OPERADS, MOPERADS, AND BIOPERADS

J. PETER MAY

Contents

Introduction

Operads appear everywhere in mathematics and have been used by many people in many fields since their introduction in the early 1970's. In particular, there is an operad $\mathscr{P}$ in the category **CAT** of (small) categories that captures familiar structures. Recall that a permutative category is a symmetric strict monoidal category, so that its product is strictly associative with a strict unit. Coherence theory tells us that permutative categies are the strictest possible structured categories equivalent to symmetric monoidal categories.

2020 *Mathematics Subject Classification.* Primary 55P42, 55P43, 55P91; Secondary 18A25, 18E30, 55P48, 55U35.

thm1
Theorem 0.1. *The category $\mathscr{P}[\mathbf{Cat}]$ of $\mathscr{P}$-algebras in* **CAT** *is isomorphic to the category of permutative categories. The category $\mathscr{P}_{ps}[\mathbf{Cat}]$ of $\mathscr{P}$-pseudoalgebras in* **CAT** *is equivalent to the category of symmetric monoidal categories.*

In the first case, we require morphisms to be strict monoidal. In the second we only require them to be lax monoidal. We shall not go into detail about pseudo-functors in this paper. There are other ways of dealing directly with general (small) symmetric monoidal categories, but I prefer to rely on the standard categorical result that symmetric monoidal categories can be functorially strictified to permutative categories.

Symmetric bimonoidal categories $\mathscr{V}$, with two products $\oplus$ and $\otimes$ related as in the common examples, such as sets (or spaces or categories) with disjoint union and cartesian products, that is coproducts and products, or modules over a commutative ring with direct sums and tensor products, are also ubiquitous. The categorical literature about them is far less extensive than their importance warrrants. The only treatments that I have found are LaPlaza's from 1972 [Lap72], a 2020 generalization (from symmetric to braided) of Blass and Gurevich [BG20], and the encyclopedic recent work of Johnson and Yau [JY].

The main purpose of this paper is to introduce the new concepts of multiplicative operads, or "moperads", and of "bioperads". We shall define bioperads and their algebras, produce a canonical bioperad $\mathscr{P}^{bi}$ in **Cat**, and prove the following result. Here symmetric bimonoidal categories are as first defined by LaPlaza (with mild caveats as in JY), and bipermutative categories are the strictest possible equivalent analog, as first defined in [May77, Definition VI.3.3].

thm2
Theorem 0.2. *The category $\mathscr{P}^{bi}[\mathbf{Cat}]$ of $\mathscr{P}^{bi}$-algebras in* **CAT** *is isomorphic to the category of bipermutative categories. The category $\mathscr{P}^{bi}_{ps}[\mathbf{Cat}]$ of $\mathscr{P}^{bi}$-pseudo-algebras in* **CAT** *is equivalent to the category of symmetric monoidal bicategories.*

It is very easy to generalize moperads and bioperads equivariantly. We give the first definition of bipermutative and symmetric bimonoidal G-categories here. Our focus is on finite groups, but the definition is general.

We then generalize bioperads to "moperad pairs", which consist of an additive classical operad and a moperad, related by a distributivity diagram. In a sequel [May], we shall use this general categorical theory to (re)develop multiplicative equivariant infinite loop space theory. There we shall use [KMZ24] to give a quite elementary construction of E_∞ ring G-spectra from bipermutative G-categories. For example, for a Galois extension L of K with Galois group G, this gives an E_∞ ring G-spectrum $R_{L/K}$ whose H fixed point spectrum $R^H_{L/K}$ is the classical $\mathbb{K}$-theory spectrum of the fixed field L^H. This achingly elementary operadic construction should lend itself to calculations.

We have separated out the very general categorical definitions given here from their use in algebraic topology since the history of operads in mathematics strongly suggests that the general theory will have many examples and applications that have nothing to do with algebraic topology.

The new topological theory is more general and, even nonequivariantly, much simpler than the earlier theory, which used operad pairs. Those consist of two classical operads related by a distributivity diagram. Both notions feed into the axiomatically redeveloped foundations of infinite loop space theory developed by

Hana Kong, Foling Zou, and myself [KMZ24]. It is the passage from bipermutative bicategories to a new version of E_∞ ring spaces that simplifies. We warn the reader that the new notions of E_∞ ring spaces and E_∞ ring spectra have not yet been fully compared with the old notions. We prove in the sequel that the new notion of an E_∞ ring spectrum is equivalent to the one first introduced in the 1970's and therefore to all of its equivalent modernizations. In the ringlike case, it will follow that the two notions of E_∞ ring space are also equivalent. However, more direct "exponential and logarithmic" categorical comparisons are not yet in place.

We think of the symmetric groups as giving the starting model of our theory. Taking $\mathscr{M}(j) = \mathbf{Asso}(j) = \Sigma_j$ and using the product $\Sigma_m \times \Sigma_n \longrightarrow \Sigma_{m+n}$, we obtain the associativity operad $\mathscr{M}$ in the category **SET** of sets. Its algebras are the monoids. Writing n for the set $\{1, \cdots, n\}$, that product between symmetric groups is obtained by identifying $m \amalg n$ with $m + n$.

Using lexicographic ordering of pairs, we can also identify $m \times n$ with mn. That gives a product $\Sigma_m \times \Sigma_n \longrightarrow \Sigma_{mn}$.[1] With this as a guide, we define moperads by replacing sums by products in the definition of operads and making a number of other changes to make sense of the new notion. In particular, that leads to the moperad $\mathscr{M}^\times$. Adding in distributivity, we obtain the bioperad $\mathscr{M}^{bi}$. These are all in **SET**.

We categorify by applying the functor $\mathscr{E}\colon \mathbf{SET} \longrightarrow \mathbf{CAT}$ that sends a set S to the indiscrete, or chaotic, category with objects the elements of S and a unique morphism between each pair of objects. Since every object of $\mathscr{E}(S)$ is initial and terminal, the classifying space functor given by $B\mathscr{E}(S) = |\mathbb{N}\mathscr{E}(S)|$, where $\mathbb{N}$ is the nerve functor from **CAT** to the category **sSET** of simplicial sets, lands in contractible spaces. Visibly, $B\mathscr{E}(\Sigma_j)$ has a free (right) action by Σ_j.

In more detail, the functors $\mathscr{E}$ and $\mathbb{N}$ are right adjoints and $|-|$ also preserves products. Therefore $\mathscr{E}$ takes operadic structures in **SET** to operadic structures in **CAT**, N takes them from **CAT** to **sSET**, and $|-|$ takes them from **sSET** to **TOP**. If Σ_j acts freely on the jth term of an operad in **SET**, then Σ_j acts freely on the jth term of the resulting operads in **CAT**, **sSET**, and **TOP**. Since these terms are all contractible, we obtain E_∞ operads, moperads, and bioperads $\mathscr{P}$, $\mathscr{P}^\times$, and $\mathscr{P}^{bi}$ in each of **CAT**, **sSET**, and **TOP** by applying our functors to $\mathscr{M}$, $\mathscr{M}^\times$, and $\mathscr{M}^{bi}$. (When it seems necessary for clarity, we add a superscript to indicate where $\mathscr{P}$ lives; the default is **CAT**). While it is a bit finicky to write down the full algebraic structures explicitly, they are clearly there since the symmetric groups are there.

We generalize equivariantly by applying the functor $\mathbf{CAT}(\mathscr{E}(G), -)$ from categories to G-categories before applying N and $|-|$. Applied to $\mathscr{P}$, $\mathscr{P}^\times$, and $\mathscr{P}^{bi}$, this gives equivariant variants $\mathscr{P}_G$, $\mathscr{P}_G^\times$, and $\mathscr{P}_G^{bi}$. It has long been understood that $\mathscr{P}_G$-algebras give the right notion of a permutative G-category. We therefore define bipermutative G-categories to be $\mathscr{P}_G^{bi}$-algebras. Applied to permutative and bipermutative categories, the functor $\mathbf{CAT}(\mathscr{E}(G), -)$ gives lots of examples. Applying the machine of [May], this will give lots of concrete constructions of G-spectra and E_∞ ring G-spectra, including those of equivariant algebraic $\mathbb{K}$-theory.

1. A review of operads and their algebras

opsandacts

We recall the classical definition of an operad, which we think of as "additive", before defining the new analog that we think of as "multiplicative". In this section

[1]This product was used operadically in [GM11], but not in the way we shall use it here.

and the next, we let $\mathscr{V}$ be a symmetric monoidal category. However, we return to this assumption later, just noting here that it needs thought when $\mathscr{V}$ has more than one symmetric monoidal structure. We denote the product on $\mathscr{V}$ by $\otimes$, again provisionally, and its unit by u. Additively, we will later think of u as 0.

operads

1.1. **The definition of operads.**

operad

Definition 1.1. An operad $\mathscr{C}$ in $\mathscr{V}$ consists of objects $\mathscr{C}(j)$, $j \geq 0$, of $\mathscr{V}$, a unit $\mathrm{id}\colon u \to \mathscr{C}(1)$, a right action by the symmetric group Σ_j on $\mathscr{C}(j)$ for each j, and product maps

$$\gamma : \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_+)$$

for $k \geq 0$ and $j_r \geq 0$, where $j_+ = j_1 + \cdots + j_k$. Usually $\mathscr{C}(0) = u$, and when $k = 0$ we then interpret γ as the identity map of $\mathscr{C}(0)$. The γ are required to be unital, associative, and equivariant in the following senses.

(i) The following unit diagrams commute:

$$\begin{array}{ccc} \mathscr{C}(k) \otimes u^k & \xrightarrow{\cong} & \mathscr{C}(k) \\ {\scriptstyle \mathrm{Id}\otimes \mathrm{id}^k}\downarrow & \nearrow{\scriptstyle \gamma} & \\ \mathscr{C}(k) \otimes \mathscr{C}(1)^k & & \end{array} \qquad \begin{array}{ccc} u \otimes \mathscr{C}(j) & \xrightarrow{\cong} & \mathscr{C}(j) \\ {\scriptstyle \mathrm{id}\otimes \mathrm{Id}}\downarrow & \nearrow{\scriptstyle \gamma} & \\ \mathscr{C}(1) \otimes \mathscr{C}(j) & & \end{array} .$$

(ii) The following associativity diagrams commute. Here we reorder the finite set $\{t | 1 \leq t \leq j_+\}$ in blocks as $\{(r,q) | 1 \leq r \leq k,\ 1 \leq q \leq j_r\}$ and we set

$$\Sigma_{r=1}^{k} j_r = j_+,\ \Sigma_{t=1}^{j} i_t = i_+ = \Sigma_{r=1}^{k} h_r, \quad \text{where} \quad h_r = \Sigma_{q=1}^{j_r} i_{r,q}$$

$$\begin{array}{ccc} \mathscr{C}(k) \otimes \big(\otimes_{r=1}^{k} \mathscr{C}(j_r)\big) \otimes \big(\otimes_{t=1}^{j_+} \mathscr{C}(i_t)\big) & \xrightarrow{\gamma\otimes\mathrm{Id}} & \mathscr{C}(j) \otimes \big(\otimes_{t=1}^{j_+} \mathscr{C}(i_t)\big) \\ & & \downarrow{\scriptstyle \gamma} \\ \downarrow{\scriptstyle \mathrm{Id}\otimes\mathrm{shuffle}} & & \mathscr{C}(i_+) \\ & & \uparrow{\scriptstyle \gamma} \\ \mathscr{C}(k) \otimes \big(\otimes_{r=1}^{k} \big(\mathscr{C}(j_r) \otimes (\otimes_{q=1}^{j_r} \mathscr{C}(i_{r,q}))\big)\big) & \xrightarrow[\mathrm{Id}\otimes\gamma^k]{} & \mathscr{C}(k) \otimes \big(\otimes_{r=1}^{k} \mathscr{C}(h_r)\big). \end{array}$$

(iii) The following equivariance diagrams commute, where $\sigma \in \Sigma_k$ and $\tau_r \in \Sigma_{j_r}$. The permutation $\sigma(j_1, \ldots, j_k) \in \Sigma_j$ permutes k blocks of letters as σ permutes k letters and $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_j$ is the block sum (with some historical ambiguity in the phraseology sorted out in in Remark 1.2 below):

$$\begin{array}{ccc} \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathscr{C}(k) \otimes \mathscr{C}(j_{\sigma(1)}) \otimes \cdots \otimes \mathscr{C}(j_{\sigma(k)}) \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \gamma} \\ \mathscr{C}(j) & \xrightarrow[\sigma(j_{\sigma(1)},\ldots,j_{\sigma(k)})]{} & \mathscr{C}(j) \end{array}$$

Here σ acts from the right on $\mathscr{C}(k)$ but acts from the left on tuples.

$$\begin{array}{ccc}
\mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\cdots\otimes\mathscr{C}(j_k) & \xrightarrow{\mathrm{Id}\otimes\tau_1\otimes\cdots\otimes\tau_k} & \mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\cdots\otimes\mathscr{C}(j_k) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\
\mathscr{C}(j) & \xrightarrow[\tau_1\oplus\cdots\oplus\tau_k]{} & \mathscr{C}(j).
\end{array}$$

con1

Remark 1.2. The original definition of an operad [May72, Definition 1.1] used equations more than diagrams and in particular used the equation

$$\gamma(c;d_1,\cdots,d_k)=\gamma(c;\delta_{\sigma^{-1}(1)},\cdots,\delta_{\sigma^{-1}(k)})\sigma(j_1,\cdots,j_k)$$

for the first equivariance diagram. Our transcription to a diagram follows [May97, Definition 1]. The permutations $\sigma(j_1,\cdots,j_k)$ and $\tau_1\oplus\cdots\oplus\tau_k$ are precisely defined by commutative diagrams of finite sets, in which all actions of permutations are from the right:

$$\begin{array}{ccc}
j_{\sigma^{-1}(1)}\amalg\cdots\amalg j_{\sigma^{-1}(k)} & \xrightarrow{\sigma} & j_1\amalg\cdots\amalg j_k \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
j_+ & \xrightarrow[\sigma(j_1,\cdots,j_k)]{} & j_+
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
j_1\amalg\cdots\amalg j_k & \xrightarrow{\tau_1\amalg\cdots\amalg\tau_k} & j_1\amalg\cdots\amalg j_k \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
j_+ & \xrightarrow[\tau_1\oplus\cdots\oplus\tau_k]{} & j_+
\end{array}$$

Here the vertical arrows are the respective block sum identifications. With the right action by σ at the top of the left diagram here converted to a left action by σ^{-1} and with the target then converted to seeing $(1,\ldots,k)$ rather than $(j_{\sigma^{-1}(1)},\cdots,j_{\sigma^{-1}(k)})$, we obtain the first equivariance diagram as written in Definition 1.1.[2]

Using just right actions, these diagrams say that γ is Σ_k and $(\Sigma_{j_1}\times\cdots\times\Sigma_{j_k})$-equivariant. Conceptually, we are using block sums to transport the coproduct on the Σ-category of finite ordered sets to a coproduct on its canonical skeleton, where Σ is the category $\coprod\Sigma_n$ of symmetric groups.

opact

1.2. **The definition of algebras over an operad.** Thinking of elements as operations, we think of $\gamma(c;d_1,\cdots,d_k)$ as the composite of the operation c with the "product" of the operations d_r. Let X^j denote the j-fold $\otimes$ power of an object X, with Σ_j acting on the left. By convention, $X^0=u$.

Addact

Definition 1.3. Let $\mathscr{C}$ be an operad. A $\mathscr{C}$-algebra is an (unbased) object X of $\mathscr{V}$ together with maps

$$\theta:\mathscr{C}(j)\otimes X^j\to X$$

for $j\geq 0$ that are unital, associative, and equivariant in the following senses. We now require $\mathscr{C}$ to be unital (alias reduced) meaning that $\mathscr{C}(0)=u$. Then $\theta\colon\mathscr{C}(0)\otimes u\longrightarrow X$ gives X a base element $u\longrightarrow X$, which we will later think of as 0; if X comes with a base element, we insist that it is the base element built in by the operad action.

[2]Our definition of $\sigma(j_1,\cdots,j_k)$ follows the history and is consistent, but a renaming of $\sigma(j_1,\cdots,j_k)$ here as $\sigma(j_{\sigma^{-1}(1)},\cdots,j_{\sigma^{-1}(k)})$ would result in $\sigma(j_1,\cdots,j_k)$ rather than $\sigma(j_{\sigma(1)},\cdots,j_{\sigma(k)})$ appearing in the first equivariance diagram. I thank Isiaiah Daily for helping me sort this out.

(i) The following unit diagram commutes:

$$\begin{array}{ccc} u\otimes X & \xrightarrow{\cong} & X \\ {\scriptstyle \mathrm{id}\otimes \mathrm{Id}}\downarrow & \nearrow{\scriptstyle\theta} & \\ \mathscr{C}(1)\otimes X. & & \end{array}$$

(ii) The following associativity diagram commutes, where $j_+ = \sum j_r$:

$$\begin{array}{ccc} \mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\cdots\otimes\mathscr{C}(j_k)\otimes X^{j_+} & \xrightarrow{\gamma\otimes\mathrm{Id}} & \mathscr{C}(j_+)\otimes X^{j_+} \\ & & \downarrow{\scriptstyle\theta} \\ {\scriptstyle \mathrm{Id}\otimes\mathrm{shuffle}}\downarrow & & X \\ & & \uparrow{\scriptstyle\theta} \\ \mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes X^{j_1}\otimes\cdots\otimes\mathscr{C}(j_k)\otimes X^{j_k} & \xrightarrow{\mathrm{Id}\otimes\theta^k} & \mathscr{C}(k)\otimes X^k. \end{array}$$

(iii) The following equivariance diagrams commute, where $\sigma\in\Sigma_j$:

$$\begin{array}{ccc} \mathscr{C}(j)\otimes X^j & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathscr{C}(j)\otimes X^j \\ {\scriptstyle\gamma}\searrow & & \swarrow{\scriptstyle\gamma} \\ & X. & \end{array}$$

The following elementary example is central.

Example 1.4. For the associativity operad $\mathscr{M} = \mathbf{ASSO}$, we take $\mathscr{V} = \mathbf{SET}$ to be the cartesian monoidal category $\mathbf{SET}$ and take $\mathscr{M}(j) = \Sigma_j$, thought of as recording all permutations of j-fold iteration of an associative product on an $\mathscr{M}$-algebra, alias monoid, X. Here the "product" $\oplus$ is given by maps

$$\Sigma_{j_1}\times\cdots\times\Sigma_{j_k}\longrightarrow\Sigma_{j_1+\cdots+j_k}=\Sigma_{j_+}. \tag{1.5}$$

This product gives $\gamma(e_k;\tau_1,\cdots,\tau_k)$. The $\gamma(\sigma;\tau_1,\cdots,\tau_k)$ for other $\sigma\in\Sigma_k$ are then determined by the first equivariance formula.

Remark 1.6. Conceptually, we are thinking of permutations as automorphisms of finite sets, and passage from sets to their automorphisms sends disjoint unions to cartesian products. This point of view allows us to think of $\mathscr{M}$ as "additive".

2. Multiplicative operads, alias "moperads", and their algebras

2.1. **The definition of moperads.** The definition of multiplicative operads is motivated in part by the multiplicative analog of Example 1.4 to be given in Example 2.9. The definition is nearly as simple as that of an operad and should be just as old, but I believe that it is new. The cited analog uses cartesian products rather than disjoint unions of finite sets with j_r elements, together with the identification of the lexicographically ordered product of such sets with the ordered set $j_\times = j_1\times\cdots\times j_r$. That gives a "product", which we denote by $\otimes$,

$$\Sigma_{j_1}\times\cdots\times\Sigma_{j_k}\longrightarrow\Sigma_{j_1\cdots j_k}=\Sigma_{j_\times}. \tag{2.1}$$

In precise analogy with the additive theory, we define permutations $\sigma < j_1, \cdots, j_k >$ and $\tau_1 \otimes \cdots \otimes \tau_k$ by commutative diagrams of finite sets, with all group actions from the right[3]:

$$\begin{array}{ccc} j_{\sigma^{-1}(1)} \times \cdots \times j_{\sigma^{-1}(k)} & \xrightarrow{\ \sigma\ } & j_1 \times \cdots \times j_k \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ j_\times & \xrightarrow[\sigma < j_1, \cdots, j_k >]{} & j_\times \end{array} \quad \text{and} \quad \begin{array}{ccc} j_1 \times \cdots \times j_k & \xrightarrow{\tau_1 \times \cdots \times \tau_k} & j_1 \times \cdots \times j_k \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ uyj_\times & \xrightarrow[\tau_1 \otimes \cdots \otimes \tau_k]{} & j_\times \end{array}$$

The vertical arrows are the respective lexicographic identifications.

Let $(\mathscr{V}, \otimes, v)$ be a symmetric monoidal category. We have changed the name of the unit object to v, and we will later think of it as 1. To handle unit conditions in moperads, we also require an analog of the contravariant functor $\mathscr{C}\colon \Lambda \longrightarrow \mathscr{V}$ that appears in handling basepoints in operad theory.

opmop

Definition 2.2. Let $\Lambda_{>0}$ denote the category of unbased finite sets $\mathbf{n} = \{1, \cdots, n\}$ and injections. A $\Lambda_{>0}$-functor $\mathscr{C}^\times$ in a symmetric monoidal category $\mathscr{V}$ is a covariant functor $\Lambda_{>0} \longrightarrow \mathscr{V}$.[4] For $k \geq 1$ and $1 \leq r \leq k+1$, let ι_r be the ordered injection $\mathbf{k} \longrightarrow \mathbf{k+1}$ that misses the rth letter. Then $\mathscr{C}^\times$ has maps $\iota_r\colon \mathscr{C}^\times(k) \longrightarrow \mathscr{C}^\times(k+1)$ that, together with (right) actions of symmetric groups, generate $\mathscr{C}^\times$ as a functor. Note that we do not require and do not want objects $\mathscr{C}^\times(0)$ in this definition.

Not1

Notation 2.3. Suppose given a $\Lambda_{>0}$-functor $\mathscr{C}^\times$ and a unit map $\iota_0\colon v \longrightarrow \mathscr{C}(1)$. We need implied maps to express the unit and associativity conditions in the following definition. For any j, we write $\iota\colon v \longrightarrow \mathscr{C}^\times(j)$ for any (ordered) injection obtained by composing ι_0 with iteration of maps ι_r.

For $1 \leq r \leq k+1$, in addition to the maps $\iota_r\colon \mathscr{C}^\times(k) \longrightarrow \mathscr{C}^\times(k+1)$, we define

$$\iota_{r,0}\colon \otimes_{s=1}^k \mathscr{C}^\times(j_s) \longrightarrow \otimes_{s=1}^{r-1} \mathscr{C}^\times(j_s) \otimes \mathscr{C}^\times(1) \otimes \otimes_{s=r+1}^k \mathscr{C}^\times(j_s)$$

by inserting $\iota\colon v \longrightarrow \mathscr{C}(1)$ in the rth slot.

Suppose given numbers $i_{r,q}$ for $1 \leq r \leq k$ and $1 \leq q \leq j_r$. Identify the numbers j_+ and $j_\times$ as the ordered sets $j_1 \amalg \cdots \amalg j_k$ and $j_1 \times \cdots \times j_k$, with block sum and lexicographic ordering. Assume that $j_+ \leq j_\times$, as always holds if all $j_r > 1$. Observe that the lexicographically ordered set of k-tuples

$$Q = (q_1, \cdots, q_k) = \big((1, q_1), \cdots, (k, q_k)\big), \quad 1 \leq q_s \leq j_s,$$

can be identified with the ordered set $\{1 \leq t \leq j_x\}$. With these notations, the evident ordered inclusion of j_+ in $j_\times$ sends (r, q) to the sequence $(1^{r-1}, q, 1^{k-r})$. We then obtain maps, denoted $\iota_{r,q}$,

$$\begin{array}{c} \mathscr{C}^\times(i_{r,q}) = v^{r-1} \otimes \mathscr{C}^\times(i_{r,q}) \otimes v^{k-r} \\ \downarrow{\scriptstyle \iota_{r,q} = \iota^{r-1} \otimes \mathrm{Id} \otimes \iota^{k-r}} \\ \otimes_{s=1}^{r-1} \mathscr{C}(i_{s,1}) \otimes \mathscr{C}(i_{r,q}) \otimes \otimes_{s=r+1}^k \mathscr{C}(i_{s,1}). \end{array}$$

We allow any choices of the maps ι. Again allowing any choices of the maps ι, we also have maps

$$\iota = \iota^k\colon v = v^k \longrightarrow \otimes_{s=1}^k \mathscr{C}^\times(i_{s,q_s})$$

[3] As in the previous footnote, we could clean up the first equivariance diagram by renaming the permutation $\sigma < j_1, \cdots, j_k >$.

[4] Covariance gives a left action of Σ_n on $\mathscr{C}(n)$, but we use the corresponding right action.

for each sequence Q. For each (r,q), write $q = q_r$. For $1 \leq r \leq k$ and $1 \leq q_r \leq j_r$, we have maps ι_{r,q_r}. Using these maps on indexing sequences $Q = (1^{r-1}, (r,q_r), 1^{k-r})$ and using maps ι on all other index sequences Q, we obtain maps

$$\iota\colon\ \otimes_{r=1}^{k} \otimes_{q=1}^{j_r} \mathscr{C}^\times(i_{r,q}) \longrightarrow \otimes_{r=1}^{k} \otimes_{Q=(q_1,\cdots,q_k)} \mathscr{C}(i_{1,q_1}) \otimes \cdots \otimes \mathscr{C}(i_{k,q_k}).$$

The intuition is that the various maps ι insert identity operations in appropriate positions.

Definition 2.4. A multiplicative operad, or moperad,[5] $\mathscr{C}^\times$ in $\mathscr{V}$ consists of a $\Lambda_{>0}$-functor $\mathscr{C}^\times$ together with a unit map $\iota_0\colon v \to \mathscr{C}^\times(1)$ and product maps

$$\gamma^\times : \mathscr{C}^\times(k) \otimes \mathscr{C}^\times(j_1) \otimes \cdots \otimes \mathscr{C}^\times(j_k) \to \mathscr{C}^\times(j_\times) \tag{2.5}$$

for $k \geq 1$ and $j_r \geq 1$, where $j_\times = j_1 \cdots j_k$. (We do not allow k or any j_r to be zero). Our maps must be unital, associative, and equivariant in the following senses.

(i) The following unit and unit element diagrams commute:

$$\begin{array}{ccc} \mathscr{C}^\times(1)\otimes v & \xrightarrow{\ \cong\ } & \mathscr{C}^\times(1) \\ {\scriptstyle \mathrm{Id}\otimes\iota_0}\big\downarrow & \nearrow{\scriptstyle \gamma^\times} & \\ \mathscr{C}^\times(1)\otimes\mathscr{C}^\times(1) & & \end{array} \qquad \begin{array}{ccc} v\otimes\mathscr{C}^\times(j) & \xrightarrow{\ \cong\ } & \mathscr{C}^\times(j) \\ {\scriptstyle \iota_0\otimes\mathrm{Id}}\big\downarrow & \nearrow{\scriptstyle \gamma^\times} & \\ \mathscr{C}^\times(1)\otimes\mathscr{C}^\times(j) & & \end{array}$$

$$\begin{array}{ccc} \mathscr{C}^\times(k)\otimes\otimes_{s=1}^{k}\mathscr{C}^\times(j_s) & \xrightarrow{\ \gamma^\times\ } & \mathscr{C}^\times(j_\times) \\ {\scriptstyle \iota_r\otimes\iota_{0,r}}\big\downarrow & \nearrow{\scriptstyle \gamma^\times} & \\ \mathscr{C}^\times(k+1)\otimes\big(\otimes_{s=1}^{r-1}\mathscr{C}^\times(j_s)\big)\otimes\mathscr{C}^\times(1)\otimes\big(\otimes_{s=r+1}^{k}\mathscr{C}^\times(j_s)\big) & & \end{array}$$

(ii) The following associativity diagrams commute when $j_+ \leq j_\times$.

$$\begin{array}{ccc} \mathscr{C}^\times(k)\otimes\big(\otimes_{r=1}^{k}\mathscr{C}^\times(j_r)\big)\otimes\big(\otimes_Q\otimes_{r=1}^{k}\mathscr{C}(i_{r,q_r})\big) & \xrightarrow{\ \gamma^\times\otimes\mathrm{Id}\ } & \mathscr{C}^\times(j_\times)\otimes\otimes_{t=1}^{j_\times}\mathscr{C}^\times(i_t) \\ {\scriptstyle \mathrm{Id}\otimes(\otimes_r\mathrm{Id}\otimes\iota)}\big\uparrow & & \big\downarrow{\scriptstyle \gamma^\times} \\ \mathscr{C}^\times(k)\otimes\big(\otimes_{r=1}^{k}\mathscr{C}^\times(j_r)\big)\otimes\big(\otimes_{r=1}^{k}\otimes_{q=1}^{j_r}\mathscr{C}^\times(i_{r,q})\big) & & \mathscr{C}^\times(i_\times) \\ {\scriptstyle \mathrm{Id}\otimes\mathrm{shuffle}}\big\uparrow & & \big\uparrow{\scriptstyle \gamma^\times} \\ \mathscr{C}^\times(k)\otimes\otimes_{r=1}^{k}\mathscr{C}^\times(j_r)\otimes\otimes_{q=1}^{j_r}\mathscr{C}^\times(i_{r,q}) & \xrightarrow[\ \mathrm{Id}\otimes(\gamma^\times)^k\ ]{} & \mathscr{C}^\times(k)\otimes\otimes_{r=1}^{k}\mathscr{C}^\times(i_{r,\times}) \end{array}$$

We can use the third diagram in (i) to get around the limitation $j_+ \leq j_\times$. In effect, that diagram increases j_+ without changing $j_\times$ or $\gamma^\times$. We can use ι_r in that diagram for any r, iterating as needed.

(iii) The following equivariance diagrams commute:

$$\begin{array}{ccc} \mathscr{C}^\times(k)\otimes\mathscr{C}^\times(j_1)\otimes\cdots\otimes\mathscr{C}^\times(j_k) & \xrightarrow{\ \sigma\otimes\sigma^{-1}\ } & \mathscr{C}^\times(k)\otimes\mathscr{C}^\times(j_{\sigma(1)})\otimes\cdots\otimes\mathscr{C}^\times(j_{\sigma(k)}) \\ {\scriptstyle \gamma^\times}\big\downarrow & & \big\downarrow{\scriptstyle \gamma^\times} \\ \mathscr{C}^\times(j_\times) & \xrightarrow[\ \sigma<j_{\sigma(1)},\cdots,j_{\sigma(k)}>\ ]{} & \mathscr{C}^\times(j_\times) \end{array}$$

[5] Clumsy attempt at wit: moperads help mop up the multiplicative theory.

and

$$\begin{array}{ccc}
\mathscr{C}^{\times}(k) \otimes \mathscr{C}^{\times}(j_1) \otimes \cdots \otimes \mathscr{C}^{\times}(j_k) & \xrightarrow{\operatorname{Id} \otimes \tau_1 \otimes \cdots \otimes \tau_k} & \mathscr{C}^{\times}(k) \otimes \mathscr{C}^{\times}(j_1) \otimes \cdots \otimes \mathscr{C}^{\times}(j_k) \\
\downarrow{\scriptstyle \gamma^{\times}} & & \downarrow{\scriptstyle \gamma^{\times}} \\
\mathscr{C}^{\times}(j_{\times}) & \xrightarrow[\tau_1 \otimes \cdots \otimes \tau_k]{} & \mathscr{C}^{\times}(j_{\times}).
\end{array}$$

mactions

2.2. **The definition of algebras over a moperad.** Again let X^j denote the j-fold $\otimes$ power of an object X, with Σ_j acting on the left. We need an analog of Notation 2.3 to obtain the associativity diagram in the following definition.

Not2

Notation 2.6. Let $X \in \mathscr{V}$ be an object with a unit map $\iota_0 \colon v \longrightarrow X$. Viewing X^{j_+} as $\otimes_{r=1}^{k} X^{j_r}$ and $X^{j_\times}$ as the $\otimes$ product of copies of X indexed on the lexicographically ordered set of k-tuples $Q = (q_1, \cdots, q_k)$, where $1 \leq q_r \leq j_r$, we define $\iota \colon X^{j_+} \longrightarrow X^{j_\times}$ as follows. For $1 \leq r \leq k$, define Q_r to be the ordered set of those k-tuples Q such that $q_s = 1$ if $s \neq r$. Since $1 \leq q_r \leq j_r$, this set has j_r elements. We define X_r to be the product of copies of X indexed on the ordered set Q_r. This gives an identification $\nu_r \colon X^{j_r} \longrightarrow X_r$. For any choices of j_r, define ι by extending the resulting identification

$$\otimes_r \nu_r \colon \otimes_r X^{j_r} \longrightarrow \otimes_r X_r$$

by taking the coordinates of $X^{j_\times}$ indexed on Q not in any Q_r to be 1^k.

Multact

Definition 2.7. Let $\mathscr{C}^{\times}$ be a moperad. A $\mathscr{C}^{\times}$-algebra is an (unbased) object X of $\mathscr{V}$ together with maps

$$\iota_0 \colon v \longrightarrow X \quad \text{and} \quad \theta^{\times} : \mathscr{C}^{\times}(j) \otimes X^j \to X$$

for $j \geq 1$ that are unital, associative, and equivariant in the following senses.

(i) The following unit and unit element diagrams commute:

$$\begin{array}{ccc}
v \otimes v \cong v & \xrightarrow{\operatorname{id}} & X \\
\downarrow{\scriptstyle \iota_0 \otimes \iota_0} & \nearrow{\scriptstyle \theta^{\times}} & \\
\mathscr{C}^{\times}(1) \otimes X & &
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{C}^{\times}(j) \otimes X^j & \xrightarrow{\theta^{\times}} & X \\
\downarrow{\scriptstyle \iota_r \otimes \iota_r} & \nearrow{\scriptstyle \theta^{\times}} & \\
\mathscr{C}^{\times}(j+1) \otimes X^{j+1}. & &
\end{array}$$

Here $1 \leq r \leq j+1$ and ι_r inserts $\operatorname{id} \colon v \longrightarrow X$ in the rth coordinate.

(ii) The following associativity diagram commutes, where $j_+ \leq j_\times$.

$$\begin{array}{ccc}
\mathscr{C}^{\times}(k) \otimes \mathscr{C}^{\times}(j_1) \otimes \cdots \otimes \mathscr{C}^{\times}(j_k) \otimes X^{j_\times} & \xrightarrow{\gamma^{\times} \otimes \operatorname{Id}} & \mathscr{C}^{\times}(j_\times) \otimes X^{j_\times} \\
\uparrow{\scriptstyle \operatorname{Id} \otimes \iota} & & \downarrow{\scriptstyle \theta^{\times}} \\
\mathscr{C}^{\times}(k) \otimes \mathscr{C}^{\times}(j_1) \otimes \cdots \otimes \mathscr{C}^{\times}(j_k) \otimes X^{j_+} & & X \\
\uparrow{\scriptstyle \text{shuffle}} & & \uparrow{\scriptstyle \theta^{\times}} \\
\mathscr{C}^{\times}(k) \otimes \mathscr{C}^{\times}(j_1) \otimes X^{j_1} \otimes \cdots \otimes \mathscr{C}^{\times}(j_k) \otimes X^{j_k} & \xrightarrow[\operatorname{Id} \otimes (\theta^{\times})^k]{} & \mathscr{C}^{\times}(k) \otimes X^k.
\end{array}$$

We can use the second diagram in (i) to get around the limitation $j_+ \leq j_\times$. In effect, that diagram increases j_+ without changing $\theta^{\times}$. We can use ι_r in that diagram for any r, iterating as needed.

(iii) The following equivariance diagram commutes, where $\sigma \in \Sigma_j$:

$$\begin{array}{ccc} \mathscr{C}^{\times}(j) \otimes X^j & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathscr{C}^{\times}(j) \otimes X^j \\ & {\scriptstyle\theta^{\times}}\searrow \qquad \swarrow{\scriptstyle\theta^{\times}} & \\ & X. & \end{array}$$

Remark 2.8. The associativity diagram in (ii) can be viewed as a parametrization of the equality

$$x_{1,1}\cdots x_{1,j_1}\cdots x_{k,1}\cdots x_{k,j_k} = (x_{1,1}\cdots x_{1,j_1})\cdots(x_{k,1}\cdots x_{k,j_k})$$

for elements $(x_{r,1}\cdots x_{r,j_r})$ for $1 \leq r \leq k$ in a monoid. We see the product on the left by going up, right, and down from the bottom left vertex and see the product on the right by going right and up. In particular, when $k = 2$ and $j_1 = j_2 = 2$, the diagram can be thought of as parametrizing the equation $wxyz = (wx)(yz)$, and then (i) shows that we can take any of the variables to be 1 to obtain a parametrized associativity relation. Note the distinction between the unit element $\iota_0 = 1 \in X$ and the identity operation coming from $\iota_0 \colon v \longrightarrow \mathscr{C}^{\times}(1)$.

Example 2.9. We define $\mathscr{M}^{\times}(j) = \Sigma_j$. The product (2.1) defines $\gamma^{\times}(e_k;\tau_1,\cdots,\tau_k)$ and the first equivariance formula extends it to give $\gamma^{\times}(\sigma;\tau_1,\cdots,\tau_k)$ for $\sigma \in \Sigma_k$. The remaining axioms are verified by the elementary verifications that motivated Definition 2.4 in the first place. Thinking of operations on sets, we are identifying the partially ordered cartesian product of finite sets with j_r elements, again denoted j_r, with the ordered finite set $j_{\times}$ with $j_1\cdots j_k$ elements. Here $\iota_r \colon \Sigma_k \longrightarrow \Sigma_{k+1}$ is induced by the ordered injection $k \longrightarrow k+1$ that skips r.

3. Bioperads and their algebras

3.1. **The definition of bioperads.** As said before, the definition of bioperads is inspired by the earlier definition of an operad pair, which is recalled in Section 8 for comparison. Our theme now, however, is the optimal melding of operads and moperads into single structures. In particular, we extend $\mathscr{M}$ to a bioperad $\mathscr{M}^{bi}$ by combining $\mathscr{M}$ and $\mathscr{M}^{\times}$. We now work in a cartesian monoidal category $\mathscr{V}$, such as **SET**, **CAT**, or **TOP**, since distributivity requires use of both diagonal maps Δ and maps to u, which appear automatically when u is a terminal object. We will need the following notations to define the permutativity diagram[6] central to the definition.

Notation 3.1. As in Notations 2.3 and 2.6, let Q run over the set of sequences $(q_1,\cdots,q_k)$ such that $1 \leq q_r \leq j_r$, ordered lexicographically. There are $j_{\times}$ such Q. Given non-negative $i_{r,q}$ for $1 \leq r \leq k$ and $1 \leq q \leq j_r$, let

$$i_Q = \times_{r=1}^{k} i_{r,q_r} \quad\text{and}\quad h_r = \Sigma_{q=1}^{j_r} i_{r,q}$$

and let $\nu = \nu(\{k, j_r, i_{r,q}\})$ be that permutation of

$$\Sigma_Q\, i_Q = \times_{1\leq r\leq k}\, h_r$$

[6]I thank Isaiah Daily for helping work out the diagram from a formula.

elements which corresponds under block sum and lexicographic identifications on the left and right to the distributivity isomorphism

$$\coprod_{Q} \Big(\prod_{1 \le r \le k} i_{r,q_r} \Big) \cong \prod_{1 \le r \le k} \Big(\coprod_{1 \le q \le j_r} i_{r,q} \Big). \tag{3.2}$$

Here we again write i for both the number i and the set $\{1, 2, \cdots, i\}$. Note that for each q, $1 \le q \le j_r$, the number $i_{r,q}$ appears once on the right but $n(r) = j_\times / j_r$ times on the left. Let $i_{+\times}$ denote the number of elements in the sets displayed in (3.2). In specifying ν, both sets are identified with the set $i_{+\times}$.

Definition 3.3. A bioperad $\mathscr{C}$ in $\mathscr{V}$ is an additive operad $\mathscr{C}$ as in Definition 1.1 and a multiplicative operad $\mathscr{C}^\times$ as in Definition 2.4 such that $\mathscr{C}(j) = \mathscr{C}^\times(j)$ for all $j \ge 1$ together with an isomorphism $0\colon u \cong \mathscr{C}(0)$. We write γ and $\gamma^\times$ for the structure maps of $\mathscr{C}$ and $\mathscr{C}^\times$. We regard $\gamma^\times$ as both the multiplicative product and a distributivity map, when it might be denoted λ.[7] (Anticipating Remark 4.3 below, we can also interpret $\mathscr{C}^\times(0)$ to be u). We codify our remaining conditions.

(i) Under γ, the additive operad conditions of Definition 1.1 hold.
(ii) Under $\gamma^\times$, the multiplicative operad conditions of Definition 2.4 hold.
(iii) The following distributivity diagram commutes:

$$\begin{array}{ccc}
\mathscr{C}^\times(k) \times \big(\times_{r=1}^{k} \mathscr{C}(j_r)\big) \times \big(\times_Q \mathscr{C}(i_Q)\big) & \xrightarrow{\gamma^\times \times \mathrm{Id}} & \mathscr{C}(j_\times) \times \big(\times_Q \mathscr{C}(i_Q)\big) \\
\uparrow{\scriptstyle \mathrm{Id} \times (\gamma^\times)^{j_\times}} & & \downarrow{\scriptstyle \gamma} \\
\mathscr{C}^\times(k) \times \big(\times_{r=1}^{k} \mathscr{C}(j_r)\big) \times \big(\times_Q \mathscr{C}^\times(k) \times_{r=1}^{k} \mathscr{C}(i_{r,q_r})\big) & & \mathscr{C}(i_{+\times}) \\
\uparrow{\scriptstyle \text{shuffle}} & & \downarrow{\scriptstyle \nu} \\
\mathscr{C}^\times(k)^{j_\times + 1} \times \big(\times_{r=1}^{k} \big(\mathscr{C}(j_r) \times (\times_{q=1}^{j_r} \mathscr{C}(i_{r,q})^{n(r)})\big)\big) & & \mathscr{C}(i_{+\times}) \\
\uparrow{\scriptstyle \Delta \times (\times_r \mathrm{Id} \times \Delta)} & & \uparrow{\scriptstyle \gamma^\times} \\
\mathscr{C}^\times(k) \times \big(\times_{r=1}^{k} \big(\mathscr{C}(j_r) \times (\times_{q=1}^{j_r} \mathscr{C}(i_{r,q}))\big)\big) & \xrightarrow[\mathrm{Id} \times \gamma^k]{} & \mathscr{C}^\times(k) \times \big(\times_{r=1}^{k} \mathscr{C}(h_r)\big)
\end{array}$$

Our motivation comes from the following concrete example.

Example 3.4. With $\mathscr{M}$ and $\mathscr{M}^\times$ as in Examples 1.4 and 2.9, $\mathscr{M}$ is a bioperad. We use the notation $\mathscr{M}^{bi}$ for $\mathscr{M}$ regarded as a bioperad.

Proof. We need only verify the distributivity law, and that holds by combinatorial inspection. Intuitively, this is just passage to skeleta from the evident distributivity law connecting disjoint unions and cartesian products of finite sets. □

3.2. **The definition of algebras over a bioperad.** The following definition parallels the earlier definition of an algebra over an operad pair, which is recalled in Definition 8.3 below. Here X^j denotes the cartesian product of j copies of X. Bipermutative categories will give examples.

Definition 3.5. An action of a bioperad $\mathscr{C}^{bi}$ on X consists of an action θ of $\mathscr{C}$ on X as in Definition 1.3 (with basepoint 0) and an action $\theta^\times$ of $\mathscr{C}^\times$ on X as in Definition 2.7 (with unit element 1) for which 0 is a strict 0, so that $\theta^\times(g; y) = 0$ if

[7] I thank Agnes Beaudry for helping me see that this is correct.

any coordinate of $y \in X^j$ is 0, and for which the following distributivity diagram commutes:

$$\begin{array}{ccc}
\mathscr{C}^\times(k) \times \big(\times_{r=1}^{k} \mathscr{C}(j_r)\big) \times X^{j_\times} & \xrightarrow{\lambda \times \mathrm{Id}} & \mathscr{C}(j_\times) \times X^{j_\times} \\
\uparrow{\scriptstyle \mathrm{Id} \times (\theta^\times)^{j_\times}} & & \downarrow{\scriptstyle \theta} \\
\mathscr{C}^\times(k) \times \big(\times_{r=1}^{k} \mathscr{C}(j_r)\big) \times \big(\times_Q \big(\mathscr{C}^\times(k) \times X^k\big)\big) & & X \\
\uparrow{\scriptstyle \text{shuffle}} & & \downarrow{\scriptstyle =} \\
\mathscr{C}^\times(k)^{j_\times + 1} \times \big(\times_{r=1}^{k} \mathscr{C}(j_r) \times X^{j_\times}\big) & & X \\
\uparrow{\scriptstyle \Delta \times \times_r \mathrm{Id} \times \Delta} & & \uparrow{\scriptstyle \theta^\times} \\
\mathscr{C}^\times(k) \times \big(\times_{r=1}^{k} \mathscr{C}(j_r) \times X^{j_r}\big) & \xrightarrow[\mathrm{Id} \times \theta^k]{} & \mathscr{C}^\times(k) \times X^k
\end{array}$$

DistDia

Remark 3.6. The last diagram is a parametrized version of the right distributivity law. In a rig space X, for variables $(x_{r,1}, \cdots, x_{r,j_r}) \in X^{j_r}$, $1 \le r \le k$, we set $z_r = x_{r,1} + \cdots + x_{r,j_r}$ and find that

$$z_1 \cdots z_k = \sum_Q x_{1,q_1} \cdots x_{k,q_k},$$

where the sum runs over the set of sequences $Q = (q_1, \cdots, q_k)$ such that $1 \le q_r \le j_r$, ordered lexicographically as above. The bottom two left vertical arrows in the diagram of Definition 3.5 give an operadic parametrization of the implicit map

$$\delta \colon X^{j_+} \longrightarrow \times_Q X^k \cong (X^k)^{j_\times} \tag{3.7}$$

whose Qth coordinate sends an element of $X^{j_+} = \times_{1 \le r \le k} X^{j_r}$ with rth coordinate $(x_{r,1}, \cdots x_{r,j_r})$ to the element with Qth coordinate $(x_{1,q_1}, \cdots, x_{k,q_k})$.

4. Monads associated to operads, moperads, and bioperads

Operads were originally defined so that an operad $\mathscr{C}$ in $\mathscr{V}$ has an associated monad $\mathbb{C}$ in the category $\mathscr{V}_*$ of based objects in $\mathscr{V}$. Again let Λ be the category of based finite sets $\underline{n} = \{0, 1, \cdots, n\}$, with basepoint 0, and based injections. Then $\mathscr{C}$ restricts to a contravariant functor $\Lambda \longrightarrow \mathscr{V}$, insertions of basepoints gives a covariant functor X^* with nth value X^n, and $\mathbb{C}X$ is the categorical tensor product $\mathscr{C} \otimes_\Lambda X^*$. The unit η and product μ of $\mathbb{C}$ are induced by the unit and structure maps of $\mathscr{C}$. The unit and associativity diagrams required of a monad are then immediate from the unit and associativity diagrams in Definition 1.1. Similarly, the unit and action maps of a $\mathscr{C}$-algebra X determine the unit and structure maps of a $\mathbb{C}$-algebra structure on X, and the categories of $\mathscr{C}$-algebras and $\mathbb{C}$-algebras are isomorphic. This was a key motivation for the definition of operads, and the name operad is a portmanteau word obtained from operation and monad.

We claim that an analogous, but perhaps less obvious, story applies to moperads in a cocomplete symmetric monoidal category $\mathscr{V}$.

mopmoncon

Construction 4.1. Just as for $\mathbb{C}X$, we construct a monad $\mathbb{C}^\times$ from a moperad $\mathscr{C}^\times$. We again start with $\coprod_j \mathscr{C}^\times(j) \otimes X^j$. Both $\mathscr{C}^\times$ and X^* are covariant functors of the category $\Lambda_>$, hence so is $\mathscr{C}^\times \otimes X^* = \coprod_j \mathscr{C}^\times(j) \otimes X^j$. Writing $\mathbf{n} = \{\mathbf{1}, \cdots ., \mathbf{n}\}$, the action by $\Lambda_>$ is defined by sending a morphism $\lambda\colon \mathbf{m} \longrightarrow \mathbf{n}$ of $\Lambda_>$ to the morphism $\lambda \otimes \lambda\colon \mathscr{C}^\times(m) \otimes X^m \longrightarrow \mathscr{C}^\times(n) \otimes X^n$ in $\mathscr{V}$, using the identity on other summands $\mathscr{C}(j) \times X^j$. Regarding the identity functor of $\mathscr{C}^\times \otimes X^*$ as the contravariant functor on $\Lambda_>$ that sends every morphism λ to the identity morphism, we can form the tensor product of Id and the covariant functor $\mathscr{C}^\times \otimes X^*$. We regard this as a kind of categorical orbit functor $(\mathscr{C}^\times \otimes X^*)/\Lambda_>$, and we denote it by $\mathbb{C}^\times X$. Concretely $\mathbb{C}^\times X$ is $\coprod_j \mathscr{C}(j) \otimes_{\Sigma_j} X^j$ modulo unit identifications that identify $c \otimes y$ with $\iota_r(c) \otimes \iota_r(y)$ for all $c \in \mathscr{C}^\times(n)$ and $y \in X^n$ for $1 \leq r \leq n+1$.

mopmonprop

Proposition 4.2. *With unit $\eta^\times$ and product $\mu^\times$ induced by the unit $v \longrightarrow \mathscr{C}^\times(1)$ and the structure maps $\gamma^\times$ of $\mathscr{C}^\times$, $\mathbb{C}^\times$ is a monad on $\mathscr{V}$. The category of algebras over the moperad $\mathscr{C}^\times$ is isomorphic to the category of algebras over the monad $\mathbb{C}^\times$.*

Proof. In more detail, the unit $\eta\colon X \longrightarrow \mathbb{C}^\times X$ is $\iota_0 \times \mathrm{Id}\colon X \cong v \otimes X \longrightarrow \mathscr{C}^\times(1) \otimes X$ and the product $\mu\colon \mathbb{C}^\times\mathbb{C}^\times X \longrightarrow \mathbb{C}^\times X$ is induced by the composite

$$\mathscr{C}^\times(k) \otimes \mathscr{C}^\times(j_1) \otimes X^{j_1} \otimes \cdots \otimes \mathscr{C}^\times(j_k) \otimes X^{j_k} \longrightarrow \mathscr{C}^\times(j_\times) \otimes X^{j_\times}$$

from the bottom left to the top right in the diagram of Definition 2.7(ii). That diagram then gives the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{C}^\times\mathbb{C}^\times X & \xrightarrow{\mu} & \mathbb{C}^\times X \\ {\scriptstyle \mathbb{C}^\times\theta^\times}\downarrow & & \downarrow{\scriptstyle \theta^\times} \\ \mathbb{C}^\times X & \xrightarrow[\theta^\times]{} & X \end{array}$$

The unit diagram $\mu\eta = \mathrm{Id}\colon X \longrightarrow X$ is immediate from Definition 2.7(i). The diagrams showing that $\mathbb{C}^\times$ is a monad are less obvious. It is immediate that the left triangle commutes in the unit diagram

$$\begin{array}{ccccc} \mathbb{C}^\times X & \xrightarrow{\eta} & \mathbb{C}^\times\mathbb{C}^\times X & \xleftarrow{\mathbb{C}^\times\eta} & \mathbb{C}^\times X \\ & {\scriptstyle \mathrm{Id}}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{Id}} & \\ & & \mathbb{C}^\times X & & \end{array}$$

However, since the sum of k copies of 1 is k and the product of k copies of 1 is 1, the commutation of the right triangle requires use of the unit identification in the definition of $\mathbb{C}^\times X$, together with the third of the unit diagrams in Definition 2.4(i). We insert

$$(\iota_1)^k \otimes (\iota_1)^k\colon \mathscr{C}(1) \otimes X \longrightarrow \mathscr{C}(k) \otimes X^k$$

on the first factor $\mathscr{C}(1) \otimes X$ that we see in the target $\mathbb{C}^\times(k) \otimes (\mathbb{C}(1) \otimes X)^k$ of the restriction of $\mathbb{C}^\times\eta$ to the kth summand $\mathscr{C}^\times(k) \otimes_{\Sigma_k} X^k$ of $\mathbb{C}^\times X$. We could equally well use any other factor. The associativity relation $\mu \circ \mu = \mu \circ \mathbb{C}^\times\mu$ is implied by a rather horrendous diagram chase using instances of Definition 2.4(ii). Again, the identification in Construction 4.1 is needed to handle the cases where $j_+ > j_\times$ and the identification ensures that these cases present no problems. We have seen that a $\mathscr{C}^\times$-algebra gives rise to a $\mathbb{C}^\times$-algebra. Conversely, if $(X, \theta^\times)$ is a $\mathbb{C}^\times$-algebra, it is a $\mathscr{C}^\times$-algebra with action induced by the evident maps

$$\mathscr{C}^\times(j) \otimes X^j \longrightarrow \mathbb{C}^\times X \xrightarrow{\theta^\times} X.$$

The unit and associativity diagrams commute by the corresponding monadic diagrams. □

Remark 4.3. We can modify Construction 4.1 by adding disjoint basepoints 0 to all $\mathscr{C}^\times(j)$, taking $\mathscr{C}^\times(0) \cong u$, thought of as 0. This will lead to new examples of Beck's two monad distributivity theory [Bec69, May09]. We then interpret $\gamma^\times$ on $\mathscr{C}^\times(0)$ to be the map $u \longrightarrow \mathscr{C}^\times(1)$ that sends u to 0 and allow $j_r = 0$ in the definition of $\gamma^\times$; when any $j_r = 0$, we require $\gamma^\times$ to take the value 0 in $\mathscr{C}^\times(0)$. In $\mathbb{C}_0^\times$-algebras, or $\mathscr{C}^\times$-algebras with 0, we require X to have a strict 0 element (formally $0\colon u \longrightarrow X$), so that $\theta^\times(g; y) = 0$ if any coordinate of $y \in X^j$ is 0.

Definition 3.5 and Remark 3.6 lead to the following result. Passing to monads, it says that $(\mathbb{C}, \mathbb{C}_0^\times)$ is a monad pair in the sense of Beck [Bec69]; see also [May09, Appendix B] or [KMZ24, Section 8.1].

Theorem 4.4. *Let X be a $\mathscr{C}^{bi}$-algebra. As k and the j_r vary, the composites from the bottom left to the top right in the distributivity diagram of Definition 3.5 induce a natural action $\theta^\times$ of $\mathscr{C}^\times$ on $\mathbb{C}X$ such that θ is a map of $\mathscr{C}^\times$-algebras. The monad $\mathbb{C}$ on the category $\mathscr{V}$ restricts to a monad on the category of $\mathscr{C}^\times$-algebras in $\mathscr{V}$.*

Proof. The cited distributivity diagram induces the commutative diagram

$$\begin{array}{ccc} \mathscr{C}^\times(k)\times(\mathbb{C}X)^k & \xrightarrow{\theta^\times} & \mathbb{C}X \\ {\scriptstyle \mathrm{Id}\times\theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ \mathscr{C}^\times(k)\times X^k & \xrightarrow[\theta^\times]{} & X \end{array} \tag{4.5}$$

Diagram chasing proves that the top arrow as k varies gives $\mathbb{C}X$ an action of $\mathscr{C}_0^\times$ such that, with the added 0 as in Remark 4.3, θ is a map of $\mathscr{C}_0^\times$-algebras, as claimed. Monadically, the unit $\eta\colon X \longrightarrow \mathbb{C}X$ and product $\mu\colon \mathbb{C}\mathbb{C}X \longrightarrow \mathbb{C}X$ are seen to be maps of $\mathbb{C}_0^\times$-algebras by further diagram chasing. □

5. Permutative and bipermutative categories

We prove Theorem 0.2 here, and we first construct the bioperad $\mathscr{P}^{bi}$ in **CAT**. Again let $\mathscr{E}\colon \mathbf{SET} \longrightarrow \mathbf{CAT}$ be the chaotic (or indiscrete) category functor, so that $\mathscr{E}(S)$ is the category with object set S and a unique morphism between each pair of objects. We abuse notation by letting $\mathscr{E}(S)$ also denote its classifying space $B\mathscr{E}(S) = |N(\mathscr{E}(S)|$, which is contractible. The functor $\mathscr{E}$ is right adjoint to the object functor $\mathscr{O}b$ that sends a category to its set of objects. As said before, $\mathscr{E}$ sends operads in **SET** to operads in **CAT** and B sends those to operads in spaces.[8] This is also true for moperads and bioperads.

Definition 5.1. Let $\mathscr{P} = \mathscr{E}(\mathscr{M})$, let $\mathscr{P}^\times = \mathscr{E}(\mathscr{M}^\times)$, and let $\mathscr{P}^{bi}$ denote $\mathscr{P}$ regarded as a bioperad. Then $\mathscr{P}^{top} = B\mathscr{P}$ is an E_∞ operad since for each j the free Σ_j-action on $\mathscr{M}$ gives a free Σ_j-action on the contractible space $\mathscr{P}^{top}(j)$.

A permutative category is a symmetric strict monoidal category, and Theorem 0.1 is standard, maybe first noticed in [May74]. The idea is that $\mathscr{P}$ sends a permutative category $\mathscr{A}$ to its "unbiased" version with a canonical j-fold product

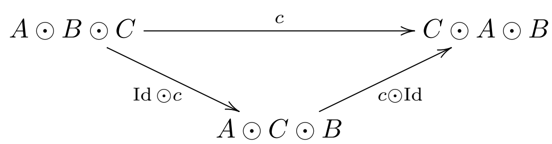
[8] Spaces are understood to be compactly generated and weak Hausdorff.

for each j. Then $B\mathscr{A}$ is an E_∞-space, hence has an associated spectrum $\mathbb{E}B\mathscr{A}$ whose zeroth space is a group completion of $B\mathscr{A}$ [May74, GM17, KMZ24].

Before turning to bipermutative categories, we recall the precise definition of a permutative category and prove a multiplicative analog of Theorem 0.1.

perm

Definition 5.2. A permutative category $(\mathscr{A}, *, c)$ is a category with a product functor $\odot\colon \mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{A}$ which is strictly associative and unital (with unit $*$) and has a natural symmetry isomorphism $c\colon A\odot B\longrightarrow B\odot A$ such that c is the identity $A = A\odot * \longrightarrow *\odot A = A$, $c\circ c = \mathrm{Id}$, and the following diagram commutes for objects A, B, C.



thm1mult

Theorem 5.3. *The category $\mathscr{P}^\times[\mathbf{Cat}]$ of $\mathscr{P}^\times$-algebras in* **CAT** *is isomorphic to the category of permutative categories. The category $\mathscr{P}^\times_{ps}[\mathbf{Cat}]$ of $\mathscr{P}^\times$-pseudo-algebras in* **CAT** *is equivalent to the category of symmetric monoidal categories.*

Proof. We prove Theorems 0.1 and 5.3 the same way. The idea is that $\mathscr{P}$-algebras and $\mathscr{P}^\times$-algebras give two different unbiased descriptions of permutative categories. In the first, the product $\odot$ is given on objects A, B by $\theta(e; A, B)$, with unit $* = \theta(0)$, where $0\in\mathscr{P}(0)$; we will think of $*$ as 0 later. In the second, the product is given by $\theta^\times(e; A, B)$, with unit $* = \theta(1)$, where $1\in\mathscr{P}^\times(1)$; we will think of $*$ as 1 later. The strict unit and associativity conditions are built into the definitions of actions, with Remark 2.8 explaining this in the case of $\mathscr{P}^\times$. The permutativity isomorphism

$$c\colon A\odot B = \theta(e; A, B)\longrightarrow \theta(\sigma; A, B) = \theta(e; B, A) = B\odot A$$

is induced by the unique morphism $e\to\sigma$ in $\mathscr{P}(2)$ or $\mathscr{P}^\times(2)$, where $\sigma\in\Sigma_2$ is the non-identity element. The uniqueness of morphisms between objects of the $\mathscr{P}(j)$ and $\mathscr{P}^\times(j)$ ensures that c has the required properties.

Starting instead with a permutative category $\mathscr{A}$, we construct actions of $\mathscr{P}$ and $\mathscr{P}^\times$ on $\mathscr{A}$ such that $\theta(e; A_1,\cdots,A_j)$ and $\theta^\times(e; A_1,\cdots,A_j)$ are both the j-fold $\odot$-product of the A_i, in order. For $\sigma\in\Sigma_j$, the actions on $(\sigma; A_1,\cdots,A_j)$ are determined by equivariance. The diagrams ensuring that these give a $\mathscr{P}$-algebra and a $\mathscr{P}^\times$-algebra again follow from the uniqueness of morphisms between objects of the $\mathscr{P}(j)$ and $\mathscr{P}^\times(j)$. Said another way, they result from coherence isomorphisms in $\mathscr{A}$. If we apply the construction back to **CAT**, we get $\mathscr{A}$ back. Similarly, if we start from an algebra, construct a permutative category from it, and construct an algebra from that, what we get is isomorphic to the algebra we started with.

Using pseudofunctors and understanding that we require our coherence morphisms to be isomorphisms, the pseudoalgebra version is proven analogously but again, we shall not go into detail here. □

To prove Theorem 0.2, we first recall the definition of bipermutative categories from [May77, Definition VI.3.3].

biperm

Definition 5.4. A bipermutative category $\mathscr{A}$ is a pair of permutative categories $(\mathscr{A}, \oplus, 0, c)$ and $(\mathscr{A}, \otimes, 1, \tilde{c})$, where c and $\tilde{c}$ are the commutativity isomorphisms, which satisfy the following three conditions.

(i) 0 is a strict zero object for $\otimes$.
(ii) The right distributivity law is strictly satisfied and the following diagram commutes for objects A, B, C:

$$\begin{array}{ccc} (A\oplus B)\otimes C & = & (A\otimes C)\oplus(B\otimes C)\\ {\scriptstyle c\otimes \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle c}\\ (B\oplus A)\otimes C & = & (B\otimes C)\oplus(A\otimes C)\end{array}$$

(iii) Define a left distributivity isomorphism ℓ by the commutative diagram

$$\begin{array}{ccc} (A\oplus B)\otimes C & = & (A\otimes C)\oplus(B\otimes C)\\ {\scriptstyle \tilde{c}}\uparrow & & \downarrow{\scriptstyle \tilde{c}\otimes\tilde{c}}\\ C\otimes(A\oplus B) & \xrightarrow[\ell]{} & (C\otimes A)\oplus(C\otimes B)\end{array}$$

Then the following diagram commutes for objects A, B, C, D.

$$\begin{array}{ccc} (A\oplus B)\otimes(C\oplus D) & = & (A\otimes(C\oplus D))\oplus(B\otimes(C\oplus D))\\ & & \downarrow{\scriptstyle \ell\otimes\ell}\\ {\scriptstyle \ell}\downarrow & & (A\otimes C)\oplus(A\otimes D)\oplus(B\otimes C)\oplus(B\otimes D)\\ & & \downarrow{\scriptstyle \mathrm{Id}\oplus c\oplus \mathrm{Id}}\\ ((A\oplus B)\otimes C)\oplus((A\oplus B)\otimes C) & = & (A\otimes C)\oplus(B\otimes C)\oplus(A\otimes D)\oplus(B\otimes D)\end{array}$$

It should be clear that both distributivity laws will not generally hold strictly. Our choice is dictated by our use of lexicographic ordering in earlier definitions.

Proof of Theorem 0.2. We first show that if $\mathscr{A}$ is a $\mathscr{P}^{bi}$-algebra, then it is a bipermutative category. Its permutative categories $(\mathscr{A}, \oplus, 0, c)$ and $(\mathscr{A}, \otimes, 1, \tilde{c})$ are given by Theorems 0.1 and 5.3. Consider the diagram in Definition 3.5 with $k = 2$, $j_1 = 2$ and $j_2 = 1$ and with operadic coordinate objects $e \in \mathscr{P}(2)$ and $e \in \mathscr{P}(1)$. Writing $A\otimes B = AB$, we find for objects A, B, C that $(A\oplus B)C = AC\oplus BC$. Taking $k = 2$, $j_1 = 1$ and $j_2 = 2$, we get the left distributivity law and taking $k = 2$, $j_1 = 2$, and $j_2 = 2$, we get the four variable diagram in Definition 5.4. If $\mathscr{A}$ is a bipermutative category, then coherence gives that the unbiased redefinition gives an action of $\mathbb{P}$ on $\mathscr{A}$. Again, these operations are inverse to each other. □

6. Equivariant permutative and bipermutative categories

Gpiperm

Let G be a group. Our interest is in finite groups but the categorical ideas work in general. We start work in any cartesian monoidal category $\mathscr{V}$, taking $G\mathscr{V}$ to be the cartesian monoidal category of G-objects in $\mathscr{V}$ and G-maps between them. Let $\mathbf{CAT}(\mathscr{V})$ be the category of categories internal to $\mathscr{V}$; it generalizes categories enriched in $\mathscr{V}$ by allowing object and morphism objects of $\mathscr{V}$ and insisting that source, target, identity, and composition be maps in $\mathscr{V}$. Theorems 0.1, 5.3 and 0.2 directly generalize to characterizations of permutative and bipermutative categories internal to $\mathscr{V}$.

Definition 6.1. Define G-categories in $\mathscr{V}$ to be categories internal to $G\mathscr{V}$, giving us the category $G\mathbf{CAT}(\mathscr{V}) = \mathbf{CAT}(G\mathscr{V})$ of G-categories in $\mathscr{V}$. We omit $\mathscr{V}$ from the notation when $\mathscr{V} = \mathscr{U}$ is the category of spaces. Define classical[9] permutative and bipermutative G-categories to be permutative and bipermutative categories internal to $G\mathscr{V}$. Then Theorems 0.1, 5.3 and 0.2 show that these can be viewed as either $\mathscr{P}$-algebras or $\mathscr{P}^{\times}$-algebras in $G\mathbf{CAT}\mathscr{V}$ and as $\mathscr{P}^{bi}$-algebras in $G\mathbf{CAT}\mathscr{V}$.

When $\mathscr{V} = \mathscr{U}$, these give rise to classical (alias naive) G-spectra or classical E_∞-ring G-spectra as in [May82, May09, KMZ24]. In [GM17, Section 4], we defined a genuine permutative G-category to be a $\mathscr{P}_G$-category $\mathscr{A}$ for a certain operad $\mathscr{P}_G$ in $GCAT$. We gave categorical justification for the definition, showing for example that $\mathscr{P}$ is a suboperad of $\mathscr{P}_G$ such that $\mathscr{A}^G$ is a $\mathscr{P}$-algebra. We gave topological justification by proving that $\mathscr{P}_G$-algebras give rise to genuine G-spectra.

Using bioperads, we can now give a precisely parallel definition of (genuine) bipermutative G-categories. We formalize using the following "genuinification" functor, which is discussed in greater detail in [GM17, Section 3.2]. In particular, its close relationship to equivariant bundle theory is discussed there. I believe it was first defined in [Tho83], where it plays a central role.

Definition 6.2. Define $\mathscr{G}\colon G\mathbf{CAT} \longrightarrow G\mathbf{CAT}$ by

$$\mathscr{G}(\mathscr{A}) = \mathbf{CAT}(\mathscr{E}(G), \mathscr{A}).$$

The target is the category of functors and natural transformations from the chaotic category $\mathscr{E}G$ to $\mathscr{A}$, with G acting by conjugation. Via

$$\mathbf{CAT}(\mathscr{C} \times \mathscr{E}(G), \mathscr{A}) \cong \mathbf{CAT}(\mathscr{C}, \mathscr{G}(A)),$$

$\mathscr{G}$ is a right adjoint. The projection $\mathscr{E}(G) \longrightarrow *$ induces a natural map

$$\iota\colon \mathscr{A} \longrightarrow \mathscr{G}(\mathscr{A}).$$

When $\mathscr{A} = \mathscr{E}(S)$ for a set S, regarded as a G-trivial G-set, ι is the inclusion of the G-fixed category $(\mathscr{G}(\mathscr{E}(S)))^G = G\mathbf{CAT}(\mathscr{E}(G), \mathscr{E}(S))$, and this holds more generally for spaces [GS16, Lemma 4.3].

Since $\mathscr{G}$ preserves products, it preserves operadic structures.

Definition 6.3. Define $\mathscr{P}_G = \mathscr{G}(\mathscr{P})$, $\mathscr{P}_G^{\times} = \mathscr{G}(\mathscr{P}^{\times})$, and $\mathscr{P}_G^{bi} = \mathscr{G}(\mathscr{P}^{bi})$. These are an operad, a moperad, and a bioperad in $G\mathbf{CAT}$. Define genuine bipermutative G-categories to be $\mathscr{P}_G^{bi}$-algebras in $G\mathbf{CAT}$. We can define genuine symmetric bimonoidal G-categories to be $\mathscr{P}_G^{bi}$-pseudoalgebras in $G\mathbf{CAT}$, but I will not go into that here. The map ι induces on inclusion $\iota\colon \mathscr{P}^{bi} = (\mathscr{P}_G^{bi})^G \longrightarrow \mathscr{P}_G^{bi}$ of bioperads of G-categories, hence the G-fixed subcategory of a genuine bipermutative G-category is a bipermutative category.

As in [GM17, Proposition 4.6], application of $\mathscr{G}$ gives lots of examples, although presumably not all.

Proposition 6.4. *The action of $\mathscr{P}^{bi}$ on a classical permutative G-category $\mathscr{A}$ induces an action of $\mathscr{P}_G^{bi}$ on $\mathscr{G}(\mathscr{A})$. Therefore $\mathscr{G}$ restricts to a functor from classical bipermutative G-categories to genuine bipermutative G-categories.*

In particular, since the given action of G can be trivial, $\mathscr{G}$ gives a functor from nonequivariant bipermutative categories to (genuine) bipermutative G-categories.

[9]The word naive was used in [GM17, Section 4]. Classical seems preferable.

7. Moperad pairs

moppairs

Bioperads and their algebras are special cases of moperad pairs and their algebras. Examples of the more general notion will be given in [May], where they will be central to a new approach to multiplicative infinite loop space theory.

pair

Definition 7.1. A moperad pair $(\mathscr{C},\mathscr{J})$ consists of a classical operad $\mathscr{C}$ as in Definition 1.1 and a moperad $\mathscr{J}$ as in Definition 2.4 together with maps

eqj3

$$\lambda\colon \mathscr{J}(k)\times\mathscr{C}(j_1)\times\cdots\times\mathscr{C}(j_k)\to\mathscr{C}(j_\times) \tag{7.2}$$

which satisfy the following properties.

(i) The following unit diagram for λ commutes.

$$\begin{array}{ccc}
\mathscr{J}(k)\times u^k & \longrightarrow & u\\
{\scriptstyle \mathrm{Id}\otimes \mathrm{id}^k}\downarrow & & \downarrow{\scriptstyle id}\\
\mathscr{J}(k)\times\mathscr{C}(1)^k & \xrightarrow[\lambda]{} & \mathscr{C}(1).
\end{array}$$

(ii) Taking $\mathscr{C}(0)$ be an object $0\cong u$, λ maps $\mathscr{J}(k)\times\times_{r=1}^k\mathscr{C}(j_r)$ to 0 if any $j_r=0$.

(iii) Taking $\mathscr{J}(0)$ to be an object $1\cong u$, λ is interpreted as $\lambda(1)=\mathrm{id}\colon 1\longrightarrow\mathscr{C}(1)$.

(iv) For the associativity law, we order the finite set $\{t|1\le t\le j_\times\}$ as the lexicographically set of tuples $\{(r,q)|1\le r\le k,\ 1\le q\le j_r\}$ and we set

$$\times_{t=1}^{j_\times} i_t = i_\times = \times_{r=1}^k h_r, \quad \text{where} \quad h_r=(i_r)_\times=\times_{q=1}^{j_r} i_{r,q}.$$

The following diagram commutes.

$$\begin{array}{ccc}
\mathscr{J}(k)\times\big(\times_{r=1}^k\mathscr{J}(j_r)\big)\times\big(\times_{t=1}^{j_\times}\mathscr{C}(i_t)\big) & \xrightarrow{\gamma^\times\times\mathrm{Id}} & \mathscr{J}(j_\times)\times\big(\times_{t=1}^{j_\times}\mathscr{C}(i_t)\big)\\
 & & \downarrow{\scriptstyle\lambda}\\
{\scriptstyle\text{shuffle}}\downarrow & & \mathscr{C}(i_\times)\\
 & & \uparrow{\scriptstyle\lambda}\\
\mathscr{J}(k)\times\big(\times_{r=1}^k\big(\mathscr{J}(j_r)\times(\times_{q=1}^{j_r}\mathscr{C}(i_{r,q}))\big)\big) & \xrightarrow[\mathrm{Id}\times\lambda^k]{} & \mathscr{J}(k)\times\big(\times_{r=1}^k\mathscr{C}(h_r)\big).
\end{array}$$

(v) For the distributivity law, reusing the notations of Definition 3.3(v), the following analogous diagram commutes.

$$\begin{array}{ccc}
\mathscr{J}(k)\times\big(\times_{r=1}^k\mathscr{C}(j_r)\big)\times\big(\times_Q\mathscr{C}(i_Q)\big) & \xrightarrow{\lambda\times\mathrm{Id}} & \mathscr{C}(j_\times)\times\big(\times_Q\mathscr{C}(i_Q)\big)\\
{\scriptstyle\mathrm{Id}\times\lambda^{j_\times}}\uparrow & & \downarrow{\scriptstyle\gamma}\\
\mathscr{J}(k)\times\big(\times_{r=1}^k\mathscr{C}(j_r)\big)\times\big(\times_Q\mathscr{J}(k)\times_{r=1}^k\mathscr{C}(i_{r,q_r})\big) & & \mathscr{C}(i_{+\times})\\
{\scriptstyle\text{shuffle}}\uparrow & & \downarrow{\scriptstyle\nu}\\
\mathscr{J}(k)^{j_\times+1}\times\big(\times_{r=1}^k\big(\mathscr{C}(j_r)\times(\times_{q=1}^{j_r}\mathscr{C}(i_{r,q})^{n(r)})\big)\big) & & \mathscr{C}(i_{+\times})\\
{\scriptstyle\Delta\times(\times_r\mathrm{Id}\times\Delta)}\uparrow & & \uparrow{\scriptstyle\lambda}\\
\mathscr{J}(k)\times\big(\times_{r=1}^k\big(\mathscr{C}(j_r)\times(\times_{q=1}^{j_r}\mathscr{C}(i_{r,q}))\big)\big) & \xrightarrow[\mathrm{Id}\times\gamma^k]{} & \mathscr{J}(k)\times\big(\times_{r=1}^k\mathscr{C}(h_r)\big)
\end{array}$$

(vi) The following equivariance diagrams for λ commute:

$$\begin{CD}
\mathscr{J}(k)\times\mathscr{C}(j_1)\times\cdots\times\mathscr{C}(j_k) @>{\sigma\times\sigma^{-1}}>> \mathscr{J}(k)\times\mathscr{C}(j_{\sigma(1)})\times\cdots\times\mathscr{C}(j_{\sigma(k)})\\
@V{\lambda}VV @VV{\lambda}V\\
\mathscr{C}(j_\times) @>>{\sigma<j_{\sigma(1)},\cdots,j_{\sigma(k)}>}> \mathscr{C}(j_\times)
\end{CD}$$

and

$$\begin{CD}
\mathscr{J}(k))\times\mathscr{C}(j_1)\times\cdots\times\mathscr{C}(j_k) @>{\mathrm{id}\times\tau_1\times\cdots\times\tau_k}>> \mathscr{J}(k)\times\mathscr{C}(j_1)\times\cdots\times\mathscr{C}(j_k)\\
@V{\lambda}VV @VV{\lambda}V\\
\mathscr{C}(j_\times) @>{\tau_1\otimes\cdots\otimes\tau_k}>> \mathscr{C}(j_\times).
\end{CD}$$

Pairact

Definition 7.3. An action of $(\mathscr{C},\mathscr{J})$ on X consists of an action θ of $\mathscr{C}$ on X (with basepoint 0) and an action ξ of $\mathscr{J}$ on X (with unit element 1) for which 0 is a strict zero, so that $\xi(g;y)=0$ if any coordinate of y is 0, and for which the following parametrized left distributivity law holds.

$$\begin{CD}
\mathscr{J}(k)\times\big(\times_{r=1}^{k}\mathscr{C}(j_r)\big)\times X^{j_\times} @>{\lambda\times\mathrm{Id}}>> \mathscr{C}(j_\times)\times X^{j_\times}\\
@A{\mathrm{Id}\times\xi^{j_\times}}AA @VV{\theta}V\\
\mathscr{J}(k)\times\big(\times_{r=1}^{k}\mathscr{C}(j_r)\big)\times\big(\times_Q\big(\mathscr{J}(k)\times X^k\big)\big) @. X\\
@A{\text{shuffle}}AA @VV{=}V\\
\mathscr{J}(k)^{j_\times+1}\times\big(\times_{r=1}^{k}\mathscr{C}(j_r)\times X^{j_\times}\big) @. X\\
@A{\Delta\times(\times_r\mathrm{Id}\times\Delta)}AA @AA{\xi}A\\
\mathscr{J}(k)\times\big(\times_{r=1}^{k}\mathscr{C}(j_r)\times X^{j_r}\big) @>>{\mathrm{Id}\times\theta^k}> \mathscr{J}(k)\times X^k
\end{CD}$$

The bottom two left vertical arrows build in the map δ defined in Remark 3.6.

8. Operad pairs

oppair

Part of the motivation for bioperads and moperad pairs comes from the much earlier understanding of operad pairs. The reader can skip this section if it is viewed as a digression, but the (still mysterious) comparison is at the heart of our ideas.[10]

pair2

Definition 8.1. An operad pair $(\mathscr{C},\mathscr{J})$ consists of classical operads $\mathscr{C}$ and $\mathscr{J}$, both as in Definition 1.1, together with maps

eqi3too

$$\lambda\colon\ \mathscr{J}(k)\times\mathscr{C}(j_1)\times\cdots\times\mathscr{C}(j_k)\to\mathscr{C}(j_\times) \tag{8.2}$$

which satisfy the following properties.

[10]We used the notation $(\mathscr{C},\mathscr{G})$ in earlier work, but that leads to conflicts in equivariant situations. We also used formulas rather than diagrams, out of sheer author laziness.

(i) The following unit diagrams commute:

$$\begin{array}{ccc} \mathscr{J}(k)\times u^k & \longrightarrow & u \\ {\scriptstyle \mathrm{Id}\otimes \mathrm{id}^k}\downarrow & & \downarrow{\scriptstyle id} \\ \mathscr{J}(k)\times \mathscr{C}(1)^k & \xrightarrow[\gamma]{} & \mathscr{C}(1) \end{array} \qquad \begin{array}{ccc} u\times \mathscr{J}(j) & \xrightarrow{\cong} & \mathscr{J}(j)\ . \\ {\scriptstyle \mathrm{id}\times \mathrm{Id}}\downarrow & \nearrow{\scriptstyle \gamma} & \\ \mathscr{J}(1)\times \mathscr{J}(j) & & \end{array}$$

(ii) Taking $\mathscr{C}(0)$ be an object $0\cong u$, λ maps $\mathscr{C}(k)\times\times_{r=1}^{k}\mathscr{C}(j_r)$ to 0 if any $j_r=0$.

(iii) Taking $\mathscr{J}(0)$ to be an object $1\cong u$, λ is interpreted as $\lambda(1)=\mathrm{id}\colon 1\longrightarrow\mathscr{C}(1)$.

(iv) For the associativity law, we order the finite set $\{t|1\le t\le j_+\}$ as the set of tuples $\{(r,q)|1\le r\le k,\ 1\le q\le j_r\}$ ordered as k blocks of j_r letters, and we set

$$\Sigma_{r=1}^{k} j_r = j_+,\ \ \times_{t=1}^{j_+} i_t = i_\times = \times_{r=1}^{k} h_r, \quad \text{where} \quad h_r=\times_{q=1}^{j_r} i_{r,q}.$$

$$\begin{array}{ccc} \mathscr{J}(k)\times\big(\times_{r=1}^{k}\mathscr{J}(j_r)\big)\times\big(\times_{t=1}^{j_+}\mathscr{C}(i_t)\big) & \xrightarrow{\gamma\times\mathrm{Id}} & \mathscr{J}(j_+)\times\big(\times_{t=1}^{j_+}\mathscr{C}(i_t)\big) \\ & & \downarrow{\scriptstyle\lambda} \\ {\scriptstyle \text{shuffle}}\downarrow & & \mathscr{C}(i_\times) \\ & & \uparrow{\scriptstyle\lambda} \\ \mathscr{J}(k)\times\big(\times_{r=1}^{k}\big(\mathscr{J}(j_r)\times(\times_{q=1}^{j_r}\mathscr{C}(i_{r,q}))\big)\big) & \xrightarrow[\mathrm{Id}\times\lambda^k]{} & \mathscr{J}(k)\times\big(\times_{r=1}^{k}\mathscr{C}(h_r)\big). \end{array}$$

(v) With notations as in Definition 3.3, the distributivity diagram that must commute is similar to that there:

$$\begin{array}{ccc} \mathscr{J}(k)\times\big(\times_{r=1}^{k}\mathscr{C}(j_r)\big)\times\big(\times_Q\mathscr{C}(i_Q)\big) & \xrightarrow{\lambda\times\mathrm{Id}} & \mathscr{C}(j_\times)\times\big(\times_Q\mathscr{C}(i_Q)\big) \\ {\scriptstyle \mathrm{Id}\times\lambda^{j_\times}}\uparrow & & \downarrow{\scriptstyle\gamma} \\ \mathscr{J}(k)\times\big(\times_{r=1}^{k}\mathscr{C}(j_r)\big)\times\big(\times_Q\mathscr{J}(k)\times_{r=1}^{k}\mathscr{C}(i_{r,q_r})\big) & & \mathscr{C}(i_{+\times}) \\ {\scriptstyle \text{shuffle}}\uparrow & & \downarrow{\scriptstyle\nu} \\ \mathscr{J}(k)^{j_\times+1}\times\big(\times_{r=1}^{k}\big(\mathscr{C}(j_r)\times(\times_{q=1}^{j_r}\mathscr{C}(i_{r,q})^{n(r)})\big)\big) & & \mathscr{C}(i_{+\times}) \\ {\scriptstyle \Delta\times(\times_r\mathrm{Id}\times\Delta)}\uparrow & & \uparrow{\scriptstyle\lambda} \\ \mathscr{J}(k)\times\big(\times_{r=1}^{k}\big(\mathscr{C}(j_r)\times(\times_{q=1}^{j_r}\mathscr{C}(i_{r,q}))\big)\big) & \xrightarrow[\mathrm{Id}\times\gamma^k]{} & \mathscr{J}(k)\times\big(\times_{r=1}^{k}\mathscr{C}(h_r)\big) \end{array}$$

(vi) The following equivariance diagrams for λ commute:

$$\begin{array}{ccc} \mathscr{J}(k)\times\mathscr{C}(j_1)\times\cdots\times\mathscr{C}(j_k) & \xrightarrow{\sigma\times\sigma^{-1}} & \mathscr{J}(k)\times\mathscr{C}(j_{\sigma(1)})\times\cdots\times\mathscr{C}(j_{\sigma(k)}) \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda} \\ \mathscr{C}(j_\times) & \xrightarrow[\sigma<j_{\sigma(1)},\cdots,j_{\sigma(k)}>]{} & \mathscr{C}(j_\times) \end{array}$$

and

$$\begin{array}{ccc}
\mathscr{J}(k)) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) & \xrightarrow{\mathrm{id} \times \tau_1 \times \cdots \times \tau_k} & \mathscr{J}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \\
\Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow{\scriptstyle \lambda} \\
\mathscr{C}(j_\times) & \xrightarrow{\tau_1 \otimes \cdots \otimes \tau_k} & \mathscr{C}(j_\times).
\end{array}$$

Pairact2

Definition 8.3. An action of $(\mathscr{C}, \mathscr{J})$ on X consists of an action θ of $\mathscr{C}$ on X (with basepoint 0) and an action ξ of $\mathscr{J}$ on X (with unit element 1) for which 0 is a strict zero, so that $\xi(g; y) = 0$ if any coordinate of y is 0, and for which the following parametrized distributivity diagram commutes:

$$\begin{array}{ccc}
\mathscr{J}(k) \times \big(\times_{r=1}^{k} \mathscr{C}(j_r)\big) \times X^{j_\times} & \xrightarrow{\lambda \times \mathrm{Id}} & \mathscr{C}(j_\times) \times X^{j_\times} \\
\Big\uparrow{\scriptstyle \mathrm{Id} \times \xi^{j_\times}} & & \Big\downarrow{\scriptstyle \theta} \\
\mathscr{J}(k) \times \big(\times_{r=1}^{k} \mathscr{C}(j_r)\big) \times \big(\times_Q \big(\mathscr{J}(k) \times X^k\big)\big) & & X \\
\Big\uparrow{\scriptstyle \text{shuffle}} & & \Big\downarrow{\scriptstyle =} \\
\mathscr{J}(k)^{j_\times + 1} \times \big(\times_{r=1}^{k} \mathscr{C}(j_r) \times X^{j_\times}\big) & & X \\
\Big\uparrow{\scriptstyle \Delta \times (\times_r \mathrm{Id} \times \Delta)} & & \Big\uparrow{\scriptstyle \xi} \\
\mathscr{J}(k) \times \big(\times_{r=1}^{k} \mathscr{C}(j_r) \times X^{j_r}\big) & \xrightarrow[\mathrm{Id} \times \theta^k]{} & \mathscr{J}(k) \times X^k
\end{array}$$

9. Monads associated to moperad pairs and operad pairs

For a moperad pair $(\mathscr{C}, \mathscr{J})$, we have the following generalization of Theorem 4.4.

YES2

Theorem 9.1. *As k and the j_r vary, the composites from the bottom left to the top right in the distributivity diagrams of Definitions 7.1 and 7.3 induce a natural action ξ of the moperad $\mathscr{J}$ on $\mathbb{C}X$ for $\mathscr{J}$-algebras X such that if X is a $(\mathscr{C}, \mathscr{J})$-algebra, then $\theta\colon \mathbb{C}X \longrightarrow X$ is a map of $\mathscr{J}$-algebras. The monad $\mathbb{C}$ on the category $\mathscr{V}$ restricts to a monad on the category of $\mathscr{J}$-algebras in $\mathscr{V}$.*

Proof. The distributivity diagram induces the commutative diagram

WOW2

$$\begin{array}{ccc}
\mathscr{J}(k) \times (\mathbb{C}X)^k & \xrightarrow{\xi} & \mathbb{C}X \\
\Big\downarrow{\scriptstyle \mathrm{Id} \times \theta} & & \Big\downarrow{\scriptstyle \theta} \\
\mathscr{J}(k) \times X^k & \xrightarrow[\xi]{} & X
\end{array} \tag{9.2}$$

□

The following precise analog of Theorems 4.4 and 9.1 was the starting point of the earlier multiplicative theory. Just as above, it was expressed in terms of pairs of monads as in [Bec69, May09]. This old result is why the claims above are plausible. The two theorems of this section explain why the general multiplicative theory of [KMZ24] applies to both moperad pairs and operad pairs, as we will see in [May].

Theorem 9.3. *As k and the j_r vary, the composites from the bottom left to the top right in the distributivity diagrams of Definitions 8.1 and 8.3 induce a natural action ξ of the operad $\mathscr{J}$ on $\mathbb{C}X$ for $\mathscr{J}$-algebras X such that if X is a $(\mathscr{C}, \mathscr{J})$-algebra, then $\theta\colon \mathbb{C}X \longrightarrow X$ is a map of $\mathscr{J}$-algebras. The monad $\mathbb{C}$ on the category $\mathscr{V}$ restricts to a monad on the category of $\mathscr{J}$-algebras in $\mathscr{V}$.*

Proof. The distributivity diagram induces the commutative diagram

(9.4)
$$\begin{array}{ccc} \mathscr{J}(k) \times (\mathbb{C}X)^k & \xrightarrow{\ \xi\ } & \mathbb{C}X \\ {\scriptstyle \mathrm{Id}\times\theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ \mathscr{J}(k) \times X^k & \xrightarrow[\ \xi\ ]{} & X \end{array}$$

□

10. Endomorphism operads, moperads, and bioperads

10.1. **The endomorphism operad** $\mathrm{End}(X)$**.** As said before, we think of elements of the components $\mathscr{C}(j)$ of an operad as operations and think of $\gamma(c \otimes d_1 \otimes \cdots \otimes d_k)$ as the composite of the operation c with the tensor product of the operations d_s. Classically, one way of expressing this is in terms of endomorphism operads. We assume that $\mathscr{V}$ has an internal Hom functor, denoted by $\mathscr{V}(-,-)$. In the separate contexts of operads and of moperads, there is only one product $\otimes$ in sight, and it is natural to assume that we have the tensor-hom adjunction

(10.1)
$$\mathscr{V}(X \otimes Y, Z) \cong \mathscr{V}(X, \mathrm{Hom}(Y, Z)).$$

Remark 10.2. However, as we shall exploit later, endomorphism operads make sense more generally. The definition of the operad requires only that we use the same product $\otimes$ in taking powers X^j and in the operations denoted $\otimes$ below. For example, $\otimes$ might in practice be the addition $\oplus$ of a symmetric bimonoidal category $\mathscr{V}$, in which case we would not have the adjunction (10.1).

For $X \in \mathscr{V}$, we set
$$\mathrm{End}(X)(j) = \mathscr{V}(X^j, X).$$
The unit is given by the identity map $X \longrightarrow X$, the right actions by symmetric groups are induced by their left actions on tensor powers, and the maps γ are the composites

(10.3)
$$\begin{array}{c} \mathrm{End}(X)(k) \otimes \mathrm{End}(X)(j_1) \otimes \cdots \otimes \mathrm{End}(X)(j_k) \\ \downarrow{\scriptstyle \mathrm{Id}\otimes(\otimes)} \\ \mathscr{V}(X^k, X) \otimes \mathscr{V}(X^{j_+}, X^k) \\ \downarrow{\scriptstyle \circ} \\ \mathscr{V}(X^{j_+}, X) = \mathrm{End}(X)(j_+). \end{array}$$

Here X^j denotes the j-fold $\otimes$-power of X. However other choices are possible.

Rmod

Example 10.4. Take $\mathscr{V}$ to be modules over a commutative ring, regarded as symmetric monoidal under $\oplus$ rather than $\otimes$. Then applying $\oplus$ to tuples of modules of R-homomorphisms gives a product $\oplus$. Thus in this case we have two endomorphism operads, $\mathrm{End}^{\otimes}(X)$, defined using only $\otimes$, and $\mathrm{End}^{\oplus}(X)$, defined using only $\oplus$. This dichotomy is present for any symmetric bimonoidal category $\mathscr{V}$.

Given the adjunction (10.1), the map

$$\otimes\colon \mathscr{V}(X^{j_1}, X) \otimes \cdots \otimes \mathscr{V}(X^{j_k}, X) \longrightarrow \mathscr{V}(X^{j_+}, X^k)$$

is adjoint to the evident map given by $\otimes$ applied to evaluation maps (composed with a shuffle isomorphism). The following result is then a standard consequence of (10.1) and the definitions.

endoplus

Proposition 10.5. *Assuming (10.1), an action of $\mathscr{C}$ on X can be redefined in adjoint form as a morphism of operads*

$$\mathscr{C} \longrightarrow End(X).$$

Moreover, the evaluation maps $\varepsilon\colon End(X)(j) \otimes X^j \longrightarrow X$ of the adjunction (10.1) specify an action of $End(X)$ on X such that an action of $\mathscr{C}$ on X is the pullback of the action of $End(X)$.

10.2. **The endomorphism moperad $\mathrm{End}^{\times}(X)$.** We also have endomorphism moperads. Letting $\mathscr{V}$ be symmetric monoidal as in Section 2.1 and again letting $\mathscr{V}$ have an internal hom functor $\mathscr{V}(-,-)$, we now also let

$$\mathrm{End}^{\times}(X)(j) = \mathscr{V}(X^j, X).$$

We assume further that X is a unital and augmented object of $\mathscr{V}$, meaning that there are maps $\mathrm{id}\colon v \longrightarrow X$ and $\varepsilon\colon X \longrightarrow v$ such that $\varepsilon \circ \mathrm{id} = \mathrm{id}$. The existence of ε holds trivially when v is a terminal object, as holds if $\mathscr{V}$ is cartesian monoidal.

Define $\iota_r\colon X^j \longrightarrow X^{j+1}$ by inserting $\iota_0\colon v \longrightarrow X$ in the rth coordinate and define

$$\iota\colon X^{j_+} \longrightarrow X^{j_\times}$$

as in Definition 2.7(ii). The augmentation ε applied to the rth coordinate of X^j gives a map $\zeta_r\colon X^{j+1} \longrightarrow X^j$ right inverse to ι_r, and we have an analogous right inverse $\zeta\colon X^{j_\times} \longrightarrow X^{j_+}$ to ι.

Applied to the domain variable of $\mathscr{V}(-,-)$, the ζ_r induce maps

$$\iota_r\colon \mathrm{End}^{\times}(X)(k-1) \longrightarrow \mathrm{End}^{\times}(X)(k)$$

that turn $\mathrm{End}^{\times}$ into a covariant functor $\Lambda_{>0} \longrightarrow \mathscr{V}$. Similarly, we use ζ in the domain variable to define the maps $\gamma^{\times}$ as the composites mopgamma

$$
\begin{array}{c}
\mathrm{End}^{\times}(X)(k) \otimes \mathrm{End}^{\times}(X)(j_1) \otimes \cdot \otimes \mathrm{End}^{\times}(X)(j_k) \\
\big\downarrow {\scriptstyle \mathrm{Id} \otimes (\otimes)} \\
\mathscr{V}(X^k, X) \otimes \mathscr{V}(X^{j_+}, X^k) \\
\big\downarrow {\scriptstyle \mathrm{Id} \otimes \mathscr{V}(\zeta, \mathrm{Id})} \\
\mathscr{V}(X^k, X) \otimes \mathscr{V}(X^{j_\times}, X^k) \\
\big\downarrow {\scriptstyle \circ} \\
\mathscr{V}(X^{j_\times}, X) = \mathrm{End}^{\times}(X)(j_\times)
\end{array}
\tag{10.6}
$$

Inspection of definitions shows that these specifications give a moperad $\mathrm{End}^{\times}(X)$, and we have the following analog of Proposition 10.5.

endotimes

Proposition 10.7. *Assuming (10.1), an action of a moperad $\mathscr{C}^{\times}$ on X can be redefined as a morphism of moperads*

$$\mathscr{C}^{\times} \longrightarrow \mathit{End}^{\times}(X).$$

Moreover, the evaluation maps $\varepsilon\colon \mathit{End}^{\times}(X)(j) \otimes X^j \longrightarrow X$ of (10.1) specify an action of $\mathit{End}^{\times}(X)$ on X such that an action of $\mathscr{C}^{\times}$ on X is the pullback of the action of $\mathit{End}(X)$.

Proof. With $\mathrm{End}^{\times}(X)(k)$ playing the role of $\mathscr{C}^{\times}(k)$ and ε playing the role of $\theta^{\times}(X)$, direct inspection shows that the unit, associativity, and equivariance diagrams of Definition 2.7 commute. The verification of the associativity diagram may be illuminating. □

We can compare γ for the endomorphism operad $\mathrm{End}(X)$ with $\gamma^{\times}$ for the endomorphism moperad $\mathrm{End}^{\times}(X)$, where both are defined using only the product $\otimes$ of $\mathscr{V}$, both for $\otimes$-powers X^j and for the structure maps as in (10.3) and (10.6).

Proposition 10.8. *The endomorphism operad $\mathit{End}(X)$ is a restriction of the endomorphism moperad $\mathit{End}^{\times}(X)$, and the moperad $\mathit{End}^{\times}(X)$ is an extension of the operad $\mathit{End}(X)$.*

Proof. Defining $\zeta^* = \mathscr{V}(\zeta, \mathrm{Id})$, we see that the following diagram commutes.

$$
\begin{array}{ccc}
\mathscr{V}(X^k, X) \otimes \mathscr{V}(X^{j_+}, X^k) & \xrightarrow{\ \circ\ } & \mathscr{V}(X^{j_+}, X) \\
{\scriptstyle \mathrm{Id} \otimes \zeta^*} \big\downarrow & & \big\downarrow {\scriptstyle \zeta^*} \\
\mathscr{V}(X^k, X) \otimes \mathscr{V}(X^{j_\times}, X^k) & \xrightarrow[\ \circ\ ]{} & \mathscr{V}(X^{j_\times}, X)
\end{array}
$$

Comparing with (10.3) and (10.6), we see that the map $\gamma^{\times}$ is the composite $\zeta^*\gamma$, hence $\gamma = \mathscr{V}(\iota, \mathrm{Id})\gamma^{\times}$. Chasing diagrams, we find compatibility diagrams relating the operad $\mathrm{End}(X)$ and the moperad $\mathrm{End}^{\times}(X)$. Given either, we can reconstruct the other from it. □

The intuition is that we are looking at the same multiplicative structure in two ways. This applies, for example, when we start with the multiplication $\otimes$ of a symmetric bimonoidal category $\mathscr{V}$. Bringing in the addition $\oplus$, the picture changes.

10.3. **The endomorphism moperad pairs $(\mathbf{End}(X), \mathbf{End}^{\times}(X))$.** Here we work in a symmetric bimonoidal category $\mathscr{V}$ with operations $\oplus$ with unit object u, now renamed 0, and $\otimes$ with unit object v, now renamed 1. Example 10.4 gives a typical algebraic example. Other examples are sets, categories, simplicial sets, and spaces, with disjoint union as the sum and cartesian product as the product. Note for contrast that, with finitely many variables as in our structures here, cartesian products appear as direct sums in the category of R-modules. In some algebraic situations we can use that the tensor product is the categorical product in a category of cocommutative coalgebras [GMR].

We assume that $\otimes$ is the categorical product, so we use the notation $\times$. We write ${}^kX$ for the sum $\oplus$ of k copies of X and write $X^{\times k}$ for the product $\times$ of k copies of X. We write $\iota\colon 1 \longrightarrow X$ and $\zeta\colon X \longrightarrow 1$ for the unique maps and have $\zeta \circ \iota = \mathrm{id}$. We also assume that X has a zero base object $0\colon 0 \longrightarrow X$. We have an iterated diagonal map $\Delta^k\colon X \longrightarrow X^{\times k}$ for each k.

We define the endomorphism operad $\mathrm{End}(X)$ using only $\oplus$. Notice that use of $\oplus$ destroys the original $(\otimes, Hom)$ adjunction motivation of (10.1). That has the effect that an action of a moperad pair will not be a map of moperad pairs into an endomorphism moperad pair. We define the endomorphism moperad $\mathrm{End}^{\times}(X)$ using only $\times$. To form a moperad pair we need an action λ of $\mathrm{End}^{\times}(X)$ on $\mathrm{End}(X)$. We start from the natural distributivity isomorphism

$$ {}^{j_1}X \ \times \cdots \times \ {}^{j_k}X \cong \ {}^{j_\times}(X^{\times k}) $$

and exploit the diagonal map $\Delta\colon X \longrightarrow X^{\times k}$.

Definition 10.9. Define λ to be the following composite:

$$
\begin{array}{c}
\mathrm{End}^{\times}(X)(k) \times \mathrm{End}(X)(j_1) \times \cdots \times \mathrm{End}(X)(j_k) \\
\Big\downarrow {\scriptstyle \mathrm{Id} \times (\times)} \\
\mathscr{V}(X^{\times k}, X) \times \mathscr{V}(\ {}^{j_\times}(X^{\times k}), X^{\times k}) \\
\Big\downarrow {\scriptstyle \mathscr{V}(\ {}^{j_\times}(\Delta^k), \mathrm{Id})} \\
\mathscr{V}(X^{\times k}, X) \times \mathscr{V}(\ {}^{j_\times}X, X^{\times k}) \\
\Big\downarrow {\scriptstyle \circ} \\
\mathscr{V}(\ {}^{j_\times}X, X) = \mathrm{End}(X)(j_\times).
\end{array}
$$

Proposition 10.10. *With this definition, $(End(X), End^{\times}(X))$ is a moperad pair.*

Proof. We must check the properties specified in Definition 7.1. Here (i) is an easy verification and (ii) and (iii) are just interpretations; (iv) and (v) are tedious diagram chases. We check (v) to give the idea. For any j, we can identify $\mathscr{V}({}^jX, X)$ with the product of j copies of $\mathscr{V}(X, X)$ □

11. Appendix: alternative associativity conditions for moperads

The associativity conditions (ii) in Definitions 2.4 and 2.7 admit alternatives that were found by Nico Marin Gamboa before the current conditions had been found. We explain those alternatives here.

Remark 11.1. We can replace the associativity diagram of Definition 2.4 with the following diagram. Fix a sequence $J_k = (j_1, \cdots, j_k)$; write

$$J_{k-1} = (j_1, \cdots, j_{k-1}) \text{ and } \mathscr{C}^\times[J_{k-1}] = \mathscr{C}^\times(j_1) \times \cdots \times \mathscr{C}^\times(j_{k-1}).$$

For $1 \le r \le k$, define $J_{k,r} = (j_1, \cdots, j_{r-1}, 1, j_r, \cdots j_{k-1})$, define $\mathscr{C}^\times[J_{k,r}]$ by inserting $\mathscr{C}^\times(1)$ into $\mathscr{C}^\times[J_{k-1}]$ in the rth slot, and define $id_r\colon \mathscr{C}^\times[J_{k-1}] \longrightarrow \mathscr{C}[J_{k,r}]$ by inserting the unit map $\text{id}\colon u \longrightarrow \mathscr{C}^\times(1)$ in the rth slot.

$$\begin{array}{ccc}
\mathscr{C}^\times(k-1) \times \mathscr{C}^\times[J_{k-1}] & \xrightarrow{\gamma^\times} & \mathscr{C}^\times(j_\times) \\
\Big\downarrow{\scriptstyle \iota_r \times \text{id}_r} & & \Big\downarrow{\scriptstyle =} \\
\mathscr{C}^\times(k) \times \mathscr{C}^\times[J_{k,r}] & \xrightarrow[\gamma^\times]{} & \mathscr{C}^\times(j_\times)
\end{array}$$

Fix another sequence $I_{j_\times} = (i_1, \cdots, i_{j_\times})$ and break it into $(j_1 \cdots j_{k-1})$-blocks by setting

$$\mathscr{C}^\times[I_{j_\times}] = \mathscr{C}^\times(i_1) \times \cdots \times \mathscr{C}^\times(i_{j_\times}) \text{ and } \mathscr{C}^\times[I_{k,q}] = \mathscr{C}^\times(i_{(q-1)j_k+1}) \times \cdots \times \mathscr{C}^\times(i_{qj_k})$$

for $1 \le q \le i_1 \cdots i_{k-1}$. Also set $h_q = i_{(q-1)j_k+1} \cdots i_{qj_k}$.

$$\begin{array}{ccc}
\mathscr{C}^\times(k) \times \mathscr{C}^\times[J_{k-1}] \times \mathscr{C}^\times(j_k) \times \mathscr{C}^\times[I_{j_\times}] & \xrightarrow{\gamma^\times \times \text{Id}} & \mathscr{C}^\times(j_\times) \times \mathscr{C}^\times[I_{j_\times}] \\
\Big\downarrow{\scriptstyle \text{Id} \times \text{id}_r \times \Delta \times \text{Id}} & & \Big\downarrow{\scriptstyle \gamma^\times} \\
\mathscr{C}^\times(k) \times \mathscr{C}^\times[J_{k,r}] \times \mathscr{C}^\times(j_k)^{j_1 \cdots j_{k-1}} \times \mathscr{C}^\times[I_{j_\times}] & & \mathscr{C}(i_\times) \\
\Big\downarrow{\scriptstyle \gamma^\times \times \text{shuffle}} & & \Big\uparrow{\scriptstyle \gamma^\times} \\
\mathscr{C}^\times(j_1 \cdots j_{k-1}) \times \times_{q=1}^{j_1 \cdots j_{k-1}} \big(\mathscr{C}^\times(j_k) \times \mathscr{C}^\times[I_{k,q}]\big) & \xrightarrow[\text{Id} \times \times_q \gamma^\times]{} & \mathscr{C}^\times(j_1 \cdots j_{k-1}) \times \times_q \mathscr{C}^\times(h_q)
\end{array}$$

We call this last diagram the Marin Gamboa diagram.[11] Using the equivariance, we can give slots s, $1 \le s \le k-1$, the privileged role here given to $s = k$.

P: Permutations? Check whether this definition is consistent, and correct with added permutations if necessary.

Remark 11.2. We can replace the associativity diagram of Definition 2.7 with the following diagram; here we use notations from above and abbreviate notation by letting $h = j_1 \cdots j_{k-1}$.

$$\begin{array}{ccc}
\mathscr{C}^\times(k) \otimes \mathscr{C}^\times(j_1) \otimes \cdots \otimes \mathscr{C}^\times(j_k) \otimes X^{j_\times} & \xrightarrow{\gamma^\times \otimes \text{Id}} & \mathscr{C}^\times(j_\times) \otimes X^{j_\times} \\
\Big\downarrow{\scriptstyle \text{Id} \otimes \text{id}_r \otimes \Delta \otimes \text{Id}} & & \Big\downarrow{\scriptstyle \theta^\times} \\
\mathscr{C}^\times(k) \otimes \mathscr{C}^\times[J_{k,r}] \otimes \mathscr{C}^\times(j_k)^h \otimes X^{j_\times} & & X \\
\Big\downarrow{\scriptstyle \gamma^\times \otimes \ \text{shuffle}} & & \Big\uparrow{\scriptstyle \theta^\times} \\
\mathscr{C}^\times(h) \otimes (\mathscr{C}^\times(j_k) \otimes X^{j_k})^h & \xrightarrow[\text{Id} \otimes (\theta^\times)^h]{} & \mathscr{C}^\times(h) \otimes X^h
\end{array}$$

[11]This ingeniuous diagram is due to Nico Marin Gamboa, author of this appendix.

Remark 11.3. This associativity diagram can be viewed as a parametrization of the equality ass2

$$x_{1,1}\cdots x_{1,j_1}\cdots x_{k,1}\cdots x_{k,j_k} = x_{1,1}\cdots x_{1,j_1}\cdots x_{k-1,1}\cdots x_{k-1,j_{k-1}}(x_{k,1}\cdots x_{k,j_k})$$

for elements $(x_{r,1}\cdots x_{r,j_r})$ for $1 \le r \le k$ in a monoid. We see the product on the left by going right, and down from the top left vertex and see the product on the right by going down, then right, and then up. This gives a start to comparison of this diagram with that of Definition 2.7. By downwards induction on k, we reach a parametrization identical to that of Remark 2.8.

Spell out the trivial case $k = 1$ first. Spell out inductive conclusion. Check for consistency with permutations. Then check the precise relationship with the associativity diagrams of Definitions 2.4 and 2.7.

12. Appendix: towards a multicategorical generalization (old notes)

Curiously, it might be easier and clearer to first define a "parametrized multicategory" and then specialize to define algebras over bioperads and, more generally, over moperad pairs. Let $\mathscr{C} = \mathscr{C}^{bi}$ be a bioperad in a cartesian monoidal category $\mathscr{V}$, writing $\mathscr{C}^{\times}$ when its multiplicative structure is relevant. We define the multicategory of $\mathscr{C}$-algebras. Its objects are the (additive) $\mathscr{C}$-algebras X in $\mathscr{V}$. Its k-morphisms for $k \ge 1$ are the maps in $\mathscr{V}$ (intuitively, the "maps of $\mathscr{C}$-algebras")

$$\xi\colon \mathscr{C}^{\times}(k) \times X_1 \times \cdots \times X_k \longrightarrow Y$$

such that ξ takes the value 0 if any $x_r = 0$ and the following distributivity diagram commutes:

$$\begin{array}{ccc}
\mathscr{C}^{\times}(k) \times \big(\times_{r=1}^{k} \mathscr{C}(j_r)\big) \times Y^{j_\times} & \xrightarrow{\lambda\times \mathrm{Id}} & \mathscr{C}(j_\times) \times Y^{j_\times} \\
\uparrow{\scriptstyle \mathrm{Id}\times \xi^{j_\times}} & & \downarrow{\scriptstyle \theta} \\
\mathscr{C}^{\times}(k) \times \big(\times_{r=1}^{k} \mathscr{C}(j_r)\big) \times \big(\times_Q \big(\mathscr{C}^{\times}(k) \times X_1 \times \cdots \times X_k\big)\big) & & Y \\
\uparrow{\scriptstyle \text{shuffle}} & & \downarrow{\scriptstyle =} \\
\mathscr{C}^{\times}(k)^{j_\times+1} \times \big(\times_{r=1}^{k} \mathscr{C}(j_r) \times X_r^{j_\times}\big) & & Y \\
\uparrow{\scriptstyle \Delta\times(\times_r \mathrm{Id}\times\Delta)} & & \uparrow{\scriptstyle \xi} \\
\mathscr{C}^{\times}(k) \times \big(\times_{r=1}^{k} \mathscr{C}(j_r) \times X_r^{j_r}\big) & \xrightarrow[\mathrm{Id}\times\theta^k]{} & \mathscr{C}^{\times}(k) \times X_1 \times \cdots \times X_k
\end{array}$$

The lower two left vertical maps build in the appropriate generalization

$$X_1^{j_1} \times \cdots \times X_k^{j_k} \longrightarrow \times_Q X_1 \times \cdots \times X_k$$

of the lexicographic map $X^{j_+} \longrightarrow \times_Q X^k$.

Composites require more thought. Suppose given

$$\xi_r\colon \mathscr{C}^{\times}(j_r) \times X_{r,1} \times \cdots \times X_{r,j_r} \longrightarrow Y_r, \quad 1 \le r \le k,$$

and

$$\xi\colon \mathscr{C}^{\times}(k) \times Y_1 \times \cdots \times Y_k \longrightarrow Z$$

We build definitions (of $\mathscr{C}$-multicategories say) so that the following diagram makes sense:

$$\begin{array}{ccc}
\mathscr{C}^{\times}(k) \times \times_{r=1}^{k}\big(\mathscr{C}(j_r) \times \times_{s=1}^{j_r} X_{r,s}\big) & \xrightarrow{\mathrm{Id}\times\times_r \xi_r} & \mathscr{C}^{\times}(k) \times \times_{r=1}^{k} Y_r \\
\Big\downarrow{\scriptstyle \mathrm{Id}\times\times_r \mathrm{Id}\times\Delta} & & \Big\downarrow{\scriptstyle \xi} \\
\mathscr{C}^{\times}(k) \times \times_{r=1}^{k}\big(\mathscr{C}(j_r) \times \times_{s=1}^{j_r} X_{r,s}^{n(r)}\big) & & Z \\
\Big\downarrow{\scriptstyle \mathrm{shuffle}} & & \Big\uparrow{\scriptstyle \xi} \\
\mathscr{C}^{\times}(k) \times \times_{r=1}^{k}\mathscr{C}^{\times}(j_r) \times \times_Q X_{1,q_1} \times \cdots \times X_{k,q_k} & \xrightarrow[\gamma^{\times}\times\mathrm{Id}]{} & \mathscr{C}^{\times}(j_{\times}) \times \times_Q X_{1,q_1} \times \cdots X_{k,q_k}
\end{array}$$

Then its right upward arrow ξ wants to define composition. It seems possible via the Gamboa diagrams or something that that arrow might somehow be determined by the given ξ_r. Then need to check associativity, etc

References

[Bec69] Jon Beck. Distributive laws. In *Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*, pages 119–140. Springer, Berlin, 1969.

[BG20] Andreas Blass and Yuri Gurevich. Braided distributivity. *Theoret. Comput. Sci.*, 807:73–94, 2020.

[GM11] Bertrand Guillou and J Peter May. Models of g-spectra as presheaves of spectra. *arXiv preprint arXiv:1110.3571*, 2011.

[GM17] B. J. Guillou and J. P. May. Equivariant iterated loop space theory and permutative G–categories. *Algebr. Geom. Topol.*, 17(6):3259–3339, 2017.

[GMR] B. J. Guillou, J. P. May, and J. Rubin. Enriched model categories in equivariant contexts. To appear in Algebraic Geometry and Topology.

[GS16] Richard Garner and Michael Shulman. Enriched categories as a free cocompletion. *Adv. Math.*, 289:1–94, 2016.

[JY] Niles Johnson and Donald Yao. Bimonoidal categories, e_n-monoidal categories, and algebraic k-theory. Available at https://nilesjohnson.net/En-monoidal.html.

[KMZ24] Hana Jia Kong, J. Peter May, and Foling Zou. Group completions and the homotopical monadicity theorem. 2024.

[Lap72] Miguel L. Laplaza. Coherence for distributivity. In *Coherence in categories*, pages 29–65. Lecture Notes in Math., Vol. 281. Springer, Berlin, 1972.

[May] J. P. May. Equivariant multiplicative infinite loop space theory. In preparation.

[May72] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin, 1972. Lectures Notes in Mathematics, Vol. 271.

[May74] J. P. May. E_∞ spaces, group completions, and permutative categories. In *New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972)*, pages 61–93. London Math. Soc. Lecture Note Ser., No. 11. Cambridge Univ. Press, London, 1974.

[May77] J. Peter May. *E_∞ ring spaces and E_∞ ring spectra*. Lecture Notes in Mathematics, Vol. 577. Springer-Verlag, Berlin, 1977. With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave.

[May82] J. P. May. Multiplicative infinite loop space theory. *J. Pure Appl. Algebra*, 26(1):1–69, 1982.

[May97] J. P. May. Definitions: operads, algebras and modules. In *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995)*, volume 202 of *Contemp. Math.*, pages 1–7. Amer. Math. Soc., Providence, RI, 1997.

[May09] J. P. May. The construction of E_∞ ring spaces from bipermutative categories. In *New topological contexts for Galois theory and algebraic geometry (BIRS 2008)*, volume 16 of *Geom. Topol. Monogr.*, pages 283–330. Geom. Topol. Publ., Coventry, 2009.

[Tho83] R. W. Thomason. The homotopy limit problem. In *Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982)*, volume 19 of *Contemp. Math.*, pages 407–419. Amer. Math. Soc., Providence, RI, 1983.

Department of Mathematics, The University of Chicago, Chicago, IL 60637
Email address: may@math.uchicago.edu