

# HOMOTOPY COHERENT ADJUNCTIONS AND THE FORMAL THEORY OF MONADS

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**ABSTRACT.** In this paper, we introduce a cofibrant simplicial category that we call the free homotopy coherent adjunction and characterise its  $n$ -arrows using a graphical calculus that we develop here. The hom-spaces are appropriately fibrant, indeed are nerves of categories, which indicates that all of the expected coherence equations in each dimension are present. To justify our terminology, we prove that any adjunction of quasi-categories extends to a homotopy coherent adjunction and furthermore that these extensions are homotopically unique in the sense that the relevant spaces of extensions are contractible Kan complexes.

We extract several simplicial functors from the free homotopy coherent adjunction and show that quasi-categories are closed under weighted limits with these weights. These weighted limits are used to define the homotopy coherent monadic adjunction associated to a homotopy coherent monad. We show that each vertex in the quasi-category of algebras for a homotopy coherent monad is a codescent object of a canonical diagram of free algebras. To conclude, we prove the quasi-categorical monadicity theorem, describing conditions under which the canonical comparison functor from a homotopy coherent adjunction to the associated monadic adjunction is an equivalence of quasi-categories. Our proofs reveal that a mild variant of Beck’s argument is “all in the weights”—much of it independent of the quasi-categorical context.

## CONTENTS

1. Introduction	2
1.1. Adjunction data	2
1.2. The free homotopy coherent adjunction	4
1.3. Weighted limits and the formal theory of monads	5
1.4. Acknowledgments	7
2. Simplicial computads	7
2.1. Simplicial categories and simplicial computads	7
2.2. Simplicial subcomputads	10
3. The generic adjunction	12
3.1. A graphical calculus for the simplicial category $\underline{\text{Adj}}$	12
3.2. The simplicial category $\underline{\text{Adj}}$ as a 2-category	21
3.3. The 2-categorical universal property of $\underline{\text{Adj}}$	23

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4.	Adjunction data	28
4.1.	Fillable arrows	28
4.2.	Parental subcomputads	30
4.3.	Homotopy Coherent Adjunctions	36
4.4.	Homotopical uniqueness of homotopy coherent adjunctions	42
5.	Weighted limits in $\mathbf{qCat}_\infty$	50
5.1.	Weighted limits and colimits	50
5.2.	Weighted limits in the quasi-categorical context	53
5.3.	The collage construction	55
6.	The formal theory of homotopy coherent monads	58
6.1.	Weighted limits for the formal theory of monads.	59
6.2.	Conservativity of the monadic forgetful functor	63
6.3.	Colimit representation of algebras	65
7.	Monadicity	71
7.1.	Comparison with the monadic adjunction	72
7.2.	The monadicity theorem	74
	References	78

## 1. INTRODUCTION

Quasi-categories, introduced by Boardman and Vogt [3], are now recognised as a convenient model for  $(\infty, 1)$ -categories, i.e., categories weakly enriched over spaces. The basic category theory of quasi-categories has been developed by Joyal [11, 13, 12], Lurie [18, 19] (under the name  $\infty$ -categories), ourselves [25], and others. Ordinary category theory can be understood to be a special case: categories form a full subcategory of quasi-categories and this full inclusion respects all (quasi-)categorical definitions. As a consequence, we find it productive to identify a category with its nerve, the corresponding quasi-category.

This paper is a continuation of [25], references to which will have the form I.x.x.x, which develops the category theory of quasi-categories using 2-categorical techniques applied to the (strict) 2-category of quasi-categories  $\mathbf{qCat}_2$ . First studied by Joyal [12],  $\mathbf{qCat}_2$  can be understood to be a quotient of the simplicially enriched category  $\mathbf{qCat}_\infty$  of quasi-categories;  $\mathbf{qCat}_2$  is defined by replacing each hom-space of  $\mathbf{qCat}_\infty$  by its homotopy category. While our discussion focuses on quasi-categories, as was the case in [25] our proofs generalise without change to other similar “ $\infty$ -cosmoi”, subcategories of fibrant objects in model categories enriched over the Joyal model structure in which all fibrant objects are cofibrant. Examples include complete Segal objects (“Rezk objects”) in model categories permitting Bousfield localisation with sufficiently many cofibrant objects. This perspective will be explored more fully in the forthcoming [27].

**1.1. Adjunction data.** In [25], we develop the theory of adjunctions between quasi-categories, defined to be adjunctions in  $\mathbf{qCat}_2$ . Examples include adjunctions between ordinary, topological, or locally Kan simplicial categories; simplicial Quillen adjunctions;

and adjunctions constructed directly on the quasi-categorical level from the existence of appropriate limits or colimits. Explicitly, the data of an adjunction

$$A \begin{array}{c} \xleftarrow{f} \\ \perp \\ \xrightarrow{u} \end{array} B \quad \eta: \text{id}_B \Rightarrow uf \quad \epsilon: fu \Rightarrow \text{id}_A$$

consists of two quasi-categories  $A, B$ ; two functors  $f, u$  (maps of simplicial sets between quasi-categories); and two natural transformations  $\eta, \epsilon$  represented by simplicial maps

$$\begin{array}{ccc} \begin{array}{ccc} B & & \\ \downarrow i_0 & \searrow & \\ B \times \Delta^1 & \xrightarrow{\eta} & B \\ \uparrow i_1 & & \uparrow u \\ B & \xrightarrow{f} & A \end{array} & \text{and} & \begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow i_0 & & \downarrow f \\ A \times \Delta^1 & \xrightarrow{\epsilon} & A \\ \uparrow i_1 & \swarrow & \\ A & & \end{array} \end{array}$$

Because the hom-spaces  $B^A$  and  $A^B$  are quasi-categories, for any choices of 1-simplices representing the unit and counit, there exist 2-simplices

$$A \times \Delta^2 \xrightarrow{\alpha} B \quad \text{and} \quad B \times \Delta^2 \xrightarrow{\beta} A$$

which witness the triangle identities in the sense that their boundaries have the form

$$\begin{array}{ccc} & ufu & \\ \eta u \nearrow & & \searrow u\epsilon \\ u & \xrightarrow{\text{id}_u} & u \end{array} \quad \begin{array}{ccc} & fuf & \\ f\eta \nearrow & & \searrow \epsilon f \\ f & \xrightarrow{\text{id}_f} & f \end{array} \quad (1.1.1)$$

This elementary definition of an adjunction between quasi-categories has a very different form from the definition given by Lurie in [18], but they are equivalent (see I.4.4.5 for one implication and [27] for the converse).

This 0-, 1-, and 2-dimensional data suffices to establish that the functors  $f$  and  $u$  form an adjunction of quasi-categories, but higher dimensional adjunction data certainly exists. For example, the 1-simplices  $fu\epsilon$  and  $\epsilon$  in  $A^A$  can be composed, defining a 1-simplex we might call  $\epsilon\epsilon$ . The 2-simplex witnessing this composition, the 2-simplex  $\epsilon \cdot \sigma^0$ , and the 2-simplex  $f\alpha$  combine to form a  $(3, 2)$ -horn in  $A^A$

$$\begin{array}{ccc} & fufu & \\ f\eta u \nearrow & \downarrow fu\epsilon & \searrow \epsilon\epsilon \\ fu & \xrightarrow{\epsilon} & \text{id}_A \\ \swarrow & & \nearrow \epsilon \\ & fu & \end{array} \quad (1.1.2)$$

which may be filled to define a 3-simplex  $\omega$  and 2-simplex  $\mu$  witnessing that  $f\eta u$  composed with  $\epsilon\epsilon$  is  $\epsilon$ .

An analogous construction replaces  $fue$  with  $\epsilon fu$  and  $f\alpha$  with  $\beta u$ . On account of the commutative diagram in the homotopy category  $h(A^A)$

$$\begin{array}{ccc} fu fu & \xrightarrow{fue} & fu \\ \epsilon fu \downarrow & \searrow^{\epsilon\epsilon} & \downarrow \epsilon \\ fu & \xrightarrow{\epsilon} & id_A \end{array} \quad (1.1.3)$$

we may choose the same 1-simplex  $\epsilon\epsilon$  as the composite of  $\epsilon fu$  with  $\epsilon$ . Filling the  $(3, 2)$ -horn in  $A^A$

$$\begin{array}{ccc} & fu fu & \\ f\eta u \nearrow & \downarrow \epsilon fu & \searrow \epsilon\epsilon \\ fu & \xrightarrow{\epsilon} & id_A \\ \parallel & \downarrow \epsilon & \nearrow \epsilon \\ & fu & \end{array} \quad (1.1.4)$$

produces a 3-simplex  $\tau$  together with another 2-simplex witnessing that  $\epsilon = \epsilon\epsilon \cdot f\eta u$ . But it is not immediately clear whether  $\omega$  and  $\tau$  may be chosen compatibly, i.e., with common 2<sup>nd</sup> face.

As a consequence of our first main theorem, we will see that the answer is yes and, furthermore, compatible choices always exist “all the way up”. To state this result we require a new definition. To that end recall that in [28], Schanuel and Street introduce the free adjunction: a strict 2-category  $\underline{\text{Adj}}$  which has the universal property that 2-functors  $\underline{\text{Adj}} \rightarrow \mathcal{K}$  stand in bijective correspondence to adjunctions in the 2-category  $\mathcal{K}$ . In honour of this result, their 2-category  $\underline{\text{Adj}}$  is called the *free adjunction*.

Inspired by their pioneering insight, we define a simplicial category, which we also call  $\underline{\text{Adj}}$ , for which we prove the following result:

**4.3.9, 4.3.11. Theorem.** *Any adjunction of quasi-categories extends to a homotopy coherent adjunction: any choice of low-dimensional adjunction data for an adjunction between quasi-categories can be extended to a simplicial functor  $\underline{\text{Adj}} \rightarrow \mathbf{qCat}_\infty$ .*

We then show, in theorems 4.4.11 and 4.4.18, that suitably defined spaces of all such extensions are contractible. These existence and homotopy uniqueness results provide us with the appropriate homotopy theoretic generalisation of the Schanuel-Street result to the quasi-categorical context. Consequently, we feel justified in calling our simplicial category  $\underline{\text{Adj}}$  the *free homotopy coherent adjunction*.

**1.2. The free homotopy coherent adjunction.** We can say more about the relationship between our simplicial category  $\underline{\text{Adj}}$  and the Schanuel-Street free adjunction. Indeed it is a somewhat unexpected and perhaps a little remarkable fact that these two are actually one

and the same gadget. More precisely, if we look upon the Schanuel-Street free adjunction as a simplicial category, by applying the fully faithful nerve functor to each of its hom-categories, then it is isomorphic to our free homotopy coherent adjunction. This result, which appears here as corollary 3.3.5, explains our adoption of the common notation  $\underline{\text{Adj}}$  to name both of these structures.

Now observe that, as a 2-category, the hom-spaces of our simplicial category  $\underline{\text{Adj}}$  are all quasi-categories, a “fibrancy” condition that indicates all possible composites of coherence data are present in  $\underline{\text{Adj}}$  and thus picked out by a simplicial functor with this domain. But of course, these hom-spaces, as nerves of categories, have *unique* fillers for all inner horns, which says furthermore that this coherence data is “minimally chosen” or “maximally coherent” in some sense.

What is unexpected from this definition, and yet essential in order to prove the “freeness” of the homotopy coherent adjunction, is that  $\underline{\text{Adj}}$  is also cofibrant in the sense of being a cofibrant object in the Bergner model structure on simplicial categories [1]. The cofibrant simplicial categories are exactly the *simplicial computads*, which we describe in section 2. This notion has antecedents in the computads of Street [30] and the description of the cofibrant objects in the model structure described by Dwyer and Kan on the category simplicial categories with fixed object set [7].

In section 3, we prove that  $\underline{\text{Adj}}$  is a simplicial computad by presenting an explicit simplicial subcomputad filtration that is then employed in the proof that any quasi-categorical adjunction underlies a homotopy coherent adjunction. Our approach is somewhat roundabout. We defined  $\underline{\text{Adj}}$  first as a simplicial category by introducing a graphical calculus for its  $n$ -arrows. This graphical calculus seamlessly encodes all the necessary simplicial structure, while highlighting a set of *atomic* arrows that freely generate the arrows in each dimension under horizontal composition. We then prove that this simplicial category is isomorphic to the 2-category  $\underline{\text{Adj}}$  under the embedding  $2\text{-Cat} \hookrightarrow \text{sSet-Cat}$ .

In section 4, we prove that any adjunction of quasi-categories extends to a homotopy coherent adjunction and moreover that homotopy coherent adjunctions  $\underline{\text{Adj}} \rightarrow \text{qCat}_\infty$  extending a given adjunction of quasi-categories are “homotopically unique”. In fact, as our use of the subcomputad filtration of  $\underline{\text{Adj}}$  makes clear, there are many extension theorems, distinguished by what we take to be the initial data of the adjunction of quasi-categories. We define spaces of extensions from a single left adjoint functor; from choices of both adjoints and a representative for the counit; from choices of both adjoints, representatives for the unit and counit, and a representative for one of the triangle identities; and so on, proving that each of these defines a contractible Kan complex.

**1.3. Weighted limits and the formal theory of monads.** The 2-category  $\underline{\text{Adj}}$  has two objects, which we denote “+” and “−”; their images specify the objects spanned by the adjunction. The hom-category  $\underline{\text{Adj}}(+, +)$  is  $\Delta_+$ —the “algebraist’s delta”—the category of finite ordinals and order-preserving maps. Ordinal sum makes  $\Delta_+$  a strict monoidal category; indeed, it is the free strict monoidal category containing a monoid. Hence, a 2-functor whose domain is the one-object 2-category with hom-category  $\Delta_+$  is exactly a monad in the target 2-category. The hom-category  $\underline{\text{Adj}}(-, -)$  is  $\Delta_+^{\text{op}}$ . In this way, the

restrictions of the free adjunction  $\underline{\text{Adj}}$  to the subcategories spanned by one endpoint or the other define the free monad and the free comonad.

Restrictions of a homotopy coherent adjunction to the subcategories  $\underline{\text{Mnd}}$  and  $\underline{\text{Cmd}}$  spanned by  $+$  and  $-$  respectively define a *homotopy coherent monad* and a *homotopy coherent comonad*. Unlike the case for adjunctions, (co)monads in  $\underline{\text{qCat}}_2$  are not automatically homotopy coherent; a monad is an algebraically-defined structure whereas an adjunction encodes a universal property. However, as a corollary of our extension theorem, any monad arising from an adjunction extends to a homotopy coherent monad, a simplicial functor with domain  $\underline{\text{Mnd}} \hookrightarrow \underline{\text{Adj}}$ . In the body of this paper, except when discussing monads, we frequently omit the appellation “homotopy coherent” because in the other settings this interpretation is automatic: categories regarded as quasi-categories via their nerves automatically define homotopy coherent diagrams.

In the second half of this paper, we present a “formal” re-proof of the quasi-categorical monadicity theorem that also illuminates the classical categorical argument. The starting insight is a characterisation of the quasi-category of algebras for a homotopy coherent monad as a weighted limit. In this context, a *weight* is a functor describing the “shape” of a generalised cone over a diagram indexed by a fixed small category. An object representing the set of cones described by a particular weight is called a *weighted limit*. The use of weighted limits can provide a useful conceptual simplification because calculations involving the weights reveal the reason why these results are true; cf. the expository paper [24].

To make use of weighted limits in the quasi-categorical context a preliminary result is needed because the simplicial subcategory  $\underline{\text{qCat}}_\infty \hookrightarrow \underline{\text{sSet}}$  is not complete. It is however closed under weighted limits whose weights are cofibrant in the projective model structure on the appropriate  $\underline{\text{sSet}}$ -valued diagram category. We prove this result and provide a general review of the theory of weighted limits in section 5. We anticipate other uses of the fact that quasi-categories are closed under weighted limits with projectively cofibrant weights than those given here. In a supplemental paper [26], we prove that for any diagram of quasi-categories admitting (co)limits of shape  $X$  and functors that preserve these colimits, the weighted limit again admits (co)limits of shape  $X$ .

In section 6, we define the quasi-category of algebras  $B[t]$  associated to a homotopy coherent monad  $t$  on a quasi-category  $B$  as a limit weighted by the restriction along  $\underline{\text{Mnd}} \hookrightarrow \underline{\text{Adj}}$  of the covariant simplicial functor represented by the object  $-$ . The homotopy coherent monadic adjunction  $f^t \dashv u^t : B[t] \rightarrow B$  is then defined formally: it is simply a reflection in  $\underline{\text{qCat}}_\infty$  of an adjunction between the weights whose limits identify the two quasi-categories involved. In particular, the monadic forgetful functor  $u^t$  is induced from a natural transformation between weights that is a projective cofibration and “constant on dimension zero” in an appropriate sense. It follows that the induced map of weighted limits is conservative (reflects isomorphisms).

We give an explicit description of the vertices in the quasi-category of algebras for a homotopy coherent monad, unpacking the weighted limit formula. A calculation on weights—reminiscent of our proof in [25] that for any simplicial object in a quasi-category admitting an augmentation and a splitting the augmentation defines the colimit—proves that these vertices are “codescent objects”.

**6.3.17. Theorem.** *Any vertex in the quasi-category  $B[t]$  of algebras for a homotopy coherent monad is the colimit of a canonical  $u^t$ -split simplicial object of free algebras.*

In section 7, we compare a general homotopy coherent adjunction extending  $f \dashv u : A \rightarrow B$  with the induced monadic adjunction defined from its homotopy coherent monad  $t = uf$ . A map between weights, this time indexed on the simplicial category  $\underline{\text{Adj}}$ , induces a canonical simplicial natural transformation from the homotopy coherent adjunction to the monadic adjunction. The monadicity theorem gives conditions under which the non-identity component of this map is an equivalence of quasi-categories.

**7.2.4,7.2.7. Theorem.** *There is a canonical comparison functor defining the component of a simplicial natural transformation between any homotopy coherent adjunction  $f \dashv u$  and its monadic homotopy coherent adjunction  $f^t \dashv u^t$ .*

$$\begin{array}{ccc}
 A & \overset{\text{---}}{\dashrightarrow} & B[t] \\
 \swarrow u & & \nearrow f^t \\
 & B & \\
 \searrow f & & \swarrow u^t
 \end{array}$$

*If  $A$  admits colimits of  $u$ -split simplicial objects, then the comparison functor admits a left adjoint. If  $u$  preserves colimits of  $u$ -split simplicial objects and reflects isomorphisms, then this adjunction defines an adjoint equivalence  $A \simeq B[t]$ .*

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## 2. SIMPLICIAL COMPUTADS

Cofibrant simplicial categories are simplicial computads, a definition we introduce in §2.1 together with some important examples. The notion of simplicial computad provides a direct characterisation of those simplicial categories that are cofibrant that is useful for inductive arguments: a simplicial functor whose domain is a simplicial computad is defined by specifying images of the atomic non-degenerate  $n$ -arrows. In §2.2, we study simplicial subcomputads in order to describe what will be needed for the “induction steps” in the proofs of section 4, which require extensions along simplicial subcomputad inclusions.

### 2.1. Simplicial categories and simplicial computads.

**2.1.1. Notation** (simplicial categories). It will be convenient to identify simplicially enriched categories, *simplicial categories* henceforth, as simplicial objects in  $\underline{\text{Cat}}$ . The category of simplicial categories is isomorphic to the full subcategory of  $\underline{\text{Cat}}^{\Delta^{\text{op}}}$  of those simplicial objects  $\mathbf{A} : \Delta^{\text{op}} \rightarrow \underline{\text{Cat}}$  for which the simplicial set obtained by composing with the

object functor  $\text{obj}: \underline{\text{Cat}} \rightarrow \underline{\text{Set}}$  is constant. In other words, a simplicial object in  $\underline{\text{Cat}}$  is a simplicial category just when each of the categories in the diagram has the same set of objects and each of the functors is the identity on objects.

Given a simplicial category  $\mathbf{A}: \Delta^{\text{op}} \rightarrow \underline{\text{Cat}}$ , an  $n$ -arrow is an arrow in  $\mathbf{A}_n$ ; an  $n$ -arrow  $f: a \rightarrow b$  is precisely an  $n$ -simplex in the simplicial set  $\mathbf{A}(a, b)$ . We write  $\emptyset$  for the initial simplicial category on no objects and  $\mathbb{1}$  for the terminal simplicial category on a single object. We adopt the same terminology for large simplicial categories  $\mathcal{K}$ , using the size conventions detailed in I.2.0.1.

**2.1.2. Notation** (whiskering in a simplicial category). For each vertex in a simplicial set and for each  $n > 0$ , there is a unique degenerate  $n$ -simplex on that vertex obtained by acting via the simplicial operator  $[n] \rightarrow [0]$ . If  $f: a \rightarrow b$  is an  $n$ -arrow in a simplicial category  $\mathbf{A}$ , and  $x: a' \rightarrow a$  and  $y: b \rightarrow b'$  are 0-arrows, we write  $fx: a' \rightarrow b$  and  $yf: a \rightarrow b'$  for the  $n$ -arrows obtained by degenerating  $x$  and  $y$  and composing in  $\mathbf{A}$ . We refer to this operation as *whiskering* the  $n$ -arrow  $f$  with  $x$  or  $y$ ; in the special case where the simplicial category is a 2-category, this coincides with the usual notion.

**2.1.3. Example** (the generic  $n$ -arrow). For any simplicial set  $X$ , let  $\mathcal{2}[X]$  denote the simplicial category with two objects 0 and 1 and whose only non-trivial hom-space is  $\mathcal{2}[X](0, 1) := X$ . Here we define  $\mathcal{2}[X](1, 0) = \emptyset$  and  $\mathcal{2}[X](0, 0) = \mathcal{2}[X](1, 1) = *$ .

For any simplicial category  $\mathcal{K}$ , a simplicial functor  $F: \mathcal{2}[X] \rightarrow \mathcal{K}$  is completely determined by the following data:

- a pair of objects  $B$  and  $A$  in  $\mathcal{K}$  and
- a simplicial map  $f: X \rightarrow \mathcal{K}(B, A)$ .

On account of the canonical bijection between simplicial functors  $\mathcal{2}[\Delta^n] \rightarrow \mathcal{K}$  and  $n$ -arrows of  $\mathcal{K}$ , we refer to the simplicial category  $\mathcal{2}[\Delta^n]$  as the *generic  $n$ -arrow*.

**2.1.4. Definition** ((relative) simplicial computads). The class of *relative simplicial computads* is the class of all simplicial functors which can be expressed as a transfinite composite of pushouts of coproducts of

- the unique simplicial functor  $\emptyset \hookrightarrow \mathbb{1}$ , and
- the inclusion simplicial functor  $\mathcal{2}[\partial\Delta^n] \hookrightarrow \mathcal{2}[\Delta^n]$  for  $n \geq 0$ .

A simplicial category  $\mathbf{A}$  is a *simplicial computad* if and only if the unique functor  $\emptyset \hookrightarrow \mathbf{A}$  is a relative simplicial computad.

**2.1.5. Observation** (an explicit characterisation of simplicial computads). An arrow  $f$  in an unenriched category is *atomic* if it is not an identity and it admits no non-trivial factorisations, i.e., if whenever  $f = g \circ h$  then one or other of  $g$  and  $h$  is an identity. A category is *freely generated* (by a reflexive directed graph) if and only if each of its non-identity arrows may be uniquely expressed as a composite of atomic arrows. In this case, the generating graph is precisely the subgraph of atomic arrows.

An extension of this kind of characterisation gives an explicit description of the simplicial computads. Specifically,  $\mathbf{A}$  is a simplicial computad if and only if:

- each non-identity  $n$ -arrow  $f$  of  $\mathbf{A}_n$  may be expressed uniquely as a composite  $f_1 \circ f_2 \circ \dots \circ f_\ell$  in which each  $f_i$  is atomic, and
- if  $f$  is an atomic  $n$ -arrow in  $\mathbf{A}_n$  and  $\alpha: [m] \rightarrow [n]$  is a degeneracy operator in  $\Delta$  then the degenerated  $m$ -arrow  $f \cdot \alpha$  is atomic in  $\mathbf{A}_m$ .

On combining this characterisation with the Eilenberg-Zilber lemma, we find that  $\mathbf{A}$  is a simplicial computad if and only if all of its non-identity arrows  $f$  can be expressed uniquely as a composite

$$f = (f_1 \cdot \alpha_1) \circ (f_2 \cdot \alpha_2) \circ \dots \circ (f_\ell \cdot \alpha_\ell) \tag{2.1.6}$$

in which each  $f_i$  is non-degenerate and atomic and each  $\alpha_i \in \Delta$  is a degeneracy operator.

2.1.7. *Observation.* A simplicial functor is called a *trivial fibration* of simplicial categories if it has the right lifting property with respect to the generating set of simplicial functors of definition 2.1.4. A simplicial functor  $P: \mathbf{E} \rightarrow \mathbf{B}$  is a trivial fibration if and only if it is surjective on objects and its action  $\mathbf{E}(A, B) \rightarrow \mathbf{B}(PA, PB)$  on each hom-space is a trivial fibration of simplicial sets. A simplicial functor is said to be a *cofibration* of simplicial categories if it is a retract of a relative simplicial computad. These are the classes appearing in Bergner’s model structure on simplicial categories [1].

The characterisation of observation 2.1.5 reveals that all retracts of simplicial computads are again simplicial computads and hence that the cofibrant objects in Bergner’s model structure are precisely the simplicial computads: no retracts are needed.

2.1.8. **Example.** The simplicial categories  $\mathcal{2}[X]$  defined in 2.1.3 are simplicial computads, with every simplex in  $X$  an atomic arrow.

2.1.9. **Example** (free simplicial resolutions define simplicial computads). There is a free-forgetful adjunction

$$\underline{\text{Cat}} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \underline{\text{Gph}}$$

between small categories and reflexive directed graphs inducing a comonad  $FU$  on  $\underline{\text{Cat}}$ . The comonad resolution associated to a small category  $\mathbf{C}$  is a simplicial computad  $FU \bullet \mathbf{C}$

$$FUC \begin{array}{c} \xleftarrow{\epsilon FU} \\ \xrightarrow{F\eta U} \\ \xleftarrow{FU\epsilon} \end{array} FUFUC \begin{array}{c} \xleftarrow{\epsilon FUFU} \\ \xrightarrow{FU\epsilon FU} \\ \xleftarrow{FUF\eta U} \\ \xrightarrow{FU FU\epsilon} \end{array} FUFUFUC \dots$$

called the *standard resolution* of  $\mathbf{C}$  in [7]. The category  $FUC$  is the free category on the underlying graph of  $\mathbf{C}$ . Its arrows are (possibly empty) strings of composable non-identity arrows of  $\mathbf{C}$ . The atomic 0-arrows are the non-identity arrows of  $\mathbf{C}$ . An  $n$ -arrow is a string of composable arrows in  $\mathbf{C}$  with each arrow in the string enclosed in exactly  $n$  pairs of parentheses. The atomic  $n$ -arrows are those strings enclosed in a single pair of “outermost” parentheses.

2.1.10. **Example.** The simplicial computad  $FU_{\bullet}\mathbf{C}$  is isomorphic to the image of the nerve of  $\mathbf{C}$  under the left adjoint to the homotopy coherent nerve

$$\text{sSet-Cat} \begin{array}{c} \xleftarrow{\mathfrak{C}} \\ \perp \\ \xrightarrow{N} \end{array} \text{sSet}$$

cf. [21, 6.7]. Indeed, for any simplicial set  $X$ ,  $\mathfrak{C}X$  is a simplicial computad. This follows from the fact that  $\mathfrak{C} \dashv N$  defines a Quillen equivalence between the model structures of Bergner and Joyal, but we prefer to give a direct proof.

The arrows in  $\mathfrak{C}X$  admit a simple geometric characterisation due to Dugger and Spivak [5]:  $n$ -arrows in  $\mathfrak{C}X$  are *necklaces* in  $X$ , i.e., maps  $\Delta^{n_1} \vee \Delta^{n_2} \vee \dots \vee \Delta^{n_k} \rightarrow X$  from a sequence of standard simplices joined head-to-tail, together with a nested sequence of  $n-1$  sets of vertices. The atomic arrows are precisely those whose necklace consists of a single simplex.

2.1.11. **Example.** In particular, the simplicial category  $\mathfrak{C}\Delta^n$  whose objects are integers  $0, 1, \dots, n$  and whose hom-spaces are the cubes

$$\mathfrak{C}\Delta^n(i, j) = \begin{cases} (\Delta^1)^{j-i-1} & i < j \\ \Delta^0 & i = j \\ \emptyset & i > j \end{cases}$$

is a simplicial computad. In each hom-space, the atomic arrows are precisely those whose simplices contain the initial vertex in the poset whose nerve defines the simplicial cube.

2.2. **Simplicial subcomputads.** The utility of the notion of simplicial computad is the following: if  $\mathbf{A}$  is a simplicial computad and  $\mathcal{K}$  is any simplicial category, a simplicial functor  $\mathbf{A} \rightarrow \mathcal{K}$  can be defined inductively simply by specifying images for the non-degenerate, atomic  $n$ -arrows in a way that is compatible with previously chosen faces. Let us now make this idea precise.

2.2.1. **Definition** (simplicial subcomputad). If  $\mathbf{A}$  is a simplicial computad then a *simplicial subcomputad*  $\mathbf{B}$  of  $\mathbf{A}$  is a simplicial subcategory that is closed under factorisations: i.e.,

- if  $g$  and  $f$  are composable arrows in  $\mathbf{A}$  and  $g \circ f$  is in  $\mathbf{B}$ , then both  $g$  and  $f$  are in  $\mathbf{B}$ .

This condition is equivalent to postulating that  $\mathbf{B}$  is a simplicial computad and that every arrow which is atomic in  $\mathbf{B}$  is also atomic in  $\mathbf{A}$ .

2.2.2. *Observation* (simplicial subcomputads and relative simplicial computads). If  $\mathbf{B}$  is a simplicial subcomputad of the simplicial computad  $\mathbf{A}$ , then the inclusion functor  $\mathbf{B} \hookrightarrow \mathbf{A}$  is a relative simplicial computad. Indeed, every relative simplicial computad may be obtained as a composite of pushouts of simplicial subcomputad inclusion. Furthermore, if  $\mathbf{C}$  is a simplicial subcategory of  $\mathbf{B}$  then  $\mathbf{C}$  is a simplicial subcomputad of  $\mathbf{B}$  if and only if it is a simplicial subcomputad of  $\mathbf{A}$ .

2.2.3. **Example.** If  $X$  is a simplicial subset of  $Y$ , then  $\mathfrak{2}[X]$  is a simplicial subcomputad of  $\mathfrak{2}[Y]$ , and  $\mathfrak{C}X$  is a simplicial subcomputad of  $\mathfrak{C}Y$ .

**2.2.4. Definition.** The *simplicial subcomputad generated by* a set of arrows  $X$  in a simplicial computad  $\mathbf{A}$  is the intersection  $\overline{X}$  of all of the simplicial subcomputads of  $\mathbf{A}$  which contain  $X$ . Note that arbitrary intersections of simplicial subcomputads are again simplicial subcomputads. The simplicial subcomputad  $\overline{X}$  can be formed by inductively closing  $X$  up to the smallest subset of  $\mathbf{A}$  containing it which satisfies the closure properties:

- if  $f \in \overline{X}$  and  $\alpha$  is a simplicial operator then  $f \cdot \alpha \in \overline{X}$ , and
- if  $g \circ f$  is a composite in  $\mathbf{A}$ , then  $g \circ f \in \overline{X}$  if and only if  $g$  and  $f$  are both in  $\overline{X}$ .

**2.2.5. Example** ((co)skeleta of simplicial categories). The  $r$ -skeleton  $\text{sk}_r \mathbf{A}$  ( $r \geq -1$ ) of a simplicial category  $\mathbf{A}$  is the smallest simplicial subcategory of  $\mathbf{A}$  which contains all of its arrows of dimension less than or equal to  $r$ . We say that a simplicial category  $\mathbf{A}$  is  $r$ -skeletal if  $\text{sk}_r \mathbf{A} = \mathbf{A}$ , i.e., when all of its arrows of dimension greater than  $r$  can be expressed as composites of degenerate arrows. When  $\mathbf{A}$  is a simplicial computad, an arrow  $f$  is in  $\text{sk}_r(\mathbf{A})$  if and only if each arrow  $f_i$  in the decomposition of (2.1.6) has dimension at most  $r$ . In this case, the skeleton  $\text{sk}_r \mathbf{A}$  is the simplicial subcomputad of  $\mathbf{A}$  generated by its set of  $r$ -arrows. By convention, we write  $\text{sk}_{-1} \mathbf{A}$  for the discrete simplicial subcategory which contains all of the objects of  $\mathbf{A}$ .

Each  $r$ -skeleton functor has a right adjoint

$$\text{sSet-Cat} \begin{array}{c} \xleftarrow{\text{sk}_r} \\ \perp \\ \xrightarrow{\text{cosk}_r} \end{array} \text{sSet-Cat}$$

which as ever we call the  $r$ -coskeleton. The 0-coskeleton  $\text{cosk}_{-1} \mathbf{A}$  is the chaotic simplicial category on the objects of  $\mathbf{A}$ . The  $r$ -coskeleton of  $\mathbf{A}$  is defined by applying the usual simplicial  $r$ -coskeleton functor  $\text{cosk}_r: \text{sSet} \rightarrow \text{sSet}$  to each hom-space  $\mathbf{A}(A, B)$ . The consequent hom-spaces  $(\text{cosk}_r \mathbf{A})(A, B) := \text{cosk}_r(\mathbf{A}(A, B))$  inherit a compositional structure from that of  $\mathbf{A}$  by dint of the fact that the simplicial  $r$ -coskeleton functor is right adjoint and thus preserves all finite products. A simplicial category  $\mathbf{A}$  is  $r$ -coskeletal if and only if the adjoint transpose  $\mathbf{A} \rightarrow \text{cosk}_r \mathbf{A}$  of the inclusion  $\text{sk}_r \mathbf{A} \hookrightarrow \mathbf{A}$  is an isomorphism. So a simplicial category is  $(-1)$ -coskeletal when all of its hom-spaces are isomorphic to the one point simplicial set  $\Delta^0$  and it is  $r$ -coskeletal precisely when each of its hom-spaces is  $r$ -coskeletal in the usual sense for simplicial sets.

**2.2.6. Proposition.** *If  $\mathbf{A}$  is a simplicial computad, a simplicial functor  $F: \mathbf{A} \rightarrow \mathcal{K}$  is uniquely specified by choosing*

- an object  $F(A)$  in  $\mathcal{K}$  for each of the objects  $A$  of  $\mathbf{A}$ , and
- an arrow  $F(f)$  in  $\mathcal{K}$  for each of the non-degenerate and atomic arrows  $f$  of  $\mathbf{A}$

*subject to the conditions that*

- *these choices are made compatibly with the dimension, domain, and codomain operations of  $\mathbf{A}$  and  $\mathcal{K}$ , and*
- *whenever  $f$  is a non-degenerate and atomic arrow and its face  $f \cdot \delta^i$  is decomposed in terms of non-degenerate and atomic arrows  $f_i$  as in (2.1.6) then the face  $F(f) \cdot \delta^i$  is the corresponding composite of degenerate images of the  $F(f_i)$ .*

*Proof.* We induct over the skeleta of  $\mathbf{A}$ . The specification of objects defines  $F: \text{sk}_{-1} \mathbf{A} \rightarrow \mathcal{K}$ . If  $F: \text{sk}_{r-1} \mathbf{A} \rightarrow \mathcal{K}$  is a simplicial functor, then extensions of  $F$  to a simplicial functor  $\text{sk}_r \mathbf{A} \rightarrow \mathcal{K}$  are uniquely specified by the following data:

- an  $r$ -arrow  $F(f)$  in  $\mathcal{K}$  for each atomic non-degenerate  $r$ -arrow  $f$  in  $\mathbf{A}$  subject to the condition that  $F(f) \cdot \delta^i = F(f \cdot \delta^i)$  for each elementary face operator  $\delta^i: [r-1] \rightarrow [r]$  in  $\Delta$ .

This condition on the faces of the chosen arrow  $F(f)$  makes sense because the faces  $f \cdot \delta^i$  of any  $r$ -arrow are  $(r-1)$ -arrows and are thus elements of  $\text{sk}_{r-1} \mathbf{A}$  to which we may apply the un-extended simplicial functor  $F: \text{sk}_{r-1} \mathbf{A} \rightarrow \mathcal{K}$ .

To construct a simplicial functor from this data we simply decompose each arrow  $f$  of  $\text{sk}_r(\mathbf{A})$  as in (2.1.6) and then observe that  $F(f_i)$  is defined for each component of that decomposition either because  $f_i$  is in  $\text{sk}_{r-1} \mathbf{A}$  or because it is a non-degenerate and atomic  $r$ -arrow and thus has an image in  $\mathcal{K}$  given by the extra data supplied above. This then provides us with a value for  $F(f)$  given by the composite

$$F(f) := (F(f_1) \cdot \alpha_1) \circ (F(f_2) \cdot \alpha_2) \circ \cdots \circ (F(f_\ell) \cdot \alpha_\ell) \quad (2.2.7)$$

and it is easily checked, using the uniqueness of these decompositions in  $\mathbf{A}$ , that this action is functorial and that it respects simplicial actions. In other words, this result tells us that we may build the skeleton  $\text{sk}_r(\mathbf{A})$  from the skeleton  $\text{sk}_{r-1}(\mathbf{A})$  by glueing on copies of the category  $\mathcal{Z}[\Delta^r]$  along functors  $\mathcal{Z}[\partial\Delta^r] \rightarrow \text{sk}_{r-1}(\mathbf{A})$ , one for each atomic non-degenerate  $r$ -arrow of  $\mathbf{A}$ .  $\square$

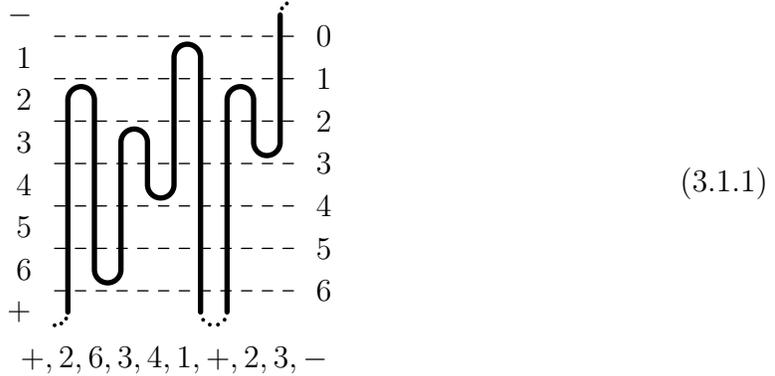
### 3. THE GENERIC ADJUNCTION

In this section, we introduce a simplicial category  $\underline{\text{Adj}}$  via a graphical calculus developed in §3.1, from which definition it will be immediately clear that we have defined a simplicial computad. This result, when combined with proposition 2.2.6, will make it relatively easy to construct simplicial functors whose domain is  $\underline{\text{Adj}}$ . In §3.2 and §3.3, we then show that  $\underline{\text{Adj}}$  is isomorphic to the simplicial category obtained by applying the nerve to each hom-category in the free 2-category containing an adjunction [28]. To emphasise the interplay between 2-categories and simplicial categories, an important theme of our work, our proof strategy is somewhat indirect. In §3.2, we show that the hom-spaces of  $\underline{\text{Adj}}$  satisfy the Segal condition; thus  $\underline{\text{Adj}}$  is isomorphic to some 2-category under the embedding  $2\text{-Cat} \hookrightarrow \text{sSet-Cat}$ . In §3.3, we show that  $\underline{\text{Adj}}$  has the same universal property as the Schanuel and Street 2-category, proving that these gadgets are isomorphic and justifying our decision not to notationally distinguish between them.

**3.1. A graphical calculus for the simplicial category  $\underline{\text{Adj}}$ .** To define a small simplicial category, thought of as an identity-on-objects simplicial object in  $\underline{\text{Cat}}$ , it suffices to specify

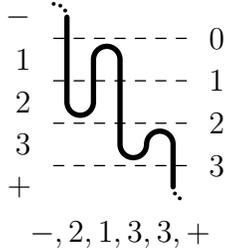
- a set of objects,
- for each  $n \geq 0$ , a set of  $n$ -arrows with (co)domains among the specified object set,
- a right action of the morphisms in  $\Delta$  on this graded set,
- a “horizontal” composition operation for  $n$ -arrows with compatible (co)domains that preserves the simplicial action.

We will define  $\underline{\text{Adj}}$  to be the simplicial category with two objects, denoted “+” and “-”, and whose  $n$ -arrows will be certain graphically inspired *strictly undulating squiggles on  $n + 1$  lines*. We will provide a formal account of these squiggles presently, but we prefer to start by engaging the reader’s intuition with a picture. For example, the diagram below depicts a 6-arrow in the hom-space  $\underline{\text{Adj}}(-, +)$ :



Here we have drawn the  $n + 1$  lines ( $n = 6$  in this case) which support this squiggle as horizontal dotted lines numbered 0 to  $n$  down the right hand side, and these lines separate  $n + 2$  levels which are labelled down the left hand side. The levels which sit between a pair of lines, sometimes called *gaps*, are labelled 1 to  $n$  while the top and bottom levels are labelled - and + respectively.

Each *turning point* of the squiggle itself lies entirely within a single level. The qualifier “strict undulation” refers to the requirement that the levels of adjacent turning points should be distinct and that they should oscillate as we proceed from left to right. For example, the following is not a strictly undulating squiggle



because its last two turning points occur on the same level.

The data of such a squiggle can be encoded by a string  $\underline{a} = (a_0, a_1, \dots, a_r)$  of letters in the set  $\{-, 1, 2, \dots, n, +\}$ , corresponding to the levels of each successive turning point, subject to conditions that we will enumerate shortly. The string corresponding to our 6-arrow (3.1.1) is displayed along the bottom of that picture. As we shall see, composition of  $n$ -arrows in  $\underline{\text{Adj}}$  will correspond to a coalesced concatenation operation on these strings, and so it is natural to read them from right to left. Consequently, the domain and codomain of such a squiggle are naturally taken to be its last and first letters respectively; in particular, the domain of the 6-arrow (3.1.1) is - and its codomain is +.

**3.1.2. Definition** (strictly undulating squiggles). We write  $\underline{a} = (a_0, a_1, \dots, a_r)$  for a non-empty string of letters in  $\{-, 1, 2, \dots, n, +\}$ , intended to represent a squiggle on  $n + 1$  lines, with domain  $\text{dom}(\underline{a}) := a_r$  and codomain  $\text{cod}(\underline{a}) := a_0$ . We define the *width* of this squiggle to be the number  $w(\underline{a}) := r$ , that is, the number of letters in its string minus 1. The *interior* of such a string is the sub-list  $a_1, \dots, a_{r-1}$  of all of its letters except for those at its end points  $a_0$  and  $a_r$ .

We say that a string  $\underline{a}$  represents a *strictly undulating squiggle on  $n + 1$  lines* if it satisfies the conditions that:

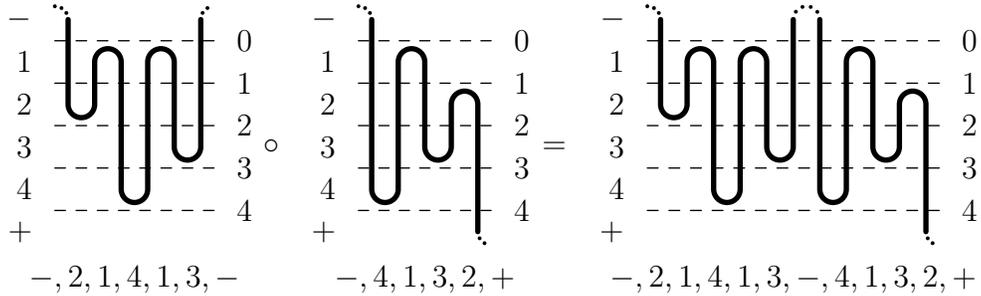
- (i)  $a_0, a_{w(\underline{a})} \in \{-, +\}$ , and
- (ii) if  $a_0 = -$  (resp.  $a_0 = +$ ) then for all  $0 \leq i < w(\underline{a})$  we have  $a_i < a_{i+1}$  whenever  $i$  is even (resp. odd) and  $a_i > a_{i+1}$  whenever  $i$  is odd (resp. even).

We also say that  $\underline{a}$  is simply an *undulating squiggle on  $n + 1$  lines* if it satisfies condition (i) above but only satisfies the weaker condition

- (ii)' if  $a_0 = -$  (resp.  $a_0 = +$ ) then for all  $0 \leq i < w(\underline{a})$  we have  $a_i \leq a_{i+1}$  whenever  $i$  is even (resp. odd) and  $a_i \geq a_{i+1}$  whenever  $i$  is odd (resp. even).

in place of condition (ii).

**3.1.3. Definition** (composing squiggles). Two such  $n$ -arrows  $\underline{b}$  and  $\underline{a}$  are composable when  $b_{w(\underline{b})} = a_0$  and their composite is described graphically as the kind of horizontal glueing depicted in the following picture:



More formally, the composite  $\underline{b} \circ \underline{a}$  is given by the string  $(b_0, \dots, b_{w(\underline{b})} = a_0, a_1, \dots, a_{w(\underline{a})})$  constructed by dropping the last letter of  $\underline{b}$  and concatenating the resulting string with  $\underline{a}$ . It is easily seen that this composition operation is associative and that it has the  $n$ -arrows  $(-)$  and  $(+)$  as identities. In other words, these operations make the collection of  $n$ -arrows into a category with objects  $-$  and  $+$ .

**3.1.4. Observation** (atomic  $n$ -arrows). Notice that an  $n$ -arrow  $\underline{c}$  of  $\underline{\text{Adj}}$  may be expressed as a composite  $\underline{b} \circ \underline{a}$  of non-identity  $n$ -arrows precisely when there is some  $0 < k < w(\underline{c})$  such that  $c_k \in \{-, +\}$ . Specifically,  $\underline{b} := (c_0, \dots, c_k)$  and  $\underline{a} := (c_k, \dots, c_{w(\underline{c})})$  are strictly undulating squiggles whose composite is  $\underline{c}$ . It follows that an  $n$ -arrow  $\underline{c}$  is atomic, in the sense of definition 2.1.4, if and only if the letters  $-$  and  $+$  do not appear in its interior.

We now describe how the simplicial operators act on the arrows of  $\underline{\text{Adj}}$ .

**3.1.5. Observation** (simplicial action on strictly undulating squiggles). The geometric idea behind the simplicial action is simple: given a simplicial operator  $\alpha: [m] \rightarrow [n]$  and a

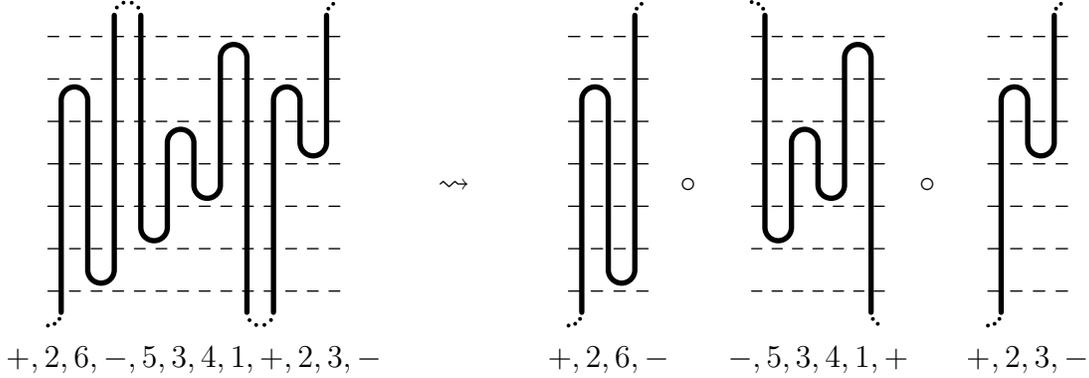








*Proof.* A squiggle in  $\underline{\text{Adj}}$  may be uniquely decomposed into a sequence of atomic arrows by splitting it at each successive  $-$  or  $+$  letter in its interior.



The operation of degenerating an arrow does not introduce any extra  $+$  or  $-$  letters into its interior, from which it follows that degenerated atomic arrows are again atomic.  $\square$

3.1.11. **Example** (adjunction data in  $\underline{\text{Adj}}$ ). For later use, we name some of the low dimensional non-degenerate atomic arrows in  $\underline{\text{Adj}}$ . There are exactly two non-degenerate atomic 0-arrows in  $\underline{\text{Adj}}$ , these being:

$$\underline{f} := \text{[diagram of f]} \quad \text{and} \quad \underline{u} := \text{[diagram of u]}$$

Since  $\underline{\text{Adj}}$  is a simplicial computad, all of its other 0-arrows may be obtained as a unique alternating composite of those two, for example:

$$\underline{f}\underline{u}\underline{f}\underline{u}\underline{f} = \text{[diagram of composite arrow fufuf]}$$

One convenient aspect of our string notation for arrows is that the act of whiskering an  $n$ -arrow  $\underline{a}$  with one the arrows  $\underline{f}$  or  $\underline{u}$ , as described in notation 2.1.2, simply amounts to appending or prepending one of the symbols  $-$  or  $+$  as follows:

$$\begin{aligned} \underline{f}\underline{a} &= \underline{a} \text{ with } - \text{ prepended,} & \underline{a}\underline{f} &= \underline{a} \text{ with } + \text{ appended,} \\ \underline{u}\underline{a} &= \underline{a} \text{ with } + \text{ prepended, and} & \underline{a}\underline{u} &= \underline{a} \text{ with } - \text{ appended.} \end{aligned}$$

There are also exactly two non-degenerate atomic 1-arrows in  $\underline{\text{Adj}}$ , these being:

$$\underline{\eta} := \text{[diagram of eta]} \quad \text{and} \quad \underline{\epsilon} := \text{[diagram of epsilon]}$$

Writing these 1-arrows as if they were 1-cells in a 2-category, they clearly take a form reminiscent of the unit  $\underline{\eta}: \text{id}_- \Rightarrow \underline{u}\underline{f}$  and counit  $\underline{\epsilon}: \underline{f}\underline{u} \Rightarrow \text{id}_+$  of an adjunction. Here again, since  $\underline{\text{Adj}}$  is a simplicial computad all of its 1-arrows are uniquely expressible as a composite



**3.2. The simplicial category  $\underline{\text{Adj}}$  as a 2-category.** Our blanket identification of categories with their nerves leads to a corresponding identification of 2-categories with simplicial categories, obtained by applying the nerve functor hom-wisely. In this section, we will show that the simplicial category  $\underline{\text{Adj}}$  is a 2-category in this sense, i.e., that its hom-spaces  $\underline{\text{Adj}}$  are nerves of categories, a “fibrancy” result. Indeed, we show in §3.3 that  $\underline{\text{Adj}}$  is isomorphic to the *generic* or *walking* adjunction, the 2-category freely generated by an adjunction.

Using the graphical calculus, it is not difficult to sketch a direct proof that  $\underline{\text{Adj}}$  is isomorphic to the generic adjunction, whose concrete description recalled in remark 3.3.8 below. However, we find it more illuminating to first verify that the hom-spaces in  $\underline{\text{Adj}}$  satisfy the Segal condition, showing that  $\underline{\text{Adj}}$  is isomorphic to some 2-category, and then prove that this 2-category has the universal property that defines the walking adjunction.

**3.2.1. Recall (Segal condition).** A simplicial set  $X$  is the nerve of a category if and only if it satisfies the (strict) Segal condition, which states that for all  $n, m \geq 1$  the commutative square

$$\begin{array}{ccc} X_{n+m} & \xrightarrow{-\cdot\{0,\dots,n\}} & X_n \\ -\cdot\{n,\dots,n+m\} \downarrow & \lrcorner & \downarrow -\cdot\{n\} \\ X_m & \xrightarrow{-\cdot\{0\}} & X_0 \end{array}$$

is a pullback. This condition says that if  $x$  is an  $n$ -simplex and  $y$  is an  $m$ -simplex in  $X$  for which the last vertex  $x \cdot \{n\}$  of  $x$  is equal to the first vertex  $y \cdot \{0\}$  of  $y$  then there exists a unique  $(n + m)$ -simplex  $z$  for which  $z \cdot \{0, \dots, n\} = x$  and  $z \cdot \{n, \dots, m + n\} = y$ .

**3.2.2. Proposition.** *Each hom-space of the simplicial category  $\underline{\text{Adj}}$  is the nerve of a category.*

*Proof.* To prove proposition 3.2.2, it suffices to verify that the arrows in each hom-space of  $\underline{\text{Adj}}$  satisfy the Segal condition. We convey the intuition with a specific example provided by the following pair of squiggles in the hom-space  $\underline{\text{Adj}}(+, +)$ :

$$\begin{array}{c} \underline{a} = \\ \begin{array}{c} - \\ 1 \\ 2 \\ + \\ +, 1, 2, -, +, 2, +, 1, 2, 1, + \end{array} \end{array} \quad \begin{array}{c} \underline{b} = \\ \begin{array}{c} - \\ 1 \\ 2 \\ + \\ +, 1, 2, -, 2, -, 2, 1, +, -, + \end{array} \end{array} \quad (3.2.3)$$

Counting the crossings of the bottom line in the first of these and the crossings of the top line in the second, as discussed in observation 3.1.9, we see that the last vertex of  $\underline{a}$  is the same as the first vertex of  $\underline{b}$ . Thus,  $\underline{a}$  and  $\underline{b}$  are a pair of arrows to which premise of the Segal condition applies.

Since these crossings match up we may “splice” these two squiggles together by identifying the bottom line of  $\underline{a}$  and the top line of  $\underline{b}$  and then fusing each string which passes through



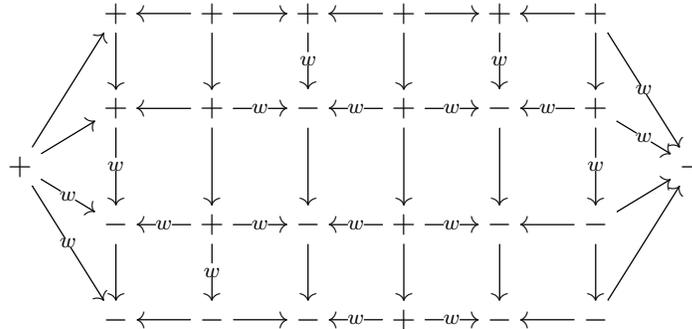
We give a sketch of the proof employing our graphical calculus. Consider an  $k$ -arrow

(3.2.6)

and draw vertical lines bisecting each undulation point (but not otherwise intersecting the squiggle) and also one vertical line to the left and to the right of the diagram. Label each intersection of a vertical and horizontal line in this picture with a “+” if it is in the region “above” the squiggle and a “-” if it is “below”; the squiggle on the left of (3.2.6) gives rise to the figure on the right.

Reading down a vertical line we get a sequence  $+, \dots, +, -, \dots, -$  which we interpret as a composable sequence of arrows comprised of identities at  $+$ , the arrow  $w$ , and then identities at  $-$ , all of which are weak equivalences. Note the leftmost (resp. rightmost) vertical line is comprised of a sequence of identities at the codomain (resp. domain) of the  $k$ -arrow (3.2.6). By “pinching” these sequences of identities, we obtain the starting and ending points of the hammock.

Reading across a horizontal line from right (the domain) to left (the codomain), we get a sequence of objects “+” or “-”, which we interpret as a zig-zag of identities together with forwards (left-pointing) and backwards (right-pointing) instances of  $w$ . With these conventions, the squiggle (3.2.6) represents the hammock:



On account of our conventions for the direction of horizontal composition, the hammock described here is a reflection of the  $k$ -arrow displayed on [6, p. 19] in a vertical line, with “backwards” arrows point to the right. Each column will contain at least one “ $w$ ”, and every “ $w$ ” in a given column will point in the same direction. This dictates the direction of the identities in that column. We leave it to the reader to verify that the hammocks corresponding to strictly undulating squiggles are “reduced” in the sense of [6, 2.1].

**3.3. The 2-categorical universal property of  $\underline{\text{Adj}}$ .** This section is devoted to relating our 2-category  $\underline{\text{Adj}}$  to the generic adjunction 2-category as first studied by Schanuel and Street in [28]. Our approach will be to show that the 2-category  $\underline{\text{Adj}}$  established by proposition 3.2.2 enjoys the universal property they used to characterise their 2-category.

It follows then that these 2-categories must be isomorphic. This observation provides us with an alternative description of  $\underline{\text{Adj}}$  in terms of the structure of  $\mathbb{A}$ , which we expound upon in remark 3.3.8.

3.3.1. *Observation* (2-categories as simplicial categories). When we regard a 2-category  $\mathcal{K}$  as a simplicial category then

- its 1-cells and 2-cells respectively define the 0-arrows and 1-arrows,
- if  $\phi_0$ ,  $\phi_1$ , and  $\phi_2$  are three 1-arrows (2-cells) in some hom-space  $\mathcal{K}(A, B)$  then there exists a unique 2-arrow  $\phi$  in  $\mathcal{K}(A, B)$  with  $\phi \cdot \delta^i = \phi_i$  for  $i = 0, 1, 2$  if and only if  $\phi_0 \cdot \phi_2 = \phi_1$  in the category  $\mathcal{K}(A, B)$ , and
- the rest of the structure of each hom-space  $\mathcal{K}(A, B)$  is completely determined by the fact that it is 2-coskeletal, as is the nerve of any category.

In the terminology of example 2.2.5 the last of these observations tells us that  $\mathcal{K}$  is a 2-coskeletal simplicial category. It follows by adjunction that every simplicial functor  $F: \text{sk}_2 \mathcal{L} \rightarrow \mathcal{K}$  admits a unique extension to a simplicial functor  $F: \mathcal{L} \rightarrow \mathcal{K}$ .

3.3.2. *Observation* (the adjunction in  $\underline{\text{Adj}}$ ). In example 3.1.11, we noted that the low dimensional data encoded in the simplicial category  $\underline{\text{Adj}}$  was reminiscent of that associated with an adjunction. The reason that we did not commit ourselves fully to that point of view there was that at that stage we did not actually know that  $\underline{\text{Adj}}$  was a 2-category. Proposition 3.2.2 allows us to cross this Rubicon and observe that the 2-category  $\underline{\text{Adj}}$  does indeed contain a genuine adjunction:

$$- \begin{array}{c} \xleftarrow{f} \\ \text{---} \perp \text{---} \\ \xrightarrow{u} \end{array} + \quad \eta: \text{id}_+ \Rightarrow \underline{u}f \quad \epsilon: f\underline{u} \Rightarrow \text{id}_- \quad (3.3.3)$$

It turns out that this is the *generic* or *universal* adjunction, in the sense made precise in the following proposition:

3.3.4. **Proposition** (a 2-categorical universal property of  $\underline{\text{Adj}}$ ). *Suppose that  $\mathcal{K}$  is a 2-category containing an adjunction*

$$A \begin{array}{c} \xleftarrow{f} \\ \text{---} \perp \text{---} \\ \xrightarrow{u} \end{array} B \quad \eta: \text{id}_B \Rightarrow \underline{u}f \quad \epsilon: f\underline{u} \Rightarrow \text{id}_A.$$

*Then there exists a unique 2-functor  $\underline{\text{Adj}} \rightarrow \mathcal{K}$  which carries the adjunction depicted in (3.3.3) to the specified adjunction in  $\mathcal{K}$ .*

*Proof.* We know that  $\underline{\text{Adj}}$  is a simplicial category and that its non-degenerate and atomic 0-arrows and 1-arrow are  $\underline{u}$ ,  $\underline{f}$ ,  $\underline{\epsilon}$ , and  $\underline{\eta}$ . Of course  $\text{sk}_{-1} \underline{\text{Adj}}$  is the discrete simplicial category with objects  $-$  and  $+$ , so we can define a simplicial functor  $F: \text{sk}_{-1} \underline{\text{Adj}} \rightarrow \mathcal{K}$  simply by setting  $F(-) := A$  and  $F(+):= B$ . Now we can apply proposition 2.2.6 to extend this to a simplicial functor  $F: \text{sk}_0 \underline{\text{Adj}} \rightarrow \mathcal{K}$  which is uniquely determined by the equalities  $F(\underline{u}) = u$  and  $F(\underline{f}) = f$  and then extend that, in turn, to a simplicial functor  $F: \text{sk}_1 \underline{\text{Adj}} \rightarrow \mathcal{K}$  which is uniquely determined by the further equalities  $F(\underline{\epsilon}) = \epsilon$  and  $F(\underline{\eta}) = \eta$ .

Applying proposition 2.2.6 one more time, we see that extensions of the simplicial functor we've defined thus far to a simplicial functor  $F: \text{sk}_2 \underline{\text{Adj}} \rightarrow \mathcal{K}$  are uniquely and completely determined by specifying how it should act on the families of 2-arrows  $\underline{\alpha}^{(n)}$  and  $\underline{\beta}^{(n)}$  introduced in example 3.1.11. As discussed in observation 3.3.1, we know that there exists a unique 2-arrow  $F(\underline{\alpha}^{(n)})$  in the 2-category  $\mathcal{K}$  which satisfies the boundary conditions required by proposition 2.2.6 if and only if the 2-cell equation  $F(\underline{\alpha}^{(n)} \cdot \delta^0) \cdot F(\underline{\alpha}^{(n)} \cdot \delta^2) = F(\underline{\alpha}^{(n)} \cdot \delta^1)$  holds in  $\mathcal{K}$ . We may compute the 2-cells that occur in these equations using the equalities listed in (3.1.12) and the simplicial functoriality of  $F$  on  $\text{sk}_1 \underline{\text{Adj}}$  to give

$$\begin{aligned} F(\underline{\alpha}^{(2r)} \cdot \delta^2) &= \eta^r u & F(\underline{\alpha}^{(2r)} \cdot \delta^1) &= u & F(\underline{\alpha}^{(2r)} \cdot \delta^0) &= u\epsilon^r \\ F(\underline{\alpha}^{(2r+1)} \cdot \delta^2) &= \eta^{r+1} & F(\underline{\alpha}^{(2r+1)} \cdot \delta^1) &= \eta & F(\underline{\alpha}^{(2r+1)} \cdot \delta^0) &= u\epsilon^r f \end{aligned}$$

and so those conditions reduce to:

$$u\epsilon^r \cdot \eta^r u = u \quad u\epsilon^r f \cdot \eta^{r+1} = \eta$$

The following middle four calculation

$$u\epsilon^r f \cdot \eta^{r+1} = u\epsilon^r f \cdot \eta^r u f \cdot \eta = (u\epsilon^r \cdot \eta^r u) f \cdot \eta$$

reveals that the second of these equations follows from the first. Furthermore, the middle four computation

$$u\epsilon^{r+1} \cdot \eta^{r+1} u = u\epsilon^r \cdot (uf)^r u\epsilon \cdot \eta^r u f u \cdot \eta u = u\epsilon^r \cdot \eta^r u \cdot u\epsilon \cdot \eta u$$

shows that we can reduce the  $(r+1)^{\text{th}}$  instance of the first equation to a combination of its  $r^{\text{th}}$  instance and the triangle identity  $u\epsilon \cdot \eta u = u$ . Consequently it follows, inductively, that all of these equations follow from that one triangle identity. The dual argument shows that the equalities that arise from the family  $\underline{\beta}^{(n)}$  all reduce to the other triangle identity  $\epsilon f \cdot f \eta = f$ .

We have shown that there exists a unique simplicial functor  $F: \text{sk}_2 \underline{\text{Adj}} \rightarrow \mathcal{K}$  which carries the canonical adjunction in  $\underline{\text{Adj}}$  to the specified adjunction. Observation 3.3.1 allows us to extend uniquely to a simplicial functor  $F: \underline{\text{Adj}} \rightarrow \mathcal{K}$ . The desired universal property is established because a 2-functor of 2-categories is no more nor less than a simplicial functor between the corresponding simplicial categories.  $\square$

**3.3.5. Corollary.** *The simplicial category  $\underline{\text{Adj}}$  is isomorphic to the Schanuel and Street 2-category of [28].*

*Proof.* Proposition 3.3.4 tells us that our 2-category  $\underline{\text{Adj}}$  satisfies the same universal property that Schanuel and Street used to characterise their 2-category.  $\square$

In [28] Schanuel and Street build their 2-category  $\underline{\text{Adj}}$  directly from  $\Delta_+$  and they appeal to Lawvere's characterisation of  $\Delta_+$  as the free strict monoidal category containing a monoid [17] in order to establish its universal property. While those authors were not the first to discuss the existence of a 2-category whose structure encapsulates the algebraic properties of adjunctions, their paper was the first to provide an explicit and computationally convenient presentation of this structure. We review their construction of  $\underline{\text{Adj}}$  here as it will be useful to pass between our presentation and theirs in the sequel.

3.3.6. *Observation* (adjunctions in  $\Delta_+$ ). As is common practice, we shall identify each poset  $P$  with a corresponding category whose objects are the elements  $p$  of  $P$  and which possesses a unique arrow  $p \rightarrow q$  if and only if  $p \leq q$  in  $P$ . Under this identification, order preserving maps are identified with functors and two order preserving maps  $f, g: P \rightarrow Q$  are related by a unique 2-cell  $f \Rightarrow g$  if and only if  $f \leq g$  under the pointwise ordering. In particular, we may regard  $\Delta_+$  as being a full sub-2-category of  $\underline{\text{Cat}}$  under the pointwise ordering of simplicial operators.

It is easily demonstrated that a simplicial operator  $\alpha: [n] \rightarrow [m]$  admits a left adjoint  $\alpha^l \dashv \alpha$  (respectively right adjoint  $\alpha \dashv \alpha^r$ ) in the 2-category  $\Delta_+$  if and only if it carries the top element  $n$  (respectively the bottom element 0) of  $[n]$  to the top element  $m$  (respectively the bottom element 0) of  $[m]$ . In particular, between the ordinals  $[n-1]$  and  $[n]$  we have the following sequence of adjunctions

$$\delta_n^n \dashv \sigma_{n-1}^{n-1} \dashv \delta_n^{n-1} \dashv \sigma_{n-1}^{n-2} \dashv \dots \dashv \sigma_{n-1}^1 \dashv \delta_n^1 \dashv \sigma_{n-1}^0 \dashv \delta_n^0 \quad (3.3.7)$$

of elementary operators. We shall use the notation  $\Delta_\infty$  (respectively  $\Delta_{-\infty}$ ) to denote the sub-category of  $\Delta$  consisting of those simplicial operators which preserve the top (respectively bottom elements) in each ordinal.

Of course, each identity operator stands as its own left and right adjoint. Furthermore, since adjunctions compose, we know that if  $\alpha: [n] \rightarrow [m]$  and  $\beta: [m] \rightarrow [r]$  both admit left (respectively right) adjoints then so does their composite and  $(\beta \circ \alpha)^l = \alpha^l \circ \beta^l$  (respectively  $(\beta \circ \alpha)^r = \alpha^r \circ \beta^r$ ). It follows that the act of taking adjunctions provides us with a pair of mutually inverse contravariant functors

$$\Delta_\infty^{\text{op}} \begin{array}{c} \xleftarrow{(-)^r} \\ \xrightarrow{(-)^l} \end{array} \Delta_{-\infty}$$

whose action on elementary operators may be read off from (3.3.7) as

$$\begin{array}{ll} (\delta_n^i)^l = \sigma_{n-1}^i & \text{when } 0 \leq i < n & (\delta_n^i)^r = \sigma_{n-1}^{i-1} & \text{when } 0 < i \leq n \\ (\sigma_{n-1}^i)^l = \delta_n^{i+1} & \text{when } 0 \leq i \leq n-1 & (\sigma_{n-1}^i)^r = \delta_n^i & \text{when } 0 \leq i \leq n-1. \end{array}$$

Notice here that these formulae cover all cases because an elementary face operator  $\delta_n^i$  is in  $\Delta_\infty$  if and only if  $i < n$  and is in  $\Delta_{-\infty}$  if and only if  $0 < i$  whereas every elementary degeneracy operator is in both of these subcategories.

3.3.8. *Remark* (the Schanuel and Street 2-category  $\underline{\text{Adj}}$ ). Schanuel and Street define  $\underline{\text{Adj}}$  to be a 2-category with two objects, which we shall again call  $-$  and  $+$ , and with hom-categories given by

$$\begin{array}{ll} \underline{\text{Adj}}(+, +) := \Delta_+, & \underline{\text{Adj}}(-, -) := \Delta_+^{\text{op}}, \\ \underline{\text{Adj}}(-, +) := \Delta_\infty \cong \Delta_{-\infty}^{\text{op}}, \text{ and} & \underline{\text{Adj}}(+, -) := \Delta_{-\infty} \cong \Delta_\infty^{\text{op}} \end{array}$$

or more evocatively depicted as

$$\begin{array}{ccc} \Delta_+^{\text{op}} \curvearrowright - & \begin{array}{c} \xrightarrow{\Delta_{-\infty} \cong \Delta_{\infty}^{\text{op}}} \\ \xleftarrow{\Delta_{\infty} \cong \Delta_{-\infty}^{\text{op}}} \end{array} & + \curvearrowright \Delta_+ \end{array}$$

The isomorphisms  $\Delta_{\infty} \cong \Delta_{-\infty}^{\text{op}}$  are those of observation 3.3.6.

It should come as no surprise that  $\Delta_+$  features as the endo-hom-category on  $+$  in this structure. In essence this fact follows directly from Lawvere's result, since the monad generated by an adjunction is a monoid in a category of endofunctors under the strict monoidal structure given by composition. Schanuel and Street define the composition operations that hold between the hom-categories of  $\underline{\text{Adj}}$  in terms of the ordinal sum bifunctor

$$\begin{array}{ccc} \Delta_+ \times \Delta_+ & \xrightarrow{-\oplus-} & \Delta_+ \\ [n], [m] & \longmapsto & [n + m + 1] \\ \alpha, \beta \downarrow & \mapsto & \downarrow \alpha \oplus \beta \\ [n'], [m'] & \longmapsto & [n' + m' + 1] \end{array} \quad \alpha \oplus \beta(i) := \begin{cases} \alpha(i) & i \leq n \\ \beta(i - n - 1) + n' + 1 & i > n \end{cases}$$

which defines the strict monoidal structure on  $\Delta_+$ .

Ordinal sum, regarded as a bifunctor on  $\Delta_+$  and on its dual  $\Delta_+^{\text{op}}$ , provide the compositions

$$\underline{\text{Adj}}(+, +) \times \underline{\text{Adj}}(+, +) \xrightarrow{\circ} \underline{\text{Adj}}(+, +) \quad \underline{\text{Adj}}(-, -) \times \underline{\text{Adj}}(-, -) \xrightarrow{\circ} \underline{\text{Adj}}(-, -)$$

on endo-hom-categories. Ordinal sum restricts to the subcategories  $\Delta_{\infty}$  and  $\Delta_{-\infty}$  to give bifunctors

$$\Delta_+ \times \Delta_{\infty} \xrightarrow{\oplus} \Delta_{\infty} \quad \Delta_{-\infty} \times \Delta_+ \xrightarrow{\oplus} \Delta_{-\infty}$$

which provide the composition operations

$$\underline{\text{Adj}}(+, +) \times \underline{\text{Adj}}(-, +) \xrightarrow{\circ} \underline{\text{Adj}}(-, +) \quad \underline{\text{Adj}}(+, -) \times \underline{\text{Adj}}(+, +) \xrightarrow{\circ} \underline{\text{Adj}}(+, -)$$

respectively. Furthermore, the isomorphic presentations of  $\underline{\text{Adj}}(+, -)$  and  $\underline{\text{Adj}}(-, +)$  in terms of  $\Delta_{\infty}^{\text{op}}$  and  $\Delta_{-\infty}^{\text{op}}$  ensure that these restricted ordinal sum bifunctors on the duals  $\Delta_+^{\text{op}}$ ,  $\Delta_{\infty}^{\text{op}}$  and  $\Delta_{-\infty}^{\text{op}}$  may also be used to provide composition operations

$$\underline{\text{Adj}}(-, -) \times \underline{\text{Adj}}(+, -) \xrightarrow{\circ} \underline{\text{Adj}}(+, -) \quad \underline{\text{Adj}}(-, +) \times \underline{\text{Adj}}(-, -) \xrightarrow{\circ} \underline{\text{Adj}}(-, +)$$

respectively. We shall simply write

$$\Delta_{\infty} \times \Delta_+^{\text{op}} \xrightarrow{\oplus} \Delta_{\infty} \quad \Delta_+^{\text{op}} \times \Delta_{-\infty} \xrightarrow{\oplus} \Delta_{-\infty}$$

to denote these transformed instances of the join operation, since the order of the factors in the domain along with the dual that occurs there will disambiguate our usage.

Finally, observe that the following restriction of the ordinal sum bifunctor

$$\Delta_{-\infty} \times \Delta_{\infty} \xrightarrow{\oplus} \Delta_+$$

carries a pair of simplicial operators to a simplicial operator which preserves both top and bottom elements. So it follows that this bifunctor factors through the interval representation  $\text{ir}: \Delta_+^{\text{op}} \rightarrow \Delta_+$  to give a bifunctor

$$\Delta_{-\infty} \times \Delta_{\infty} \xrightarrow{\oplus} \Delta_+^{\text{op}}$$

which we use to provide the last two composition actions:

$$\underline{\text{Adj}}(-, +) \times \underline{\text{Adj}}(+, -) \xrightarrow{\circ} \underline{\text{Adj}}(+, +) \quad \underline{\text{Adj}}(+, -) \times \underline{\text{Adj}}(-, +) \xrightarrow{\circ} \underline{\text{Adj}}(-, -)$$

A copy of the object  $[0]$  resides in each of the hom-categories  $\underline{\text{Adj}}(-, +)$  and  $\underline{\text{Adj}}(+, -)$  and that these correspond to the 0-arrows  $\underline{u}$  and  $\underline{f}$  respectively in our presentation of  $\underline{\text{Adj}}$ . Furthermore the copies of the face operator  $\delta^0: [-1] \rightarrow [0]$  that reside in  $\underline{\text{Adj}}(+, +)$  and  $\underline{\text{Adj}}(-, -)$  correspond to our unit  $\underline{\eta}$  and counit  $\underline{\epsilon}$  respectively.

#### 4. ADJUNCTION DATA

Recall an adjunction of quasi-categories is an adjunction in  $\mathbf{qCat}_2$ . The basic theory of adjunctions is developed in section I.4. In this section, we filter the free homotopy coherent adjunction  $\underline{\text{Adj}}$  by a sequence of “parental” subcomputads and use this filtration to prove that any adjunction of quasi-categories—or, more precisely, any diagram indexed by a parental subcomputad—extends to a homotopy coherent adjunction, a diagram  $\underline{\text{Adj}} \rightarrow \mathbf{qCat}_{\infty}$ . Our proof is essentially constructive, enumerating the choices necessary to make each stage of the extension. We conclude by proving that the appropriate spaces of extensions, defined here, are contractible Kan complexes, the usual form of a “homotopical uniqueness” statement in the quasi-categorical context.

In §4.1, we introduce the notion of fillable arrow, which will be used in §4.2 to define parental subcomputads. Our aim in this section is to prove proposition 4.2.15, which shows that any nested pair of parental subcomputads may be filtered as a countable sequence of such, where each subcomputad is generated relative to the previous one by a finite set of fillable arrows. In §4.3, we apply this result to prove that any adjunction extends to a homotopy coherent adjunction. In §4.4, we give precise characterisations of the homotopical uniqueness of such extensions.

##### 4.1. Fillable arrows.

4.1.1. **Definition.** An arrow  $\underline{a}$  of  $\underline{\text{Adj}}$  is (*left*) *fillable* if and only if

- it is non-degenerate and atomic,
- its codomain  $a_0 = -$ , and
- $a_i \neq a_1$  for all  $i > 1$ .

Write  $\text{Atom}_n \subset \underline{\text{Adj}}_n$  for the subset of all atomic and non-degenerate  $n$ -arrows and write  $\text{Fill}_n \subset \text{Atom}_n$  for the subset of fillable  $n$ -arrows.

Our proof inductively specifies the data in the image of a homotopy coherent adjunction by choosing fillers for horns corresponding to fillable arrows. We will see that the unique

fillable 0-arrow  $\underline{f} = (-, +)$  behaves somewhat differently; nonetheless it is linguistically convenient to include it among the fillable arrows.

4.1.2. *Observation* (fillable arrows and distinguished faces). Any fillable  $n$ -arrow  $\underline{a}$  with  $n > 0$  has width greater than or equal to 2 and a distinguished codimension-1 face whose index is  $k(\underline{a}) := a_1$ . Note here that  $1 \leq k \leq n$  so this distinguished face may have any index except for 0. On account of the graphical calculus, in which a fillable arrow  $\underline{a}$  corresponds to a squiggle descending from “ $-$ ” on the left to make its first turn at level  $k(\underline{a})$ , we refer to  $k(\underline{a})$  as the *depth* of  $\underline{a}$ .

The fillability of  $\underline{a}$  implies that no reduction steps are then required in the process of forming the distinguished face  $\underline{a} \cdot \delta^{k(\underline{a})}$ . Consequently, this distinguished face is again non-degenerate and has the same width as  $\underline{a}$ . To further analyse this distinguished face, we need to consider two cases:

- **case**  $k(\underline{a}) < n$ :  $\underline{a} \cdot \delta^{k(\underline{a})}$  is also atomic. However, it is not fillable because non-degeneracy of  $\underline{a}$  requires that there is some  $i > 1$  such that  $a_i = a_1 + 1$ , whence the entries of  $\underline{a} \cdot \delta^{k(\underline{a})}$  at indices 1 and  $i$  are both equal to  $k(\underline{a})$ .
- **case**  $k(\underline{a}) = n$ :  $\underline{a} \cdot \delta^{k(\underline{a})}$  decomposes as  $\underline{f}\underline{b}$  where  $\underline{b}$  is non-degenerate and atomic, has width one less than that of  $\underline{a}$ , and has  $b_0 = +$ .

We shall use the notation  $\underline{a}^\diamond$  to denote the non-degenerate, atomic, and non-fillable  $(n - 1)$ -arrow given by:

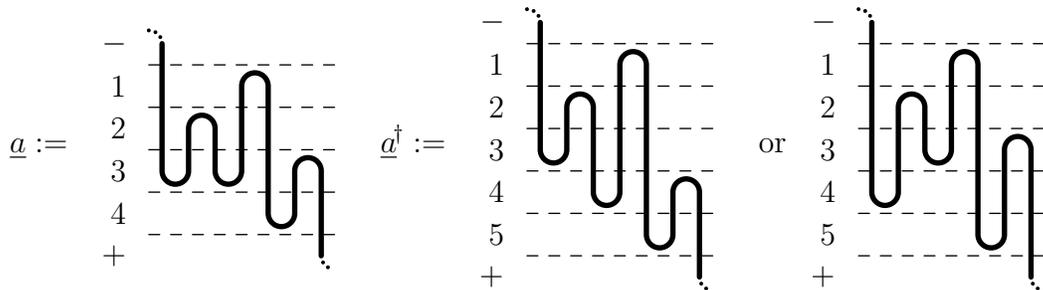
$$\underline{a}^\diamond := \begin{cases} \underline{a} \cdot \delta^{k(\underline{a})} & \text{when } k(\underline{a}) < n, \text{ and} \\ \underline{b} & \text{when } k(\underline{a}) = n \text{ and } \underline{a} \cdot \delta^{k(\underline{a})} = \underline{f}\underline{b}. \end{cases}$$

4.1.3. **Lemma.** *Let  $\underline{a}$  be a non-degenerate and atomic  $n$ -arrow of  $\underline{\text{Adj}}$  with  $a_0 = -$ . Then either it is:*

- a fillable arrow, or
- the codimension-1 face of exactly two fillable  $(n + 1)$ -arrows of the same width.

In the second case, both fillable  $(n + 1)$ -arrows have  $\underline{a}$  as the  $a_1^{\text{th}}$  face. One of these fillable arrows, which we shall denote by  $\underline{a}^\dagger$ , has depth  $a_1$  and the other has depth  $a_1 + 1$ . Consequently,  $\underline{a}^\dagger$  is the unique fillable  $(n + 1)$ -arrow with the property that  $(\underline{a}^\dagger)^\diamond = \underline{a}$ .

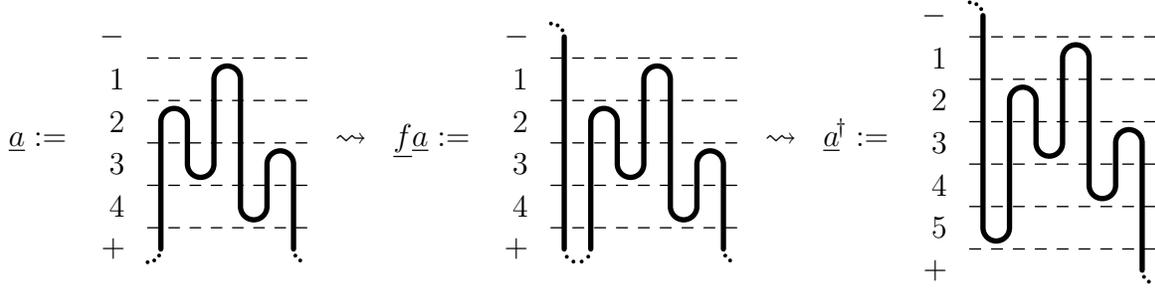
*Proof.* If  $\underline{a}$  is not fillable, then  $a_j = a_1$  for some  $j > 1$ . Any arrow  $\underline{b}$  admitting  $\underline{a}$  as a codimension-1 face is obtained by inserting an extra line. If the arrow  $\underline{b}$  is to be fillable and of the same width as  $\underline{a}$ , then this line must be inserted in the  $k^{\text{th}}$  level and used to separate  $a_1$  from the other  $a_j$ . There are exactly two ways to do this, as illustrated below:



□

**4.1.4. Lemma.** *Let  $\underline{a}$  be a non-degenerate and atomic  $n$ -arrow of  $\underline{\text{Adj}}$  with  $a_0 = +$ . Then the composite arrow  $\underline{fa}$  is a codimension-1 face of exactly one fillable  $(n+1)$ -arrow  $\underline{a}^\dagger$ . The  $(n+1)$ -arrow  $\underline{a}^\dagger$  has width one greater than that of  $\underline{a}$ ,  $\underline{a}_1^\dagger = n+1$ , and  $\underline{fa} = \underline{a}^\dagger \cdot \delta^{n+1}$ . In other words,  $\underline{a}^\dagger$  is the unique fillable  $(n+1)$ -arrow with the property that  $(\underline{a}^\dagger)^\circ = \underline{a}$ .*

*Proof.* The construction of  $\underline{a}^\dagger$  from  $\underline{a}$  is illustrated in the following sequence of diagrams:



That is, we “add” an extra loop to the left and then “insert” an extra line into the bottom most space. The uniqueness of this  $(n+1)$ -arrow is clear. □

## 4.2. Parental subcomputads.

**4.2.1. Definition** (fillable parents). When  $\underline{a}$  is a non-degenerate and atomic  $n$ -arrow in  $\underline{\text{Adj}}$  which is not fillable then we define its *fillable parent*  $\underline{a}^\dagger$  to be the fillable  $(n+1)$ -arrow introduced in lemma 4.1.3 in the case where  $a_0 = -$  and in lemma 4.1.4 in the case where  $a_0 = +$ . These lemmas tell us that  $\underline{a}^\dagger$  is the unique fillable  $(n+1)$ -arrow with the property that  $(\underline{a}^\dagger)^\circ = \underline{a}$ . Consequently, the fillable parent relation provides us with a canonical bijection between the set  $\text{Fill}_{n+1}$  of all fillable  $(n+1)$ -arrows and the set  $\text{Atom}_n \setminus \text{Fill}_n$  of all non-degenerate and atomic  $n$ -arrows which are not fillable.

**4.2.2. Definition** (parental subcomputads of  $\underline{\text{Adj}}$ ). We say that a subcomputad  $\mathbf{A}$  of  $\underline{\text{Adj}}$  is *parental* if it contains at least one non-identity arrow and satisfies the condition that

- if  $\underline{a}$  is a non-degenerate and atomic arrow in  $\mathbf{A}$  then either it is fillable or its fillable parent  $\underline{a}^\dagger$  is also in  $\mathbf{A}$ .

These conditions imply that any parental subcomputad must contain a fillable arrow. By observation 3.1.9, the  $0^{\text{th}}$  vertex of any fillable arrow may be decomposed as a composite  $\underline{fb}$ . Hence, any parental subcomputad contains the 0-arrow  $\underline{f}$ .

**4.2.3. Example.** The 0-arrow  $\underline{f}$  is fillable and the subcomputad  $\overline{\{\underline{f}\}} \subset \underline{\text{Adj}}$  that it generates, as described in definition 2.2.4, has  $\underline{f}$  as its only non-degenerate and atomic arrow, so is trivially a parental subcomputad. Since every parental subcomputad must contain  $\underline{f}$ , this is the minimal such.

The counit 1-arrow  $\underline{\epsilon}$  is fillable and the generated subcomputad  $\overline{\{\underline{\epsilon}\}} \subset \underline{\text{Adj}}$  has  $\underline{f}$ ,  $\underline{u}$ , and  $\underline{\epsilon}$  as its non-degenerate and atomic arrows. Now  $\underline{f}$  is fillable and  $\underline{\epsilon}$  is the fillable parent of  $\underline{u}$ , so  $\overline{\{\underline{\epsilon}\}}$  is a parental subcomputad.

The triangle identity 2-arrow  $\underline{\beta}$  is fillable and the generated subcomputad  $\overline{\{\underline{\beta}\}} \subset \underline{\text{Adj}}$  has  $\underline{f}$ ,  $\underline{u}$ ,  $\underline{\epsilon}$ ,  $\underline{\eta}$ , and  $\underline{\beta}$  as its non-degenerate and atomic arrows. Now  $\underline{f}$  is fillable,  $\underline{\epsilon}$  is the fillable parent of  $\underline{u}$ , and  $\underline{\beta}$  is the fillable parent of  $\underline{\eta}$ , so  $\overline{\{\underline{\beta}\}}$  is a parental subcomputad.

**4.2.4. Example** (a non-example). The subcomputad  $\overline{\{\underline{\alpha}, \underline{\beta}\}} \subset \underline{\text{Adj}}$  generated by the triangle identity 2-arrows has  $\underline{f}$ ,  $\underline{u}$ ,  $\underline{\epsilon}$ ,  $\underline{\eta}$ ,  $\underline{\beta}$ , and  $\underline{\alpha}$  as its non-degenerate and atomic arrows. This is not parental, as witnessed by the fact that the 3-arrow  $\underline{\omega}$  of example 3.1.11 is the fillable parent of the 2-arrow  $\underline{\alpha}$  but it is not an arrow in  $\overline{\{\underline{\alpha}, \underline{\beta}\}}$ .

**4.2.5. Example.** Example 3.1.11 names the 3-arrows  $\underline{\omega}$  and  $\underline{\tau}$  and the 2-arrow  $\underline{\mu}$  which featured in the discussion of adjunction data in section 1.1. Observe that the arrows  $\underline{\omega}$  and  $\underline{\tau}$  are both fillable and that the subcomputad  $\overline{\{\underline{\omega}, \underline{\tau}\}} \subset \underline{\text{Adj}}$  which they generate has  $\underline{f}$ ,  $\underline{u}$ ,  $\underline{\epsilon}$ ,  $\underline{\eta}$ ,  $\underline{\beta}$ ,  $\underline{\alpha}$ ,  $\underline{\tau}$ ,  $\underline{\omega}$ , and  $\underline{\mu}$  as its non-degenerate and atomic arrows. Since  $\underline{\tau}$  is the fillable parent of  $\underline{\mu}$ ,  $\overline{\{\underline{\omega}, \underline{\tau}\}}$  is also a parental subcomputad.

Examples 4.2.3 and 4.2.5 establish a chain of parental subcomputad inclusions

$$\overline{\{\underline{f}\}} \subset \overline{\{\underline{\epsilon}\}} \subset \overline{\{\underline{\beta}\}} \subset \overline{\{\underline{\omega}, \underline{\tau}\}} \subset \underline{\text{Adj}}.$$

Our aim in the remainder of this section is to filter a general parental subcomputad inclusion  $\mathbf{A} \subset \mathbf{A}'$  as a countable tower of parental subcomputads, with each sequential inclusion presented as the pushout of an explicit map. To describe each “attaching step,” we turn our attention to certain families of simplicial categories. Recall the simplicial categories  $\mathfrak{2}[X]$  introduced in 2.1.3.

**4.2.6. Notation.** Let  $\mathfrak{3}[X]$  denote the simplicial category with objects 0, 1, and 2, non-trivial hom-sets  $\mathfrak{3}[X](0, 1) := X$ ,  $\mathfrak{3}[X](1, 2) := \Delta^0$ ,  $\mathfrak{3}[X](0, 2) := X \star \Delta^0$ , and whose only non-trivial composition operation is defined by the canonical inclusion:

$$\mathfrak{3}[X](1, 2) \times \mathfrak{3}[X](0, 1) = \Delta^0 \times X \cong X \hookrightarrow X \star \Delta^0 = \mathfrak{3}[X](0, 2)$$

Here we define  $\mathfrak{3}[X](2, 1) = \mathfrak{3}[X](1, 0) = \mathfrak{3}[X](2, 0) = \emptyset$  and  $\mathfrak{3}[X](0, 0) = \mathfrak{3}[X](1, 1) = \mathfrak{3}[X](2, 2) = *$ . A simplicial functor  $F: \mathfrak{3}[X] \rightarrow \mathcal{K}$  is determined by the following data:

- a 0-arrow  $f: B \rightarrow A$  and an object  $C$  of  $\mathcal{K}$  and
- a pair of simplicial maps  $g: X \rightarrow \mathcal{K}(C, B)$  and  $h: X \star \Delta^0 \rightarrow \mathcal{K}(C, A)$  such that the following square commutes:

$$\begin{array}{ccc} X & \longrightarrow & X \star \Delta^0 \\ g \downarrow & & \downarrow h \\ \mathcal{K}(C, B) & \xrightarrow{\mathcal{K}(C, f)} & \mathcal{K}(C, A) \end{array} \quad (4.2.7)$$

The map  $h: X \star \Delta^0 \rightarrow \mathcal{K}(C, A)$  may be described in terms of Joyal’s slicing construction of definition I.2.4.2, by giving a 0-arrow  $a: C \rightarrow A$  (the image of the  $\Delta^0$ ) and a simplicial map  $\bar{h}: X \rightarrow \mathcal{K}(C, A)_{/a}$ . The commutative square (4.2.7) transposes to the commutative

square:

$$\begin{array}{ccc}
 X & \xrightarrow{\bar{h}} & \mathcal{K}(C, A)_{/a} \\
 g \downarrow & & \downarrow \pi \\
 \mathcal{K}(C, B) & \xrightarrow{\mathcal{K}(C, f)} & \mathcal{K}(C, A)
 \end{array} \tag{4.2.8}$$

This data may be captured by a single map  $l: X \rightarrow \mathcal{K}(C, f)_{/a}$  whose codomain is the pullback of  $\mathcal{K}(C, A)_{/a}$  along  $\mathcal{K}(C, f)$ , i.e., the slice  $\mathcal{K}(C, f)_{/a}$  of the map  $\mathcal{K}(C, f)$  over  $a$  as defined in remark I.2.4.14. Thus, a simplicial functor  $F: \mathfrak{B}[X] \rightarrow \mathcal{K}$  is determined by the following data:

- a pair of 0-arrows  $f: B \rightarrow A$  and  $a: C \rightarrow A$  of  $\mathcal{K}$  and
- a simplicial map  $l: X \rightarrow \mathcal{K}(C, f)_{/a}$ .

**4.2.9. Definition.** A fillable  $n$ -arrow  $\underline{a}$  gives rise to a corresponding simplicial functor  $F_{\underline{a}}$  into  $\underline{\text{Adj}}$  defined as follows:

- If  $a_1 < n$  define  $F_{\underline{a}}: \mathfrak{2}[\Delta^n] \rightarrow \underline{\text{Adj}}$  to be the simplicial functor induced out of the generic  $n$ -arrow  $\mathfrak{2}[\Delta^n]$  by  $\underline{a}$ ; cf. 2.1.3.
- If  $a_1 = n$  define  $F_{\underline{a}}: \mathfrak{3}[\Delta^{n-1}] \rightarrow \underline{\text{Adj}}$  so that it:
  - maps the objects 1 to  $+$ , 2 to  $-$ , and 0 to  $a_{w(\underline{a})}$ , the domain of  $\underline{a}$ ,
  - maps the hom-set  $\mathfrak{3}[\Delta^{n-1}](1, 2) \cong \Delta^0$  to  $\underline{\text{Adj}}(+, -)$  by the unique simplicial map which corresponds to the 0-arrow  $\underline{f}$ ,
  - maps the hom-set  $\mathfrak{3}[\Delta^{n-1}](0, 1) \cong \Delta^{n-1}$  to  $\underline{\text{Adj}}(a_{w(\underline{a})}, +)$  by the unique simplicial map which corresponds to the  $(n-1)$ -arrow  $\underline{a}^\diamond$ , and
  - maps the hom-set  $\mathfrak{3}[\Delta^{n-1}](0, 2) \cong \Delta^n$  to  $\underline{\text{Adj}}(a_{w(\underline{a})}, -)$  by the unique simplicial map which corresponds to the  $n$ -arrow  $\underline{a}$  itself.

The relation  $\underline{a} \cdot \delta_n = \underline{f} \underline{a}^\diamond$  implies that these actions are compatible with the composition structures of  $\mathfrak{3}[\Delta^{n-1}]$  and  $\underline{\text{Adj}}$ .

**4.2.10. Lemma** (extending parental subcomputads). *Suppose that  $\mathbf{A}$  is a parental subcomputad of  $\underline{\text{Adj}}$  and that  $\underline{a}$  is a fillable  $n$ -arrow of depth  $k := a_1$  which is not a member of  $\mathbf{A}$ , and let  $\mathbf{A}'$  be the subcomputad of  $\underline{\text{Adj}}$  generated by  $\mathbf{A} \cup \{\underline{a}\}$ . Suppose also that the codimension-1 face  $\underline{a} \cdot \delta^i$  is a member of  $\mathbf{A}$  for each  $i \in [n]$  with  $i \neq k$ . Then  $\mathbf{A}'$  is also a parental subcomputad, and we may restrict the the functor  $F_{\underline{a}}$  of definition 4.2.9 to express the inclusion  $\mathbf{A} \hookrightarrow \mathbf{A}'$  as a pushout*

$$\begin{array}{ccc}
 \mathfrak{2}[\Delta^{n,k}] & \hookrightarrow & \mathfrak{2}[\Delta^n] \\
 F_{\underline{a}} \downarrow & & \downarrow F_{\underline{a}} \\
 \mathbf{A} & \hookrightarrow & \mathbf{A}'
 \end{array} \tag{4.2.11}$$

when  $k < n$  and as a pushout

$$\begin{array}{ccc}
 \mathfrak{Z}[\partial\Delta^{n-1}] & \hookrightarrow & \mathfrak{Z}[\Delta^{n-1}] \\
 F_{\underline{a}} \downarrow & & \downarrow F_{\underline{a}} \\
 \mathbf{A} & \hookrightarrow & \mathbf{A}'
 \end{array} \quad (4.2.12)$$

when  $k = n$ .

*Proof.* Because  $\underline{a}$  is the fillable parent of  $\underline{a}^\diamond$ , this non-degenerate, atomic, and non-fillable  $(n-1)$ -arrow cannot be an element of the parental subcomputad  $\mathbf{A}$ . Now by assumption all of the faces  $\underline{a} \cdot \delta^i$  with  $i \neq k$  are contained in  $\mathbf{A}$ , so  $\underline{a}$  and  $\underline{a}^\diamond$  are the only two atomic arrows which are in  $\mathbf{A}'$  but are not in  $\mathbf{A}$ . The first of these is fillable and the second has the first as its fillable parent; hence,  $\mathbf{A}'$  is again parental.

To verify that the squares given in the statement are pushouts of simplicial categories, observe that extensions of  $F: \mathbf{A} \rightarrow \mathcal{K}$  to a simplicial functor  $F': \mathbf{A}' \rightarrow \mathcal{K}$  are completely determined by specifying what the atomic arrows  $\underline{a}$  and  $\underline{a}^\diamond$  should be mapped to in  $\mathcal{K}$ , subject to domain, codomain, and face conditions imposed by the simplicial functor  $F$ . Specifically, to make this extension we must provide:

- **case  $k < n$ :** an  $n$ -arrow  $g$  in  $\mathcal{K}$  with the property that  $g \cdot \delta^i = F(\underline{a} \cdot \delta^i)$  for all  $i \neq k$ , i.e., a simplicial functor  $g: \mathfrak{Z}[\Delta^n] \rightarrow \mathcal{K}$  which makes the following square commute:

$$\begin{array}{ccc}
 \mathfrak{Z}[\Delta^{n,k}] & \hookrightarrow & \mathfrak{Z}[\Delta^n] \\
 g \downarrow & & \downarrow g \\
 \mathbf{A} & \xrightarrow{F} & \mathcal{K}
 \end{array}$$

- **case  $k = n$ :** an  $(n-1)$ -arrow  $g$  and an  $n$ -arrow  $h$  in  $\mathcal{K}$  with the property that  $g \cdot \delta^i = F(\underline{a}^\diamond \cdot \delta^i)$  and  $h \cdot \delta^i = F(\underline{a} \cdot \delta^i)$  for all  $i \neq n$ , and also that  $h \cdot \delta^n = (Ff)g$ . In other words, by 4.2.6, we require a simplicial functor  $h: \mathfrak{Z}[\Delta^{n-1}] \rightarrow \mathcal{K}$  which makes the following square commute:

$$\begin{array}{ccc}
 \mathfrak{Z}[\partial\Delta^{n-1}] & \hookrightarrow & \mathfrak{Z}[\Delta^{n-1}] \\
 h \downarrow & & \downarrow h \\
 \mathbf{A} & \xrightarrow{F} & \mathcal{K}
 \end{array}$$

□

By iterating lemma 4.2.10, we have proven:

**4.2.13. Corollary.** *Suppose that  $\mathbf{A}$  is a parental subcomputad of  $\underline{\text{Adj}}$  and that  $X$  is a set of fillable arrows, each of which is not in  $\mathbf{A}$  but has the property that every face except the one indexed by its depth is in  $\mathbf{A}$ . Let  $\mathbf{A}'$  be the subcomputad of  $\underline{\text{Adj}}$  generated by  $\mathbf{A} \cup X$ . Then*

$\mathbf{A}'$  is a parental subcomputad and the inclusion  $\mathbf{A} \hookrightarrow \mathbf{A}'$  may be expressed as a pushout

$$\begin{array}{ccc}
\left( \coprod_{\substack{\underline{a} \in X \\ a_1 < \dim(\underline{a})}} 2[\Lambda^{\dim(\underline{a}), a_1}] \right) \sqcup \left( \coprod_{\substack{\underline{a} \in X \\ a_1 = \dim(\underline{a})}} 3[\partial \Delta^{\dim(\underline{a})-1}] \right) & \hookrightarrow & \left( \coprod_{\substack{\underline{a} \in X \\ a_1 < \dim(\underline{a})}} 2[\Delta^{\dim(\underline{a})}] \right) \sqcup \left( \coprod_{\substack{\underline{a} \in X \\ a_1 = \dim(\underline{a})}} 3[\Delta^{\dim(\underline{a})-1}] \right) \\
\downarrow \langle F_{\underline{a}} \rangle_{\underline{a} \in X} & & \downarrow \langle F_{\underline{a}} \rangle_{\underline{a} \in X} \\
\mathbf{A} & \hookrightarrow & \mathbf{A}'
\end{array} \tag{4.2.14}$$

of simplicial categories.

**4.2.15. Proposition.** *Suppose that  $\mathbf{A}$  and  $\mathbf{A}'$  are parental subcomputads of  $\underline{\text{Adj}}$  and that  $\mathbf{A} \subseteq \mathbf{A}'$ . Then we may filter this inclusion as a countable tower of parental subcomputads  $\mathbf{A}_0 \subseteq \mathbf{A}_1 \subseteq \mathbf{A}_2 \subseteq \dots$  ( $\mathbf{A} = \mathbf{A}_0$  and  $\mathbf{A}' = \bigcup_{i \geq 0} \mathbf{A}_i$ ) in such a way that for each  $i \geq 1$  there is a non-empty and finite set  $X_i$  of arrows such that*

- (i) *each arrow in  $X_i$  is fillable, is not contained in  $\mathbf{A}_{i-1}$ , but has the property that every face except the one indexed by its depth is in  $\mathbf{A}_{i-1}$ , and*
- (ii) *the subcomputad  $\mathbf{A}_i$  is generated by  $\mathbf{A}_{i-1} \cup X_i$ .*

Hence, the inclusion map  $\mathbf{A} \hookrightarrow \mathbf{A}'$  may be expressed as a countable composite of inclusions all of which may be constructed as pushouts of the form (4.2.14).

*Proof.* Let  $X$  denote the set of all fillable arrows which are in  $\mathbf{A}'$  and are not in  $\mathbf{A}$ , and let  $X_{w,k,n}$  denote the subset of those arrows which have width  $w$ , depth  $k$ , and dimension  $n$ . Now any non-degenerate arrow of  $\underline{\text{Adj}}$  must have dimension which is strictly less than its width, and it is clear there can only be a finite number of non-degenerate arrows of any given width. The depth of any fillable arrow is always less than or equal to its dimension, so it follows that  $X_{w,k,n}$  is always finite and that it is empty unless  $k \leq n < w$ .

Now order those index triples  $(w, k, n)$  which have  $k \leq n < w$  under the lexicographic ordering: for  $i \geq 1$ , let  $(w_i, k_i, n_i)$  index the subsequence of that linear ordering of those index triples for which  $X_{w,k,n}$  is non-empty, and write  $X_i := X_{w_i, k_i, n_i}$ . Let  $\mathbf{A}_i$  be the subcomputad of  $\underline{\text{Adj}}$  generated by  $\mathbf{A} \cup (\bigcup_{j=1}^i X_j)$  and observe that this family filters the inclusion  $\mathbf{A} \subseteq \mathbf{A}'$  since, by construction, the union of the subcomputads  $\mathbf{A}_i$  is  $\mathbf{A}'$ , the subcomputad generated by  $\mathbf{A} \cup (\bigcup_{i \geq 1} X_i)$ .

We complete our proof by induction on the index  $i$ , starting from the parental subcomputad  $\mathbf{A}_0 = \mathbf{A}$ . Adopt the inductive hypothesis that for all indices  $j < i$  the subcomputad  $\mathbf{A}_j$  is parental and condition (i) holds. For the inductive step, it suffices to check that all of the arrows in  $X_i$  satisfy condition (i) with respect to  $\mathbf{A}_{i-1}$ ; this amounts to verifying that the appropriate codimension-1 faces are in  $\mathbf{A}_{i-1}$ . Applying corollary 4.2.10 to the set  $X_i$  and the subcomputad  $\mathbf{A}_{i-1}$ , which is parental by the inductive hypothesis, it follows that  $\mathbf{A}_i$  is again parental.

Observe that  $\mathbf{A}_{i-1}$  is the smallest subcomputad of  $\underline{\text{Adj}}$  which contains  $\mathbf{A}$  and all fillable simplices in  $\mathbf{A}'$  which have

- width less than  $w_i$ , or

- width  $w_i$  and depth less than  $k_i$ , or
- width  $w_i$ , depth  $k_i$ , and dimension less than  $n_i$ .

Consider an arrow  $\underline{a}$  in  $X_i$ , which has width  $w_i$ , depth  $k_i$ , and dimension  $n_i$ . To complete the inductive step, it remains only to show that the face  $\underline{a} \cdot \delta^l$  is a member of  $\mathbf{A}_{i-1}$  for every  $l \in [n_i]$  which is not equal to the depth  $k_i$ . The details in each case, while tedious, are entirely straightforward.

- **case  $l \neq k_i - 1, l \neq k_i$ :** Under this condition the line numbered  $l$  is not one of those separating the level  $a_1$  from the other entries of  $\underline{a}$ . It follows that the removal of line  $l$  will not cause entry  $a_1$  to be eliminated by a reduction step, to become  $-$  or  $+$ , or to end up in the same space as another later entry. In other words,  $\underline{a} \cdot \delta^l$  is fillable if it is non-degenerate and atomic with depth  $k_i - 1$  if  $l < k_i - 1$  and depth  $k_i$  if  $l > k_i$ .

Since  $\underline{\text{Adj}}$  is a simplicial computad,  $\underline{a} \cdot \delta^l$  may be expressed uniquely as a composite  $(\underline{b}^1 \cdot \alpha_1) \circ (\underline{b}^2 \cdot \alpha_2) \circ \dots \circ (\underline{b}^r \cdot \alpha_r)$  in which each  $\underline{b}^j$  is non-degenerate and atomic arrow of  $\mathbf{A}'$  and each  $\alpha_j$  is a degeneracy operator. For the reasons just observed,  $\underline{b}^1$  is a fillable arrow of width less than or equal to  $w_i$ , depth less than or equal to  $k_i$ , and dimension strictly less than  $n_i$ . Hence,  $\underline{b}^1$  is a member of the subcomputad  $\mathbf{A}_{i-1}$ .

Furthermore,  $\underline{b}^1$  has width greater or equal to 2 (because  $b_1 \neq -$  and  $b_1 \neq +$ ) so the width of each  $\underline{b}^j$  with  $j > 1$  is less than or equal to  $w_i - 2$ . Consequently, when  $j > 1$ , then  $\underline{b}^j$  is either a fillable arrow of width less than or equal to  $w_i - 2$  or it has a fillable parent of width less than or equal to  $w_i - 1$ . In either case,  $\underline{b}^j$  is a member of the subcomputad  $\mathbf{A}_{i-1}$ . As  $\underline{a} \cdot \delta^l$  is a composite of degenerate images of arrows which are all in  $\mathbf{A}_{i-1}$ , it too lies in  $\mathbf{A}_{i-1}$ .

- **case  $l = k_i - 1$ :** As observed in 4.2.2, the parental subcomputad  $\mathbf{A}_{i-1}$  contains the fillable 0-arrow  $\underline{f}$ . Since  $\underline{a}$  is not in  $\mathbf{A}_{i-1}$ , we know that it must have width greater than or equal to 2 and dimension greater than or equal to 1. The only fillable arrow of width 2 is  $\underline{\epsilon} = (-, 1, -)$ , for which the depth  $k_i = 1, l = k_i - 1 = 0$ , and we have  $\underline{\epsilon} \cdot \delta^0 = -$  an identity, which is certainly in  $\mathbf{A}_{i-1}$ . So from hereon we may assume that  $w_i > 2$ . With this assumption,  $a_1 > a_2 \neq -$  and thus  $k_i \geq 2$ , from which it follows that on removing line  $l = k_i - 1$  the resulting face  $\underline{a} \cdot \delta^l$  must again be atomic. Now we have two subcases:
  - **case  $a_1 = a_2 + 1$ :** The line  $l = k_i - 1 = a_1 - 1$  separates the levels of  $a_1$  and  $a_2$ , so when we remove it to form the face  $\underline{a} \cdot \delta^l$  we must also perform at least one reduction step. This implies that  $\underline{a} \cdot \delta^l$  has width less than or equal to  $w_i - 2$  and that this face is possibly degenerate. There is a unique atomic and non-degenerate arrow  $\underline{b}$  and a unique degeneracy operator  $\alpha$  such that  $\underline{a} \cdot \delta^l = \underline{b} \cdot \alpha$ . The arrow  $\underline{b}$  is either a fillable arrow of width less than or equal to  $w_i - 2$  or it has a fillable parent of width less than or equal to  $w_i - 1$ . In either case, it follows that  $\underline{b}$ , and thus its degenerated partner  $\underline{a} \cdot \delta^l$ , is a member of  $\mathbf{A}_{i-1}$ .
  - **case  $a_1 > a_2 + 1$ :** The line  $l$  separates the level  $a_1$  from the level immediately above it, which contains neither  $a_0$  nor  $a_2$ . So when we remove that line to form the face  $\underline{a} \cdot \delta^l$  no reduction steps are required and this face must again be non-degenerate. However, since the arrow  $\underline{a}$  is non-degenerate there must be some  $j > 2$  such that  $a_j = a_1 - 1$  and hence  $\underline{a} \cdot \delta^l$  is not fillable. Now lemma 4.1.3 implies that the fillable

parent  $(\underline{a} \cdot \delta^l)^\dagger$  has depth  $a_1 - 1$ , which is one less than the depth  $k_i = a_1$  of  $\underline{a}$ . As  $\underline{a} \cdot \delta^l$  is a member of the parental subcomputad  $\mathbf{A}'$ , its fillable parent is also in  $\mathbf{A}'$ , and we may apply by the characterisation of  $\mathbf{A}_{i-1}$  to conclude that this fillable parent  $(\underline{a} \cdot \delta^l)^\dagger$ , and thus its face  $\underline{a} \cdot \delta^l$ , is in the parental subcomputad  $\mathbf{A}_{i-1}$ .  $\square$

**4.3. Homotopy Coherent Adjunctions.** In this section, we use proposition 4.2.15 to show that every adjunction of quasi-categories gives rise to a simplicial functor  $\underline{\text{Adj}} \rightarrow \underline{\text{qCat}}_\infty$  which carries the canonical adjunction in  $\underline{\text{Adj}}$  to the chosen adjunction of quasi-categories.

4.3.1. *Recall* (2-categories from quasi-categorically enriched categories). Recall, from observation I.3.1.2, that the homotopy category construction  $h: \underline{\text{qCat}} \rightarrow \underline{\text{Cat}}$  gives rise to a functor  $h_*: \underline{\text{qCat-Cat}} \rightarrow \underline{2\text{-Cat}}$  which reflects the category of quasi-categorically enriched categories  $\underline{\text{qCat-Cat}}$  into its full sub-category of 2-categories  $\underline{2\text{-Cat}}$ . The 2-category  $h_*\mathcal{K}$  is constructed by applying  $h$  to each of the hom-spaces of  $\mathcal{K}$ . We write  $\mathcal{K}_2 := h_*\mathcal{K}$  for the 2-category associated to a quasi-categorically enriched category  $\mathcal{K}$ . When  $F: \mathcal{K} \rightarrow \mathcal{L}$  is a simplicial functor of quasi-categorically enriched categories, we write  $F_2 := h_*F: \mathcal{K}_2 \rightarrow \mathcal{L}_2$  for the associated 2-functor. We shall also adopt the notation  $Q_{\mathcal{K}}: \mathcal{K} \rightarrow \mathcal{K}_2$  for the manifest quotient simplicial functor, the component at  $\mathcal{K}$  of the unit of the reflection  $h_*$ .

Given this relationship, we shall use the 2-cell notation  $\phi: f \Rightarrow g$  to denote a 1-arrow with 0-arrow faces  $f = \phi \cdot \delta^1$  and  $g = \phi \cdot \delta^0$ . This notation is consistent with the corresponding usage in the 2-category  $\mathcal{K}_2$ , since  $\phi$  is a representative of a genuine 2-cell  $\phi: f \Rightarrow g$  in there.

4.3.2. *Observation* (adjunctions in a quasi-categorically enriched category). Suppose that  $\mathcal{K}$  is a quasi-categorically enriched category. An adjunction  $f \dashv u: A \rightarrow B$  in the 2-category  $\mathcal{K}_2$  may be presented by the following information in  $\mathcal{K}$  itself:

- a pair of 0-arrows  $u \in \mathcal{K}(A, B)$  and  $f \in \mathcal{K}(B, A)$ ,
- a pair of 1-arrows  $\eta \in \mathcal{K}(B, B)$  and  $\epsilon \in \mathcal{K}(A, A)$  which represent the unit and counit 2-cells in  $\mathcal{K}_2$  and whose boundaries are depicted in the following pictures

$$\text{id}_B \xrightarrow{\eta} u f \qquad f u \xrightarrow{\epsilon} \text{id}_A ,$$

and

- a pair of 2-arrows  $\alpha \in \mathcal{K}(A, B)$  and  $\beta \in \mathcal{K}(B, A)$  which witness the triangle identities and whose boundaries are depicted in the following pictures:

$$\begin{array}{ccc} & u f u & \\ \eta u \nearrow & & \searrow u \epsilon \\ u & \xrightarrow{\alpha} & u \\ & u \cdot \sigma^0 & \end{array} \qquad \begin{array}{ccc} & f u f & \\ f \eta \nearrow & & \searrow \epsilon f \\ f & \xrightarrow{\beta} & f \\ & f \cdot \sigma^0 & \end{array}$$

This information is not uniquely determined by our adjunction since it involves choices of representative 1-arrows for its unit and counit 2-cells and choices of witnessing 2-arrows for its triangle identities.

This data used to present an adjunction in  $\mathcal{K}$  uniquely determines a simplicial functor  $T: \{\underline{\alpha}, \underline{\beta}\} \rightarrow \mathcal{K}$  whose domain is the subcomputad of  $\underline{\text{Adj}}$  generated by the triangle identity

2-arrows, as in example 4.2.4, and whose action on non-degenerate and atomic arrows is given by  $T(\underline{f}) = f$ ,  $T(\underline{u}) = u$ ,  $T(\underline{\epsilon}) = \epsilon$ ,  $T(\underline{\eta}) = \eta$ ,  $T(\underline{\beta}) = \beta$ , and  $T(\underline{\alpha}) = \alpha$ .

Since adjunctions are defined equationally in a 2-category, they are preserved by any 2-functor. It follows, therefore, that adjunctions are preserved by the 2-functor  $F_2: \mathcal{K}_2 \rightarrow \mathcal{L}_2$  associated with any simplicial functor  $F: \mathcal{K} \rightarrow \mathcal{L}$  of quasi-categorically enriched categories. Explicitly, the adjunction displayed above transports along  $F$  to give an adjunction  $F(f) \dashv F(u)$  in  $\mathcal{L}$  which is presented by unit and counit 1-arrows  $F(\eta)$  and  $F(\epsilon)$  and 2-arrows  $F(\alpha)$  and  $F(\beta)$  which witness its triangle identities.

**4.3.3. Notation.** For the remainder of this section we shall assume that  $\mathcal{K}$  and  $\mathcal{L}$  are quasi-categorically enriched categories. Furthermore, we shall assume that we have been given a simplicial functor  $P: \mathcal{K} \rightarrow \mathcal{L}$  which is a *local isofibration* in the sense that its action  $P: \mathcal{K}(A, B) \rightarrow \mathcal{L}(PA, PB)$  on each hom-space is an isofibration of quasi-categories. We will also fix an adjunction

$$A \begin{array}{c} \xleftarrow{f} \\ \perp \\ \xrightarrow{u} \end{array} B$$

in  $\mathcal{K}_2$  that is presented in  $\mathcal{K}$  by unit and counit 1-arrows  $\eta: \text{id}_B \Rightarrow uf$  and  $\epsilon: fu \Rightarrow \text{id}_A$  and 2-arrows  $\alpha$  and  $\beta$  which witness its triangle identities as in observation 4.3.2. To remind the reader of our standing hypotheses, we might write “suppose  $\mathcal{K}$  has an adjunction  $(f \dashv u, \epsilon)$ ,” listing in parentheses the data in  $\mathcal{K}$  chosen to present an adjunction in  $\mathcal{K}_2$ .

**4.3.4. Observation** (the internal universal property of the counit). Suppose that  $C$  is an arbitrary object of  $\mathcal{K}$ . The representable simplicial functor  $\mathcal{K}(C, -): \mathcal{K} \rightarrow \underline{\text{qCat}}_\infty$  carries our adjunction  $f \dashv u$  in  $\mathcal{K}$  to an adjunction

$$\mathcal{K}(C, A) \begin{array}{c} \xleftarrow{\mathcal{K}(C, f)} \\ \perp \\ \xrightarrow{\mathcal{K}(C, u)} \end{array} \mathcal{K}(C, B)$$

of quasi-categories with unit  $\mathcal{K}(C, \eta)$  and counit  $\mathcal{K}(C, \epsilon)$ . On applying proposition I.4.4.8 to this adjunction of quasi-categories, we find that if  $a$  is a 0-arrow in  $\mathcal{K}(C, A)$  then  $\epsilon a: fua \Rightarrow a$  may be regarded as being an object of the slice quasi-category  $\mathcal{K}(C, f)_{/a}$  wherein it is a terminal object. So, in particular, it follows that if  $\partial\Delta^{n-1} \rightarrow \mathcal{K}(C, f)_{/a}$  is a simplicial map which carries the vertex  $\{n-1\}$  of  $\partial\Delta^{n-1}$  to the object  $\epsilon a$  then it may be extended along the inclusion  $\partial\Delta^{n-1} \hookrightarrow \Delta^{n-1}$  to a simplicial map  $\Delta^{n-1} \rightarrow \mathcal{K}(C, f)_{/a}$ .

On consulting 4.2.6, we discover that simplicial maps  $\partial\Delta^{n-1} \rightarrow \mathcal{K}(C, f)_{/a}$  (respectively  $\Delta^{n-1} \rightarrow \mathcal{K}(C, f)_{/a}$ ) which carry  $\{n-1\}$  to  $\epsilon a$  stand in bijective correspondence to simplicial functors  $\mathfrak{B}[\partial\Delta^{n-1}] \rightarrow \mathcal{K}$  (respectively  $\mathfrak{B}[\Delta^{n-1}] \rightarrow \mathcal{K}$ ) which carry the 0-arrow  $\{0\}$  of  $\mathfrak{B}[\partial\Delta^{n-1}](1, 2) = \Delta^0$  to  $f$ , the 0-arrow  $\{n-1\}$  of  $\mathfrak{B}[\partial\Delta^{n-1}](0, 1)$  to  $ua$ , and the 1-arrow  $\{n-1, n\}$  of  $\mathfrak{B}[\partial\Delta^{n-1}](0, 2) = \Delta^{n-1} \star \Delta^0 \cong \Delta^n$  to  $\epsilon a$ . It follows that the universal property of the counit 1-arrow  $\epsilon$  discussed above simply posits the existence of the lift  $T'$  in the

following diagram

$$\begin{array}{ccc} \mathfrak{B}[\partial\Delta^{n-1}] & \xrightarrow{T} & \mathcal{K} \\ \downarrow & \nearrow T' & \\ \mathfrak{B}[\Delta^{n-1}] & & \end{array}$$

so long as  $T(\{0\}: 1 \rightarrow 2) = f$ ,  $T(\{n-1\}: 0 \rightarrow 1) = ua$ , and  $T(\{n-1, n\}: 0 \rightarrow 2) = \epsilon a$ .

To prove a relative version of this result, we require the following lemma:

**4.3.5. Lemma** (a relative universal property of terminal objects). *Suppose that  $E$  and  $B$  are quasi-categories which possess terminal objects and that  $p: E \twoheadrightarrow B$  is an isofibration which preserves terminal objects, in the sense that if  $t$  is terminal in  $E$  then  $pt$  is terminal in  $B$ . Then any lifting problem*

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{u} & E \\ \downarrow & & \downarrow p \\ \Delta^n & \xrightarrow{v} & B \end{array}$$

with  $n > 0$  has a solution so long as  $u$  carries the vertex  $\{n\}$  to a terminal object in  $E$ .

*Proof.* Using the universal property of the terminal object  $t := u\{n\}$  in  $E$  we may extend the map  $u: \partial\Delta^n \rightarrow E$  to a map  $w: \Delta^n \rightarrow E$ . Now we have two maps  $pw, v: \Delta^n \rightarrow B$  both of which restrict to the boundary  $\partial\Delta^n$  to give the same map  $pu: \partial\Delta^n \rightarrow B$ . From these we can construct a map  $h: \partial\Delta^{n+1} \rightarrow B$  with  $h\delta^{n+1} = pw$  and  $h\delta^n = v$  by starting with the degenerate simplex  $pw\sigma^n: \Delta^{n+1} \rightarrow B$ , restricting to its boundary, and then replacing the  $n^{\text{th}}$  face in this sphere with  $v: \Delta^n \rightarrow B$ . Of course  $h$  maps the object  $\{n+1\}$  to the object  $pt$  which is terminal in  $B$ , so it follows that we may extend it to a map  $k: \Delta^{n+1} \rightarrow B$ . We may also construct a map  $g: \Lambda^{n+1, n} \rightarrow E$  by restriction from the degenerate simplex  $w\sigma^n: \Delta^{n+1} \rightarrow E$  and observe that we may decompose the commutative square of the statement into the following composite of commutative squares:

$$\begin{array}{ccccc} \partial\Delta^n & \xrightarrow{\delta^n} & \Lambda^{n+1, n} & \xrightarrow{g} & E \\ \downarrow & & \downarrow & \nearrow l & \downarrow p \\ \Delta^n & \xrightarrow{\delta^n} & \Delta^{n+1} & \xrightarrow{k} & B \end{array}$$

Since the central vertical of this square is an inner horn inclusion and its right hand vertical is an isofibration of quasi-categories, it follows that the lifting problem on the right has a solution  $l: \Delta^{n+1} \rightarrow E$  as marked. Now it is clear that the map  $l\delta^n: \Delta^n \rightarrow E$  provides a solution to the original lifting problem.  $\square$

**4.3.6. Proposition** (the relative internal universal property of the counit). *If  $\mathcal{K}$  has an adjunction  $(f \dashv u, \epsilon)$ , then the following lifting problem has a solution*

$$\begin{array}{ccc} \mathfrak{B}[\partial\Delta^{n-1}] & \xrightarrow{T} & \mathcal{K} \\ \downarrow & & \downarrow P \\ \mathfrak{B}[\Delta^{n-1}] & \xrightarrow{S} & \mathcal{L} \end{array}$$

provided that  $T(\{0\}: 1 \rightarrow 2) = f$ ,  $T(\{n-1\}: 0 \rightarrow 1) = ua$ , and  $T(\{n-1, n\}: 0 \rightarrow 2) = \epsilon a$  for some 0-arrow  $a \in \mathcal{K}(C, A)$ .

*Proof.* On consulting 4.2.6, we see that we may translate the lifting problem of the statement into a lifting problem of the following form

$$\begin{array}{ccc} \partial\Delta^{n-1} & \xrightarrow{t} & \mathcal{K}(C, f)_{/a} \\ \downarrow & & \downarrow P \\ \Delta^{n-1} & \xrightarrow{s} & \mathcal{L}(PC, Pf)_{/Pa} \end{array}$$

in simplicial sets. The upper horizontal map  $t: \partial\Delta^{n-1} \rightarrow \mathcal{K}(C, f)_{/a}$  carries the vertex  $\{n-1\}$  to the object  $\epsilon a$  of  $\mathcal{K}(C, f)_{/a}$ , which is terminal in there by observation 4.3.4. Furthermore, using the local isofibration property of the simplicial functor  $P$  it is easily verified that the vertical map on the right of this square is an isofibration of quasi-categories. This map carries the terminal object  $\epsilon a$  of  $\mathcal{K}(C, f)_{/a}$  to the object  $P(\epsilon a) = (P\epsilon)(Pa)$  of  $\mathcal{L}(PC, Pf)_{/Pa}$ , which is again terminal since  $P\epsilon$  is the counit of the transported adjunction  $Pf \dashv Pu$  in  $\mathcal{L}$ . Applying lemma 4.3.5, we obtain the desired lift.  $\square$

**4.3.7. Observation.** As an easier observation of a similar ilk, note that the fact that our simplicial functor  $P: \mathcal{K} \rightarrow \mathcal{L}$  is a local isofibration implies that that we may solve the lifting problem

$$\begin{array}{ccc} \mathfrak{Z}[\Lambda^{n,k}] & \xrightarrow{T} & \mathcal{K} \\ \downarrow & & \downarrow P \\ \mathfrak{Z}[\Delta^n] & \xrightarrow{S} & \mathcal{L} \end{array}$$

whenever  $n \geq 2$  and  $0 < k < n$ .

Combining these observations with the results of §4.2, we obtain the following lifting result:

**4.3.8. Theorem.** *Suppose that  $\mathbf{A}$  and  $\mathbf{A}'$  are parental subcomputads of  $\underline{\text{Adj}}$  and that  $\mathbf{A} \subseteq \mathbf{A}'$ . Furthermore, assume that  $\mathbf{A}$  contains the 0-arrows  $\underline{u}$  and  $\underline{f}$  and the 1-arrow  $\underline{\epsilon}$ . Then*

if  $\mathcal{K}$  has an adjunction  $(f \dashv u, \epsilon)$ , we may solve the lifting problem

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{T} & \mathcal{K} \\ \downarrow & & \downarrow P \\ \mathbf{A}' & \xrightarrow{S} & \mathcal{L} \end{array}$$

so long as  $T(\underline{f}) = f$ ,  $T(\underline{u}) = u$ , and  $T(\underline{\epsilon}) = \epsilon$ .

*Proof.* We know, by proposition 4.2.15, that we may filter the inclusion  $\mathbf{A} \subseteq \mathbf{A}'$  as a countable sequence of inclusions all of which may be constructed as pushouts of the form (4.2.11) or (4.2.12). It follows that we may reduce this result to the case where  $\mathbf{A}'$  is a parental subcomputad which extends  $\mathbf{A}$  by the addition of a single fillable  $n$ -arrow  $\underline{a}$  of depth  $k := a_1$  as discussed in the statement of lemma 4.2.10.

Now consider the two cases identified in lemma 4.2.10. The first of these is the easy case  $k < n$ , in which situation we have the following commutative diagram

$$\begin{array}{ccccc} \mathfrak{2}[\Lambda^{n,k}] & \xrightarrow{F_{\underline{a}}} & \mathbf{A} & \xrightarrow{T} & \mathcal{K} \\ \downarrow & & \downarrow & & \downarrow P \\ \mathfrak{2}[\Delta^n] & \xrightarrow{F_{\underline{a}}} & \mathbf{A}' & \xrightarrow{S} & \mathcal{L} \end{array}$$

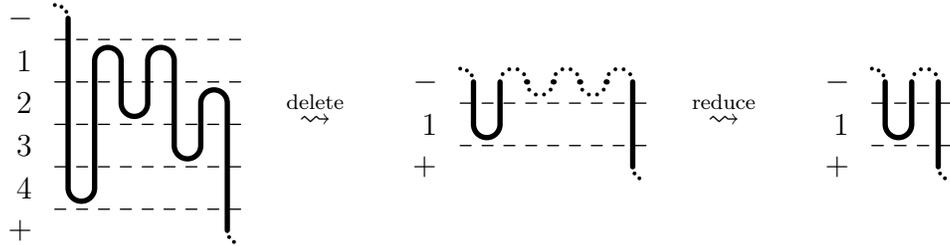
where the square on the left is the pushout of (4.2.11). Now observation 4.3.7 provides us with a solution for lifting problem which is the composite of these two squares. Then we may use that lift and the universal property of the pushout on the left to construct the solution we seek for the lifting problem on the right.

In the case where  $k = n$  our argument is a little more involved, but here again we start with a commutative diagram

$$\begin{array}{ccccc} \mathfrak{3}[\partial\Delta^{n-1}] & \xrightarrow{F_{\underline{a}}} & \mathbf{A} & \xrightarrow{T} & \mathcal{K} \\ \downarrow & & \downarrow & & \downarrow P \\ \mathfrak{3}[\Delta^{n-1}] & \xrightarrow{F_{\underline{a}}} & \mathbf{A}' & \xrightarrow{S} & \mathcal{L} \end{array}$$

where the square on the left is the pushout of (4.2.12). Consulting the definition of the simplicial functor  $F_{\underline{a}}: \mathfrak{3}[\Delta^{n-1}] \rightarrow \underline{\text{Adj}}$ , as given in 4.2.9, we see that it maps the 0-arrow  $\{0\}: 1 \rightarrow 2$  to  $f$  and it maps the  $n$ -arrow  $\text{id}_{[n]}: 0 \rightarrow 2$  to  $\underline{a}$ , so it maps  $\{n-1, n\}: 0 \rightarrow 2$  to  $\underline{a} \cdot \{n-1, n\}$ . To calculate this edge we delete the lines numbered  $0, 1, \dots, n-2$  and then reduce. However  $\underline{a}$  is a fillable  $n$ -arrow of depth  $k = n$  so it has  $a_0 = -$ , it is atomic so  $a_i \neq -, +$  for  $0 < i < w(\underline{a})$ , and  $a_i \neq a_1$  for all  $i > 1$ ; these facts together imply that  $n = a_1 > a_i$  for  $1 < i < w(\underline{a})$ . There are now two cases to consider, depending on whether the domain  $a_{w(\underline{a})}$  of  $\underline{a}$  is  $-$  or  $+$ . In the first of these, the removal of lines  $0, 1, \dots, n-2$  leaves a string of the form  $(-, 1, -, -, \dots, -)$  in which the sequence of trailing  $-$  symbols

is of odd length, so this reduces to  $\underline{\epsilon} = (-, 1, -)$ . In the second, the removal of those lines leaves a string of the form  $(-, 1, -, -, \dots, -, +)$  where again the sequence of  $-$  symbols is of odd length, so this reduces to  $\underline{\epsilon}f = (-, 1, -, +)$ . The second of these computations is illustrated in the following sequence of squiggle pictures:



By assumption,  $T$  maps  $f$  to the 0-arrow  $f: B \rightarrow A$  and it maps  $\underline{\epsilon}$  to the 1-arrow  $\epsilon$ , so it follows that  $TF_{\underline{a}}(\{n-1, n\}: 0 \rightarrow 2)$  is equal to  $\epsilon$  when  $a_{w(\underline{a})}$  is  $-$  and is equal to  $\epsilon f$  when  $a_{w(\underline{a})}$  is  $+$ . In either case the map  $TF_{\underline{a}}: \mathfrak{B}[\partial\Delta^{n-1}] \rightarrow \mathcal{K}$  conforms to the conditions of proposition 4.3.6, with  $a = \text{id}_A$  in the first case and  $a = f$  in the second. Applying that result, we may find a solution for the lifting problem expressed by this composite rectangle. Then we may use that lift and the universal property of the pushout on the left to construct the solution we seek for the lifting problem on the right.  $\square$

Having cleared the heavy lifting, we are now in a position to prove that it is possible to extend every adjunction in  $\mathcal{K}_2$  to a homotopy coherent adjunction in  $\mathcal{K}$ .

**4.3.9. Theorem** (homotopy coherence of adjunctions I). *If  $\mathcal{K}$  has an adjunction  $(f \dashv u, \epsilon)$ , then there exists a simplicial functor  $H: \underline{\text{Adj}} \rightarrow \mathcal{K}$  for which  $H(f) = f$ ,  $H(u) = u$ , and  $H(\underline{\epsilon}) = \epsilon$ .*

*Proof.* By example 4.2.3, the subcomputad  $\overline{\{\underline{\epsilon}\}} \subset \underline{\text{Adj}}$  is a parental subcomputad whose non-degenerate and atomic arrows are  $\underline{f}$ ,  $\underline{u}$ , and  $\underline{\epsilon}$ . Consequently, there exists a simplicial functor  $T: \overline{\{\underline{\epsilon}\}} \rightarrow \mathcal{K}$  which is uniquely determined by the fact that it maps those generators to the corresponding arrows  $f$ ,  $u$ , and  $\epsilon$  in  $\mathcal{K}$  respectively. Now we have a lifting problem

$$\begin{array}{ccc}
 \overline{\{\underline{\epsilon}\}} & \xrightarrow{T} & \mathcal{K} \\
 \downarrow & \nearrow H & \downarrow ! \\
 \underline{\text{Adj}} & \xrightarrow{!} & \mathbb{1}
 \end{array}$$

where  $\mathbb{1}$  denotes the terminal simplicial category whose only hom-set is  $\Delta^0$ . Because each hom-space of  $\mathcal{K}$  is a quasi-category, the right hand vertical in this square is a local isofibration. Applying theorem 4.3.8, we obtain the dashed lift  $H: \underline{\text{Adj}} \rightarrow \mathcal{K}$  which, by construction, has the properties asked for in the statement.  $\square$

**4.3.10. Remark.** Applying theorem I.6.1.4 to the characterisation of adjunctions found in example I.5.0.4, we see that a functor  $f: B \rightarrow A$  is a left adjoint if and only if the slice quasi-category  $f_{/a}$  has a terminal object for each vertex  $a \in A$ . In this case, theorem I.6.1.4

supplies a right adjoint  $u$  and counit  $\epsilon: fu \Rightarrow \text{id}_A$  in  $\mathbf{qCat}_2$ . On choosing any representative 1-arrow for that counit, theorem 4.3.9 extends this data to a simplicial functor  $H: \underline{\mathbf{Adj}} \rightarrow \mathbf{qCat}_\infty$ .

**4.3.11. Theorem** (homotopy coherence of adjunctions II). *If  $\mathcal{K}$  has an adjunction  $(f \dashv u, \epsilon, \eta, \beta)$ , there exists a simplicial functor  $H: \underline{\mathbf{Adj}} \rightarrow \mathcal{K}$  for which  $H(\underline{f}) = f$ ,  $H(\underline{u}) = u$ ,  $H(\underline{\epsilon}) = \epsilon$ ,  $H(\underline{\eta}) = \eta$ , and  $H(\underline{\beta}) = \beta$ .*

*Proof.* We follow the same pattern of argument as in the proof of theorem 4.3.9. This starts by observing that example 4.2.3 tells us that the subcomputad  $\overline{\{\beta\}} \subset \underline{\mathbf{Adj}}$  is a parental subcomputad whose non-degenerate and atomic arrows are  $\underline{f}$ ,  $\underline{u}$ ,  $\underline{\epsilon}$ ,  $\underline{\eta}$ , and  $\underline{\beta}$ . It follows then that there exists a simplicial functor  $T: \overline{\{\beta\}} \rightarrow \mathcal{K}$  which is uniquely determined by the fact that it maps those generators to the corresponding arrows  $f$ ,  $u$ ,  $\epsilon$ ,  $\eta$ , and  $\beta$  in  $\mathcal{K}$  respectively. Applying theorem 4.3.8, we again construct a simplicial functor  $H: \underline{\mathbf{Adj}} \rightarrow \mathcal{K}$  which, by construction, satisfies the conditions of the statement.  $\square$

**4.3.12. Remark.** Note that theorem 4.3.11 does not impose any conditions concerning the action of the simplicial functor  $H: \underline{\mathbf{Adj}} \rightarrow \mathcal{K}$  on the other triangle identity 2-arrow  $\underline{\alpha}$ . In general, while  $H(\underline{\alpha})$  is a 2-arrow which witnesses the other triangle identity of  $f \dashv u$  there is no reason why it should be equal to the particular witness  $\alpha$  that we fixed in observation 4.3.2. Indeed it is possible that there may be no simplicial functor  $H: \underline{\mathbf{Adj}} \rightarrow \mathcal{K}$  which simultaneously maps both of the 2-arrows  $\underline{\alpha}$  and  $\underline{\beta}$  to that chosen pair of witnesses for the triangle identities.

**4.3.13. Definition.** We know by proposition 3.3.4 that there exists a unique 2-functor  $F: \underline{\mathbf{Adj}} \rightarrow \mathcal{K}_2$  which carries the canonical adjunction in  $\underline{\mathbf{Adj}}$  to the chosen adjunction  $f \dashv u$  in  $\mathcal{K}_2$ . If this 2-functor lifts through the quotient simplicial functor from  $\mathcal{K}$  to  $\mathcal{K}_2$  as in the following diagram

$$\begin{array}{ccc} \underline{\mathbf{Adj}} & \xrightarrow{H} & \mathcal{K} \\ & \searrow F & \downarrow Q_{\mathcal{K}} \\ & & \mathcal{K}_2 \end{array}$$

then we say that the dashed simplicial functor  $H: \underline{\mathbf{Adj}} \rightarrow \mathcal{K}$  is a lift of our adjunction  $f \dashv u$  to a *homotopy coherent adjunction* in  $\mathcal{K}$ . More explicitly,  $H$  is any simplicial functor which maps  $\underline{u}$  and  $\underline{f}$  to the corresponding 0-arrows of the adjunction  $f \dashv u$  and which maps  $\underline{\epsilon}$  and  $\underline{\eta}$  to representatives for the unit and counit of that adjunction. As an immediate consequence of theorem 4.3.11, every adjunction in  $\mathcal{K}_2$  lifts to a homotopy coherent adjunction in  $\mathcal{K}$ .

**4.4. Homotopical uniqueness of homotopy coherent adjunctions.** We conclude this section by proving that the space of all lifts of an adjunction to a homotopy adjunction is not only non-empty, as guaranteed by theorem 4.3.11, but is also a contractible Kan complex. In other words, this result says that lifts of an adjunction to a homotopy coherent adjunction are “homotopically unique”.

4.4.1. *Observation* (simplicial enrichment of simplicial categories). We may apply the product preserving exponentiation functor  $(-)^X: \mathbf{sSet} \rightarrow \mathbf{sSet}$  to the hom-spaces of any simplicial category  $\mathcal{K}$  to obtain a simplicial category  $\mathcal{K}^X$ . This construction defines a bifunctor  $\mathbf{sSet}^{\text{op}} \times \mathbf{sSet}\text{-Cat} \rightarrow \mathbf{sSet}\text{-Cat}$ , and there exist canonical natural isomorphisms  $\mathcal{K}^{\Delta^0} \cong \mathcal{K}$  and  $\mathcal{K}^{X \times Y} \cong (\mathcal{K}^X)^Y$  which obey manifest coherence conditions.

Given an action of this kind of  $\mathbf{sSet}$  on  $\mathbf{sSet}\text{-Cat}$ , we may construct an enrichment of the latter to a (large) simplicial category. Specifically, we take the  $n$ -arrows between simplicial categories  $\mathcal{K}$  and  $\mathcal{L}$  to be simplicial functors  $F: \mathcal{K} \rightarrow \mathcal{L}^{\Delta^n}$ . The action of  $\Delta$  on these is given by  $F \cdot \alpha := \mathcal{L}^\alpha \circ F$ , and we compose  $F$  with a second such  $n$ -simplex  $G: \mathcal{L} \rightarrow \mathcal{M}^{\Delta^n}$  by forming the composite:

$$\mathcal{K} \xrightarrow{F} \mathcal{L}^{\Delta^n} \xrightarrow{G^{\Delta^n}} (\mathcal{M}^{\Delta^n})^{\Delta^n} \cong \mathcal{M}^{(\Delta^n \times \Delta^n)} \xrightarrow{\mathcal{M}^\nabla} \mathcal{M}^{\Delta^n}$$

The associativity and identity rules for this composition operation are direct consequences of the fact that for any  $X$  the diagonal map  $\nabla: X \rightarrow X \times X$  and the unique map  $!: X \rightarrow \Delta^0$  obey the co-associativity and co-identity rules. Under this enrichment by (possibly large) simplicial sets, the construction  $\mathcal{K}^X$  becomes the simplicial cotensor of  $\mathcal{K}$  by  $X$ .

We write  $\text{icon}(\mathcal{K}, \mathcal{L})$  to denote the (possibly large) simplicial hom-space between simplicial categories. The notation “icon” is chosen here because a 1-simplex in  $\text{icon}(\mathcal{K}, \mathcal{L})$  should be thought of as analogous to an *identity component oplax natural transformation* in 2-category theory, as defined by Lack [16]. In particular, the simplicial functors  $\mathcal{K} \rightarrow \mathcal{L}$  serving as the domain and the codomain of a 1-simplex in  $\text{icon}(\mathcal{K}, \mathcal{L})$  agree on objects.

The universal property of  $\mathcal{L}^X$  as a cotensor may be expressed as a natural isomorphism  $\text{icon}(\mathcal{K}, \mathcal{L})^X \cong \text{icon}(\mathcal{K}, \mathcal{L}^X)$  and, in particular, it provides a natural bijection between simplicial maps  $X \rightarrow \text{icon}(\mathcal{K}, \mathcal{L})$  and simplicial functors  $\mathcal{K} \rightarrow \mathcal{L}^X$ .

We will be interested in fibers of maps  $\text{icon}(\mathbf{A}', \mathcal{K}) \rightarrow \text{icon}(\mathbf{A}, \mathcal{K})$  associated to an identity-on-objects inclusion  $\mathbf{A} \hookrightarrow \mathbf{A}'$  between small simplicial categories. It is easily checked, using the fact that the hom-spaces of  $\mathcal{K}$  are all small simplicial sets, that any such fibre will be a small simplicial set.

For the remainder of this section we shall assume that  $\mathcal{K}$  and  $\mathcal{L}$  denote quasi-categorically enriched categories. The icon enrichment of  $\mathbf{sSet}\text{-Cat}$  is homotopically well-behaved with respect to local isofibrations and relative simplicial computads, in the precise sense formalised in the next lemma.

4.4.2. **Lemma.** *Suppose that  $P: \mathcal{K} \rightarrow \mathcal{L}$  is a simplicial functor which is a local isofibration, and suppose that  $I: \mathbf{A} \hookrightarrow \mathbf{B}$  is relative simplicial computad. Furthermore assume either that  $P$  is surjective on objects or that  $I$  acts bijectively on objects. Then the Leibniz simplicial map*

$$\widehat{\text{icon}}(I, P): \text{icon}(\mathbf{B}, \mathcal{K}) \longrightarrow \text{icon}(\mathbf{A}, \mathcal{K}) \times_{\text{icon}(\mathbf{A}, \mathcal{L})} \text{icon}(\mathbf{B}, \mathcal{L})$$

*is a fibration in Joyal’s model structure.*

*Proof.* We wish to prove that every lifting problem

$$\begin{array}{ccc} X & \longrightarrow & \text{icon}(\mathbf{B}, \mathcal{K}) \\ \downarrow i & & \downarrow \widehat{\text{icon}(I, P)} \\ Y & \longrightarrow & \text{icon}(\mathbf{A}, \mathcal{K}) \times_{\text{icon}(\mathbf{A}, \mathcal{L})} \text{icon}(\mathbf{B}, \mathcal{L}) \end{array}$$

whose left-hand vertical  $i: X \hookrightarrow Y$  is a trivial cofibration in the Joyal model structure on  $\underline{\text{sSet}}$ , has a solution. This lifting problem transposes into the corresponding problem:

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathcal{K}^Y \\ \downarrow I & & \downarrow \widehat{\text{hom}(i, P)} \\ \mathbf{B} & \longrightarrow & \mathcal{K}^X \times_{\mathcal{L}^X} \mathcal{L}^Y \end{array}$$

Here the simplicial functor  $\widehat{\text{hom}}(i, P)$  on the right is surjective on objects whenever  $P$  is, and its action on each hom-set is the Leibniz map

$$\widehat{\text{hom}}(i, P): \mathcal{K}(C, D)^Y \rightarrow \mathcal{K}(C, D)^X \times_{\mathcal{L}(PC, PD)^X} \mathcal{L}(PC, PD)^Y,$$

which is a trivial fibration of quasi-categories because  $P: \mathcal{K}(C, D) \rightarrow \mathcal{L}(PC, PD)$  is an isofibration of quasi-categories and  $i: X \hookrightarrow Y$  is a trivial cofibration in Joyal's model structure.

Now by definition 2.1.4 we know that  $I: \mathbf{A} \hookrightarrow \mathbf{B}$  can be expressed as a countable composite of pushouts of inclusions  $\emptyset \hookrightarrow \mathbb{1}$  and  $2[\partial\Delta^n] \hookrightarrow 2[\Delta^n]$  for  $n \geq 0$ . Furthermore  $I$  is bijective on objects if and only if that decomposition doesn't contain any pushouts of the inclusion  $\emptyset \hookrightarrow \mathbb{1}$ . So it follows that it is enough to check that we may solve lifting problems of the forms:

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathcal{K}^Y \\ \downarrow & & \downarrow \widehat{\text{hom}(i, P)} \\ \mathbb{1} & \longrightarrow & \mathcal{K}^X \times_{\mathcal{L}^X} \mathcal{L}^Y \end{array} \qquad \begin{array}{ccc} 2[\partial\Delta^n] & \longrightarrow & \mathcal{K}^Y \\ \downarrow & & \downarrow \widehat{\text{hom}(i, P)} \\ 2[\Delta^n] & \longrightarrow & \mathcal{K}^X \times_{\mathcal{L}^X} \mathcal{L}^Y \end{array}$$

Now solutions to problems like those on the right are guaranteed by the fact that the actions of  $\widehat{\text{hom}}(i, P)$  on hom-spaces are trivial fibrations. Furthermore, if  $P$  is surjective on objects then we can solve problems like those on the left. Otherwise, when  $I$  is bijective on objects we need not solve any such problems.  $\square$

Special cases of lemma 4.4.2 imply that  $\text{icon}(I, \mathcal{K}): \text{icon}(\mathbf{B}, \mathcal{K}) \twoheadrightarrow \text{icon}(\mathbf{A}, \mathcal{K})$  is an isofibration of quasi-categories if  $\mathbf{A}$  is a simplicial computad and  $\mathbf{A} \hookrightarrow \mathbf{B}$  is a relative simplicial computad.

**4.4.3. Observation.** Translating the proof of lemma 4.4.2 to the marked model structure of I.2.3.8, we obtain a corresponding result for relative simplicial computads and local isofibrations of categories enriched in naturally marked quasi-categories. In particular,

if  $\mathcal{K}$  has naturally marked hom-spaces and  $\mathbf{A}$  is a simplicial computad, then the space  $\text{icon}(\mathbf{A}, \mathcal{K})$  is a naturally marked quasi-category.

We will be interested in the isomorphisms in these spaces of icons.

**4.4.4. Lemma.** *Suppose that  $F: \mathcal{K} \rightarrow \mathcal{L}$  is a simplicial functor which is locally conservative in the sense that its action  $F: \mathcal{K}(A, B) \rightarrow \mathcal{L}(FA, FB)$  on each hom-space reflects isomorphisms. Suppose also that  $\mathbf{A}$  is a simplicial computad and that  $I: \mathbf{A} \hookrightarrow \mathbf{B}$  is a relative simplicial computad that is bijective on objects. Then the Leibniz simplicial map*

$$\widehat{\text{icon}}(I, F): \text{icon}(\mathbf{B}, \mathcal{K}) \longrightarrow \text{icon}(\mathbf{A}, \mathcal{K}) \times_{\text{icon}(\mathbf{A}, \mathcal{L})} \text{icon}(\mathbf{B}, \mathcal{L})$$

is a conservative functor of quasi-categories.

*Proof.* We work in the marked model structure, where we know that a simplicial map is conservative if and only if it has the right lifting property with respect to the inclusion  $\mathcal{Q} \hookrightarrow \mathcal{Q}^\sharp$  of the unmarked 1-simplex into the marked 1-simplex. Transposing this lifting property, we find that it is equivalent to postulating that every lifting problem

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathcal{K}^{2^\sharp} \\ \downarrow & & \downarrow \widehat{\text{hom}}(i, F) \\ \mathbf{B} & \longrightarrow & \mathcal{K}^2 \times_{\mathcal{L}^2} \mathcal{L}^{2^\sharp} \end{array}$$

has a solution. Since  $\mathbf{A} \hookrightarrow \mathbf{B}$  is both a relative simplicial computad and bijective on objects it is expressible as a composite of pushouts of inclusions of the form  $\mathcal{Q}[\partial\Delta^n] \hookrightarrow \mathcal{Q}[\Delta^n]$ , and it suffices to consider the case when  $\mathbf{A} \hookrightarrow \mathbf{B}$  is an inclusion of this form. This amounts to showing that each lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{K}(A, B)^{2^\sharp} \\ \downarrow & & \downarrow \widehat{\text{hom}}(i, F) \\ \Delta^n & \longrightarrow & \mathcal{K}(A, B)^2 \times_{\mathcal{L}(FA, FB)^2} \mathcal{L}(FA, FB)^{2^\sharp} \end{array}$$

has a solution, which transposes to give the following lifting problem:

$$\begin{array}{ccc} (\partial\Delta^n \times \mathcal{Q}^{2^\sharp}) \cup (\Delta^n \times \mathcal{Q}) & \longrightarrow & \mathcal{K}(A, B) \\ \downarrow & & \downarrow F \\ \Delta^n \times \mathcal{Q}^{2^\sharp} & \longrightarrow & \mathcal{L}(FA, FB) \end{array}$$

The marked simplicial sets on the left have the same underlying simplicial sets and differ only in their markings. Consequently, the (unique) existence of the solution to this latter problem follows immediately from the assumption that  $F$  is locally conservative.  $\square$

**4.4.5. Definition.** The vertices of the quasi-category  $\text{icon}(\underline{\text{Adj}}, \mathcal{K})$  are precisely the homotopy coherent adjunctions in  $\mathcal{K}$ , so we write  $\text{cohadj}(\mathcal{K}) = \text{icon}(\underline{\text{Adj}}, \mathcal{K})$  for the *space of homotopy coherent adjunctions* in  $\mathcal{K}$ .

**4.4.6. Lemma.** *The space of homotopy coherent adjunctions in  $\mathcal{K}$  is a (possibly large) Kan complex.*

*Proof.* Lemma 4.4.2 implies that  $\text{cohadj}(\mathcal{K})$  is a quasi-category, so to demonstrate that it is a Kan complex we need only show that all of its arrows are isomorphisms. This problem reduces to a 2-categorical argument. Observe that the quotient simplicial functor  $Q_{\mathcal{K}}: \mathcal{K} \rightarrow \mathcal{K}_2$  is locally conservative simply because, by definition, the isomorphisms of the quasi-category  $\mathcal{K}(A, B)$  are those arrows which map to isomorphisms in the homotopy category  $\mathcal{K}_2(A, B) = h(\mathcal{K}(A, B))$ . Lemma 4.4.4 implies that the functor  $\text{cohadj}(Q_{\mathcal{K}}): \text{cohadj}(\mathcal{K}) \rightarrow \text{cohadj}(\mathcal{K}_2)$  is conservative. So it follows that the arrows of  $\text{cohadj}(\mathcal{K})$  are isomorphisms if this is the case for  $\text{cohadj}(\mathcal{K}_2)$ .

It is easy to see that  $\text{cohadj}(\mathcal{K}_2)$  is a category. The 2-categorical universal property of  $\underline{\text{Adj}}$  established in proposition 3.3.4 furnishes a concrete description of this category:

- objects are adjunctions  $(f \dashv u, \epsilon, \eta)$  in  $\mathcal{K}_2$ ,
- arrows  $(\phi, \psi): (f \dashv u, \epsilon, \eta) \rightarrow (f' \dashv u', \epsilon', \eta')$  consist of a pair of 2-cells  $\phi: f \Rightarrow f'$  and  $\psi: u \Rightarrow u'$  in  $\mathcal{K}_2$  which satisfy the equations  $\epsilon' \cdot (\phi\psi) = \epsilon$  and  $\eta' = (\psi\phi) \cdot \eta$ , and
- identities and composition are given component-wise.

The isomorphisms in this category are those pairs whose constituent 2-cells are invertible. Given any arrow  $(\phi, \psi): (f \dashv u, \epsilon, \eta) \rightarrow (f' \dashv u', \epsilon', \eta')$ , we may construct the mates of the 2-cells  $\phi$  and  $\psi$  under the the given adjunctions, that is to say the following composites:

$$\psi' := f' \xrightarrow{f'\eta} f'u f \xrightarrow{f'\psi f} f'u' f \xrightarrow{\epsilon' f} f \quad \phi' := u' \xrightarrow{\eta u'} u f u' \xrightarrow{u\phi u'} u f' u' \xrightarrow{u\epsilon'} u$$

We leave it to the reader to verify that  $\phi'$  is inverse to  $\psi$  and that  $\psi'$  is inverse to  $\phi$ .  $\square$

The following proposition strengthens lemma 4.4.2 when we restrict our attention to inclusions of parental subcomputads of  $\underline{\text{Adj}}$ .

**4.4.7. Proposition.** *Suppose that  $\mathbf{A}$  and  $\mathbf{A}'$  are parental subcomputads of  $\underline{\text{Adj}}$  with  $\mathbf{A} \subseteq \mathbf{A}'$  containing the arrows  $\underline{f}$ ,  $\underline{u}$ , and  $\underline{\epsilon}$ . Suppose that  $T: \mathbf{A} \rightarrow \mathcal{K}$  is a simplicial functor for which  $T(\underline{f}) = f$ ,  $T(\underline{u}) = u$ , and  $T(\underline{\epsilon}) = \epsilon$  define an adjunction in  $\mathcal{K}_2$ . Then the fibre  $E_T$  of the isofibration  $\text{icon}(I, \mathcal{K}): \text{icon}(\mathbf{A}', \mathcal{K}) \rightarrow \text{icon}(\mathbf{A}, \mathcal{K})$  over the vertex  $T$  is a contractible Kan complex.*

*Proof.* The vertices of  $E_T$  are simply those simplicial functors  $H: \mathbf{A}' \rightarrow \mathcal{K}$  which extend the given simplicial functor  $T: \mathbf{A} \rightarrow \mathcal{K}$ . It follows, from theorem 4.3.8, that some such extension does exist and thus that  $E_T$  is inhabited. Our task is to generalise that argument and show that if  $i: X \hookrightarrow Y$  is any inclusion of simplicial sets then any lifting problem of the form displayed in the displayed left-hand square, or equivalently the composite rectangle

$$\begin{array}{ccccc} X & \longrightarrow & E_T & \longrightarrow & \text{icon}(\mathbf{A}', \mathcal{K}) \\ \downarrow i & \nearrow & \downarrow & \lrcorner & \downarrow \text{icon}(I, \mathcal{K}) \\ Y & \longrightarrow & \Delta^0 & \xrightarrow{T} & \text{icon}(\mathbf{A}, \mathcal{K}) \end{array}$$

has a solution as illustrated by the dashed map. Transposing, we obtain an equivalent lifting problem

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{T} \mathcal{K} & \xrightarrow{\mathcal{K}^!} \mathcal{K}^Y \\ I \downarrow & & \downarrow \mathcal{K}^i \\ \mathbf{A}' & \longrightarrow & \mathcal{K}^X \end{array} \quad (4.4.8)$$

in the category of simplicial categories.

Now the vertical simplicial functor on the right of this diagram is a local isofibration; its action on the hom-space from  $C$  to  $D$  is  $\mathcal{K}(C, D)^i: \mathcal{K}(C, D)^Y \rightarrow \mathcal{K}(C, D)^X$ . Furthermore, as noted in observation 4.3.2, the simplicial functor  $\mathcal{K}^!: \mathcal{K} \rightarrow \mathcal{K}^Y$  preserves the adjunctions of  $\mathcal{K}_2$ . In particular, the upper horizontal map carries  $\underline{f}$ ,  $\underline{u}$ , and  $\underline{\epsilon}$  to the data of an adjunction in  $\mathcal{K}^Y$ , and we may apply theorem 4.3.8 to provide us with a solution to the transformed lifting problem (4.4.8) as required.  $\square$

Our first homotopical uniqueness result arises by specialising proposition 4.4.7 to the case where  $\mathbf{A} = \overline{\{\underline{\epsilon}\}}$  and  $\mathbf{A}' = \underline{\text{Adj}}$ .

4.4.9. *Observation.* We use the universal property of  $\text{icon}(\overline{\{\underline{\epsilon}\}}, \mathcal{K})$  to deduce a new description of this space. Observation 4.4.1 tells us that simplicial maps  $X \rightarrow \text{icon}(\overline{\{\underline{\epsilon}\}}, \mathcal{K})$  are in bijective correspondence with simplicial functors  $T: \overline{\{\underline{\epsilon}\}} \rightarrow \mathcal{K}^X$ . Any such  $T$  is completely and uniquely determined by giving objects  $A$  and  $B$  of  $\mathcal{K}$ , 0-arrows  $T(\underline{f}) \in \mathcal{K}(B, A)^X$  and  $T(\underline{u}) \in \mathcal{K}(A, B)^X$ , and a 1-arrow  $T(\underline{\epsilon}): T(\underline{f})T(\underline{u}) \Rightarrow \text{id}_A \in \mathcal{K}(A, A)^X$ . This amounts to giving a triple of simplicial maps  $T(\underline{f}): X \rightarrow \mathcal{K}(B, A)$ ,  $T(\underline{u}): X \rightarrow \mathcal{K}(A, B)$ , and  $T(\underline{\epsilon}): X \rightarrow \mathcal{K}(A, A)^2$  which make the following square

$$\begin{array}{ccc} X & \xrightarrow{T(\underline{\epsilon})} & \mathcal{K}(A, A)^2 \\ (!, T(\underline{f}), T(\underline{u})) \downarrow & & \downarrow (p_1, p_0) \\ \Delta^0 \times \mathcal{K}(B, A) \times \mathcal{K}(A, B) & \xrightarrow{\text{id}_A \times \circ_B} & \mathcal{K}(A, A) \times \mathcal{K}(A, A) \end{array}$$

commute. More concisely, we can express all of this information as a single map from  $X$  into the pullback of the diagram formed by the right hand vertical and lower horizontal maps of this square. On consulting I.3.3.15 or Example 5.1.9 below, we recognise that this pullback is precisely the comma quasi-category  $\circ_B \downarrow \text{id}_A$  displayed in the following diagram:

$$\begin{array}{ccc} & \circ_B \downarrow \text{id}_A & \\ p_1 \swarrow & & \searrow p_0 \\ \Delta^0 & \xleftarrow{\psi} & \mathcal{K}(B, A) \times \mathcal{K}(A, B) \\ \text{id}_A \searrow & & \swarrow \circ_B \\ & \mathcal{K}(A, A) & \end{array}$$

In conclusion, the space  $\text{icon}(\overline{\{\underline{\epsilon}\}}, \mathcal{K})$  is isomorphic to the (possibly large) coproduct, indexed by pairs  $A, B \in \text{obj } \mathcal{K}$ , of comma quasi-categories  $\circ_B \downarrow \text{id}_A$ .

4.4.10. **Definition** (the space of counits). An object of

$$\text{icon}(\overline{\{\underline{\epsilon}\}}, \mathcal{K}) \cong \coprod_{A, B \in \text{obj } \mathcal{K}} \circ_B \downarrow \text{id}_A$$

may be written as a triple  $(f, u, \epsilon)$  where  $f \in \mathcal{K}(B, A)$  and  $u \in \mathcal{K}(A, B)$  are 0-arrows and  $\epsilon: fu \Rightarrow \text{id}_A \in \mathcal{K}(A, A)$  is a 1-arrow. We are most interested in those objects  $(f, u, \epsilon)$  with the property that  $\epsilon$  represents the counit of an adjunction  $f \dashv u$  in  $\mathcal{K}_2$ ; we write  $(f \dashv u, \epsilon)$  to denote an object satisfying that condition.

Now define  $\text{counit}(\mathcal{K})$  to be the simplicial subset of  $\text{icon}(\overline{\{\underline{\epsilon}\}}, \mathcal{K})$  whose 0-simplices are the objects  $(f \dashv u, \epsilon)$ , whose 1-simplices are the isomorphisms between them, and whose higher simplices are precisely those whose vertices and edges are members of these classes. As a quasi-category whose 1-simplices are isomorphisms,  $\text{counit}(\mathcal{K})$  is a Kan complex, which we call the *space of counits* in  $\mathcal{K}$ .

Any given homotopy coherent adjunction  $H: \underline{\text{Adj}} \rightarrow \mathcal{K}$  provides us with an adjunction  $H(\underline{f}) \dashv H(\underline{u})$  in  $\mathcal{K}_2$  with counit represented by the 1-arrow  $H(\underline{\epsilon})$ ; it follows that  $H$  restricts along the inclusion  $I: \overline{\{\underline{\epsilon}\}} \hookrightarrow \underline{\text{Adj}}$  to give an object  $(H(\underline{f}) \dashv H(\underline{u}), H(\underline{\epsilon}))$  of  $\text{counit}(\mathcal{K})$ . By lemma 4.4.6, all of the arrows of  $\text{cohadj}(\mathcal{K})$  are isomorphisms; therefore, as functors preserve isomorphisms, every arrow in  $\text{cohadj}(\mathcal{K})$  maps to an arrow of  $\text{counit}(\mathcal{K})$  under the isofibration  $\text{icon}(I, \mathcal{K}): \text{icon}(\underline{\text{Adj}}, \mathcal{K}) \twoheadrightarrow \text{icon}(\overline{\{\underline{\epsilon}\}}, \mathcal{K})$ . Consequently this map factors through the space of counits to give an isofibration  $p_C: \text{cohadj}(\mathcal{K}) \twoheadrightarrow \text{counit}(\mathcal{K})$  of Kan complexes.

An object  $H: \underline{\text{Adj}} \rightarrow \mathcal{K}$  of  $\text{cohadj}(\mathcal{K})$  is in the fibre  $E_\epsilon$  of the isofibration  $\text{cohadj}(\mathcal{K}) \twoheadrightarrow \text{counit}(\mathcal{K})$  over some  $(f \dashv u, \epsilon)$  precisely if it is a lift of the adjunction  $f \dashv u$  to a homotopy coherent adjunction which happens to map  $\underline{\epsilon}$  to the chosen counit representative  $\epsilon$ . We call this fiber the *space of homotopy coherent adjunctions extending the counit  $\epsilon$* . Proposition 4.4.7 specialises to prove our first uniqueness theorem:

4.4.11. **Theorem.** *The space  $E_\epsilon$  of homotopy coherent adjunctions extending the counit  $\epsilon$  is a contractible Kan complex.*

Theorem 4.4.11 has the following extension.

4.4.12. **Proposition.** *The isofibration  $p_C: \text{cohadj}(\mathcal{K}) \twoheadrightarrow \text{counit}(\mathcal{K})$  is a trivial fibration of Kan complexes.*

*Proof.* The map  $p_C: \text{cohadj}(\mathcal{K}) \twoheadrightarrow \text{counit}(\mathcal{K})$  is an isofibration between Kan complexes, and hence a Kan fibration. The conclusion follows immediately from theorem 4.4.11 and the following standard result from simplicial homotopy theory.  $\square$

4.4.13. **Lemma.** *A Kan fibration  $p: E \twoheadrightarrow B$  is a trivial fibration if and only if its fibres are contractible.*

*Proof.* This can be proven either by appealing to the long exact sequence of a fibration or by a direct combinatorial argument (see [23, 5.4.16] or [20, 17.6.5]).  $\square$

It is also natural to ask what happens if we start only with a single 0-arrow  $f: B \rightarrow A$ , which we know has some right adjoint in  $\mathcal{K}_2$ , and consider all of its extensions to a homotopy coherent adjunction. To answer this question we start by considering the isofibration  $\text{icon}(I, \mathcal{K}): \text{icon}(\overline{\{\underline{\epsilon}\}}, \mathcal{K}) \rightarrow \text{icon}(\overline{\{\underline{f}\}}, \mathcal{K})$  of quasi-categories induced by the inclusion  $I: \overline{\{\underline{f}\}} \hookrightarrow \overline{\{\underline{\epsilon}\}}$  of subcomputads of  $\underline{\text{Adj}}$ .

4.4.14. *Observation.* Observation 4.4.1 tells us that simplicial maps  $X \rightarrow \text{icon}(\overline{\{\underline{f}\}}, \mathcal{K})$  are in bijective correspondence with simplicial functors  $T: \overline{\{\underline{f}\}} \rightarrow \mathcal{K}^X$ . Any such  $T$  is completely and uniquely specified by giving the 0-arrow  $T(\underline{f})$  of  $\mathcal{K}^X$ . This in turn corresponds to specifying a pair of objects  $A$  and  $B$  of  $\mathcal{K}$  and a simplicial map  $T(\underline{f}): X \rightarrow \mathcal{K}(B, A)$  or, in other words, to giving a simplicial map  $T(\underline{f}): X \rightarrow \coprod_{A, B \in \text{obj } \mathcal{K}} \mathcal{K}(B, A)$ . Thus, the space  $\text{icon}(\overline{\{\underline{f}\}}, \mathcal{K})$  is isomorphic to the (possibly large) coproduct  $\coprod_{A, B \in \text{obj } \mathcal{K}} \mathcal{K}(B, A)$  of hom-spaces.

Combined with observation 4.4.9, we see that the isofibration  $\text{icon}(I, \mathcal{K}): \text{icon}(\overline{\{\underline{\epsilon}\}}, \mathcal{K}) \rightarrow \text{icon}(\overline{\{\underline{f}\}}, \mathcal{K})$  is isomorphic to the coproduct of the family of projection isofibrations

$$\circ_B \downarrow \text{id}_A \xrightarrow{p_0} \mathcal{K}(B, A) \times \mathcal{K}(A, B) \xrightarrow{\pi_0} \mathcal{K}(B, A)$$

indexed by pairs of objects  $A, B \in \mathcal{K}$ . In particular, the fibre of this isofibration over an object  $f \in \mathcal{K}(B, A)$  is isomorphic to the comma quasi-category  $\mathcal{K}(A, f) \downarrow \text{id}_A$ .

4.4.15. **Definition** (the space of left adjoints). Define  $\text{leftadj}(\mathcal{K})$  to be the simplicial subset of  $\text{icon}(\overline{\{\underline{f}\}}, \mathcal{K}) \cong \coprod_{A, B \in \text{obj } \mathcal{K}} \mathcal{K}(B, A)$  whose 0-simplices are those 0-arrows  $f$  of  $\mathcal{K}$  which possess a right adjoint in  $\mathcal{K}_2$ , whose 1-simplices are the isomorphisms between them, and whose higher simplices are precisely those whose vertices and edges are members of these classes. As a quasi-category whose 1-simplices are isomorphisms,  $\text{leftadj}(\mathcal{K})$  is a Kan complex, which we call the *space of left adjoints* in  $\mathcal{K}$ .

4.4.16. *Observation.* An object  $(f \dashv u, \epsilon)$  of  $\text{counit}(\mathcal{K})$  maps to the object  $f$  of  $\text{leftadj}(\mathcal{K})$  under the isofibration  $\text{icon}(I, \mathcal{K}): \text{icon}(\overline{\{\underline{\epsilon}\}}, \mathcal{K}) \rightarrow \text{icon}(\overline{\{\underline{f}\}}, \mathcal{K})$ , which restricts to give an isofibration  $q_L: \text{counit}(\mathcal{K}) \rightarrow \text{leftadj}(\mathcal{K})$  of Kan complexes.

4.4.17. **Proposition.** *The isofibration  $q_L: \text{counit}(\mathcal{K}) \rightarrow \text{leftadj}(\mathcal{K})$  is a trivial fibration of Kan complexes.*

*Proof.* An isofibration of Kan complexes is a Kan fibration, so this result follows from lemma 4.4.13 once we show that the fibres of  $q_L$  are contractible.

If  $f \in \mathcal{K}$  is an object of  $\text{leftadj}(\mathcal{K})$  then the fibre  $F_f$  of  $q_L: \text{counit}(\mathcal{K}) \rightarrow \text{leftadj}(\mathcal{K})$  over  $f$  is isomorphic to a sub-quasi-category of the fibre  $\mathcal{K}(A, f) \downarrow \text{id}_A$  of the isofibration  $\text{icon}(I, \mathcal{K}): \text{icon}(\overline{\{\underline{\epsilon}\}}, \mathcal{K}) \rightarrow \text{icon}(\overline{\{\underline{f}\}}, \mathcal{K})$ . Its objects are pairs  $(u, \epsilon)$  which have the property that  $u$  is right adjoint to the fixed 0-arrow  $f$  with counit represented by the 1-arrow  $\epsilon: fu \Rightarrow \text{id}_A$ .

Given such an object  $(u, \epsilon)$  then we may apply the simplicial functor  $\mathcal{K}(A, -)$  to the adjunction  $f \dashv u$  to obtain an adjunction of quasi-categories  $\mathcal{K}(A, f) \dashv \mathcal{K}(A, u): \mathcal{K}(A, A) \rightarrow$

$\mathcal{K}(A, B)$  whose counit is represented by  $\mathcal{K}(A, \epsilon)$ . By I.4.4.8, the object  $(u, \epsilon)$  is a terminal object in  $\mathcal{K}(A, f) \downarrow \text{id}_A$ . Therefore, the fibre  $F_f$  is simply the full sub-quasi-category of terminal objects in  $\mathcal{K}(A, f) \downarrow \text{id}_A$  and, as such, it is contractible as required.  $\square$

Composing the trivial fibrations of propositions 4.4.12 and 4.4.17, we obtain a trivial fibration  $p_L: \text{coadj}(\mathcal{K}) \rightarrow \text{leftadj}(\mathcal{K})$  which maps each homotopy coherent adjunction  $H: \underline{\text{Adj}} \rightarrow \mathcal{K}$  to its left adjoint 0-arrow  $H(f)$ . So if  $f \in \mathcal{K}(B, A)$  is a left adjoint 0-arrow, that is to say an object in  $\text{leftadj}(\mathcal{K})$ , then the fibre  $E_f$  of  $p_L$  over that vertex has objects which are precisely those homotopy coherent adjunctions  $H: \underline{\text{Adj}} \rightarrow \mathcal{K}$  for which  $H(f) = f$ . Consequently, we call  $E_f$  the *space of homotopy coherent adjunctions extending the left adjoint  $f$* . Now lemma 4.4.13 applied to the trivial fibration  $p_L$  tells us that such extensions exist and are homotopically unique, in the sense that the fibre  $E_f$  is a contractible Kan complex, proving our second uniqueness theorem.

**4.4.18. Theorem.** *The space  $E_f$  of homotopy coherent adjunctions extending the left adjoint  $f$  is a contractible Kan complex.*  $\square$

## 5. WEIGHTED LIMITS IN $\mathbf{qCat}_\infty$

Of paramount importance to enriched category are the notions of *weighted limit* and *weighted colimit*. Here we consider only three sorts of enrichment — in sets, in categories, or in simplicial sets — so we may as well suppose that the base for enrichment is a complete and cocomplete cartesian closed category  $\mathcal{V}$ .

Our aim in §5.2 is to show that  $\mathbf{qCat}_\infty$  admits a large class of weighted limits: those whose weights are projective cofibrant simplicial functors. These will be used to develop a “formal” theory of monads in the quasi-categorical context by extending a new presentation of the analogous 2-categorical results. For the reader’s convenience, we review the basics of the theory of weighted limits in §5.1. A more thorough treatment can be found in [14] or [22]. In §5.3, we establish a correspondence between projective cofibrant simplicial functors and certain relative simplicial computads that will be exploited in section 6 to identify projective cofibrant weights.

**5.1. Weighted limits and colimits.** An ordinary limit is an object representing the Set-valued functor of cones over a fixed diagram. But in the enriched context, this Set-based universal property is insufficiently expressive. The intuition is that in the presence of extra structure on the hom-sets of a category, cones over a diagram might come in exotic “shapes.”

**5.1.1. Definition** (cotensors). For example, in the case of a diagram of shape  $\mathbb{1}$  in a  $\mathcal{V}$ -category  $\mathcal{M}$ , the shape of a cone might be an object  $V \in \mathcal{V}$ . Writing  $D \in \mathcal{M}$  for the object in the image of the diagram, the  $V$ -weighted limit of  $D$  is an object  $V \pitchfork D \in \mathcal{M}$  satisfying the universal property

$$\mathcal{M}(M, V \pitchfork D) \cong \mathcal{V}(V, \mathcal{M}(M, D))$$

where this isomorphism is meant to be interpreted in the category  $\mathcal{V}$ . For historical reasons,  $V \pitchfork D$  is called the *cotensor* of  $D \in \mathcal{M}$  by  $V \in \mathcal{V}$ . Assuming the objects with these defining universal properties exist, cotensors define a bifunctor  $- \pitchfork -: \mathcal{V}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{M}$ .

For example, a closed symmetric monoidal category is always cotensored over itself: the cotensor is simply the internal hom. The cotensor  $V \pitchfork D$  is also denoted by  $D^V$  when the context disambiguates between objects  $D \in \mathcal{M}$  and  $V \in \mathcal{V}$ .

**5.1.2. Definition** (weighted limits). Suppose  $\mathbf{A}$  and  $\mathcal{M}$  are respectively small and large  $\mathcal{V}$ -categories and write  $\mathcal{M}^{\mathbf{A}}$  for the category of  $\mathcal{V}$ -functors and  $\mathcal{V}$ -natural transformations. Suppose further that  $\mathcal{M}$  is complete and admits cotensors. Then the *weighted limit* bifunctor  $\{ -, \}__{\mathbf{A}} : (\mathcal{V}^{\mathbf{A}})^{\text{op}} \times \mathcal{M}^{\mathbf{A}} \rightarrow \mathcal{M}$  is defined by the formula

$$\begin{aligned} \{W, D\}_{\mathbf{A}} &:= \int_{a \in \mathbf{A}} W(a) \pitchfork D(a) \\ &:= \lim \left( \prod_{a \in \mathbf{A}} W(a) \pitchfork D(a) \rightrightarrows \prod_{a, b \in \mathbf{A}} (\mathbf{A}(a, b) \times W(a)) \pitchfork D(b) \right) \end{aligned} \quad (5.1.3)$$

Here, the *weight*  $W$  for the limit of a diagram  $D$  of shape  $\mathbf{A}$  is a covariant  $\mathcal{V}$ -valued functor of  $\mathbf{A}$ . We refer to the object  $\{W, D\}_{\mathbf{A}}$  as the *limit* of the diagram  $D$  *weighted by*  $W$ . It is characterised by the universal property

$$\mathcal{M}(M, \{W, D\}_{\mathbf{A}}) \cong \mathcal{V}^{\mathbf{A}}(W, \mathcal{M}(M, D)) \quad (5.1.4)$$

where the isomorphism is again interpreted in  $\mathcal{V}$ .

A map of weights  $V \rightarrow W$  induces a functor between the weighted limits  $\{W, D\}_{\mathbf{A}} \rightarrow \{V, D\}_{\mathbf{A}}$  which we refer to as the functor *derived from* the map  $V \rightarrow W$ .

**5.1.5. Example** (representable weights). Let  $\mathbf{A}_a$  denote the covariant  $\mathcal{V}$ -enriched representable of  $\mathbf{A}$  at an object  $a$ . The bifunctor (5.1.3) admits canonical isomorphisms

$$\{\mathbf{A}_a, D\}_{\mathbf{A}} \cong \int_{b \in \mathbf{A}} \mathbf{A}(a, b) \pitchfork D(b) \cong D(a) \quad (5.1.6)$$

which are natural in  $a \in \mathbf{A}$  and  $D \in \mathcal{M}^{\mathbf{A}}$ ; this result is simply a recasting of the classical Yoneda lemma. Hence, limits weighted by representables are computed simply by evaluating the diagram at the appropriate object.

**5.1.7. Observation.** The defining universal property of the weighted limit bifunctor, as expressed in the natural isomorphism of (5.1.4), provides us with an enriched adjunction

$$(\mathcal{V}^{\mathbf{A}})^{\text{op}} \begin{array}{c} \xleftarrow{\mathcal{M}(-, D)} \\ \perp \\ \xrightarrow{\{-, D\}_{\mathbf{A}}} \mathcal{M} \end{array}$$

for each fixed diagram  $D$ . Consequently, the right adjoint functor  $\{-, D\}_{\mathbf{A}}$  carries (weighted) colimits in  $\mathcal{V}^{\mathbf{A}}$  to (weighted) limits in  $\mathcal{M}$ ; weighted colimits in  $\mathcal{V}^{\mathbf{A}}$  are simply weighted limits in  $(\mathcal{V}^{\mathbf{A}})^{\text{op}}$ . In summary, the weighted limit bifunctor is cocontinuous in the weights. It follows, in particular, that weights can be “made-to-order” using colimits; that is a weight constructed as a colimit of representables will stipulate the expected universal property.

5.1.8. **Example** (diagrams). The category  $\mathcal{V}^{\mathbf{A}}$  of weights admits pointwise tensors by objects of  $\mathcal{V}$ , satisfying a universal property dual to that of definition 5.1.1. As  $\mathcal{V}$  is cartesian monoidal, we adopt the notation  $V \times W$  for the tensor of  $V \in \mathcal{V}$  by  $W \in \mathcal{V}^{\mathbf{A}}$  defined by  $(V \times W)(a) := V \times W(a)$ .

The cocontinuity of the weighted limit bifunctor in its first variable tells us that if  $D$  is a diagram in  $\mathcal{M}^{\mathbf{A}}$  and  $V$  is an object in  $\mathcal{V}$  then the weighted limit  $\{V \times W, D\}_{\mathbf{A}}$  is isomorphic to  $V \pitchfork \{W, D\}_{\mathbf{A}}$ . So, in particular, it follows from example 5.1.5 that  $\{V \times \mathbf{A}_a, D\}_{\mathbf{A}}$  is naturally isomorphic to  $V \pitchfork D(a)$ . Particularly in the quasi-categorical context appearing below, the cotensor  $V \pitchfork D(a)$  is often referred to as the *object of diagrams* of shape  $V$  in  $D(a)$ .

5.1.9. **Example** (comma quasi-categories). Let  $\mathbf{A}$  be the category  $\bullet \rightarrow \bullet \leftarrow \bullet$ . Let  $W$  be the  $\underline{\text{sSet}}$ -valued weight with this shape whose image is

$$\Delta^0 \xrightarrow{\delta^0} \Delta^1 \xleftarrow{\delta^1} \Delta^0$$

The limit of a diagram  $D$

$$B \xrightarrow{f} A \xleftarrow{g} C$$

in the simplicial category  $\mathbf{qCat}_{\infty} \hookrightarrow \underline{\text{sSet}}$  weighted by  $W$  is the comma quasi-category  $f \downarrow g$  introduced in I.3.3.15, constructed by the pullback

$$\begin{array}{ccc} f \downarrow g & \longrightarrow & A^2 \\ \downarrow \lrcorner & & \downarrow \\ C \times B & \xrightarrow{g \times f} & A \times A. \end{array}$$

5.1.10. **Example** (homotopy limits as weighted limits). The homotopy limit of a diagram of shape  $\mathbf{A}$  taking values in the fibrant objects of a simplicial model category is the limit weighted by  $\mathbf{A}/-: \mathbf{A} \rightarrow \underline{\text{sSet}}$ . This is the Bousfield Kan formula [4].

5.1.11. **Lemma** (weighted limits and Kan extensions). *Suppose given a  $\mathcal{V}$ -functor  $K: \mathbf{D} \rightarrow \mathbf{C}$ , a diagram  $D: \mathbf{C} \rightarrow \mathcal{M}$ , and a weight  $W: \mathbf{D} \rightarrow \mathcal{V}$ . The limit of the restricted diagram  $DK$  weighted by  $W$  is isomorphic to the limit of  $D$  weighted by the left Kan extension of  $W$  along  $K$ .*

$$\{W, DK\}_{\mathbf{D}} \cong \{\text{lan}_K W, D\}_{\mathbf{C}}$$

$$\begin{array}{ccc} & \mathbf{C} & \\ K \nearrow & & \text{lan}_K W \searrow \\ \mathbf{D} & \xrightarrow{W} & \mathcal{V} \\ & \uparrow \cong & \nwarrow \end{array}$$

*Proof.* The defining universal properties of these weighted limits are easily seen to coincide; see [14, 4.57].  $\square$

**5.2. Weighted limits in the quasi-categorical context.** The following proposition shows that the limit of any diagram of quasi-categories weighted by a projective cofibrant functor is again a quasi-category. The proof is very simple and indeed related conclusions have been drawn elsewhere; see for instance [8]. For the duration of this section and the next we shall assume that  $\mathbf{A}$  is a small simplicial category.

**5.2.1. Definition** (projective cofibrations). A natural transformation in  $\underline{\mathbf{sSet}}^{\mathbf{A}}$  is said to be a *projective cofibration* if and only if it has the left lifting property with respect to those natural transformations which are pointwise trivial fibrations. The maps in the set

$$\{\partial\Delta^n \times \mathbf{A}_a \rightarrow \Delta^n \times \mathbf{A}_a \mid n \geq 0, a \in \mathbf{A}\}$$

of *projective cells* are projective cofibrations. A natural transformation in  $\underline{\mathbf{sSet}}^{\mathbf{A}}$  is a pointwise trivial fibration if and only if it has the right lifting property with respect to all projective cells. We say that a natural transformation  $i: V \rightarrow W$  is a *relative projective cell complex* if it is a countable composite of pushouts of coproducts of projective cells, or equivalently, if it is a transfinite composite of pushouts of projective cells. A simplicial functor  $W$  in  $\underline{\mathbf{sSet}}^{\mathbf{A}}$  is a *projective cell complex* if the unique natural transformation  $\emptyset \rightarrow W$  is a relative projective cell complex.

By the small object argument, we may factor every natural transformation  $f: V \rightarrow W$  in  $\underline{\mathbf{sSet}}^{\mathbf{A}}$  as a composite of a relative projective cell complex  $i: V \rightarrow U$  followed by a pointwise trivial fibration  $p: U \rightarrow W$ . It follows that a map  $i: V \rightarrow W$  is a projective cofibration if and only if it is a retract of a relative projective cell complex.

**5.2.2. Proposition.** *Let  $i: V \rightarrow W$  be a projective cofibration of weights in  $\underline{\mathbf{sSet}}^{\mathbf{A}}$ , and suppose that  $p: D \rightarrow E$  is a natural transformation of diagrams in  $\underline{\mathbf{sSet}}^{\mathbf{A}}$  and a pointwise (trivial) fibration in the Joyal model structure. Then the Leibniz limit map  $\{i, p\}_{\mathbf{A}}^{\wedge}: \{W, D\}_{\mathbf{A}} \rightarrow \{W, E\}_{\mathbf{A}} \times_{\{V, E\}_{\mathbf{A}}} \{V, D\}_{\mathbf{A}}$  is also a (trivial) fibration.*

*Proof.* The natural transformation  $i: V \rightarrow W$  is a projective cofibration if and only if it is a retract of a countable composite of pushouts of coproducts of projective cells. Observation 5.1.7 tells us that the weighted limit bifunctor is cocontinuous in its first variable, so the Leibniz map  $\{i, p\}_{\mathbf{A}}^{\wedge}$  as a retract of a countable tower of pullbacks of products of Leibniz maps of the form:

$$\{i_n \times \mathbf{A}_a, p\}_{\mathbf{A}}^{\wedge}: \{\Delta^n \times \mathbf{A}_a, D\}_{\mathbf{A}} \longrightarrow \{\Delta^n \times \mathbf{A}_a, E\} \times_{\{\partial\Delta^n \times \mathbf{A}_a, E\}} \{\partial\Delta^n \times \mathbf{A}_a, D\} \quad (5.2.3)$$

Hence, it suffices to show that each of these is a (trivial) fibration. Example 5.1.8 provides a natural isomorphism  $\{X \times \mathbf{A}_a, D\}_{\mathbf{A}} \cong D(a)^X$  from which we see that the Leibniz map (5.2.3) is isomorphic to the Leibniz hom:

$$\widehat{\text{hom}}(i_n, p_a): D(a)^{\Delta^n} \longrightarrow E(a)^{\Delta^n} \times_{E(a)^{\partial\Delta^n}} D(a)^{\partial\Delta^n}$$

Now  $p_a: D(a) \rightarrow E(a)$  is a (trivial) fibration, because  $p$  is a pointwise (trivial) fibration by assumption, and so (5.2.3) is also a (trivial) fibration as a consequence of the fact that the Joyal model structure is cartesian.  $\square$

**5.2.4. Proposition.** *The full simplicial subcategory  $\mathbf{qCat}_\infty$  of quasi-categories is closed in  $\mathbf{sSet}$  under limits weighted by projective cofibrant weights in the sense that  $\mathbf{qCat}_\infty$  has and  $\mathbf{qCat}_\infty \hookrightarrow \mathbf{sSet}$  preserves such limits.*

*Proof.* Since  $\mathbf{qCat}_\infty$  is a full simplicial subcategory of  $\mathbf{sSet}$ , all we need do is show that if  $W: \mathbf{A} \rightarrow \mathbf{sSet}$  is a projective cofibrant weight and  $D: \mathbf{A} \rightarrow \mathbf{sSet}$  is a diagram whose vertices are all quasi-categories then the weighted limit  $\{W, D\}_{\mathbf{A}}$  in  $\mathbf{sSet}$  is also a quasi-category. This is a special case of proposition 5.2.2.  $\square$

**5.2.5. Remark.** Simplicial limits with projective cofibrant weights should be thought of as analogous to *flexible 2-limits*, i.e., 2-limits built out of products, inserters, equifiers, and retracts (splittings of idempotents) [2]. The flexible limits also include iso-inserters, descent objects, and comma objects. When a 2-category  $\mathbf{A}$  is regarded as a simplicial category, the change-of-base functor  $h_*: \mathbf{sSet}^{\mathbf{A}} \rightarrow \mathbf{Cat}^{\mathbf{A}}$  carries projective cofibrant weights to flexible weights. The weights for flexible limits are the cofibrant objects in a model structure on the diagram 2-category  $\mathbf{Cat}^{\mathbf{A}}$  that is enriched over the folk model structure on  $\mathbf{Cat}$ . In analogy with our result, the fibrant objects in a  $\mathbf{Cat}$ -enriched model structure are closed under flexible weighted limits [15, theorem 5.4].

The next result shows that limits weighted by projective cofibrant weights are *homotopical*, that is, preserve pointwise equivalences between diagrams in  $\mathbf{qCat}_\infty$ .

**5.2.6. Proposition.** *Let  $W: \mathbf{A} \rightarrow \mathbf{sSet}$  be projective cofibrant, and let  $D, E: \mathbf{A} \rightarrow \mathbf{qCat}_\infty$  be a pair of diagrams equipped with a natural transformation  $w: D \rightarrow E$  which is a pointwise equivalence. Then the induced map  $\{W, D\}_{\mathbf{A}} \rightarrow \{W, E\}_{\mathbf{A}}$  is an equivalence of quasi-categories.*

*Proof.* Applying the construction of Ken Brown's lemma,  $w: D \rightarrow E$  may be factored as the composite of a right inverse to a pointwise trivial fibration followed by a pointwise trivial fibration. So it suffices to show that if  $w: D \rightarrow E$  is a pointwise trivial fibration then  $\{W, D\}_{\mathbf{A}} \rightarrow \{W, E\}_{\mathbf{A}}$  is an equivalence. This is a special case of proposition 5.2.2.  $\square$

**5.2.7. Remark.** The proofs of propositions 5.2.2, 5.2.4, and 5.2.6 apply mutatis mutandis to show that the fibrant objects in any model category that is enriched over either Quillen's or Joyal's model structure on  $\mathbf{sSet}$  is closed under weighted limits with projective cofibrant weights and that these constructions are homotopical. The essential input in all cases is the closure property of the (trivial) fibrations in such model categories with respect to Leibniz cotensors by monomorphisms of simplicial sets.

**5.2.8. Example** (homotopy limits of quasi-categories). For any small category  $\mathbf{A}$ , the weight  $\mathbf{A}/-: \mathbf{A} \rightarrow \mathbf{sSet}$  is projective cofibrant [9, 14.8.5]. An  $\mathbf{A}$ -diagram of quasi-categories can be regarded as a functor  $D: \mathbf{A} \rightarrow \mathbf{msSet}$  taking values in the fibrant objects of the marked model structure of I.2.3.8. The advantage of this interpretation is that the marked model structure is a simplicial model structure [18, 3.1.4.4]. By the last remark the weighted limit of a diagram of naturally marked quasi-categories is again a naturally marked quasi-category. In this way, we see that  $\mathbf{qCat}_\infty$  is closed under the formation of homotopy limits. See [22] for more details.

5.2.9. *Remark* (2-categorical weighted limits and quasi-categorical weighted limits). Recall our convention to regard a 2-functor  $W: \mathbf{A} \rightarrow \underline{\mathbf{Cat}}$  as a simplicial functor  $W: \mathbf{A} \rightarrow \underline{\mathbf{qCat}}_\infty$  via the embedding  $\underline{2\text{-Cat}} \hookrightarrow \underline{\mathbf{sSet-Cat}}$ . It follows from the defining weighted limit formula (5.1.3) and the fact that the nerve preserves exponentials that the (2-)limit of a 2-functor  $D: \mathbf{A} \rightarrow \underline{\mathbf{Cat}}$  weighted by  $W$ , when regarded as a quasi-category, is isomorphic to the limit of the associated simplicial functor  $D: \mathbf{A} \rightarrow \underline{\mathbf{qCat}}_\infty$  weighted by the simplicial functor  $W$ . Many of the weights appearing in sections 6 and 7 are simplicial re-interpretations of 2-functors. In this way, the special case of the quasi-categorical monadicity theorem, in which the quasi-categories in question are ordinary categories, can be interpreted directly in the full subcategory  $\underline{\mathbf{Cat}}$  of the simplicial category  $\underline{\mathbf{qCat}}_\infty$ .

5.3. **The collage construction.** To apply proposition 5.2.4, it will be useful to know that certain weights constructed from simplicial computads are projective cofibrant. This follows from a recognition principle which relates projective cofibrations to retracts of relative simplicial computads between *collages*.

5.3.1. **Definition** (the collage construction). Let  $W: \mathbf{A} \rightarrow \underline{\mathbf{sSet}}$  be a simplicial functor. The *collage* of  $W$  is a simplicial category  $\text{coll } W$  containing  $\mathbf{A}$  as a full simplicial subcategory and precisely one additional object  $*$  whose endomorphism space is a point. Declare the hom-spaces from  $a \in \mathbf{A}$  to  $*$  to be empty and define

$$\text{coll } W(*, a) := W(a).$$

The action maps  $\mathbf{A}(a, b) \times W(a) \rightarrow W(b)$  provide the required compositions between these hom-spaces derived from  $W$  and the hom-spaces in the full subcategory  $\mathbf{A}$ . This construction is functorial: it carries a natural transformation  $f: V \rightarrow W$  to a simplicial functor  $\text{coll}(f): \text{coll } V \rightarrow \text{coll } W$  which acts as the identity on the copies of  $\mathbf{A}$  and  $*$  in those collages and whose actions  $\text{coll}(f): \text{coll } V(*, a) \rightarrow \text{coll } W(*, a)$  are the components  $f_a: V(a) \rightarrow W(a)$ .

5.3.2. *Observation* (a right adjoint to the collage construction). For our purposes the collage construction is simply a functor of ordinary, unenriched categories. By definition, the functor  $\text{coll}(f): \text{coll } V \rightarrow \text{coll } W$  associated with a natural transformation  $f: V \rightarrow W$  commutes with the inclusions of the coproduct  $\mathbf{A} + \{*\}$  into the collages that comprise its domain and codomain. So we may write the collage construction as a functor from  $\underline{\mathbf{sSet}}^{\mathbf{A}}$  to the slice category  $(\mathbf{A} + \{*\})/\underline{\mathbf{sSet-Cat}}$  which carries a simplicial functor  $W: \mathbf{A} \rightarrow \underline{\mathbf{sSet}}$  to the inclusion  $\mathbf{A} + \{*\} \hookrightarrow \text{coll } W$ . This functor admits a right adjoint

$$(\mathbf{A} + \{*\})/\underline{\mathbf{sSet-Cat}} \begin{array}{c} \xleftarrow{\text{coll}} \\ \perp \\ \xrightarrow{\text{wgt}} \end{array} \underline{\mathbf{sSet}}^{\mathbf{A}}$$

carrying an object  $F: \mathbf{A} + \{*\} \rightarrow \mathbf{E}$  to the simplicial functor  $\text{wgt}(\mathbf{E}, F): \mathbf{A} \rightarrow \underline{\mathbf{sSet}}$  whose action on objects is given by  $\text{wgt}(\mathbf{E}, F)(a) := \mathbf{E}(F(*), F(a))$  and whose action on hom-spaces is determined by composition in  $\mathbf{E}$  as follows:

$$\mathbf{A}(a, b) \times \mathbf{E}(F(*), F(a)) \xrightarrow{F \times \text{id}} \mathbf{E}(F(a), F(b)) \times \mathbf{E}(F(*), F(a)) \xrightarrow{\circ} \mathbf{E}(F(*), F(b)).$$

The unit of this adjunction is an isomorphism, implying that the collage construction is a fully faithful functor.

A simplicial category  $\mathbf{W}$  is of the form  $\text{coll}(W)$  if and only if it is comprised of a full simplicial subcategory isomorphic to  $\mathbf{A}$  plus one other object  $*$  satisfying the conditions that  $\mathbf{W}(*, *) = \Delta^0$  and  $\mathbf{W}(a, *) = \emptyset$  for all objects  $a$  in  $\mathbf{A}$ .

The following result characterises relative projective cell complexes.

**5.3.3. Proposition.** *A natural transformation  $i: V \rightarrow W$  in  $\text{sSet}^{\mathbf{A}}$  is a relative projective cell complex if and only if its collage  $\text{coll}(i): \text{coll} V \rightarrow \text{coll} W$  is a relative simplicial computad.*

*Proof.* Exploiting the adjunction  $\text{coll} \dashv \text{wgt}$ , there is a bijective correspondence between simplicial functors  $F: \text{coll}(X \times \mathbf{A}_a) \rightarrow \mathbf{B}$  and natural transformations  $X \times \mathbf{A}_a \rightarrow \text{wgt}(\mathbf{B}, FI)$ , where  $I: \mathbf{A} + \{*\} \hookrightarrow \text{coll}(X \times \mathbf{A}_a)$  denotes the canonical inclusion. Applying the defining property of the cotensor  $X \times \mathbf{A}_a$  and Yoneda's lemma, we see that maps of this latter kind correspond to simplicial maps  $X \rightarrow \text{wgt}(\mathbf{B}, FI)(a) = \mathbf{B}(F(*), F(a))$ . Consequently we obtain a natural bijective correspondence between simplicial functors  $F: \text{coll}(X \times \mathbf{A}_a) \rightarrow \mathbf{B}$  and pairs of simplicial functors  $F: \mathbf{A} \rightarrow \mathbf{B}$  and  $\hat{F}: \mathcal{2}[X] \rightarrow \mathbf{B}$  with the property that  $\hat{F}(1) = F(a)$ . That latter pair is obtained from  $F: \text{coll}(X \times \mathbf{A}_a) \rightarrow \mathbf{B}$  by restricting it to the subcategory  $\mathbf{A}$  and by composing it with a canonical comparison functor  $K_X: \mathcal{2}[X] \rightarrow \text{coll}(X \times \mathbf{A}_a)$  respectively. Using this characterisation, it is easy to check that if  $f: X \rightarrow Y$  is any simplicial map then the square

$$\begin{array}{ccc} \mathcal{2}[X] & \xrightarrow{K_X} & \text{coll}(X \times \mathbf{A}_a) \\ \mathcal{2}[f] \downarrow & & \lrcorner \downarrow \text{coll}(f \times \mathbf{A}_a) \\ \mathcal{2}[Y] & \xrightarrow{K_Y} & \text{coll}(Y \times \mathbf{A}_a) \end{array} \quad (5.3.4)$$

is a pushout.

Transfinite composites and pushouts are colimits of connected diagrams, so they are both preserved and reflected by the forgetful functor  $(\mathbf{A} + \{*\})/\text{sSet-Cat} \rightarrow \text{sSet-Cat}$ . Furthermore, the collage construction is a fully faithful left adjoint functor so it too preserves and reflects all small colimits. For the “only if” direction, we suppose that  $i: V \hookrightarrow W$  is a relative projective cell complex, i.e., that it can be expressed as a transfinite composite of pushouts of natural transformations of the form  $\partial\Delta^n \times \mathbf{A}_a \hookrightarrow \Delta^n \times \mathbf{A}_a$ . On applying the collage construction and projecting into  $\text{sSet-Cat}$ , we obtain a decomposition of  $\text{coll}(i): \text{coll} V \hookrightarrow \text{coll} W$  as a transfinite composite of pushouts of simplicial functors of the form  $\text{coll}(\partial\Delta^n \times \mathbf{A}_a) \hookrightarrow \text{coll}(\Delta^n \times \mathbf{A}_a)$ . Composing each of those pushouts with the pushout square (5.3.4), we can also express  $\text{coll}(i)$  as transfinite composite of pushouts of simplicial functors of the form  $\mathcal{2}[\partial\Delta^n] \hookrightarrow \mathcal{2}[\Delta^n]$ . This proves that  $\text{coll}(i)$  is a relative simplicial computad.

Conversely for the “if” direction, if  $\text{coll}(i): \text{coll}(V) \hookrightarrow \text{coll}(W)$  is a relative simplicial computad, then it can be expressed as a transfinite composite of functors  $I^\beta: \mathbf{W}^\beta \hookrightarrow \mathbf{W}^{\beta+1}$  each of which is a pushout of a functor  $\mathcal{2}[\partial\Delta^n] \hookrightarrow \mathcal{2}[\Delta^n]$  for some  $n \geq 0$ ; because  $\text{coll}(V)$

and  $\text{coll}(W)$  share the same sets of objects, we will not require the functor  $\emptyset \hookrightarrow \mathbb{1}$  of definition 2.1.4.

If  $\beta \leq \beta'$  then we may regard  $\mathbf{W}^\beta$  as being a simplicial subcategory of  $\mathbf{W}^{\beta'}$ . In particular since each  $\mathbf{W}^\beta$  sits as a simplicial subcategory between  $\text{coll}(V)$  and  $\text{coll}(W)$  it follows that it too must have  $\mathbf{A}$  as a full simplicial subcategory plus one extra object  $*$  for which  $\mathbf{W}^\beta(*, *) = \Delta^0$  and  $\mathbf{W}^\beta(a, *) = \emptyset$ . Applying the characterisation of observation 5.3.2, there exists an essentially unique simplicial functor  $W^\beta$  in  $\underline{\text{sSet}}^{\mathbf{A}}$  such that  $\text{coll}(W^\beta) \cong \mathbf{W}^\beta$ . As the collage construction is fully faithful, we also obtain induced maps  $i^{\beta, \beta'} : W^\beta \rightarrow W^{\beta'}$  whose images under the collage construction are isomorphic to the connecting functors  $I^{\beta, \beta'} : \mathbf{W}^\beta \hookrightarrow \mathbf{W}^{\beta'}$  under the chosen isomorphisms  $\text{coll}(W^\beta) \cong \mathbf{W}^\beta$ . Finally, the transfinite sequence  $i^{\beta, \beta'} : W^\beta \rightarrow W^{\beta'}$  in  $\underline{\text{sSet}}^{\mathbf{A}}$  is actually a transfinite composite because it maps under the collage construction to a sequence which is isomorphic to our chosen transfinite composite  $I^{\beta, \beta'} : \mathbf{W}^\beta \hookrightarrow \mathbf{W}^{\beta'}$  in  $\underline{\text{sSet-Cat}}$  and these colimits are reflected.

All that remains is to show that each  $i^\beta : W^\beta \rightarrow W^{\beta+1}$  is a pushout of a projective cell. We know that  $\text{coll}(i^\beta) : \text{coll} W^\beta \hookrightarrow \text{coll} W^{\beta+1}$  is isomorphic to  $I^\beta : \mathbf{W}^\beta \hookrightarrow \mathbf{W}^{\beta+1}$ , by construction, and that latter functor is a pushout of some inclusion  $\mathcal{Z}[\partial\Delta^n] \hookrightarrow \mathcal{Z}[\Delta^n]$ , so it follows that there is a pushout:

$$\begin{array}{ccc} \mathcal{Z}[\partial\Delta^n] & \xrightarrow{F^\beta} & \text{coll} W^\beta \\ \downarrow & \lrcorner & \downarrow \text{coll}(i^\beta) \\ \mathcal{Z}[\Delta^n] & \longrightarrow & \text{coll} W^{\beta+1} \end{array}$$

However,  $\text{coll} W^\beta$  and  $\text{coll} W^{\beta+1}$  can only differ in hom-spaces whose domains are  $*$  and whose codomains are objects of  $\mathbf{A}$  so the attaching simplicial functor  $F^\beta$  must map 0 to the object  $*$  and 1 to some object  $a$  of  $\mathbf{A}$ . It follows we may apply the observations of the first paragraph of this proof to factor  $F^\beta$  as a composite of the canonical comparison  $K_{\partial\Delta^n} : \mathcal{Z}[\partial\Delta^n] \rightarrow \text{coll}(\partial\Delta^n \times \mathbf{A}_a)$  and a simplicial functor  $\hat{F}^\beta : \text{coll}(\partial\Delta^n \times \mathbf{A}_a) \rightarrow \text{coll} W^\beta$ . As the collage construction is fully faithful, there must exist a unique natural transformation  $f^\beta : \partial\Delta^n \times \mathbf{A}_a \rightarrow W^\beta$  with the property that  $\text{coll}(f^\beta) = \hat{F}^\beta$ . This defines a factorisation of the pushout above through the pushout (5.3.4) to give the following diagram

$$\begin{array}{ccccc} \mathcal{Z}[\partial\Delta^n] & \xrightarrow{K_{\partial\Delta^n}} & \text{coll}(\partial\Delta^n \times \mathbf{A}_a) & \xrightarrow{\text{coll}(f^\beta)} & \text{coll} W^\beta \\ \downarrow & & \lrcorner \downarrow \text{coll}(i \times \mathbf{A}_a) & & \lrcorner \downarrow \text{coll}(i^\beta) \\ \mathcal{Z}[\Delta^n] & \xrightarrow{K_{\Delta^n}} & \text{coll}(\Delta^n \times \mathbf{A}_a) & \xrightarrow{\text{coll}(g^\beta)} & \text{coll} W^{\beta+1} \end{array}$$

in which the right hand square is a pushout by the usual cancellation argument. As the collage construction reflects pushouts, we conclude that  $i^\beta : W^\beta \hookrightarrow W^{\beta+1}$  is a pushout of the projective cell  $i \times \mathbf{A}_a : \partial\Delta^n \times \mathbf{A}_a \hookrightarrow \Delta^n \times \mathbf{A}_a$  as required.  $\square$

Sections 6 and 7 make substantial use of the following special case of proposition 5.3.3.

**5.3.5. Proposition.** *A simplicial functor  $W: \mathbf{A} \rightarrow \mathbf{sSet}$  is a projective cell complex if and only if the canonical inclusion  $\mathbf{A} \hookrightarrow \text{coll } W$  is a relative simplicial computad.*

*Proof.* Proposition 5.3.3 tells us that  $\emptyset \rightarrow W$  is a projective cofibration if and only if  $\text{coll } \emptyset \rightarrow \text{coll } W$  is a relative simplicial computad. This latter functor is simply the canonical inclusion  $\mathbf{A} + \{*\} \hookrightarrow \text{coll } W$ , which is a relative simplicial computad if and only if  $\mathbf{A} \hookrightarrow \text{coll } W$  is such.  $\square$

## 6. THE FORMAL THEORY OF HOMOTOPY COHERENT MONADS

Let  $\underline{\mathbf{Mnd}}$  denote the full sub-2-category of  $\underline{\mathbf{Adj}}$  on the object  $+$ . We know from corollary 3.3.5 and remark 3.3.8 that the hom-category of  $\underline{\mathbf{Mnd}}$  is the category  $\Delta_+$  and that its horizontal composition is given by the join operation. So Lawvere’s characterisation [17] of  $\Delta_+$  as the free monoidal category containing a monoid tells us that  $\underline{\mathbf{Mnd}}$  is the free 2-category containing a monad.

We have seen that any adjunction in the 2-category  $\mathcal{K}_2$  of a quasi-categorically enriched category  $\mathcal{K}$  extends to a homotopy coherent adjunction, a simplicial functor  $\underline{\mathbf{Adj}} \rightarrow \mathcal{K}$ . The composite simplicial functor  $\underline{\mathbf{Mnd}} \rightarrow \underline{\mathbf{Adj}} \rightarrow \mathcal{K}$  is the homotopy coherent monad generated by that adjunction. More generally, we regard any simplicial functor  $\underline{\mathbf{Mnd}} \rightarrow \mathcal{K}$  as a *homotopy coherent monad* in  $\mathcal{K}$ . In this section and the next we justify this definition by developing the theory of homotopy coherent monads in the quasi-categorical context, including Beck’s monadicity theorem.

The “formal theory of monads” plays homage to Ross Street’s paper [29], which develops a formal 2-categorical theory of monads and their associated Eilenberg-Moore and Kleisli constructions. However, our method here is not a direct generalisation of his. For example, he defines the Eilenberg-Moore object associated with a monad in a 2-category using a universal property which is expressed in terms of the associated 2-category of monads and monad morphisms. In the quasi-categorical context, we will describe the Eilenberg-Moore object, which we refer to as the *quasi-category of algebras*, as a limit of the homotopy coherent monad  $\underline{\mathbf{Mnd}} \rightarrow \mathbf{qCat}_\infty$  weighted by a projective cofibrant weight extracted from the simplicial computad  $\underline{\mathbf{Adj}}$ .

This weight for Eilenberg-Moore objects was first observed in the 2-categorical context by Lawvere [17], but he does not appear to have recognised the connection with the free adjunction. Our description and analysis of  $\underline{\mathbf{Adj}}$  will allow us to take his insights much further, applying them directly to understanding monadicity in the quasi-categorical context. One novelty in our approach is that we describe almost all constructions and computations involved our proof of Beck’s theorem in terms of the properties of weights of various kinds and of natural transformations between them. This is the topic of section 7.

A key technical point is that the weights we derive from the simplicial category  $\underline{\mathbf{Adj}}$  are all shown to be projective cofibrant, as an immediate consequence of the fact that  $\underline{\mathbf{Adj}}$  is a simplicial computad. It follows then, by proposition 5.2.4, that limits of diagrams of quasi-categories weighted by such weights are again quasi-categories. In particular, our weighted limits approach produces explicit quasi-categorical models of all key structures involved in the theory of homotopy coherent monads.

In §6.1, we introduce the weights for the quasi-category of algebras for a homotopy coherent monad. We then build the associated (monadic) adjunction by exploiting a corresponding adjunction of weights. In §6.2, we show that the monadic forgetful functor reflects isomorphisms. In §6.3, we describe how the objects in the quasi-category of algebras for a homotopy coherent monad are themselves colimits of canonically constructed simplicial objects. A full proof of this result is deferred to [26], where we prove a substantial generalization that applies to other quasi-categories defined via projective cofibrant weighted limits, but we outline that argument here.

The proofs in this section are all just formal arguments involving the weights — the quasi-category on which the homotopy coherent monad is defined need not be referenced. In analogy with the classical case, the proof of the quasi-categorical analog of Beck’s monadicity theorem in the next section will require certain hypotheses on the underlying quasi-categories.

### 6.1. Weighted limits for the formal theory of monads.

6.1.1. **Definition** (homotopy coherent monads). A *homotopy coherent monad* in a quasi-categorically enriched category  $\mathcal{K}$  is a simplicial functor  $T: \underline{\mathbf{Mnd}} \rightarrow \mathcal{K}$ . The action of this functor on the unique object  $+$  in  $\underline{\mathbf{Mnd}}$  picks out an object  $B$  of  $\mathcal{K}$ . Its action on the sole hom-space  $\Delta_+$  of  $\underline{\mathbf{Mnd}}$  is given by a functor  $t: \Delta_+ \rightarrow \mathcal{K}(B, B)$  of quasi-categories which is a monoid map relative to the join operation on  $\Delta_+$  and the composition operation on the endo-hom-space  $\mathcal{K}(B, B)$ .

6.1.2. *Remark.* To fix ideas, from here to the end of the paper we shall work in  $\mathbf{qCat}_\infty$  with respect to a fixed homotopy coherent monad  $T: \underline{\mathbf{Mnd}} \rightarrow \mathbf{qCat}_\infty$  acting on a quasi-category  $B$  via a functor  $t: \Delta_+ \rightarrow B^B$ . However, our arguments can be interpreted equally in the context of an arbitrary quasi-categorically enriched category  $\mathcal{K}$  that admits all limits weighted by projective cofibrant weights.

6.1.3. *Observation* (weights on  $\underline{\mathbf{Mnd}}$ ). The constructions that we will apply to homotopy coherent monads will be expressed as limits weighted by projective cofibrant weights in  $\mathbf{sSet}^{\underline{\mathbf{Mnd}}}$ . Any simplicial functor  $W: \underline{\mathbf{Mnd}} \rightarrow \mathbf{sSet}$  is describable as a simplicial set  $W = W(+)$  equipped with a left action  $\cdot: \Delta_+ \times W \rightarrow W$  of the simplicial monoid  $(\Delta_+, \oplus, [-1])$ . Furthermore, a map  $f: V \rightarrow W$  in  $\mathbf{sSet}^{\underline{\mathbf{Mnd}}}$  is a simplicial map which is equivariant with respect to the actions of  $\Delta_+$  on  $V$  and  $W$ , in the sense that the square

$$\begin{array}{ccc} \Delta_+ \times V & \xrightarrow{\Delta_+ \times f} & \Delta_+ \times W \\ \downarrow \cdot & & \downarrow \cdot \\ V & \xrightarrow{f} & W \end{array}$$

commutes. In particular, a cone  $c: W \rightarrow T(-)^A$  weighted by  $W$  over the diagram  $T$  with summit  $A$  is specified by giving a simplicial map  $c: W \rightarrow B^A$  which makes the following

square

$$\begin{array}{ccc} \Delta_+ \times W & \xrightarrow{t \times c} & B^B \times B^A \\ \downarrow & & \downarrow \circ \\ W & \xrightarrow{c} & B^A \end{array}$$

commute.

Of course, since homotopy coherent monads are also simplicial functors  $\underline{\mathbf{Mnd}} \rightarrow \mathbf{qCat}_\infty$ , they too may be expressed as simplicial sets with left  $\Delta_+$ -actions.

**6.1.4. Definition** (monad resolutions). Write  $W_+ : \underline{\mathbf{Mnd}} \rightarrow \underline{\mathbf{sSet}}$  for the (unique) represented simplicial functor on  $\underline{\mathbf{Mnd}}$ . When described as in observation 6.1.3,  $W_+$  is the simplicial set  $\Delta_+$  acting on itself on the left by the join operation.

By the Yoneda lemma (5.1.6), the limit of any diagram weighted by a representable simplicial functor always exists and is isomorphic to the value of the diagram at the representing object. Hence, the weighted limit  $\{W_+, T\}_{\underline{\mathbf{Mnd}}}$  is isomorphic to  $B$ , the object on which the monad operates, and the data of the limit cone is given by the action  $t : \Delta_+ \rightarrow B^B$  of the simplicial functor  $T$  on the hom-space of  $\underline{\mathbf{Mnd}}$ . We refer to this diagram as the *monad resolution* and use the following notation for the evident 1-skeletal subset of its image

$$\mathrm{id}_B \xrightarrow{\eta} t \begin{array}{c} \xrightarrow{\eta t} \\ \xleftarrow{\mu} \\ \xrightarrow{t\eta} \end{array} t^2 \begin{array}{c} \xrightarrow{\eta t t} \\ \xleftarrow{t\eta t} \\ \xrightarrow{t\eta t} \end{array} t^3 \dots \dots \in B^B.$$

Evaluating this at any object  $b$  we obtain an augmented cosimplicial object in  $B$ , which we draw as:

$$b \xrightarrow{\eta b} t b \begin{array}{c} \xrightarrow{\eta t b} \\ \xleftarrow{\mu b} \\ \xrightarrow{t\eta b} \end{array} t^2 b \begin{array}{c} \xrightarrow{\eta t t b} \\ \xleftarrow{t\eta t b} \\ \xrightarrow{t\eta t b} \end{array} t^3 b \dots \dots \in B \quad (6.1.5)$$

When  $B$  is an ordinary category, regarded as a quasi-category, remark 5.2.9 applies. A homotopy coherent monad is just an ordinary monad and the cone  $t : \Delta_+ \rightarrow B^B$  is the usual monad resolution.

Another way to describe the weight  $W_+$  is to view it as the restriction of the covariant representable  $\underline{\mathbf{Adj}}_+ : \underline{\mathbf{Adj}} \rightarrow \underline{\mathbf{sSet}}$  to the full simplicial subcategory  $\underline{\mathbf{Mnd}}$ . This suggests the following definition:

**6.1.6. Definition.** Write  $W_-$  for the restriction of the covariant representable  $\underline{\mathbf{Adj}}_- : \underline{\mathbf{Adj}} \rightarrow \underline{\mathbf{sSet}}$  to the full simplicial subcategory  $\underline{\mathbf{Mnd}}$ . When described as a left  $\Delta_+$ -simplicial set, as in observation 6.1.3,  $W_-$  has underlying simplicial set  $\Delta_\infty$  and left action  $\oplus : \Delta_+ \times \Delta_\infty \rightarrow \Delta_\infty$ .

**6.1.7. Definition** (quasi-category of algebras). The *quasi-category of (homotopy coherent) algebras* for a homotopy coherent monad  $T$  on an object  $B$  in  $\mathbf{qCat}_\infty$  is the weighted limit

$$B[t] := \{W_-, T\}_{\underline{\mathbf{Mnd}}}.$$

Once we show that the weight  $W_-$  is projective cofibrant, proposition 5.2.4 will imply that every homotopy coherent monad possesses an associated quasi-category of algebras. The proof of this fact is entirely straightforward and follows a pattern we shall see repeated for other weights below. We identify the collage  $\text{coll } W_-$  as a simplicial subcategory of  $\underline{\text{Adj}}$ , use that description to show that it is a simplicial computad, and then appeal to proposition 5.3.5.

**6.1.8. Lemma.** *The simplicial functor  $W_- : \underline{\text{Mnd}} \rightarrow \underline{\text{sSet}}$  is projective cofibrant.*

*Proof.* The collage of  $W_-$  can be identified with the (non-full) simplicial subcategory of  $\underline{\text{Adj}}$  containing the hom-spaces  $\underline{\text{Adj}}(+, +)$  and  $\underline{\text{Adj}}(-, +)$  but with the hom-spaces from  $-$  and  $+$  to  $-$  respectively trivial and empty. This is a simplicial computad whose atomic arrows are precisely those squiggles whose codomain is  $+$  and which do not contain any instances of  $+$  in their interiors. Note that the atomic arrows in  $\text{coll } W_-$  are not necessarily atomic in  $\underline{\text{Adj}}$ , as they may contain any number of occurrences of  $-$ , so  $\text{coll } W_-$  is not a simplicial subcomputad of  $\underline{\text{Adj}}$ . However, the atomic arrows of  $\underline{\text{Mnd}}$  are also those squiggles which do not contain any instances of  $+$  in their interiors, so  $\underline{\text{Mnd}}$  is a simplicial subcomputad of  $\text{coll } W_-$ , and it follows that  $\underline{\text{Mnd}} \hookrightarrow \text{coll } W_-$  is a relative simplicial computad. The conclusion now follows from proposition 5.3.5.  $\square$

**6.1.9. Corollary.** *Every homotopy coherent monad in  $\underline{\text{qCat}}_\infty$  admits a quasi-category of algebras.*

*Proof.* Immediate from lemma 6.1.8 and proposition 5.2.4.  $\square$

**6.1.10. Remark.** We unpack definition 6.1.7 to view an object of  $B[t]$  as a homotopy coherent algebra. The definition of  $B[t]$  as a limit weighted by  $W_-$  tells us that the object  $b: \Delta^0 \rightarrow B[t]$  corresponds to a  $W_-$ -weighted cone over  $T$  with summit  $\Delta^0$ . As discussed in observation 6.1.3, such a cone is simply a simplicial map  $b: \Delta_\infty \rightarrow B$  satisfying the equivariance condition that

$$\begin{array}{ccc} \Delta_+ \times \Delta_\infty & \xrightarrow{t \times b} & B^B \times B \\ \oplus \downarrow & & \downarrow \text{ev} \\ \Delta_\infty & \xrightarrow{b} & B \end{array} \quad (6.1.11)$$

commutes. Evaluating  $b$  at  $[0] \in \Delta_\infty$  we obtain an object in  $B$ , which we shall also denote by  $b$ . Evaluating at  $\sigma^0: [1] \rightarrow [0]$ , we obtain an arrow in  $B$ , which we shall denote by  $\beta: tb \rightarrow b$ .

The condition (6.1.11) implies, in particular, that the composite of  $b: \Delta_\infty \rightarrow B$  with the functor  $- \oplus [0]: \Delta_+ \rightarrow \Delta_\infty$  is equal to the resolution displayed in (6.1.5). Drawing this algebra in the way that we drew our monad resolutions (6.1.5), we obtain the following picture:

$$\begin{array}{ccccccc} b & \xrightarrow{\eta b} & tb & \xrightarrow{\eta tb} & t^2 b & \xrightarrow{\eta t^2 b} & t^3 b \dots \dots \in B \\ & \xleftarrow{\beta} & & \xleftarrow{\mu b} & & \xleftarrow{\mu t b} & \\ & & & \xleftarrow{t \eta b} & & \xleftarrow{t \eta t b} & \\ & & & \xleftarrow{t \beta} & & \xleftarrow{t t \beta} & \end{array} \quad (6.1.12)$$

The higher dimensional data of (6.1.12) implies, in particular, that  $(b, \beta)$  defines a  $h(t)$ -algebra, in the usual sense, in the homotopy category  $hB$ . However, it is not generally the case that all  $h(t)$ -algebras in  $hB$  can be lifted to homotopy coherent  $t$ -algebras in  $B$ .

6.1.13. **Example** (free monoid monad). Let  $\underline{\text{Kan}}_\infty$  denote the simplicial category of Kan complexes. We may construct the free strictly associative monoid on a Kan complex  $K$  in the usual way: take the coproduct

$$tK := \coprod_{n \geq 0} K^n$$

which is again a Kan complex and equip it with the obvious concatenation operation as its product  $tK \times tK \rightarrow tK$ . This provides us with a simplicially enriched monad on  $\underline{\text{Kan}}_\infty$  whose monad resolution  $\Delta_+ \rightarrow \underline{\text{Kan}}_\infty^{\underline{\text{Kan}}_\infty}$  may be transposed to give a left action  $\Delta_+ \times \underline{\text{Kan}}_\infty \rightarrow \underline{\text{Kan}}_\infty$  of the strict monoidal category  $(\Delta_+, \oplus, [-1])$ . Here we regard the category  $\Delta_+$  as being a simplicial category with discrete hom-spaces.

Applying the homotopy coherent nerve construction  $N: \underline{\text{sSet-Cat}} \rightarrow \underline{\text{sSet}}$ , which coincides with the usual nerve construction on discrete simplicial categories, we obtain a left action  $\Delta_+ \times N\underline{\text{Kan}}_\infty \rightarrow N\underline{\text{Kan}}_\infty$ , transposing to define a monoid map  $\Delta_+ \rightarrow N\underline{\text{Kan}}_\infty^{N\underline{\text{Kan}}_\infty}$ . This defines a homotopy coherent monad on the (large) quasi-category  $N\underline{\text{Kan}}_\infty$ .

Consulting remark 6.1.10, we see that a vertex in the associated quasi-category of coherent algebras corresponds to a functor  $\Delta_\infty \rightarrow N\underline{\text{Kan}}_\infty$  satisfying the naturality condition with respect to the left actions of  $\Delta_+$  on its domain and codomain. We can take the transpose of that map under the adjunction  $\mathfrak{C} \dashv N$  to give a simplicial functor  $\mathfrak{C}\Delta_\infty \rightarrow \underline{\text{Kan}}_\infty$ ; hence, a homotopy coherent algebra is a homotopy coherent diagram of shape  $\Delta_\infty$  in  $\underline{\text{Kan}}_\infty$ . The data in the image of this functor picks out a Kan complex  $K$ , an action map  $\beta: tK \rightarrow K$ , various composites of these as displayed in (6.1.12), and higher dimensional homotopy coherence data that relates those to the structure of the monad resolution at  $K$ . In particular, this data ensures that the action map  $\beta: tK \rightarrow K$  supplies  $K$  with the structure of a strictly associative monoid in the classical homotopy category of Kan complexes.

Importantly, weighted limits can be used not just to define the quasi-category of algebras for a homotopy coherent monad but also the full (monadic) homotopy coherent adjunction.

6.1.14. **Definition** (monadic adjunction). Composing the Yoneda embedding and the restriction along  $\underline{\text{Mnd}} \hookrightarrow \underline{\text{Adj}}$ , one obtains a simplicial functor

$$\begin{aligned} \underline{\text{Adj}}^{\text{op}} &\rightarrow \underline{\text{sSet}}^{\underline{\text{Adj}}} \rightarrow \underline{\text{sSet}}^{\underline{\text{Mnd}}} & (6.1.15) \\ + &\longmapsto W_+ \\ - &\longmapsto W_- \end{aligned}$$

We know that the weighted limit construction  $\{-, T\}_{\underline{\text{Mnd}}}$  is simplicially contravariantly functorial on the full subcategory of projective cofibrant weights in  $\underline{\text{sSet}}^{\underline{\text{Mnd}}}$ . In particular, the representable  $W_+$  and the weight for quasi-categories of algebras  $W_-$  are both projective cofibrant, so it follows that we may compose the simplicial functor (6.1.15) with

the weighted limit construction  $\{-, T\}_{\underline{\text{Mnd}}}: (\underline{\text{sSet}}_{\text{cof}}^{\underline{\text{Mnd}}})^{\text{op}} \rightarrow \underline{\text{qCat}}_{\infty}$  to obtain a homotopy coherent adjunction  $\underline{\text{Adj}} \rightarrow \underline{\text{qCat}}_{\infty}$ .

We denote the primary data involved in this adjunction as follows

$$\{W_-, T\}_{\underline{\text{Mnd}}} = B[t] \begin{array}{c} \xleftarrow{f^t} \\ \perp \\ \xrightarrow{u^t} \end{array} B \cong \{W_+, T\}_{\underline{\text{Mnd}}} \quad \begin{array}{l} \eta^t: \text{id}_B \Rightarrow u^t f^t \\ \epsilon^t: f^t u^t \Rightarrow \text{id}_{B[t]} \end{array}$$

and call this the *homotopy coherent monadic adjunction* associated with the homotopy coherent monad  $T$ .

## 6.2. Conservativity of the monadic forgetful functor.

**6.2.1. Definition** (conservative functors). We say a functor  $f: A \rightarrow B$  between quasi-categories is *conservative* if it reflects isomorphisms; that is to say, if it has the property that a 1-simplex in  $A$  is an isomorphism if and only if its image in  $B$  under  $f$  is an isomorphism. It is clear that  $f$  is conservative if and only if the corresponding functor  $h(f): hA \rightarrow hB$  of homotopy categories is conservative.

As in the categorical context, the monadic forgetful functor  $u^t: B[t] \rightarrow B$  is always conservative. This will follow from the following general result.

**6.2.2. Proposition.** *Suppose  $\mathbf{A}$  is a small simplicial category and that  $i: V \hookrightarrow W$  in  $\underline{\text{sSet}}^{\mathbf{A}}$  is a projective cofibration between projective cofibrant weights with the property that for all objects  $a \in \mathbf{A}$  the simplicial map  $i_a: V(a) \hookrightarrow W(a)$  is surjective on vertices. Then for any diagram  $D$  in  $\underline{\text{qCat}}_{\infty}^{\mathbf{A}}$ , the functor  $\{i, D\}_{\mathbf{A}}: \{W, D\}_{\mathbf{A}} \rightarrow \{V, D\}_{\mathbf{A}}$  is conservative.*

*Proof.* Applying the small object argument for the restricted set of projective cells  $\partial\Delta^n \times \mathbf{A}_a \hookrightarrow \Delta^n \times \mathbf{A}_a$  with  $n > 0$ , we factor  $i: V \hookrightarrow W$  as a composite of a natural transformation  $i': V \hookrightarrow U$  which is a transfinite composite of pushouts of those cells and a natural transformation  $p: U \rightarrow W$  which has the right lifting property with respect to those cells. This second condition means that for all objects  $a$  of  $\mathbf{A}$  the simplicial map  $p_a: U(a) \rightarrow W(a)$  has the right lifting property with respect to each  $\partial\Delta^n \hookrightarrow \Delta^n$  for  $n > 0$ . Now  $i_a = p_a i'_a$  is surjective on vertices, by assumption, so it follows that  $p_a$  is also surjective on vertices. This means that  $p$  has the right lifting property with respect to each cell  $\emptyset \cong \partial\Delta^0 \times \mathbf{A}_a \hookrightarrow \Delta^0 \times \mathbf{A}_a \cong \mathbf{A}_a$ , so  $p_a$  is actually a trivial fibration. As  $i$  is a projective cofibration by assumption, it has the right lifting property with respect to the pointwise trivial fibration  $p$ . Solving the obvious lifting problem between  $i$  and  $p$ , we conclude that  $i$  is a retract of  $i'$ . This demonstrates that  $i$  is a retract of a transfinite composite of pushouts of the restricted set of projective cells  $\{\partial\Delta^n \times \mathbf{A}_a \hookrightarrow \Delta^n \times \mathbf{A}_a \mid a \in \mathbf{A}, n > 0\}$ .

Arguing as in the proof of proposition 5.2.2, we may now express  $\{i, D\}_{\mathbf{A}}$  as a retract of a transfinite co-composite (limit of a tower) of pullbacks of functors of the form

$$D(a)^i: D(a)^{\Delta^n} \longrightarrow D(a)^{\partial\Delta^n}$$

with  $n > 0$ . The class of conservative functors is closed under transfinite co-composites, pullbacks, and splitting of idempotents: working in the marked model structure, a functor is conservative if and only if it has the right lifting property with respect to  $\mathcal{2} \hookrightarrow \mathcal{2}^{\sharp}$ . So

it is enough to show that each of those functors is conservative. This is an easy corollary of lemma I.2.3.10: pointwise equivalences in  $D(a)^{\Delta^n}$  are detected in  $D(a)^{\partial\Delta^n}$ , provided  $n > 0$ , because  $\partial\Delta^n \rightarrow \Delta^n$  is surjective on 0-simplices.  $\square$

**6.2.3. Corollary.** *The monadic forgetful 0-arrow  $u^t: B[t] \rightarrow B$  is conservative.*

*Proof.* The forgetful functor  $u^t$  is constructed by applying the contravariant weighted limit functor  $\{-, T\}_{\mathbf{A}}$  to the natural transformation which arises by applying the simplicial functor displayed in (6.1.15) to the 0-arrow  $\underline{u}$  in  $\underline{\text{Adj}}$ . In other words, this is the natural transformation  $W_+ \hookrightarrow W_-$  which acts by pre-whiskering the elements of  $W_+(+) = \underline{\text{Adj}}(+, +)$  with the 0-arrow  $\underline{u}$  to give an element of  $W_- (+) = \underline{\text{Adj}}(-, +)$ . Our graphical calculus makes clear that this pre-whiskering operation is injective, so it follows that we may use it to identify  $\text{coll } W_+$  with a simplicial subcategory of  $\text{coll } W_-$ , which we have already identified with a simplicial subcategory of  $\underline{\text{Adj}}$  in the proof of lemma 6.1.8.

Under this identification  $\text{coll } W_+$  becomes a simplicial subcategory of  $\text{coll } W_-$  which differs from it solely to the extent that its hom-space  $\text{coll } W_+(-, +)$  contains only those squiggles of  $\text{coll } W_+(-, +) = \underline{\text{Adj}}(-, +)$  that decompose as  $\underline{au}$  for some unique squiggle  $\underline{a}$  in  $\underline{\text{Adj}}(+, +)$ . In particular, every atomic arrow of  $\text{coll } W_-$  is also atomic in  $\text{coll } W_+$ . Thus,  $\text{coll } W_+ \hookrightarrow \text{coll } W_-$  is a relative simplicial computad to which we may apply proposition 5.3.3 to show that  $-\circ\underline{u}: W_+ \hookrightarrow W_-$  is a relative projective cell complex. Furthermore, every 0-arrow in  $\underline{\text{Adj}}$  is an alternating composite of the atomic 0-arrows  $\underline{u}$  and  $\underline{f}$ , so it is clear that every 0-arrow in  $W_- (+) = \underline{\text{Adj}}(-, +)$  does decompose as  $\underline{au}$  and is thus in the image of  $-\circ\underline{u}: W_+ \hookrightarrow W_-$ . Applying proposition 6.2.2, we conclude that  $u^t: B[t] \rightarrow B$  is conservative.  $\square$

**6.2.4. Observation.** In the proof of corollary 6.2.3, we observed that  $W_+ \hookrightarrow W_-$  is a relative projective cell complex. By proposition 5.3.3, we can extract an explicit presentation from the corresponding relative simplicial computad  $\text{coll } W_+ \hookrightarrow \text{coll } W_-$  via our graphical calculus. Applying the weighted limit  $\{-, T\}$  to the map  $W_+ \hookrightarrow W_-$ , this translates to a presentation of the monadic forgetful functor as a limit of a tower of isofibrations, each layer of which is defined as the pullback of a map  $B^{\Delta^n} \rightarrow B^{\partial\Delta^n}$  corresponding to an atomic  $n$ -arrow of  $\text{coll } W_-$  not in the image of  $\text{coll } W_+$ . In particular,  $u^t: B[t] \rightarrow B$  is an isofibration.

**6.2.5. Remark.** We may express the projective cofibration  $-\circ\underline{u}: W_+ \hookrightarrow W_-$  in the form of observation 6.1.3 using the representation of  $\underline{\text{Adj}}$  given in remark 3.3.8. Under that interpretation it is the simplicial map  $-\oplus[0]: \Delta_+ \hookrightarrow \Delta_\infty$ , which satisfies the required equivariance condition

$$\begin{array}{ccc} \Delta_+ \times \Delta_+ & \xrightarrow{\Delta_+ \times (-\oplus[0])} & \Delta_+ \times \Delta_\infty \\ \oplus \downarrow & & \downarrow \oplus \\ \Delta_+ & \xrightarrow{-\oplus[0]} & \Delta_\infty \end{array}$$

as an immediate consequence of the associativity of the join.

**6.3. Colimit representation of algebras.** Perhaps the key technical insight enabling Beck’s proof of the monadicity theorem is the observation that any algebra is canonically a colimit of a particular diagram of free algebras. More precisely any algebra  $(b, \beta)$  for a monad  $t$  on a category  $B$  is a  *$u^t$ -split coequaliser*

$$t^2b \begin{array}{c} \xleftarrow{\quad -\eta_{tb} \quad -} \\ \xrightarrow{\mu_b} \\ \xleftarrow{t\eta_b} \\ \xrightarrow{t\beta} \end{array} tb \begin{array}{c} \xleftarrow{\quad -\eta_b \quad -} \\ \xrightarrow{\beta} \end{array} b \tag{6.3.1}$$

Here the solid arrows are maps which respect  $t$ -algebra structures on these objects, whereas the dotted splittings are not. Split coequalisers are examples of *absolute colimits*, that is to say colimits which are preserved by any functor. In particular they are preserved by  $t: B \rightarrow B$  itself, a fact we may exploit in order to show that the forgetful functor  $u^t: B[t] \rightarrow B$  creates the canonical colimits of the form (6.3.1).

On our way to a monadicity theorem that can be applied to homotopy coherent adjunctions of quasi-categories, we demonstrate that any vertex in the quasi-category  $B[t]$  of algebras for a homotopy coherent monad  $T: \underline{\mathbf{Mnd}} \rightarrow \mathbf{qCat}_\infty$  has an analogous colimit presentation. In this context, the  $u^t$ -split coequaliser (6.3.1) is replaced by a canonical  *$u^t$ -split augmented simplicial object*. In this section, we give a precise statement of this result and a sketch of its proof. The full details are deferred to [26] because the argument, relying on our description of the quasi-category of algebras as a projective cofibrant weighted limit, applies to general quasi-categories defined as limits of this form.

As explained in section I.5, colimits in a quasi-category are encoded by absolute left lifting diagrams in  $\mathbf{qCat}_2$  of a particular form. For the reader’s convenience, we briefly recall definition I.5.2.9.

**I.5.2.9. Definition** (colimits in a quasi-category). We say a quasi-category  $A$  *admits colimits of a family of diagrams  $k: K \rightarrow A^X$  of shape  $X$*  if there is an absolute left lifting diagram in  $\mathbf{qCat}_2$

$$\begin{array}{ccc} & & A \\ & \nearrow \text{colim} & \downarrow c \\ K & \xrightarrow{k} & A^X \\ & \uparrow \lambda & \\ & & A \end{array}$$

in which  $c$ , the “constant map”, is the adjoint transpose of the projection  $\pi_A: A \times X \rightarrow A$ .

For example, the split augmented simplicial objects in a quasi-category  $B$  provide us with a family of colimit diagrams:

**I.5.3.1. Theorem.** *For any quasi-category  $B$ , the canonical diagram*

$$B^{\Delta_\infty} \begin{array}{c} \xrightarrow{\text{ev}_0} \\ \xrightarrow{\text{res}} \end{array} B^{\Delta^{\text{op}}} \begin{array}{c} \nearrow \\ \downarrow c \end{array} B \tag{6.3.2}$$

*is an absolute left lifting diagram. Hence, given any simplicial object admitting an augmentation and a splitting, the augmented simplicial object defines a colimit cone over the original simplicial object. Furthermore, such colimits are preserved by any functor.*

6.3.3. *Recall* (constructing the triangle in theorem I.5.3.1). The object  $[-1]$  is terminal in the category  $\Delta_+^{\text{op}}$ , so there exists a necessarily unique natural transformation:

$$\begin{array}{ccc} \Delta^{\text{op}} & \xrightarrow{\quad} & \Delta_+^{\text{op}} \\ & \searrow \quad \downarrow & \nearrow [-1] \\ & ! & \mathbb{1} \end{array} \quad (6.3.4)$$

Furthermore, post-composition by the arrow  $\underline{u}: - \rightarrow +$  of  $\underline{\text{Adj}}$  provides us with an embedding  $\underline{\text{Adj}}(-, -) \hookrightarrow \underline{\text{Adj}}(-, +)$ , which we denote by  $\underline{u} \circ -: \Delta_+^{\text{op}} \hookrightarrow \Delta_\infty$ . On applying the contravariant 2-functor  $B^{(-)}$  to this data we obtain the following diagram

$$\begin{array}{ccc} & & B \\ & \nearrow \text{ev}_{-1} & \downarrow c \\ B^{\Delta_\infty} & \xrightarrow{B^{\underline{u} \circ -}} B^{\Delta_+^{\text{op}}} & \xrightarrow{\text{res}} B^{\Delta^{\text{op}}} \\ & \uparrow & \downarrow \\ & & B \end{array} =: \begin{array}{ccc} & & B \\ & \nearrow \text{ev}_0 & \downarrow c \\ B^{\Delta_\infty} & \xrightarrow{\text{res}} B^{\Delta_+^{\text{op}}} & \end{array}$$

whose composite is the triangle displayed in the statement of theorem I.5.3.1.

The importance of this particular family of colimits is that their presence in  $B$  allows us to infer the existence of a more general family of colimits in the quasi-category  $B[t]$ . The following definition specifies the class of diagrams in that family:

6.3.5. **Definition** (*u-split augmented simplicial objects*). By remark 3.3.8, the image of the embedding  $\underline{u} \circ -: \Delta_+^{\text{op}} \hookrightarrow \Delta_\infty$  of recollection 6.3.3 is the subcategory of  $\Delta_\infty$  generated by all of its elementary operators except for the face operators  $\delta^0: [n-1] \rightarrow [n]$  for each  $n \geq 1$ . We call these extra face maps  $\delta^0$  *splitting operators*.

Given any functor  $u: C \rightarrow B$  of quasi-categories, a *u-split augmented simplicial object* is an augmented simplicial object  $\Delta_+^{\text{op}} \rightarrow C$  that, when mapped to  $B$  by  $u$ , comes equipped with an extension  $\Delta_\infty \rightarrow B$ , providing actions of the splitting operators and associated higher coherence data. In other words, such structures comprise pairs of horizontal functors in the following diagram

$$\begin{array}{ccc} \Delta_+^{\text{op}} & \longrightarrow & C \\ \downarrow \text{u} \circ - & \lrcorner & \downarrow u \\ \Delta_\infty & \longrightarrow & B \end{array}$$

which make that square commute.

We may define a quasi-category  $S(u)$  of *u-split augmented simplicial objects* by forming the following pullback:

$$\begin{array}{ccc} S(u) & \longrightarrow & B^{\Delta_\infty} \\ \downarrow \lrcorner & & \downarrow B^{\underline{u} \circ -} \\ C^{\Delta_+^{\text{op}}} & \xrightarrow{u^{\Delta_+^{\text{op}}}} & B^{\Delta_+^{\text{op}}} \end{array} \quad (6.3.6)$$

This is a pullback of exponentiated quasi-categories whose right hand vertical is an isofibration, as  $\underline{u} \circ - : \Delta_+^{\text{op}} \hookrightarrow \Delta_\infty$  is injective. It follows that  $S(u)$  is indeed a quasi-category. This construction may also be described as the limit of the diagram  $\mathcal{2} \rightarrow \underline{\text{qCat}}_\infty$  whose image is the functor  $u: C \rightarrow B$  weighted by the projective cofibrant weight  $\mathcal{2} \rightarrow \underline{\text{sSet}}$  whose image is  $\underline{u} \circ - : \Delta_+^{\text{op}} \hookrightarrow \Delta_\infty$ .

The following proposition motivates our consideration of this particular class of diagrams:

**6.3.7. Proposition.** *The monadic forgetful functor  $u^t: B[t] \rightarrow B$  creates colimits of  $u^t$ -split simplicial objects. It follows immediately that  $u^t$  both preserves and reflects such colimits.*

Our proof of this result relies on the following theorem which we prove as corollary 5.5 of [26], where it appears as a special case of a much more general theorem proven there:

**6.3.8. Theorem.** *The monadic forgetful functor  $u^t: B[t] \rightarrow B$  of a homotopy coherent monad creates any colimits that  $t: B \rightarrow B$  preserves.*

**6.3.9. Observation.** Before sketching a proof of theorem 6.3.8, let us expand upon its statement. It asks us to show that if  $k: K \rightarrow B[t]^X$  is a family of diagrams in  $B[t]$  whose underlying diagrams  $(u^t)^X k: K \rightarrow B^X$  admit colimits in  $B$  that are preserved by  $t: B \rightarrow B$ , then we may lift them to give colimits of the diagrams we started with in  $B[t]$ .

In other words, consider a family of diagrams  $k: K \rightarrow B[t]^X$  and an absolute left lifting diagram

$$\begin{array}{ccc}
 & & B \\
 & \nearrow^{\text{colim}} & \downarrow c \\
 K & \xrightarrow{(u^t)^X k} & B^X
 \end{array} \tag{6.3.10}$$

so that the composite diagram

$$\begin{array}{ccccc}
 & & B & \xrightarrow{t} & B \\
 & \nearrow^{\text{colim}} & \downarrow c & & \downarrow c \\
 K & \xrightarrow{(u^t)^X k} & B^X & \xrightarrow{t^X} & B^X
 \end{array}$$

is again an absolute left lifting diagram. Theorem 6.3.8 asserts that under these assumptions there is a diagram

$$\begin{array}{ccc}
 & & B[t] \\
 & \nearrow^{\overline{\text{colim}}} & \downarrow c \\
 K & \xrightarrow{k} & B[t]^X
 \end{array}$$

which lies over (6.3.10), in the sense that it satisfies the equality

$$\begin{array}{ccc}
 & B[t] & \xrightarrow{u^t} & B \\
 \text{colim} \nearrow & \downarrow c & & \downarrow c \\
 K & \xrightarrow{k} & B[t]^X & \xrightarrow{(u^t)^X} & B^X
 \end{array}
 \quad \uparrow \bar{\lambda}
 \quad = \quad
 \begin{array}{ccc}
 & B & \\
 \text{colim} \nearrow & \downarrow c & \\
 K & \xrightarrow{(u^t)^X k} & B^X
 \end{array}$$

and that any such diagram  $\bar{\lambda}$  lying over (6.3.10) in this way is itself an absolute left lifting diagram.

*Sketch proof of theorem 6.3.8.* The map  $u^t: B[t] \rightarrow B$  is induced by the inclusion of weights  $W_+ \hookrightarrow W_-$  applied to the homotopy coherent monad  $T$ . As noted in observation 6.2.4, we may extract an explicit presentation of  $W_+ \hookrightarrow W_-$  as a relative projective cell complex from our graphical calculus. Passing to weighted limits of the homotopy coherent monad  $T$ ,  $u^t: B[t] \rightarrow B$  is then the limit of a tower of isofibrations defined as pullbacks of maps of the form  $B^{\Delta^n} \rightarrow B^{\partial\Delta^n}$ ; these isofibrations arise as the limit of  $T$  weighted by a projective cell  $W_+ \times \partial\Delta^n \hookrightarrow W_+ \times \Delta^n$  indexed by an atomic  $n$ -arrow of  $\text{coll } W_-$  not in the image of  $\text{coll } W_+$ .

By proposition I.5.2.18, the cotensors of  $B$  admit geometric realisations of split augmented simplicial objects, defined pointwise in  $B$ . Because  $t$  preserves these colimits, the maps in the pullback diagrams defining the layers in the tower for  $u^t: B[t] \rightarrow B$  preserve these colimits. We conclude by arguing inductively that each pullback admits and the legs of the pullback cone preserve such colimits. The limit stage of this induction creates the desired colimits in  $B[t]$ .  $\square$

*Proof of proposition 6.3.7.* Start by applying the 2-functor  $B[t]^{(-)}$  to the 2-cell in (6.3.4) to obtain a triangle which we may combine with the canonical projection  $S(u^t) \rightarrow B[t]^{\Delta_+^{\text{op}}}$  of (6.3.6) to give a composite triangle:

$$\begin{array}{ccc}
 & & B[t] & \\
 & & \nearrow \text{ev}_{-1} & \downarrow c \\
 S(u^t) & \longrightarrow & B[t]^{\Delta_+^{\text{op}}} & \xrightarrow{\text{res}} & B[t]^{\Delta^{\text{op}}}
 \end{array}
 \quad \uparrow
 \quad (6.3.11)$$

Now, the 2-functoriality properties of exponentiation provide us with the following pasting equation

$$\begin{array}{ccc}
 & B[t] & \xrightarrow{u^t} & B \\
 \text{ev}_{-1} \nearrow & \downarrow c & & \downarrow c \\
 B[t]^{\Delta_+^{\text{op}}} & \xrightarrow{\text{res}} & B[t]^{\Delta^{\text{op}}} & \xrightarrow{(u^t)^{\Delta^{\text{op}}}} & B^{\Delta^{\text{op}}}
 \end{array}
 \quad \uparrow
 \quad = \quad
 \begin{array}{ccc}
 & B & \\
 \text{ev}_{-1} \nearrow & \downarrow c & \\
 B[t]^{\Delta_+^{\text{op}}} & \xrightarrow{(u^t)^{\Delta_+^{\text{op}}}} & B^{\Delta_+^{\text{op}}} & \xrightarrow{\text{res}} & B^{\Delta^{\text{op}}}
 \end{array}$$

which we may combine with the defining pullback square (6.3.6) to show that the composite of the triangle in (6.3.11) with the functor  $u^t$  reduces to:

$$\begin{array}{ccc}
 & & B \\
 & \nearrow \text{ev}_0 & \downarrow c \\
 S(u^t) & \longrightarrow B^{\Delta_\infty} & \xrightarrow{\text{res}} B^{\Delta^{\text{op}}}
 \end{array} \tag{6.3.12}$$

Here the triangle on the right is simply that given in the statement of theorem I.5.3.1, whose construction is described in recollection 6.3.3. In other words, we have shown that the triangle in (6.3.11) lies over the one in (6.3.12), in the sense discussed in observation 6.3.9.

The triangle in (6.3.12) is an absolute left lifting diagram, simply because it is obtained by pre-composing the absolute left lifting diagram of theorem I.5.3.1 by the functor  $S(u^t) \rightarrow B^{\Delta_\infty}$  of (6.3.6). Theorem I.5.3.1 also tells us that these colimits are preserved by all functors and so, in particular, they are preserved by  $t: B \rightarrow B$ . Consequently, we may apply theorem 6.3.8 to show that  $u^t: B[t] \rightarrow B$  creates these colimits as postulated. On consulting observation 6.3.9, we see that we have succeeded in showing that the particular triangle given in (6.3.11) is an absolute left lifting diagram which displays those colimits.  $\square$

Proposition 6.3.7 can be specialised to provide the promised representation of homotopy coherent algebras as colimits of diagrams of free such algebras. All that remains to do so is to define this particular family of diagrams of  $u^t$ -split augmented simplicial objects in  $B[t]$ .

6.3.13. *Observation* (a direct description of  $S(u^t)$ ). We can express the quasi-category  $S(u^t)$  of  $u^t$ -split simplicial objects in  $B[t]$  directly as a limit of the homotopy coherent monad  $T$  weighted by a projectively cofibrant weight. To see how this may be achieved, start by recalling that the forgetful functor  $u^t$  is constructed by applying the covariant weighted limit functor  $\{-, T\}_{\text{Mnd}}$  to the natural transformation  $- \circ \underline{u}: W_+ \hookrightarrow W_-$  described in remark 6.2.5. This features in the following pushout in  $\underline{\text{Set}}^{\text{Mnd}}$  of weights

$$\begin{array}{ccc}
 W_+ \times \Delta_+^{\text{op}} & \xleftarrow{(- \circ \underline{u}) \times \Delta_+^{\text{op}}} & W_- \times \Delta_+^{\text{op}} \\
 W_+ \times (u \circ -) \downarrow & & \downarrow \Gamma \\
 W_+ \times \Delta_\infty & \xleftarrow{\quad \quad \quad} & W_s
 \end{array} \tag{6.3.14}$$

in which the products are tensors of weights by simplicial sets. Now the contravariant simplicial functor  $\{-, T\}$  is cocontinuous, so it carries this pushout to a pullback which is easily seen to be the  $u^t$  instance of the pullback (6.3.6). Consequently,  $S(u^t)$  is canonically isomorphic to the limit  $\{W_s, T\}_{\text{Mnd}}$ .

6.3.15. *Observation*. Every homotopy coherent algebra for  $T$  gives rise to a  $u^t$ -split simplicial object in  $B[t]$ , and this construction may be encapsulated in a functor  $B[t] \rightarrow S(u^t)$  which we now describe. Using observation 6.1.3, we re-express the objects and maps that occur in the diagram whose pushout we formed in (6.3.14) in terms of the structure of the

simplicial category  $\underline{\text{Adj}}$ . Doing so we get the upper horizontal and left-hand vertical maps in the following square

$$\begin{array}{ccc} \underline{\text{Adj}}(+, +) \times \underline{\text{Adj}}(-, -) & \xrightarrow{(-\circ u) \times \underline{\text{Adj}}(-, -)} & \underline{\text{Adj}}(-, +) \times \underline{\text{Adj}}(-, -) \\ \underline{\text{Adj}}(+, +) \times (u \circ -) \downarrow & & \downarrow \circ \\ \underline{\text{Adj}}(+, +) \times \underline{\text{Adj}}(-, +) & \xrightarrow{\circ} & \underline{\text{Adj}}(-, +) \end{array}$$

whose lower right-hand vertex is the simplicial set underlying  $W_-$ . This diagram commutes by associativity of the composition in  $\underline{\text{Adj}}$ , and each of the maps respects the manifest left (post-composition) actions of  $\underline{\text{Adj}}(+, +) = \Delta_+$  on each of its nodes. Hence, this defines a cone under (6.3.14) which induces an action preserving map  $W_s \rightarrow W_-$ , and applying  $\{-, T\}$  to all of this data we obtain a commutative diagram:

$$\begin{array}{ccc} B[t] & \xrightarrow{t_\bullet} & B[t]^{\Delta_+^{\text{op}}} \\ \downarrow s_\bullet & \searrow & \downarrow u^t \\ S(u^t) & \xrightarrow{\perp} & B[t]^{\Delta_+^{\text{op}}} \\ \downarrow & & \downarrow \\ B^{\Delta_\infty} & \xrightarrow{\text{res}} & B^{\Delta_+^{\text{op}}} \end{array} \quad (6.3.16)$$

Finally, we wish to describe the diagrams in the image of the functor  $t_\bullet: B[t] \rightarrow B[t]^{\Delta_+^{\text{op}}}$ . Recall from remark 3.3.8 that the map of weights defining  $t_\bullet$  is given by the join operation  $\oplus: \Delta_\infty \times \Delta_+^{\text{op}} \rightarrow \Delta_\infty$ . The functor  $t_\bullet$  carries each homotopy coherent algebra  $b: \Delta_\infty \rightarrow B$ , presented by the diagram (6.1.12), to a functor  $\bar{b}: \Delta_+^{\text{op}} \rightarrow B[t]$ , which we now describe. As in remark 6.1.10, the equaliser formula (5.1.3) for weighted limits can be used to identify  $B[t]$  as a subobject of  $B^{\Delta_\infty}$ . Under this identification, the diagram  $\bar{b}: \Delta_+^{\text{op}} \rightarrow B[t] \hookrightarrow B^{\Delta_\infty}$  is the composite

$$\bar{b} := \Delta_\infty \times \Delta_+^{\text{op}} \xrightarrow{\oplus} \Delta_\infty \xrightarrow{b} B.$$

The vertex  $[-1] \in \Delta_+^{\text{op}}$  acts as the identity for the join operation, so the algebra  $\bar{b}[-1]$  equals  $b \in B[t]$ . The vertex  $[0] \in \Delta_+^{\text{op}}$  acts by precomposition with  $- \oplus [0]: \Delta_\infty \rightarrow \Delta_\infty$ , restricting  $b$  to the subdiagram

$$tb \begin{array}{c} \xleftarrow{\eta tb} \\ \xrightarrow{\mu b} \end{array} t^2b \begin{array}{c} \xleftarrow{\eta ttb} \\ \xrightarrow{\mu tb} \\ \xleftarrow{t\eta b} \\ \xrightarrow{t\mu b} \end{array} t^3b \begin{array}{c} \xleftarrow{\eta tttb} \\ \xrightarrow{\mu ttb} \\ \xleftarrow{t\eta ttb} \\ \xrightarrow{t\mu tb} \\ \xleftarrow{tt\eta b} \\ \xrightarrow{tt\mu b} \end{array} t^4b \dots \in B$$

that defines the free algebra  $tb \in B[t]$ . The other vertices of  $\Delta_+^{\text{op}}$  act by further restriction. In summary, the functor  $\bar{b}: \Delta_+^{\text{op}} \rightarrow B[t]$  carries the vertex  $[n] \in \Delta_+^{\text{op}}$  to  $t^{n+1}b \in B[t]$ . A similar analysis can be used to identify the morphisms in  $\bar{b}$  as components of the original

diagram  $b: \Delta_\infty \rightarrow B$ . In conclusion, we write

$$b \xleftarrow{\beta} tb \xleftarrow{\mu b} t^2b \xleftarrow{t\eta b} t^3b \cdots \in B[t]$$

$$\begin{array}{c} \xleftarrow{\mu b} \\ \xleftarrow{t\eta b} \\ \xleftarrow{t\mu b} \\ \xleftarrow{tt\eta b} \\ \xleftarrow{tt\mu b} \\ \xleftarrow{ttt\eta b} \\ \xleftarrow{ttt\mu b} \end{array}$$

for the obvious 1-skeletal subset of  $t_\bullet b = \bar{b} \in B[t]^{\Delta_+^{\text{op}}}$ .

**6.3.17. Theorem** (canonical colimit representation of algebras). *Given a homotopy coherent monad  $T: \underline{\text{Mnd}} \rightarrow \underline{\text{qCat}}_\infty$ , the functor  $t_\bullet: B[t] \rightarrow B[t]^{\Delta_+^{\text{op}}}$  of (6.3.16) encodes an absolute left lifting diagram*

$$\begin{array}{ccc} & B[t] & \\ \text{id} \nearrow & & \downarrow c \\ B[t] & \xrightarrow{t_\bullet} & B[t]^{\Delta^{\text{op}}} \end{array} = \begin{array}{ccc} & B[t] & \\ \text{ev}_{-1} \nearrow & & \downarrow c \\ B[t] & \xrightarrow{t_\bullet} B[t]^{\Delta_+^{\text{op}}} \xrightarrow{\text{res}} & B[t]^{\Delta^{\text{op}}} \end{array} \quad (6.3.18)$$

created from the  $u^t$ -split simplicial objects in  $B$

$$\begin{array}{ccc} & B[t] & \xrightarrow{u^t} B \\ \text{id} \nearrow & & \downarrow c \\ B[t] & \xrightarrow{t_\bullet} B[t]^{\Delta^{\text{op}}} & \xrightarrow{(u^t)^{\Delta^{\text{op}}}} B^{\Delta^{\text{op}}} \end{array} = \begin{array}{ccc} & B & \\ \text{ev}_0 \nearrow & & \downarrow c \\ B[t] & \xrightarrow{s_\bullet} B^{\Delta_\infty} \xrightarrow{\text{res}} & B^{\Delta^{\text{op}}} \end{array} \quad (6.3.19)$$

The colimits (6.3.18) exhibit the algebras for a homotopy coherent monad  $T$  on the quasi-category  $B$  as colimits of canonical simplicial objects whose vertices are free algebras.

*Proof.* An immediate corollary of proposition 6.3.7: simply pre-compose the absolute lifting diagrams (6.3.11) and (6.3.12) by the functor  $B[t] \rightarrow S(u^t)$  constructed in observation 6.3.15.  $\square$

## 7. MONADICITY

Our aim in this section is to provide a new proof of the quasi-categorical monadicity theorem, originally due to Lurie [19]. Given an adjunction of quasi-categories, theorem 4.3.11 extends this data to a homotopy coherent adjunction from which we can construct an associated homotopy coherent monadic adjunction, as described in definition 6.1.14. Immediately from our weighted limits definition, there is a comparison map from the original adjunction to the monadic one, defined as a component of a simplicial natural transformation between the  $\underline{\text{Adj}}$ -indexed simplicial functors but of course also interpretable in  $\underline{\text{qCat}}_2$ . The monadicity theorem provides conditions on the original adjunction under which this comparison functor is an equivalence quasi-categories.

**7.1. Comparison with the monadic adjunction.** Suppose given a homotopy coherent adjunction  $H: \underline{\text{Adj}} \rightarrow \underline{\text{qCat}}_\infty$  which restricts to a homotopy coherent monad  $T: \underline{\text{Mnd}} \rightarrow \underline{\text{qCat}}_\infty$ . By the Yoneda lemma, the limits of the diagram  $H$  weighted by the two representables  $\underline{\text{Adj}}_+$  and  $\underline{\text{Adj}}_-$  are isomorphic to the two objects in the diagram. Furthermore, the Yoneda embedding  $\underline{\text{Adj}}_*: \underline{\text{Adj}}^{\text{op}} \rightarrow \underline{\text{sSet}}^{\underline{\text{Adj}}}$  defines an adjunction between these weights, whose left adjoint is a map  $\underline{\text{Adj}}_- \rightarrow \underline{\text{Adj}}_+$ . Applying  $\{-, H\}_{\underline{\text{Adj}}}$  returns the homotopy coherent adjunction:

$$A \cong \{\underline{\text{Adj}}_-, H\}_{\underline{\text{Adj}}} \begin{array}{c} \xleftarrow{f} \\ \perp \\ \xrightarrow{u} \end{array} \{\underline{\text{Adj}}_+, H\}_{\underline{\text{Adj}}} \cong B$$

**7.1.1. Observation** (changing the index for the weights). In order to compare the monadic adjunction defined in 6.1.14 with  $H$  it will be convenient to have formulas that describe the monadic adjunction as a weighted limit indexed over the simplicial category  $\underline{\text{Adj}}$  instead of its subcategory  $\underline{\text{Mnd}}$ . As a consequence of lemma 5.1.11, this can be done by simply taking the left Kan extension of the defining weights.

Note that if  $W$  is projectively cofibrant, so is  $\text{lan } W$ : the right adjoint in the adjunction

$$\underline{\text{sSet}}^{\underline{\text{Adj}}} \begin{array}{c} \xleftarrow{\text{lan}} \\ \perp \\ \xrightarrow{\text{res}} \end{array} \underline{\text{sSet}}^{\underline{\text{Mnd}}}$$

manifestly preserves trivial fibrations, so the left adjoint preserves cofibrant objects.

**7.1.2. Definition** (quasi-category of algebras, revisited). Recall from definition 6.1.7 that the quasi-category of algebras is the limit of the homotopy coherent monad underlying  $H$  weighted by  $W_-$ , the restriction of the representable  $\underline{\text{Adj}}_-$ . By lemma 5.1.11, this quasi-category is equivalently described as the limit of the  $H$  weighted by  $\text{lan } W_-$

$$\begin{array}{ccc} & \underline{\text{Adj}} & \\ \nearrow & \uparrow \cong & \searrow \text{lan } W_- \\ \underline{\text{Mnd}} & \xrightarrow{W_-} & \underline{\text{sSet}} \end{array} \quad \text{i.e., } \{W_-, \text{res } H\}_{\underline{\text{Mnd}}} \cong \{\text{lan } W_-, H\}_{\underline{\text{Adj}}}.$$

The left Kan extension of the representable functor  $W_+$  on  $\underline{\text{Mnd}}$  along the inclusion  $\underline{\text{Mnd}} \rightarrow \underline{\text{Adj}}$  is the representable  $\underline{\text{Adj}}_+$ . Because the inclusion  $\underline{\text{Mnd}} \hookrightarrow \underline{\text{Adj}}$  is full,  $\text{lan } W_-(+) \cong W_-(+) = \Delta_\infty$ . By the standard formula for left Kan extensions, the value of  $\text{lan } W_-: \underline{\text{Adj}} \rightarrow \underline{\text{sSet}}$  at the object  $-$  is defined by the coend

$$\text{lan } W_-(-) = \int^{\underline{\text{Mnd}}} \underline{\text{Adj}}(+, -) \times \Delta_\infty = \text{coeq} \left( \Delta_{-\infty} \times \Delta_+ \times \Delta_\infty \rightrightarrows \Delta_{-\infty} \times \Delta_\infty \right)$$

This coequaliser identifies the left and right actions of  $\Delta_+$ . In accordance with our notational conventions, the categories on the right denote the quasi-categories given by their nerves, but because  $h: \underline{\text{qCat}} \rightarrow \underline{\text{Cat}}$  preserves finite products as well as colimits, our remarks apply equally to the analogous coequaliser in  $\underline{\text{Cat}}$ .

7.1.3. **Lemma.** *The coequaliser of the pair*

$$\Delta_{-\infty} \times \Delta_+ \times \Delta_{\infty} \rightrightarrows \Delta_{-\infty} \times \Delta_{\infty} \quad (7.1.4)$$

identifying the left and right actions of  $\Delta_+$  is  $\Delta^{\text{op}}$ .

*Proof.* The coequaliser (7.1.4) is computed pointwise in  $\underline{\text{Set}}$ . We will show that the obvious diagram  $\Delta_{-\infty} \times \Delta_+ \times \Delta_{\infty} \rightrightarrows \Delta_{-\infty} \times \Delta_{\infty} \rightarrow \Delta^{\text{op}}$  is a coequaliser diagram using the graphical calculus for the simplicial category  $\underline{\text{Adj}}$ .

To see that the map from the coequaliser to  $\Delta^{\text{op}}$  is surjective on  $k$ -simplices note that a  $k$ -simplex in  $\Delta_+^{\text{op}} = \underline{\text{Adj}}(-, -)$  lies in the subcategory  $\Delta^{\text{op}}$  if and only if its representing squiggle passes through the level labelled  $+$ . The squiggle to the left of this point is a  $k$ -simplex in  $\Delta_{-\infty}$  and the squiggle to the right is a  $k$ -simplex in  $\Delta_{\infty}$ . Juxtaposition defines a surjective map.

To see the map from the coequaliser is injective, suppose a squiggle representing a  $k$ -simplex in  $\Delta^{\text{op}}$  passes through  $+$  at least once and consider the two preimages in  $\Delta_{-\infty} \times \Delta_{\infty}$  corresponding to distinct subdivisions. The squiggle between the two chosen  $+$  symbols is a  $k$ -simplex in  $\Delta_+$ . Hence, the coequaliser already identifies our two chosen preimages of the given  $k$ -simplex of  $\Delta^{\text{op}}$ .  $\square$

7.1.5. *Observation.* To summarise the preceding discussion, we can give the following explicit description of the weight  $\text{lan } W_-$  as a simplicial functor  $\underline{\text{Adj}} \rightarrow \text{qCat}_{\infty}$ . The representable  $\underline{\text{Adj}}_- : \underline{\text{Adj}} \rightarrow \text{qCat}_{\infty}$  defines an adjunction

$$\underline{\text{Adj}}(-, -) \begin{array}{c} \xleftarrow{f \circ -} \\ \perp \\ \xrightarrow{u \circ -} \end{array} \underline{\text{Adj}}(-, +) \quad \rightsquigarrow \quad \Delta_+^{\text{op}} \begin{array}{c} \xleftarrow{f \circ -} \\ \perp \\ \xrightarrow{u \circ -} \end{array} \Delta_{\infty}$$

We see that the left adjoint lands in the full subcategory  $\Delta^{\text{op}} \subset \Delta_+^{\text{op}}$  by employing our graphical calculus:  $\Delta^{\text{op}} \subset \Delta_+^{\text{op}}$  is the simplicial subset consisting of squiggles from  $-$  to  $-$  that go through  $+$ . The functor  $f \circ -$  post-composes a squiggle from  $-$  to  $+$  with  $f = (-, +)$ , and hence lands in this subcategory. The restricted adjunction is  $\text{lan } W_-$ .

$$\text{lan } W_- : \underline{\text{Adj}} \rightarrow \text{qCat}_{\infty} \quad \rightsquigarrow \quad \Delta^{\text{op}} \begin{array}{c} \xleftarrow{f \circ -} \\ \perp \\ \xrightarrow{u \circ -} \end{array} \Delta_{\infty} \quad (7.1.6)$$

7.1.7. **Definition** (monadic adjunction, revisited). Enriched left Kan extension defines a simplicial functor  $\text{lan} : \underline{\text{sSet}}^{\text{Mnd}} \rightarrow \underline{\text{sSet}}^{\text{Adj}}$ . Composing with (6.1.15) yields a simplicial functor  $\underline{\text{Adj}}^{\text{op}} \rightarrow \underline{\text{sSet}}^{\text{Adj}}$  defining an adjunction between the weights  $\text{lan } W_-$  and  $\underline{\text{Adj}}_+$ . Composing with  $\{-, H\}_{\underline{\text{Adj}}}$  produces the monadic adjunction

$$B[t] \cong \{\text{lan } W_-, H\}_{\underline{\text{Adj}}} \begin{array}{c} \xleftarrow{f^t} \\ \perp \\ \xrightarrow{u^t} \end{array} \{\underline{\text{Adj}}_+, H\}_{\underline{\text{Adj}}} \cong B$$



We use the same notation  $W_s$  for the weights defined by (6.3.14) and (7.2.2); context will disambiguate these. The left-hand vertical of the pullback defining this quasi-category  $S(u)$  is a family of diagrams  $k: S(u) \rightarrow A^{\Delta^{\text{op}}}$ .

Using definition I.5.2.9, we say the quasi-category  $A$  admits colimits of  $u$ -split simplicial objects if there exists a functor  $\text{colim}: S(u) \rightarrow A$  and a 2-cell that define an absolute left lifting diagram

$$\begin{array}{ccc}
 & & A \\
 & \nearrow^{\text{colim}} & \downarrow^c \\
 S(u) & \xrightarrow{k} & A^{\Delta^{\text{op}}}
 \end{array}
 \tag{7.2.3}$$

**7.2.4. Theorem** (monadicity I). *Let  $H$  be a homotopy coherent adjunction with underlying homotopy coherent monad  $T$  and suppose that  $A$  admits colimits of  $u$ -split simplicial objects. Then the comparison functor  $R: A \rightarrow B[t]$  admits a left adjoint.*

*Proof.* We begin by defining the left adjoint  $L: B[t] \rightarrow A$ . The weight  $\text{lan } W_-$  defines a cone under (7.2.2):

$$\begin{array}{ccc}
 \underline{\text{Adj}}_+ \times \Delta^{\text{op}} & \longrightarrow & \underline{\text{Adj}}_+ \times \Delta_\infty \\
 \downarrow & & \downarrow \\
 \underline{\text{Adj}}_- \times \Delta^{\text{op}} & \xrightarrow{\ell} & \text{lan } W_-
 \end{array}
 \tag{7.2.5}$$

By the Yoneda lemma, the map  $\underline{\text{Adj}}_+ \times \Delta_\infty \rightarrow \text{lan } W_-$  is determined by a map  $\Delta_\infty \rightarrow \text{lan } W_-(+) = \Delta_\infty$ , which we take to be the identity. Similarly, the map  $\underline{\text{Adj}}_- \times \Delta^{\text{op}} \rightarrow \text{lan } W_-$  is determined by a map  $\Delta^{\text{op}} \rightarrow \text{lan } W_-(-) = \Delta^{\text{op}}$  which we also take to be the identity. The cone (7.2.5) defines a map of weights  $W_s \rightarrow \text{lan } W_-$  and hence a map  $B[t] \rightarrow S(u)$  of weighted limits. Define  $L$  to be the composite of this functor with  $\text{colim}: S(u) \rightarrow A$ . It follows from commutativity of (7.2.5) that  $L: B[t] \rightarrow A$  commutes with the left adjoints  $f^t$  and  $f$ .

The diagram (7.2.3) defining the colimits of  $u$ -split simplicial objects in  $A$  restricts to define an absolute left lifting diagram

$$\begin{array}{ccc}
 & & A \\
 & \nearrow^L & \downarrow^c \\
 B[t] & \longrightarrow & A^{\Delta^{\text{op}}}
 \end{array}
 \tag{7.2.6}$$

The diagram component  $B[t] \rightarrow A^{\Delta^{\text{op}}}$  is derived from the map  $\ell$  of weights. Recall the functor  $R: A \rightarrow B[t]$  is derived from the counit of the left Kan extension–restriction

adjunction, a map  $\nu: \text{lan } W_- \rightarrow \underline{\text{Adj}}_-$  of weights. The left-hand diagram of weights

$$\begin{array}{ccc} \underline{\text{Adj}}_- & \xleftarrow{\text{id}_{\underline{\text{Adj}}_-} \times !} & \underline{\text{Adj}}_- \times \Delta_+^{\text{op}} \\ \nu \uparrow & & \uparrow \\ \text{lan } W_- & \xleftarrow{\ell} & \underline{\text{Adj}}_- \times \Delta^{\text{op}} \end{array} \rightsquigarrow \begin{array}{ccc} A & \xrightarrow{c} & A^{\Delta_+^{\text{op}}} \\ R \downarrow & & \downarrow \text{res} \\ B[t] & \longrightarrow & A^{\Delta^{\text{op}}} \end{array}$$

commutes because the lower-left composite corresponds, via the Yoneda lemma, to the inclusion  $\Delta^{\text{op}} \hookrightarrow \Delta_+^{\text{op}}$ , as does the upper-right composite. Hence, the induced functors define a commutative diagram on weighted limits, displayed above right.

In this way we see that the canonical natural transformation

$$\begin{array}{ccc} & & A \\ & \nearrow \text{ev}_{-1} & \downarrow c \\ A^{\Delta_+^{\text{op}}} & \xrightarrow{\text{res}} & A^{\Delta^{\text{op}}} \\ & \uparrow & \end{array}$$

defined by the 2-cell (6.3.4) induces the 2-cell on the right-hand side that we call  $\epsilon: LR \Rightarrow \text{id}$  via the universal property of the absolute left lifting diagram (7.2.6):

$$\begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ c \downarrow & \parallel & \downarrow c \\ A^{\Delta_+^{\text{op}}} & \xrightarrow{\text{res}} & A^{\Delta^{\text{op}}} \end{array} \begin{array}{ccc} \uparrow & & \uparrow \\ \text{ev}_{-1} & & \uparrow \\ \uparrow & & \uparrow \end{array} =: \begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ R \downarrow & \uparrow & \downarrow c \\ B[t] & \longrightarrow & A^{\Delta^{\text{op}}} \end{array} = \begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ \exists! \uparrow \epsilon & \nearrow & \downarrow c \\ B[t] & \xrightarrow{L} & A^{\Delta^{\text{op}}} \end{array}$$

Similarly, the commutative diagram of weights induces a commutative diagram of limits

$$\begin{array}{ccc} \text{lan } W_- & & \\ \uparrow & \swarrow & \\ \underline{\text{Adj}}_- \times \Delta^{\text{op}} & \xleftarrow{\nu \times \text{id}_{\Delta^{\text{op}}}} & \text{lan } W_- \times \Delta^{\text{op}} \end{array} \rightsquigarrow \begin{array}{ccc} B[t] & & \\ \downarrow & \searrow t_\bullet & \\ A^{\Delta^{\text{op}}} & \xrightarrow{R^{\Delta^{\text{op}}}} & B[t]^{\Delta^{\text{op}}} \end{array}$$

Hence, the composite 2-cell

$$\begin{array}{ccc} A & \xrightarrow{R} & B[t] \\ L \nearrow & \downarrow c & \downarrow c \\ B[t] & \longrightarrow & A^{\Delta^{\text{op}}} \end{array} \begin{array}{ccc} \uparrow & & \uparrow \\ \text{id} & & \text{id} \\ \uparrow & & \uparrow \end{array} =: \begin{array}{ccc} B[t] & \xrightarrow{RL} & B[t] \\ \text{id} \downarrow & \uparrow & \downarrow c \\ B[t] & \longrightarrow & B[t]^{\Delta^{\text{op}}} \end{array} = \begin{array}{ccc} B[t] & \xrightarrow{RL} & B[t] \\ \exists! \uparrow \eta & \nearrow & \downarrow c \\ B[t] & \xrightarrow{\text{id}} & B[t]^{\Delta^{\text{op}}} \end{array}$$

defines the 2-cell  $\eta: \text{id} \Rightarrow RL$  using the universal property of the canonical colimit diagram of theorem 6.3.17.

It follows from a straightforward appeal to the uniqueness statements of these universal properties, left to the reader, that the 2-cells defined in this way satisfy the triangle identities and hence give rise to an adjunction  $L \dashv R$  in  $\text{qCat}_2$ .  $\square$

7.2.7. **Theorem** (monadicity II). *If  $A$  admits and  $u: A \rightarrow B$  preserves colimits of  $u$ -split simplicial objects, then the unit of the adjunction  $L \dashv R$  of theorem 7.2.4 is an isomorphism. If  $u$  is conservative, then*

$$A \begin{array}{c} \xleftarrow{L} \\ \perp \\ \xrightarrow{R} \end{array} B[t]$$

is an adjoint equivalence.

*Proof.* Recall  $u$  factors as  $u^t \cdot R$ . The first hypothesis asserts that

$$\begin{array}{ccccc} & & A & \xrightarrow{R} & B[t] & \xrightarrow{u^t} & B \\ & \nearrow L & \downarrow c & & \downarrow c & & \downarrow c \\ B[t] & \longrightarrow & A^{\Delta^{\text{op}}} & \xrightarrow{R^{\Delta^{\text{op}}}} & B[t]^{\Delta^{\text{op}}} & \xrightarrow{(u^t)^{\Delta^{\text{op}}}} & B^{\Delta^{\text{op}}} \end{array}$$

is an absolute left lifting diagram. By inspecting the defining maps of weights, we see that the bottom composite  $B[t] \rightarrow B[t]^{\Delta^{\text{op}}}$  is the map  $t_\bullet$  of (6.3.16). By theorem 6.3.17,

$$\begin{array}{ccccc} & & B[t] & \xrightarrow{u^t} & B \\ & \nearrow \text{id}_{B[t]} & \downarrow c & & \downarrow c \\ B[t] & \xrightarrow[t_\bullet]{} & B[t]^{\Delta^{\text{op}}} & \xrightarrow{(u^t)^{\Delta^{\text{op}}}} & B^{\Delta^{\text{op}}} \end{array}$$

is also an absolute left lifting diagram.

This gives us two a priori distinct absolute left lifting diagrams defining colimits for the same family of diagrams  $B[t] \rightarrow B^{\Delta^{\text{op}}}$ . Write  $\alpha$  for the first 2-cell and  $\beta$  for the second. By the definition of  $\eta$  in the proof of theorem 7.2.4, we have

$$\begin{array}{ccc} \begin{array}{ccc} B[t] & \xrightarrow{u^t} & B \\ \nearrow RL & & \downarrow c \\ B[t] & \longrightarrow & B^{\Delta^{\text{op}}} \end{array} & \xrightarrow{\alpha} & \begin{array}{ccc} B[t] & \xrightarrow{u^t} & B \\ \nearrow \text{id}_{B[t]} & & \downarrow c \\ B[t] & \longrightarrow & B^{\Delta^{\text{op}}} \end{array} \\ & & \text{= } \begin{array}{ccc} B[t] & \xrightarrow{RL} & B[t] & \xrightarrow{u^t} & B \\ \downarrow \text{id}_{B[t]} & \nearrow \eta & \nearrow \text{id}_{B[t]} & & \downarrow c \\ B[t] & \longrightarrow & B[t] & \longrightarrow & B^{\Delta^{\text{op}}} \end{array} \end{array}$$

Conversely, the universal property of  $\alpha$  can be used to define a 2-cell  $\zeta$

$$\begin{array}{ccc} \begin{array}{ccc} B & & \\ \nearrow u^t & & \downarrow c \\ B[t] & \longrightarrow & B^{\Delta^{\text{op}}} \end{array} & \xrightarrow{\beta} & \begin{array}{ccc} B & & \\ \nearrow u^t & & \downarrow c \\ B[t] & \longrightarrow & B^{\Delta^{\text{op}}} \end{array} \\ & & \text{= } \begin{array}{ccc} B[t] & \xrightarrow{u^t} & B \\ \downarrow \text{id}_{B[t]} & \nearrow \zeta & \nearrow u^t RL \\ B[t] & \longrightarrow & B[t] & \longrightarrow & B^{\Delta^{\text{op}}} \end{array} \end{array}$$

It is easy to see that  $\zeta$  and  $u^t \eta$  are inverses. But corollary 6.2.3 tells us that  $u^t: B[t] \rightarrow B$  is conservative. Because isomorphisms in quasi-categories are defined pointwise, it follows that  $(u^t)^{B[t]}: B[t]^{B[t]} \rightarrow B^{B[t]}$  is as well. Thus  $\eta$  is an isomorphism.

Isomorphisms are preserved by restricting along any functor. In particular,  $\eta_R$  is an isomorphism. By uniqueness of inverses,  $R\epsilon: RLR \Rightarrow R$  is its inverse, and is also an

isomorphism. Hence  $u\epsilon = u^t R\epsilon$  is an isomorphism. If  $u$  is conservative, it follows that  $\epsilon$  is an isomorphism, and hence that  $L \dashv R$  is an adjoint equivalence.  $\square$

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