectral sequence E_2-page

$$\hat{E}_2 = \hat{H}^*(C_{p^k}, U_*THH(R)) \Longrightarrow U_*THH(R)^{tC_{p^k}}$$

is finitely generated and free over $\mathbb{F}_p[t^{\pm 1}, \mu_{n+1}^e] = \mathbb{F}_p[t^{\pm 1}, t^e \cdot \mu_{n+1}^e]$, uniformly in $0 \leq k \leq \infty$.



Multiplication by v defines a filtration-shifting self-map, and passing to cofibers gives a (hastened) C_{p^k}-Tate spectral sequence

$$\hat{E}_2 = \hat{H}^*(C_{p^k}, U_*THH(R))/(t^e \cdot \mu_{n+1}^e) \Longrightarrow (U/v)_*THH(R)^{tC_{p^k}}$$

with an E_2-page that is finitely generated and free over $\mathbb{F}_p[t^{\pm 1}]$. Here t has (stem, filtration) equal to $(-2, 2)$, so in all sufficiently large stems $* \geq d$ the hastened C_{p^k}-Tate E_2-page (and E_∞-page) is concentrated in negative filtrations.

On the other hand, the (hastened) C_{p^k}-homotopy fixed point spectral sequence

$$E_2 = H^*(C_{p^k}, U_*THH(R))/(t^e \cdot \mu_{n+1}^e) \Longrightarrow (U/v)_*THH(R)^{hC_{p^k}}$$

is concentrated in non-negative filtrations, so the canonical map must induce the zero homomorphism $(U/v)_*\mathrm{can} = 0$ in stems $* \geq d$. □

Proposition 3.1 ([HW22, Prop. 6.1.1]). *For any type $n+1$ finite p-local spectrum U, the descent spectral sequence computing $U_*THH(R)$ by descent along $THH(R) \to THH(R/BP)$ collapses at a finite E_r-page, with a horizontal vanishing line. Hence, if U is an $\mathbb{E}_1$-ring, the kernel of*

$$\alpha \colon U_*THH(R) \longrightarrow U_*THH(R/BP)$$

is nilpotent.

Using [HPS99] and a thick subcategory argument in R-modules, this follows from the next result.

Proposition 3.2 ([HW22, Prop. 6.1.6]). *The descent spectral sequence for*

$$\mathbb{F}_p \otimes_R THH(R) \longrightarrow \mathbb{F}_p \otimes_R THH(R/BP)$$

collapses at the E_2-page, with λ_k in filtration 1 for each $1 \leq k \leq n+1$, and μ_{n+1} mapping to $\sigma^2 v_{n+1}$ in filtration 0.

The proof involves recognizing the descent E_1-term as the cobar complex of a flat Hopf algebroid, and showing that the E_2-term is of finite type and of the same size as the known abutment.

4. Proof of polynomial THH for R

Proof of Theorem 1.7 (= Thm. E). Recall that

$$THH(R/BP) = R \otimes_{R \otimes_{BP} R^{op}} R\,.$$

The bar spectral sequence

$$\begin{aligned}\mathrm{Tor}^{BP_*}(R_*, R_*^{op}) &= R_* \otimes \Lambda(\sigma v_k \mid k \geq n+1) \\ &\Longrightarrow \pi_*(R \otimes_{BP} R^{op})\end{aligned}$$

collapses. The bar spectral sequence

$$\begin{aligned}\mathrm{Tor}^{\pi_*(R \otimes_{BP} R^{op})}(R_*, R_*) &= R_* \otimes \Gamma(\sigma^2 v_k \mid k \geq n+1) \\ &\Longrightarrow \pi_* THH(R/BP)\end{aligned}$$

also collapses, but has multiplicative extensions

$$(\gamma_{p^i} \sigma^2 v_k)^p \equiv \gamma_{p^i} \sigma^2 v_{k+1}$$

for $k \geq n+1$, so that

$$\pi_* THH(R/BP) \cong R_*[\gamma_{p^i} \sigma^2 v_{n+1} \mid i \geq 0]\,.$$

These multiplicative extensions are established using naturality along $R \to \mathbb{F}_p$ and the calculation of Dyer–Lashof operations in $\mathscr{A}_*$ due to (Kristensen and) Steinberger. □

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