JOURNAL OF ALGEBRA 12, 533–568 (1969)

Matric Massey Products

J. PETER MAY

Department of Mathematics, The University of Chicago, Chicago, Illinois 60637

Communicated by Saunders MacLane

Received October 18, 1968

It has long been recognized that the homologies of differential algebras and of differential modules over differential algebras have not only products but also higher order operations, namely Massey products. These operations have largely been ignored because of the difficulty in computing them and because of their seeming lack of conceptual interest. We shall here introduce and study generalizations of these operations; the new operations will be defined on n-tuples of matrices rather than on n-tuples of elements. Of course, the generalization does nothing to simplify the computations, although the results on spectral sequences in this paper will have this effect. The larger class of operations does, however, have essential conceptual interest. There are a variety of situations in algebraic topology where the geometry naturally gives a notion of decomposability. The new operations are precisely what is required to describe these notions algebraically. For example, if G is a connected topological monoid, then the geometric notion of a decomposable element in the Pontryagin ring $H_*(G)$ is an element of the kernel of the homology suspension $\sigma_* : H_*(G) \rightarrow H_*(BG)$, and in fact $\ker \sigma_*$ is exactly the set of all elements decomposable as matric Massey products. If B is a simply connected space, the dual statement is true; if $\sigma^* : H^*(B) \rightarrow H^*(\Omega B)$ is the cohomology suspension, then $\ker \sigma^*$ is the set of all elements which are decomposable as matric Massey products. Other such situations will be given in [*10*], where the statements above are proven. Moreover, these operations will be used in [*10*] to compute the cohomologies of a wide variety of homogeneous spaces and principal bundles and to develop an algorithm for the computation of the mod 2 cohomology of any simply connected two-stage space. Statements of the results in question may be found in [*9*].

Our program in this paper is as follows. We shall define matric Massey products and prove their naturality in Section 1. We shall prove certain linearity relations satisfied by our operations and study their indeterminacy in Section 2. The main purpose of this section is to show that, at least under reasonable technical assumptions, matric Massey products are respectable

481/12/4-6

homology operations in the sense that their indeterminacies can be explicitly described and are groups, and the operations themselves are cosets of these groups. In Section 3, which overlaps in part with work of Kraines [5], we determine various associativity and commutativity relations satisfied by our operations. These relations, most of which are new even for ordinary Massey products, show that an algebra which happens to be the homology of a differential algebra will generally have a very rich internal structure.

The last section contains a detailed study of the behavior of our operations in spectral sequences. We first obtain a convergence theorem and a generalized Leibnitz formula. These results generalize the usual statement that the spectral sequence of a filtered differential algebra is a spectral sequence of differential algebras. Indeed, ordinary products are subsumed in our theory as 2-tuple Massey products. We then obtain a result which relates higher differentials to matric Massey products in the limit term and show how this result can be used to study the extension problem at the end of spectral sequences. The results of this section will be used in [11] to study the mod 2 cohomology of $B\,\mathrm{Spin}(n)$ and, applied to the spectral sequence of [8], will be a central tool in the study of the cohomology of the Steenrod algebra in [12]. In both applications, the E_2-term of the special sequence studied has the form $E_2^{p,q} = \mathrm{Ext}_A^{p,q}(K, K)$, where A is a connected algebra over a field K. While the results here apply to a much wider class of spectral sequences, they are particularly natural tools for the study of spectral sequences having such an E_2-term. This is so because, as we shall show in [10], we then have that every element of $E_2^{p,*}$ for $p > 1$ is built up via matric Massey products from elements of $E_2^{1,*}$, and the actual computation of the operations on the E_2-level is quite straightforward.

Analogs of the results of Section 4 have recently been proven for the Adams spectral sequence, with matric Toda brackets replacing matric Massey products in the limit term, by Lawrence [6], generalizing a special case due to Moss [13]. It is to be expected that the precise analogs of the remaining results of this paper are also valid for matric Toda brackets in stable homotopy.

1. The Definition of Matric Massey Products

We must first fix notations. Throughout this paper, we shall work with the following data. We suppose given a commutative ring Λ and a collection $\{R_{ij} \mid 0 \leqslant i < j \leqslant n\}$ of differential Z-graded Λ-modules. We denote gradings by superscripts and we assume that the differentials, always denoted by d, have degree $+1$. We write $\otimes$ for $\otimes_\Lambda$, and we suppose given morphisms of differential Λ-modules

$$\mu = \mu_{ijk} : R_{ij} \otimes R_{jk} \to R_{ik}, \qquad 0 \leqslant i < j < k \leqslant n,$$

such that the following diagrams commute:

$$\begin{array}{ccc} R_{ij} \otimes R_{jk} \otimes R_{kl} & \xrightarrow{1\otimes\mu} & R_{ij} \otimes R_{jl} \\ \downarrow{\scriptstyle \mu\otimes 1} & & \downarrow{\scriptstyle \mu} \\ R_{ik} \otimes R_{kl} & \xrightarrow{\mu} & R_{il} . \end{array}$$

We shall denote the pairings μ by juxtaposition of elements.

The collection of systems $R = \{R_{ij}, \mu_{ijk}\}$ form a category $\mathcal{O}\!\ell_n$. If $S = \{S_{ij}, v_{ijk}\}$ is another such system, then a morphism $f : R \to S$ is a collection of morphisms of differential graded Λ-modules $f = f_{ij} : R_{ij} \to S_{ij}$ such that the following diagrams commute:

$$\begin{array}{ccc} R_{ij} \otimes R_{jk} & \xrightarrow{\mu} & R_{ik} \\ \downarrow{\scriptstyle f\otimes f} & & \downarrow{\scriptstyle f} \\ S_{ij} \otimes S_{jk} & \xrightarrow{v} & S_{ij} \end{array}$$

Before defining our operations, we cite some of the most important situations in which they will make sense. In fact, we have the following categories and functors to the category $\mathcal{O}\!\ell_n$:

(1) The category of DGA-algebras: if U is a DGA-algebra, define $R_{ij} = U$ and let $\mu_{ijk} = \mu$ be the product on U.

(2) The category of triples (M, U, N), where U is a DGA-algebra and M is a right, N a left, differential graded U-module: if (M, U, N) is such a triple, define

$$R_{ij} = \begin{cases} M & i = 0, \quad j < n \\ U & 0 < i < j < n \\ N & 0 < i, \quad j = n \\ M \otimes_U N & i = 0, \quad j = n \end{cases}$$

The pairings $U \otimes U \to U$, $M \otimes U \to M$, $U \otimes N \to N$, and $M \otimes N \to M \otimes_U N$ are, respectively, the product in U, the module product in M and in N, and the natural epimorphism that defines $M \otimes_U N$.

(3) The category of topological spaces X: given X, let $U = C^*(X; \Lambda)$ and apply the functor of (1).

(4) The category of topological monoids G: given G, let $U = C_*(G; \Lambda)$ (regraded by nonpositive superscripts) and apply the functor of (1).

(5) The category of fibred products

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array} :$$

let $(M, U, N) = (C^*(X), C^*(B), C^*(Y))$ and apply the functor of (2).

(6) The category of associated bundles $X \times_G Y$: let

$$(M, U, N) = (C_*(X), C_*(G), C_*(Y))$$

and apply the functor of (2).

(7) The category of spaces X and subspaces A_i, $1 \leqslant i \leqslant n$, such that (A_i, A_j) is an excisive couple in X: let $A_{ij} = \bigcup_{k=i+1}^{j} A_k$ and define $R_{ij} = C^*(X, A_{ij}; A)$; the μ_{ijk} are given by the cup product.

(8) The category of coalgebras C: given C, let U be the cobar construction $F(C)$ and apply the functor of (1).

(9) The category of triples (X, C, Y) where C is a coalgebra, X a right and Y a left C-comodule: a functor to the category described in (2) can be constructed by use of the twosided cobar construction defined in [*10*].

Let $R \in \mathcal{C}l_n$. Matric Massey products will be defined on certain n-tuples of matrices $(V_1, ..., V_n)$, where V_i has entries in the homology $H(R_{i-1,i})$, and will take values in the set of matrices with entries in $H(R_{0n})$. The most important situation occurs when V_1 is a row matrix and V_n is a column matrix; then the values will be 1×1 matrices and will be regarded as elements of $H(R_{0n})$. These operations essentially determine those elements of $H(R_{0n})$ which are decomposed by the system R.

In order to make our definition precise, we shall need some preliminary technical notations. These are designed to keep track of gradings and to avoid difficulties with signs. Once these notations are fixed, both gradings and signs will generally take care of themselves in the sequel.

Notations 1.1. (a) *Matrices of integers.* Consider matrices $D = (d_{ij})$ and $D' = (d'_{ij})$ with entries in Z. Let D be a $p \times q$-matrix and D' a $p' \times q'$-matrix. We say that (D, D') is a compatible pair if $q = p'$ and if the sum $d_{ik} + d'_{kj}$ is independent of k for each i and j. We then define the $p \times q'$-matrix $D * D' = (e_{ij})$ by $e_{ij} = d_{ik} + d'_{kj}$. Let $D_1, ..., D_n$ be matrices with entries in Z. We say that $(D_1, ..., D_n)$ is a compatible system if each (D_i, D_{i+1}) is a compatible pair. If $D = (d_{ij})$ and if $m \in Z$, then we define $D^m = (d_{ij} + m)$. Let $(D_1, ..., D_n)$ be a compatible system. We then define matrices D_{ij}, $0 \leqslant i < j \leqslant n$, by the formula

$$D_{i,j} = (D_{i+1} * \cdots * D_j)^{-(j-i-1)}, \qquad 0 \leqslant i < j \leqslant n. \tag{1}$$

Inductively, each (D_{ik}, D_{kj}), $i < k < j$, is a compatible pair, and

$$D^1_{ij} = D_{ik} * D_{kj} \quad \text{for each } k \text{ such that} \quad 0 \leqslant i < k < j \leqslant n. \tag{2}$$

(b) *Matrices of degrees.* Let E be a Z-graded Λ-module. Let ME denote the set of matrices with entries in E. If $X = (x_{ij}) \in ME$, we define $D(X) = (\deg x_{ij})$. Let $Y \in MF$ for some Z-graded Λ-module F. We say that (X, Y) is a multipliable pair if $(D(X), D(Y))$ is a compatible pair. If (X, Y) is a multipliable pair and if $E \otimes F \to G$ is a morphism of Λ-modules, then we can multiply X and Y by the usual product of matrices to obtain a matrix XY with entries in G. Clearly $D(XY) = D(X) * D(Y)$. Let $X_1, \ldots, X_n$ be matrices with entries in Z-graded Λ-modules. We say that $(X_1, \ldots, X_n)$ is a multipliable system if $(D(X_1), \ldots, D(X_n))$ is a compatible system, and we then define

$$D(X_1, \ldots, X_n) = D(X_1) * \cdots * D(X_n). \tag{3}$$

Were it possible to multiply the X_i, we would obtain a matrix $X_1 \cdots X_n$ such that $D(X_1 \cdots X_n) = D(X_1, \ldots, X_n)$.

(c) *Signs.* If $x \in E$, we define $\bar{x} = (-1)^{1+\deg x} x$. If $X = (x_{ij}) \in ME$, we define $\bar{X} = (\bar{x}_{ij})$. If E is a differential Λ-molule, we define $dX = (dx_{ij})$. Let $E \otimes F \to G$ be a morphism of differential Λ-modules and let (X, Y) be a multipliable pair, $X \in ME$ and $Y \in MF$. Then we have the relations

$$d\bar{X} = -\overline{dX}; \qquad \overline{XY} = -\bar{X}\,\bar{Y}; \qquad d(XY) = d(X)\,Y - \bar{X}\,d(Y). \tag{4}$$

At this point, we are ready to define the operations. Let $R \in \mathcal{O}\!\ell_n$. Suppose that $(V_1, \ldots, V_n)$ is a multipliable system, where $V_i \in MH(R_{i-1,i})$. Matric Massey products are designed to formalize the notion that relations between relations should result in the definition of higher operations. Thus for the matric Massey product $\langle V_1, \ldots, V_n\rangle$ to be defined, it will be necessary that $0 \in \langle V_i, \ldots, V_j\rangle \subset MH(R_{i-1,j})$, $j - i < n$. Roughly, the operations are built up inductively as follows. Let $A_{i-1,i} \in MR_{i-1,i}$ be a matrix of representative cycles for V_i, abbreviated $\{A_{i-1,i}\} = V_i$. Suppose that $\bar{V}_i V_{i+1} = 0$. Here the product is of course induced from the pairing μ and the homology product $H(R_{i-1,i}) \otimes H(R_{i,i+1}) \to H(R_{i-1,i+1})$. Then there exist $A_{i-1,i+1} \in MR_{i-1,i+1}$ such that $dA_{i-1,i+1} = \bar{A}_{i-1,i}A_{i,i+1}$. But then $\bar{A}_{i-1,i}A_{i,i+2} + \bar{A}_{i-1,i+1}A_{i+1,i+2}$ is a matrix of cycles, in $MR_{i-1,i+2}$, and we say that its homology class belongs to $\langle V_i, V_{i+1}, V_{i+2}\rangle$. We can ask whether these classes are zero or not. Inductively, it makes sense to seek matrices $A_{ij} \in MR_{ij}$ for $0 \leqslant i < j \leqslant n$ and $(i, j) \neq (0, n)$ such that

$$\{A_{i-1,i}\} = V_i\,; \qquad dA_{ij} = \tilde{A}_{ij}\,,$$

where

$$\tilde{A}_{ij} = \sum_{k=i+1}^{j-1} \bar{A}_{ik} A_{kj}, \qquad 1 < j - i < n. \tag{5}$$

Condition (5) is compatible with the grading since $D(A_{ij}) = D_{ij}$ and $D(dA_{ij}) = D_{ik} * D_{kj}$, where the D_{ij} are the matrices of integers defined in (1) in terms of the matrices $D_i = D(V_i)$. Condition (5) is compatible with the formula $d^2 = 0$ in view of (4). If we are given such matrices A_{ij}, then

$$d\tilde{A}_{0n} = 0, \qquad \text{where} \qquad \tilde{A}_{0n} = \sum_{k=1}^{n-1} \bar{A}_{0k} A_{kn}. \tag{6}$$

We shall say that $\{\tilde{A}_{0n}\} \in \langle V_1, ..., V_n \rangle \subset MH(R_{0n})$. Observe that $D(\tilde{A}_{0n}) = D^1_{0n}$ by (2), and we therefore have, by (3),

$$D\langle V_1, ..., V_n \rangle = D(V_1, ..., V_n)^{2-n}. \tag{7}$$

From now on, we shall not mention grading or signs unless absolutely necessary, and we proceed to the formal definition.

Definition 1.2. Let $(V_1, ..., V_n)$ be a multipliable system of matrices, $V_i \in MH(R_{i-1,i})$. We say that the matric Massey product $\langle V_1, ..., V_n \rangle$ is defined if there exist matrices $A_{ij} \in MR_{ij}$, $0 \leqslant i < j \leqslant n$ and $(i, j) \neq (0, n)$, which satisfy (5); such a set of matrices is said to be a defining system for $\langle V_1, ..., V_n \rangle$. If $\{A_{ij}\}$ is a defining system for $\langle V_1, ..., V_n \rangle$, then we say that $\{\tilde{A}_{0n}\} \in \langle V_1, ..., V_n \rangle$, where $\tilde{A}_{0n}$ is given by (6), and $\langle V_1, ..., V_n \rangle$ is defined to be set of all homology classes so obtainable from some defining system. In particular, $\langle V_1, V_2 \rangle$ is always defined and contains only the product $\bar{V}_1 V_2$. We say that $\langle V_1, ..., V_n \rangle$ is strictly defined if each

$$\langle V_i, ..., V_j \rangle, \qquad 1 \leqslant j - i \leqslant n - 2,$$

is defined, in $MH(R_{i-1,j})$, and contains only the zero matrix. In particular, every defined triple product is strictly defined.

Let $\{A_{ij} \mid 0 \leqslant i < j \leqslant n, \ (i, j) \neq (0, n)\}$ be a defining system for $\langle V_1, ..., V_n \rangle$. We may describe the system by the strictly upper triangular block matrix $A = (A_{ij})$ with upper right corner deleted:

$$A = \begin{pmatrix} 0 & A_{01} & A_{02} & \cdots & A_{0,n-1} & * \\ & 0 & A_{12} & \cdots & A_{1,n-1} & A_{1,n} \\ & & 0 & \ddots & \vdots & \vdots \\ & \bigcirc & & & & \\ & & & & 0 & A_{n-1\,n} \\ & & & & & 0 \end{pmatrix}$$

Then $dA = \bar{A}A - \tilde{A}_{0,n}$ in the sense that if $\tilde{A}_{ij} = \sum_{k=i+1}^{j-1} \bar{A}_{ik}A_{kj}$, then $\bar{A}A = (\tilde{A}_{ij})$ and this matrix differs from dA only in the presence of the block $\tilde{A}_{0n}$ in the upper right corner. We shall abbreviate $\tilde{A}_{0n}$ to $\tilde{A}$ and, under the interpretation just given, we can use the simple formula $\bar{A}A - dA = \tilde{A}$ to describe representative matrices for elements of $\langle V_1 ,..., V_n\rangle$.

In many respects, strictly defined matric Massey products provide a more satisfactory generalization of triple products than do arbitrary matric Massey products. In fact, many of our results below will only be valid for strictly defined operations. These results depend on building defining systems $A = (A_{ij})$ by induction on $j - i$, and such arguments may fail without strict definition since one could reach a nonbounding cycle $\tilde{A}_{ij}$ even though $\langle V_1 ,..., V_n\rangle$ is defined. In all such cases, we shall appeal to the following lemma.

LEMMA 1.3. *$\langle V_1 ,..., V_n\rangle$ is strictly defined if and only if each partial defining system $\{A_{pq} \mid q - p \leqslant k\}$, $1 \leqslant k < n - 1$, can be completed to a defining system A for $\langle V_1 ,..., V_n\rangle$.*

Proof. The second condition means that if we are given matrices A_{pq} for $q - p \leqslant k$ (and any k such that $1 \leqslant k < n - 1$) which satisfy (5) and if $j - i = k + 1$, then $\tilde{A}_{ij}$ is a matrix of boundaries. Clearly this is so if and only if each $\langle V_i ,..., V_j\rangle = \{0\}$, $j - i < n$.

It is worth remarking that matric Massey products could be defined for infinite matrices having only finitely many nonzero entries. In this context, if E is a Λ-module, then ME is an infinitely graded Λ-module with one degree for each pair (i, j), $i \geqslant 1$ and $j \geqslant 1$. If F is another Λ-module, then the set of all multipliable pairs (X, Y), $X \in ME$ and $Y \in MF$, is also an infinitely graded Λ-module, with $D(X, Y) = D(X) * D(Y)$. Starting with these observations, the tensor product $ME \otimes_\Lambda MF$ can be constructed as usual and can be proven to have all the standard properties. We shall use this fact in the proof of Theorem 1.5 below.

We shall complete this section by studying the naturality of our operations. For many applications, it is not enough to have naturality on maps in the category $\mathcal{C}l_n$. Ordinary products are clearly preserved by maps of differential Λ-modules that commute with the given pairings up to chain homotopy, and we shall formulate a notion of n-homotopy multiplicativity which will guaranteee the preservation of n-tuple matric Massey products. The following definition is essentially due to Clark [*1*] and parallels geometric work of Stasheff [*14*].

DEFINITION 1.4. Let R and S be objects in $\mathcal{C}l_n$. Let $f = \{f_{ij}\}$, where $f_{ij} : R_{ij} \to S_{ij}$ is a morphism of differential Λ-modules. We say that $f : R \to S$ is an n-homotopy multiplicative map if for each l such that $1 \leqslant l \leqslant n$ and each sequence $I = (i_0, ..., i_l)$, $0 \leqslant i_0 < i_1 < \cdots < i_l \leqslant n$, there exist morphisms of Λ-modules $h_l : R_{i_0 i_1} \otimes \cdots \otimes R_{i_{l-1} i_l} \to S_{i_0 i_l}$, of degree $1 - l$, such that $h_1 = f$ and if $u_k \in R_{i_{k-1} i_k}$, $1 \leqslant k \leqslant l$, then

$$dh_l(u_1 \otimes \cdots \otimes u_l) = \sum_{k=1}^{l} h_l(\bar{u}_1 \otimes \cdots \otimes \bar{u}_{k-1} \otimes du_k \otimes u_{k+1} \otimes \cdots \otimes u_l)$$

$$- \sum_{k=1}^{l-1} h_{l-1}(\bar{u}_1 \otimes \cdots \otimes \bar{u}_{k-1} \otimes \bar{u}_k u_{k+1} \otimes u_{k+2} \otimes \cdots \otimes u_l)$$

$$+ \sum_{k=1}^{l-1} h_k(\bar{u}_1 \otimes \cdots \otimes \bar{u}_k)\, h_{l-k}(u_{k+1} \otimes \cdots \otimes u_l). \tag{7}$$

If f is a morphism in $\mathcal{O}\!\mathcal{l}_n$, we may take $h_l = 0$, $l > 1$. The existence of h_2 says that f is homotopy multiplicative since

$$dh_2 - h_2(d \otimes 1 - 1 \otimes d) = \mu(f \otimes f) - f\mu : R_{ij} \otimes R_{jk} \to S_{ik}.$$

The h_l for $l > 2$ are higher multiplicative homotopies.

THEOREM 1.5. *Let $f : R \to S$ be an n-homotopy multiplicative map. Suppose that $\langle V_1, ..., V_n \rangle$ is defined in $MH(R_{0n})$, where $V_i \in MH(R_{i-1,i})$. Then $\langle f_*(V_1), ..., f_*(V_n) \rangle$ is defined in $MH(S_{0n})$, and*

$$f_*(\langle V_1, ..., V_n \rangle) \subset \langle f_*(V_1), ..., f_*(V_n) \rangle.$$

Moreover, if each $f_* : H(R_{ij}) \to H(S_{ij})$ is an isomorphism, then equality holds.

Proof. We shall work with infinite matrices having only finitely many nonzero entries in this proof. Let $h_1 = f$ and let h_l, $2 \leqslant l \leqslant n$, be given homotopies satisfying (7). In view of the remarks preceding Definition 1.4, (7) remains valid if the u_k are replaced by matrices with entries in $Ri_{k-1}i_k$. To simplify the proof, we introduce the following notation. Suppose that

$$A^{ij} = \{A_{kl} \mid i \leqslant k < l \leqslant j, (k, l) \neq (i, j)\}$$

is a defining system for $\langle V_{i+1}, ..., V_j \rangle$. Then define

$$H(A^{ij}) = \sum_{m=2}^{j-i} \sum_{I} h_m(A_{i_0 i_1} \otimes \cdots \otimes A_{i_{m-1} i_m}),$$

where the second sum is taken over all sequences $I = (i_0, ..., i_m)$ such that $i = i_0 < \cdots < i_m = j$. If A_{ij} such that $dA_{ij} = \tilde{A}_{ij}$ is also given, define

$F(A^{ij}) = f(A_{ij}) + H(A^{ij})$, and define $F(A^{k-1,k}) = f(A_{k-1,k})$. Then a straightforward computation proves that

$$\text{(i)} \qquad dH(A^{ij}) = \sum_{k=i+1}^{j-1} \bar{F}(A^{ik})F(A^{kj}) - f(\tilde{A}_{ij}),$$

and given A_{ij}, it follows (since $df = fd$) that

$$\text{(ii)} \qquad dF(A^{ij}) = \sum_{k=i+1}^{j-1} \bar{F}(A^{ik})F(A^{kj}).$$

Now if $A = (A_{ij})$ is a defining system for $\langle V_1, ..., V_n \rangle$, then, by (ii), $B = (F(A_{ij}))$ is a defining system for $\langle f_*(V_1), ..., f_*(V_n) \rangle$. By (i), with $(i, j) = (0, n)$, $\tilde{B}$ is homologous to $f(\tilde{A})$, and this proves that

$$f_*(\langle V_1, ..., V_n \rangle) \subset \langle f_*(V_1), ..., f_*(V_n) \rangle.$$

For the opposite inclusion, assume that each $f_* : H(R_{ij}) \to H(S_{ij})$ is an isomorphism and let B be a given defining system for $\langle f_*(V_1), ..., f_*(V_n) \rangle$. By induction on $j - i$, we shall choose a defining system $A = (A_{ij})$ for $\langle V_1, ..., V_n \rangle$ such that $f(\tilde{A})$ is homologous to $\tilde{B}$. Let $A_{i-1,i}$ be any matrix of representative cycles for V_i. Since $f(A_{i-1,i})$ and $B_{i-1,i}$ both represent $f_*(V_i)$, we may choose matrices $C_{i-1,i} \in MS_{i-1,i}$ such that

$$dC_{i-1,i} = f(A_{i-1,i}) - B_{i-1,i}.$$

Now, for any q such that $1 < q \leqslant n - 2$, assume inductively that matrices $A_{kl} \in MR_{kl}$ and $C_{kl} \in MS_{kl}$ have been found, for each pair (k, l) such that $1 < l - k < q$, which satisfy

(iii)

$$dA_{kl} = \tilde{A}_{kl} \quad \text{and} \quad dC_{kl} = F(A^{kl}) - B_{kl} + \sum_{m=k+1}^{l-1} (\bar{C}_{km}F(A^{ml}) + B_{km}C_{ml}).$$

Let $j - i = q$. Then an easy calculation, using (ii), shows that

$$\text{(iv)} \qquad d\left(\sum_{k=i+1}^{j-1} \bar{C}_{ik}F(A^{kj}) + B_{ik}C_{kj} \right) = \tilde{B}_{ij} - \sum_{k=i+1}^{j-1} \bar{F}(A^{ik})F(A^{kj}).$$

Since $\tilde{B}_{ij}$ is a matrix of boundaries, so is $\sum_{k=i+1}^{j-1} \bar{F}(A^{ik})F(A^{kj})$. By (i), it follows that $f(\tilde{A}_{ij})$ and therefore $\tilde{A}_{ij}$ are also matrices of boundaries. Choose $A'_{ij} \in MR_{ij}$ such that $dA'_{ij} = \tilde{A}_{ij}$. Let

$$E_{ij} = f(A'_{ij}) + H(A^{ij}) - B_{ij} + \sum_{k=i+1}^{j-1} (\bar{C}_{ik}F(A^{kj}) + B_{ik}C_{kj}).$$

Then $E_{ij} \in MS_{ij}$ is a matrix of cycles. Choose matrices of cycles $D_{ij} \in MR_{ij}$ such that $f(D_{ij})$ is homologous to E_{ij} and choose $C_{ij} \in MS_{ij}$ such that $dC_{ij} = E_{ij} - f(D_{ij})$. Let $A_{ij} = A'_{ij} - D_{ij}$. Then A_{ij} and C_{ij} satisfy (iii), with (k, l) replaced by (i, j). By induction, there exist such A_{ij} and C_{ij} for all pairs (i, j), $1 < j - 1 < n$. Now formulas (i) and (iv), with (i, j) replaced by $(0, n)$, show that $f(\tilde{A})$ and $\tilde{B}$ are homologous, and this completes the proof.

The full generality of the theorem above will be used elsewhere to obtain a generalized Cartan formula relating Steenrod operations to matric Massey products. It will also be needed in [*10*] to study matric Massey products in the Eilenberg–Moore spectral sequence.

2. Indeterminacy and Linearity Relations

In this section, we shall study the indeterminacy of matric Massey products and shall establish certain linearity relations satisfied by these operations. We require the following terminology.

Definition 2.1. By the sum $\langle V_1, \dots, V_n \rangle + \langle W_1, \dots, W_n \rangle$ of two matric Massey products, we shall mean the set

$$\{x + y \mid x \in \langle V_1, \dots, V_n \rangle \text{ and } y \in \langle W_1, \dots, W_n \rangle\}.$$

The use of this notation will imply the extra hypothesis $D(V_1, \dots, V_n) = D(W_1, \dots, W_n)$ needed for the stated sums to be compatible with the grading. If $\lambda \in A$, we define $\lambda\langle V_1, \dots, V_n \rangle$ to be the set $\{\lambda x \mid x \in \langle V_1, \dots, V_n \rangle\}$. With these notations, we can define the indeterminacy $\mathrm{In}\langle V_1, \dots, V_n \rangle$ by the formula

$$\begin{aligned} \mathrm{In}\langle V_1, \dots, V_n \rangle &= \langle V_1, \dots, V_n \rangle - \langle V_1, \dots, V_n \rangle \\ &= \{x - y \mid x, y \in \langle V_1, \dots, V_n \rangle\}. \end{aligned}$$

The following lemma will be needed in the study of the indeterminacy of our operations. It shows that the set $\langle V_1, \dots, V_n \rangle$ is independent of the choice of representative cycles for the V_i.

Lemma 2.2. *If $\langle V_1, \dots, V_n \rangle$ is defined, then the entire set $\langle V_1, \dots, V_n \rangle$ can be obtained from defining systems A which start with any fixed chosen set of matrices $A_{i-1,i}$ of representative cycles for the V_i.*

Proof. Let B be any defining system for $\langle V_1, \dots, V_n \rangle$ and choose matrices

$C_{i-1,i}$ such that $dC_{i-1,i} = A_{i-1,i} - B_{i-1,i}$. Define A_{ij} for $1 < j - i$, by induction on $j - i$, via the formula

$$A_{ij} = B_{ij} - \bar{C}_{i,i+1}A_{i+1,j} - B_{i,j-1}C_{j-1,j}.$$

Then A is a defining system for $\langle V_1, ..., V_n \rangle$ such that

$$d(\bar{C}_{01}A_{1n} + B_{0,n-1}C_{n-1,n}) = \tilde{B} - \tilde{A}.$$

Using the lemma, we can show that any element of the indeterminacy of an n-tuple matric Massey product is an element of an appropriate $(n - 1)$-tuple matric Massey product. Actually, as the form of the matrices W_k defined in the next proposition will show, we must first slightly generalize the notion of matric Massey product to allow symbols $\langle V_1, ..., V_n \rangle$ where the V_i are block matrices with blocks having entries in various $H(R_{ij})$ and where the blocks are so arranged that the pairings μ_{ijk} allow the formation of the products $\bar{V}_i V_{i+1}$. Since the principle should be clear and precise formulation is awkward, we do not give the explicit definition. For application to all examples, except (7), of Section 1, the generalization is in fact unnecessary. The following result is due to D. Kraines (unpublished). It generalizes the well-known fact that the indeterminacy of an ordinary triple product is the set of all sums $\bar{v}_1 x_2 + \bar{x}_1 v_3 = (\bar{v}_1, \bar{x}_1)\binom{x_2}{v_3}$ which have the same degree as $\langle v_1, v_2, v_3 \rangle$.

PROPOSITION 2.3. *Let $\langle V_1, ..., V_n \rangle$ be defined in $MH(R_{0n})$, $V_i \in MH(R_{i-1,i})$. Then* $\mathrm{In}\langle V_1, ..., V_n \rangle \subset \bigcup_{(X_1, ..., X_{n-1})} \langle W_1, ..., W_{n-1} \rangle$, *where*

$$W_1 = (V_1, X_1), \qquad W_k = \begin{pmatrix} V_k & X_k \\ 0 & V_{k+1} \end{pmatrix} \quad \text{if} \quad 2 \leqslant k \leqslant n - 2,$$

$$W_{n-1} = \begin{pmatrix} X_{n-1} \\ V_n \end{pmatrix};$$

the union is taken over all $(n - 1)$-tuples of matrices $(X_1, ..., X_{n-1})$ such that $X_k \in MH(R_{k-1,k+1})$ and $D(X_k) = D(V_k, V_{k+1})^{-1}$. Moreover, if $n = 3$, then equality holds:

$$\mathrm{In}\langle V_1, V_2, V_3 \rangle = \bigcup_{(X_1, X_2)} \langle W_1, W_2 \rangle = \bigcup_{(X_1, X_2)} (\bar{V}_1 X_2 + \bar{X}_1 V_3).$$

Proof. Let $A = (A_{ij})$ and $A' = (A'_{ij})$ be any two defining systems for $\langle V_1, ..., V_n \rangle$. By Lemma 2.2, we may assume that $A_{i-1,i} = A'_{i-1,i}$ for all i. Define matrices $B_{ij} = A_{i,j+1} - A'_{i,j+1} \in MR_{i,j+1}$, $0 \leqslant i < j \leqslant n - 1$, $(i, j) \neq (0, n - 1)$. Each $B_{k-1,k}$ is a matrix of cycles, and we define

$X_k = \{B_{k-1,k}\} \in MH(R_{k-1,k+1})$. If $n = 3$, then $\tilde{A} - \tilde{A}' = \bar{A}_{01}B_{12} + \bar{B}_{01}A_{12}$, which represents $\langle W_1, W_2 \rangle$, and it is clear that by adding matrices of cycles to A_{01} and to A_{13} we can obtain any element of the form $\bar{V}_1 X_2 + \bar{X}_1 V_3$ as an element of $\text{In}\langle V_1, V_2, V_3 \rangle$. If $n > 3$, we define C_{ij}, $0 \leqslant i < j \leqslant n-1$, $(i,j) \neq (0, n-1)$, by

$$C_{ij} = \begin{cases} (A_{0j}, B_{0j}) & \text{if } i = 0 \\ \begin{pmatrix} A_{ij} & B_{ij} \\ 0 & A'_{i+1,j+1} \end{pmatrix} & \text{if } 1 \leqslant i < j \leqslant n-2 \\ \begin{pmatrix} B_{i,n-1} \\ A'_{i+1,n} \end{pmatrix} & \text{if } j = n-1 \end{cases}$$

Then it is easy to verify that $C = (C_{ij})$ is a defining system for $\langle W_1, ..., W_{n-1} \rangle$ such that $\tilde{C} = \tilde{A} - \tilde{A}'$.

In view of the fact that $\langle V_1, V_2 \rangle = \bar{V}_1 V_2$, an alternative way of writing the indeterminacy of a triple product is

$$\text{In}\langle V_1, V_2, V_3 \rangle = \bigcup_{(X_1, X_2)} \langle X_1, V_3 \rangle + \langle V_1, X_2 \rangle.$$

We shall generalize this formula to certain strictly defined n-tuple matric Massey products. It will be most expeditious to first obtain upper and lower bounds for the indeterminacy of arbitrary strictly defined operations.

PROPOSITION 2.4. *Let $\langle V_1, ..., V_n \rangle$ be strictly defined.*

(i) *If $X_k \in MH(R_{k-1,k+1})$ and $D(X_k) = D(V_k, V_{k+1})^{-1}$, $1 \leqslant k \leqslant n-1$, then $\langle V_1, ..., V_{k-1}, X_k, V_{k+2}, ..., V_n \rangle$ is strictly defined and is contained in* $\text{In}\langle V_1, ..., V_n \rangle$.

(ii) $$\text{In}\langle V_1, ..., V_n \rangle \subset \bigcup_{(X_1, ..., X_{n-1})} \sum_{k=1}^{n-1} \langle V_1, ..., V_{k-1}, X_k, V_{k+2}, ..., V_n \rangle.$$

In particular, $\text{In}\langle V_1, ..., V_n \rangle = \{0\}$ *if and only if each*

$$\langle V_1, ..., V_{k-1}, X_k, V_{k+2}, ..., V_n \rangle$$

consists only of the zero matrix. Moreover, if $n = 4$, then equality holds in (ii).

Proof. (i). We proceed by induction on n, the result being obvious if $n = 3$. Assume (i) for $l < n$. Then

$$\langle V_1, ..., V_{k-1}, X_k, V_{k+2}, ..., V_j \rangle \subset \text{In}\langle V_i, ..., V_j \rangle = \{0\}$$

for $i \leqslant k \leqslant j-1$ and $j-i<n-1$, hence $\langle V_1, \ldots, V_{k-1}, X_k, V_{k+2}, \ldots, V_n\rangle$ is strictly defined. Let D be any defining system for this operation. We may write D in the form

$$\text{(a)} \qquad D_{ij}^k = \begin{cases} A_{ij}^k, & j<k \\ B_{ij}^k, & i \leqslant k-1 \leqslant j-1 \\ A_{i+1,j+1}^k, & i \geqslant k. \end{cases}$$

Then the A_{ij}^k form a partial defining system for $\langle V_1, \ldots, V_n\rangle$ which by Lemma 1.3, can be extended to a complete defining system A^k. Define C^k by

$$\text{(b)} \qquad C_{ij}^k = A_{ij}^k \quad \text{if} \quad j \leqslant k \quad \text{or} \quad i \geqslant k;$$

$$C_{ij}^k = A_{ij}^k + B_{i,j-1}^k \quad \text{if} \quad i \leqslant k-1 < j-1.$$

Then C^k is a defining system for $\langle V_1, \ldots, V_n\rangle$ such that $\tilde{C}^k - \tilde{A}^k = \tilde{D}^k$, and this proves statement (i).

(ii). Let A and A' be any two defining systems for $\langle V_1, \ldots, V_n\rangle$. By Lemma 2.2, we may assume that $A_{k-1,k} = A'_{k-1,k}$. Let

$$B_{k-1,k}^k = A'_{k-1,k+1} - A_{k-1,k+1}, \qquad 1 \leqslant k \leqslant n-1;$$

$B_{k-1,k}^k$ is a matrix of cycles which represents an element $X_k \in MH(R_{k-1,k+1})$. Let $A^1 = A$. By induction on k, we shall obtain defining systems A^k for $\langle V_1, \ldots, V_n\rangle$ and matrices B_{ij}^k for $i \leqslant k-1 \leqslant j-1$ and $1 \leqslant k \leqslant n-1$ such that

$$\text{(c)} \qquad A_{ij}^k = A'_{ij} \quad \text{for} \quad j \leqslant k; \qquad B_{ik}^k = A'_{i,k+1} - A_{i,k+1}^k; \qquad \text{and}$$

$$\text{(d)} \quad dB_{ij}^k = \sum_{m=i+1}^{k-1} \bar{A}_{im}^k B_{mj}^k + \sum_{m=k}^{j-1} \bar{B}_{im}^k A_{m+1,j+1}^k, \qquad 1<j-i<n-1.$$

In fact, suppose given A^k. If B_{ik}^k is defined by (c), then (d) is satisfied for $j=k$. The remaining B_{ij}^k, $i \leqslant k-1<j-1$, are obtained by induction on $j-i$ as follows. Condition (d) is precisely what is required for formula (a) to give a defining system D^k for $\langle V_1, \ldots, X_k, \ldots, V_n\rangle$. Since the latter product is strictly defined and since, given B_{pq}^k for $q-p<j-i$, the right side of (d) represents an element of $\langle V_i, \ldots, X_k, \ldots, V_{j+1}\rangle = \{0\}$, we can indeed find the B_{ij}^k. Given the B_{ij}^k, we define C^k by formula (b) and let $A^{k+1} = C^k$. Formulas (b) and (c) then imply that $A_{ij}^{k+1} = A'_{ij}$ for $j \leqslant k+1$. Moreover, we now have that $C^{n-1} = A'$. Putting these facts together, we see that

$$\tilde{A}' - \tilde{A} = \tilde{C}^{n-1} - \tilde{A}^1 = \sum_{k=1}^{n-1} (\tilde{C}^k - \tilde{A}^k) = \sum_{k=1}^{n-1} \tilde{D}^k,$$

and this proves (ii). The fact that equality holds in (ii) when $n = 4$ will follow from a more general result (Theorem 2.8 below).

The preceding result is unsatisfactory in that it fails to show that $\text{In}\langle V_1, ..., V_n\rangle$ is a Λ-module and that $\langle V_1, ..., V_n\rangle$ is a coset. To obtain maximum precision in the statement of hypotheses which guarantee these conclusions, among others, we require the following technical definition and lemma.

DEFINITION 2.5. Let $\langle V_1, ..., V_n\rangle$ be strictly defined, and consider the strictly defined operations $\langle V_1, ..., V_{l-1}, X_l, V_{l+2}, ..., V_n\rangle$, where $X_l \in MH(R_{l-1,l+1})$ and $D(X_l) = D(V_l, V_{l+1})^{-1}$. We say that $\langle V_1, ..., V_n\rangle$ is k-rigidly defined, $1 \leqslant k \leqslant n$, provided that $\langle V_1, ..., X_l, ..., V_n\rangle = \{0\}$ for all X_l if $1 \leqslant l \leqslant k-2$ or $k+1 \leqslant l \leqslant n-1$ and for $X_l = 0$ if $l = k-1$ or $l = k$. This condition is certainly satisfied if $\text{In}\langle V_1, ..., V_n\rangle = \{0\}$, but is more general. For example, since $\langle V_1, 0\rangle = \{0\} = \langle 0, V_3\rangle$, any defined triple product is 2-rigidly defined. Clearly $\langle V_1, ..., V_n\rangle$ is both k and k' rigidly defined, $k+1 < k'$, if and only if $\text{In}\langle V_1, ..., V_n\rangle = \{0\}$.

LEMMA 2.6. *Let $\langle V_1, ..., V_n\rangle$ be k-rigidly defined, and let A_{ij}, $k \leqslant i$ and $j < k$, be any given partial defining system for $\langle V_1, ..., V_n\rangle$. Let $x \in \langle V_1, ..., V_n\rangle$. Then there exists a defining system A for $\langle V_1, ..., V_n\rangle$ which extends the given partial defining system and satisfies $\{\tilde{A}\} = x$.*

Proof. Let A' be a defining system for $\langle V_1, ..., V_n\rangle$ such that $\{\tilde{A}'\} = x$. We may assume that $A'_{i-1,i} = A_{i-1,i}$ if $i < k$ or $i \geqslant k+1$, and we may define

(a) $A_{k-1,k} = A'_{k-1,k}$; $\quad A_{k-1,k+1} = A'_{k-1,k+1}$; $\quad A_{k-2,k} = A'_{k-2,k}$.

Let $B^l_{l-1,l} = A'_{l-1,l+1} - A_{l-1,l+1}$, $1 \leqslant l \leqslant n-1$. Then $B^l_{l-1,l}$ is a matrix of cycles, and we let $X_l = \{B^l_{l-1,l}\}$. Clearly $X_{k-1} = 0$ and $X_k = 0$. Now extend the given A_{ij} (including those defined in (a) to a defining system A for $\langle V_1, ..., V_n\rangle$. The proof of (ii) of the preceding proposition, together with the definition of k-rigidity, immediately imply that $\tilde{A}$ is homologous to $\tilde{A}'$, and this proves the result.

Before completing our study of indeterminacy, we obtain a result describing the linearity of matric Massey products. This result will then be used to show that $\text{In}\langle V_1, ..., V_n\rangle$ is a Λ-module in favorable cases.

PROPOSITION 2.7. *Let $\langle V_1, ..., V_n\rangle$ be defined, and let $1 \leqslant k \leqslant n$.*

(i) *If V_k is the zero matrix, then $0 \in \langle V_1, ..., V_n\rangle$.*

(ii) *If $\lambda \in \Lambda$, then $\lambda\langle V_1, ..., V_n\rangle \subset \langle V_1, ..., \lambda V_k, ..., V_n\rangle$ and equality holds if λ is invertible in Λ.*

(iii) *If* $V_k = V_k' + V_k''$ *and* $\langle V_1, ..., V_k', ..., V_n \rangle$ *is strictly defined, then*

$$\langle V_1, ..., V_n \rangle \subset \langle V_1, ..., V_k', ..., V_n \rangle + \langle V_1, ..., V_k'', ..., V_n \rangle$$

and equality holds if $\langle V_1, ..., V_k', ..., V_n \rangle$ *is k-rigidly defined.*

Proof. (i) follows from the case $\lambda = 0$ of (ii). To prove (ii), let A be any defining system for $\langle V_1, ..., V_n \rangle$ and define $B_{ij} = A_{ij}$ if $k \leqslant i$ or $j < k$ and $B_{ij} = \lambda A_{ij}$ if $i < k \leqslant j$. Then B is a defining system for $\langle V_1, ..., \lambda V_k, ..., V_n \rangle$ such that $\tilde{B} = \lambda \tilde{A}$. If λ is invertible, then we have

$$\langle V_1, ..., \lambda V_k, ..., V_n \rangle = \lambda\lambda^{-1}\langle V_1, ..., \lambda V_k, ..., V_n \rangle \subset \lambda\langle V_1, ..., V_n \rangle$$

and therefore equality holds. To prove (iii), let A be any defining system for $\langle V_1, ..., V_n \rangle$ and choose a defining system B for $\langle V_1, ..., V_k', ..., V_n \rangle$ such that $B_{ij} = A_{ij}$ if $k \leqslant i$ or $j < k$. Let $C_{ij} = B_{ij}$ if $k \leqslant i$ or $j < k$ and

$$C_{ij} = A_{ij} - B_{ij} \qquad \text{if} \qquad i < k \leqslant j.$$

Then C is a defining system for $\langle V_1, ..., V_k'', ..., V_n \rangle$ such that $\tilde{B} + \tilde{C} = \tilde{A}$. To prove the opposite inclusion, let C be any defining system for $\langle V_1, ..., V_k'', ..., V_n \rangle$. By the previous lemma, since $\langle V_1, ..., V_k', ..., V_n \rangle$ is k-rigidly defined, any element of this product can be obtained as $\{\tilde{B}\}$, where B satisfies $B_{ij} = C_{ij}$ if $k \leqslant i$ or $j < k$. Given any such C and B, let $A_{ij} = B_{ij}$ if $k \leqslant i$ or $j < k$ and $A_{ij} = B_{ij} + C_{ij}$ if $i < k \leqslant j$. Then A is a defining system for $\langle V_1, ..., V_n \rangle$ such that $\tilde{A} = \tilde{B} + \tilde{C}$.

We can now obtain a reasonably definitive result on the indeterminacy of strictly defined matric Massey products. The result appears to be best possible in the sense that if any hypothesis is deleted, then there is a counter-example to the conclusion.

THEOREM 2.8. *Let* $\langle V_1, ..., V_n \rangle$ *be strictly defined, and consider the strictly defined operations* $\langle V_1, ..., V_{k-1}, X_k, V_{k+2}, ..., V_n \rangle$, *where* $X_k \in MH(R_{k-1,k+1})$ *satisfies* $D(X_k) = D(V_k, V_{k+1})^{-1}$.

(i) *If each* $\langle V_1, ..., X_k, ..., V_n \rangle$, $2 \leqslant k \leqslant n-2$, *is k-rigidly defined, then*

$$\text{In}\langle V_1, ..., V_n \rangle = \bigcup_{(X_1, ..., X_{n-1})} \sum_{k=1}^{n-1} \langle V_1, ..., X_k, ..., V_n \rangle.$$

(ii) *If each* $\langle V_1, ..., X_k, ..., V_n \rangle$, $1 \leqslant k \leqslant n-1$, *is k-rigidly defined, then* $\text{In}\langle V_1, ..., V_n \rangle$ *is a* Λ*-module and* $x + \text{In}\langle V_1, ..., V_n \rangle = \langle V_1, ..., V_n \rangle$ *for each* $x \in \langle V_1, ..., V_n \rangle$.

Proof. (i) We know that

$$\operatorname{In}\langle V_1, \ldots, V_n\rangle \subset \bigcup_{(X_1,\ldots,X_{n-1})} \sum_{k=1}^{n-1} \langle V_1, \ldots, X_k, \ldots, V_n\rangle$$

and we must prove the opposite inclusion. Let D^k be a defining system for some $\langle V_1, \ldots, X_k, \ldots, V_n\rangle$, $1 \leqslant k \leqslant n-1$. We may write D^k in the form

$$\text{(a)} \qquad D^k_{ij} = \begin{cases} A^k_{ij}, & j < k \\ B^k_{ij}, & i \leqslant k-1 \leqslant j-1 \\ A^k_{i+1,j+1}, & i \geqslant k \end{cases} \qquad \begin{aligned} &0 \leqslant i < j \leqslant n-1, \\ &(i,j) = (0, n-1). \end{aligned}$$

Suppose the given A^1_{ij}, $i \geqslant 2$, have been completed to a defining system A^1 for $\langle V_1, \ldots, V_n\rangle$, and consider the following inductive definition of A^k, $2 \leqslant k \leqslant n$:

$$\text{(b)} \qquad A^{k+1}_{ij} = A^k_{ij} \quad \text{if} \quad j \leqslant k \quad \text{or} \quad i \geqslant k;$$

$$A^{k+1}_{ij} = A^k_{ij} + B^k_{i,j-1} \quad \text{if} \quad i \leqslant k-1 < j-1.$$

If the A^k_{ij} in (a) could be so chosen as to satisfy (b), then we would have

$$\tilde{A}^n - \tilde{A}^1 = \sum_{k=1}^{n-1} (\tilde{A}^{k+1} - \tilde{A}^k) = \sum_{k=1}^{n-1} \tilde{D}^k,$$

and this would imply that $\sum_{k=1}^{n-1} \{\tilde{D}^k\} \in \operatorname{In}\langle V_1, \ldots, V_n\rangle$, as desired. We claim that this can indeed be done. Observe that (b) implies the following explicit formula:

$$\text{(c)} \ A^k_{i+1,j+1} = A^1_{i+1,j+1} \quad \text{if} \quad i \geqslant k; \qquad A^k_{ij} = A^1_{ij} + \sum_{l=i+1}^{j-1} B^l_{i,j-1} \quad \text{if} \quad j < k.$$

We start with the given A^1_{ij}, $i \geqslant 2$. By Lemma 2.6, no matter how the A^1_{0j} and A^1_{1j} are chosen, we can assume that the A^k_{ij} of (a) are given by (c) for $2 \leqslant k \leqslant n-2$ since each $\langle V_1, \ldots, X_k, \ldots, V_n\rangle$ is k-rigidly defined. If we could assume that the A^{n-1}_{ij} of (a) were given by (c) for $2 \leqslant i < j \leqslant n-2$ (with A^{n-1}_{0j} and A^{n-1}_{1j} arbitrary, $j \leqslant n-2$), then we could define A^1_{0j} and A^1_{1j}, $j \leqslant n-2$, by $A^1_{ij} = A^{n-1}_{ij} - \sum_{l=i+1}^{j-1} B^l_{i,j-1}$ and (c) would hold in its entirety. We could then choose any $A^1_{1,n-1}$, $A^1_{1,n}$, and $A^1_{0,n-1}$ such that A^1 was a defining system for $\langle V_1, \ldots, V_n\rangle$ and could define the remaining A^k_{ij}, namely those for $i \leqslant k \leqslant j$ and $2 \leqslant k \leqslant n-1$, by (b) and thus complete the proof. It remains therefore to show that if $\{E_{ij} \mid 2 \leqslant i < j \leqslant n-2\}$ is any partial defining system for $\langle V_1, \ldots, V_{n-2}, X_{n-1}\rangle$, then it can be completed to a defining system E such that $\{\tilde{E}\} = \{\tilde{D}^{n-1}\}$. We may assume that

$E_{i,i+1} = D^{n-1}_{i,i+1}$ and we define $E_{ij} = D^{n-1}_{ij}$ if $j \leqslant 2$ or $i \geqslant n-3$. We complete the resulting partial defining system to a defining system E and we let $Y_l = \{E_{l-1,l+1} - D^{n-1}_{l-1,l+1}\}$. Observe that $Y_1 = 0$ and $Y_{n-2} = 0$. By the proof of (ii) of Proposition 2.4, we have that

$$\{\tilde{E} - \tilde{D}^{n-1}\} \in \langle 0, V_3, ..., V_{n-2}, X_{n-1}\rangle + \langle V_1, ..., V_{n-3}, 0\rangle$$

$$+ \sum_{l=2}^{n-3} \langle V_1, ..., V_{l-1}, Y_l, V_{l+2}, ..., V_{n-2}, X_{n-1}\rangle.$$

Since $\langle V_1, ..., V_{l-1}, Y_l, V_{l+2}, ..., V_n\rangle$ is l-rigidly defined each

$$\langle V_1, ..., V_{l-1}, Y_l, V_{l+2}, ..., V_{n-2}, X_{n-1}\rangle = \{0\}, \quad 2 \leqslant l \leqslant n-3.$$

Since $\langle V_1, ..., V_{n-3}, X_{n-2}, V_n\rangle$ is $(n-2)$-rigidly defined,

$$\langle V_1, ..., V_{n-3}, 0\rangle = \{0\}.$$

By the previous proposition, $0 \in \langle 0, V_3, ..., V_{n-2}, X_{n-1}\rangle$ and it is easy to verify that $\mathrm{In}\langle 0, V_3, ..., V_{n-2}, X_{n-1}\rangle = \{0\}$ by use of Proposition 2.4 and the k-rigidity of all $\langle V_1, ..., X_k, ..., V_n\rangle$, $2 \leqslant k \leqslant n-2$. Thus $\{\tilde{E}\} = \{\tilde{D}^{n-1}\}$ and the proof of (i) is complete.

(ii) If $x_k \in \langle V_1, ..., X_k, ..., V_n\rangle$ and $y_k \in \langle V_1, ..., Y_k, ..., V_n\rangle$, then $\lambda x_k \in \langle V_1, ..., \lambda X_k, ..., V_n\rangle$ and $x_k + y_k \in \langle V_1, ..., X_k + Y_k, ..., V_n\rangle$ by the previous proposition. It follows from (i) that $\mathrm{In}\langle V_1, ..., V_n\rangle$ is a Λ-module. Let $x \in \langle V_1, ..., V_n\rangle$. If $y \in \langle V_1, ..., V_n\rangle$, then $y = x + (y - x) \in x + \mathrm{In}\langle V_1, ..., V_n\rangle$ and it remains to prove that $x + \mathrm{In}\langle V_1, ..., V_n\rangle \subset \langle V_1, ..., V_n\rangle$. It suffices to show that $x + x_k \in \langle V_1, ..., V_n\rangle$, x_k as above. Let $x = \{\tilde{A}\}$ for some defining system A. Let $x_k = \{\tilde{D}^k\}$ for the defining system D^k given in (a). By Lemma 2.6, we may assume that $A^k_{ij} = A_{ij}$ for $j < k$ or $i \geqslant k+1$. Let $C_{ij} = A_{ij}$ if $j \leqslant k$ or $i \geqslant k$ and $C_{ij} = A_{ij} + B^k_{i,j-1}$ if $i \leqslant k-1 < j-1$. Then C is a defining system for $\langle V_1, ..., V_n\rangle$ such that $\tilde{C} = \tilde{A} + \tilde{D}^k$ and therefore $x + x_k \in \langle V_1, ..., V_n\rangle$, as was to be shown.

We should indicate the precise content of the hypotheses of the theorem. The hypotheses of (i) amount to the following requirements on $(n-2)$-tuple matric Massey products (X_k as in the theorem):

$$\langle V_1, ..., V_{k-1}, 0, V_{k+3}, ..., V_n\rangle = \{0\}, \qquad 1 \leqslant k \leqslant n-2, \tag{1}$$

and

$$\langle V_1, ..., V_{k-1}, X_k, V_{k+2}, ..., V_{l-1}, X_l, V_{l+2}, ..., V_n\rangle = \{0\}, \tag{2}$$

$$1 < k+1 < l < n \qquad \text{and} \qquad (k, l) \neq (1, n-1).$$

The hypotheses of (ii) add the further requirement

$$\langle X_1, V_2, ..., V_{n-1}, X_{n-1}\rangle = \{0\}. \tag{3}$$

Conditions (1) and (2) are vacuous if $n = 4$, and therefore Theorem 2.8 does complete the proof of Proposition 2.4.

There are various formulas relating sums of matric Massey products to matric Massey products defined by larger matrices. The relevance of larger matrices is made clear by the observation that

$$\langle V_1, V_2\rangle + \langle W_1, W_2\rangle = \left\langle (V_1, W_1), \begin{pmatrix} V_2 \\ W_2 \end{pmatrix} \right\rangle.$$

We complete this section by giving two examples of such formulas, the second of which is designed for use in Section 4. (The remarks preceding Proposition 2.3 are needed to make sense of this example.)

PROPOSITION 2.9. *Let* $\langle V_1, ..., V_n\rangle$ *and* $\langle W_1, ..., W_n\rangle$ *be defined. Then* $\langle X_1, ..., X_n\rangle$ *is defined, where*

$$X_1 = (V_1, W_1), \qquad X_i = \begin{pmatrix} V_i & 0 \\ 0 & W_i \end{pmatrix} \text{ if } 1 < i < n, \qquad X_n = \begin{pmatrix} V_n \\ W_n \end{pmatrix},$$

and $\langle V_1, ..., V_n\rangle + \langle W_1, ..., W_n\rangle \subset \langle X_1, ..., X_n\rangle$.

Proof. Let A and B be defining systems for $\langle V_1, ..., V_n\rangle$ and $\langle W_1, ..., W_n\rangle$ and define

$$C_{0j} = (A_{0j}, B_{0j}), \quad C_{ij} = \begin{pmatrix} A_{ij} & 0 \\ 0 & B_{ij} \end{pmatrix} \text{ if } 0 < i < j < n, \quad C_{in} = \begin{pmatrix} A_{in} \\ B_{in} \end{pmatrix}.$$

Then C is a defining system for $\langle X_1, ..., X_n\rangle$ such that $\tilde{C} = \tilde{A} + \tilde{B}$.

PROPOSITION 2.10. *Let* $V_i \in MH(R_{i-1,i})$. *Fix* l *such that* $0 \leqslant l \leqslant n - 2$ *and suppose given* $Y_k \in MH(R_{k-1,k+l})$ *for* $1 \leqslant k \leqslant n - l$ *such that* $\langle \bar{V}_1, ..., \bar{V}_{k-1}, Y_k, V_{k+l+1}, ..., V_n\rangle$ *is strictly defined. Then* $\langle X_1, ..., X_{n-l}\rangle$ *is defined, where*

$$X_1 = (Y_1, \bar{V}_1), \quad X_i = \begin{pmatrix} V_{i+l} & 0 \\ Y_i & \bar{V}_i \end{pmatrix} \text{ if } 1 < i < n - l, \quad X_{n-l} = \begin{pmatrix} V_n \\ Y_{n-l} \end{pmatrix},$$

and $\sum_{k=1}^{n-l} \langle \bar{V}_1, ..., \bar{V}_{k-1}, Y_k, V_{k+l+1}, ..., V_n\rangle \cap \langle X_1, ..., X_{n-l}\rangle \neq \varnothing$.

Proof. Since $\langle \bar{V}_1, ..., \bar{V}_{n-l-1}, Y_{n-l}\rangle$ and $\langle Y_1, V_{l+2}, ..., V_n\rangle$ are defined, there exist matrices A_{ij}, $j < n - l$ or $i > l$, such that $\{A_{i-1,i}\} = V_i$ and $dA_{ij} = \tilde{A}_{ij}$ for $j > i + 1$. Choose defining systems A^k for the

$$\langle \bar{V}_1, ..., \bar{V}_{k-1}, Y_k, V_{k+l+1}, ..., V_n\rangle$$

such that $A_{ij}^k = A_{i+l,j+l}$ if $i \geqslant k$ and $A_{ij}^k = \bar{A}_{ij}$ if $j < k$. Let $B_{ij} = \sum_{k=i+1}^{j} A_{ij}^k$, $0 \leqslant i < j \leqslant n - l$ and $(i,j) \neq (0, n-l)$, so that $B_{k-1,k} = A_{k-1,k}^k$, and define

$$C_{0j} = (B_{0j}, \bar{A}_{0j}), \qquad C_{ij} = \begin{pmatrix} A_{i+l,j+l} & 0 \\ B_{ij} & \bar{A}_{ij} \end{pmatrix} \quad \text{if} \quad 0 < i < j < n - l,$$

$$C_{i,n-l} = \begin{pmatrix} A_{i+l,n} \\ B_{i,n-l} \end{pmatrix}$$

Then C is a defining system for $\langle X_1, ..., X_n \rangle$ such that $\tilde{C} = \sum_{k=1}^{n-l} \tilde{A}^k$.

3. Associativity and Commutativity Relations

Matric Massey products satisfy a variety of associativity and commutativity relations, and we shall obtain some of the most useful of these here. In the following two theorems, we assume given an object $R \in \mathcal{O}_n$ and a fixed multipliable system of matrices $(V_1, ..., V_n)$, $V_i \in MH(R_{i-1,i})$.

THEOREM 3.1. *The following associativity formulas hold.*

(i) *If* $\langle V_2, ..., V_n \rangle$ *is defined and if* l, $1 < l < n$, *is given such that* $\langle V_1, ..., V_j \rangle = \{0\}$ *for* $1 < j < l$, *then*

$$V_1 \langle V_2, ..., V_n \rangle \subset -\langle \langle V_1, ..., V_l \rangle, V_{l+1}, ..., V_n \rangle.$$

(ii) *If* $\langle V_1, ..., V_{n-1} \rangle$ *is defined and if* k, $1 \leqslant k < n - 1$, *is given such that* $\langle V_{i+1}, ..., V_n \rangle = \{0\}$ *for* $k < i < n$, *then*

$$\langle \bar{V}_1, ..., \bar{V}_{n-1} \rangle V_n \subset \langle \bar{V}_1, ..., \bar{V}_k, \langle V_{k+1}, ..., V_n \rangle \rangle.$$

(iii) *If* $\langle V_2, ..., V_{n-1} \rangle$ *is defined and contains zero and if* k *and* l, $1 \leqslant k < l < n$, *are given such that* $\langle V_{i+1}, ..., V_n \rangle = \{0\}$ *for* $k < i < n$ *and* $\langle V_1, ..., V_j \rangle = \{0\}$ *for* $1 < j < l$, *then*

$$0 \in \langle \bar{V}_1, ..., \bar{V}_k, \langle V_{k+1}, ..., V_n \rangle \rangle + \langle \langle V_1, ..., V_l \rangle, V_{l+1}, ..., V_n \rangle.$$

Proof. A simple check of signs (using $\langle V_1, V_2 \rangle = \bar{V}_1 V_2$ and $\langle \bar{V}_1, ..., \bar{V}_n \rangle = -\overline{\langle V_1, ..., V_n \rangle}$) shows that (i) and (ii) are just more precise versions of (iii) in the special cases $k = 1$ and $l = n - 1$. To prove (iii), let A_{ij}, $1 \leqslant i < j \leqslant n - 1$, be matrices such that $\{A_{i-1,i}\} = V_i$ and $dA_{ij} = \tilde{A}_{ij}$ for $j > i + 1$. These exist since $0 \in \langle V_2, ..., V_{n-1} \rangle$. Choose further matrices A_{in}, $i > k$, and A_{0j}, $j < l$, such that $\{A_{ij} \mid i \geqslant k\}$ and $\{A_{ij} \mid j \leqslant l\}$ are defining systems for $\langle V_{k+1}, ..., V_n \rangle$ and for $\langle V_1, ..., V_l \rangle$. We can do this since $\langle V_{i+1}, ..., V_n \rangle = \{0\}$, $i > k$, and $\langle V_1, ..., V_j \rangle = \{0\}$, $j < l$. Now define matrices B_{ij}, $j \leqslant k+1$, and C_{ij}, $i \geqslant l-1$, by

$$B_{ij} = \bar{A}_{ij} \quad \text{if} \quad j \leqslant k; \qquad B_{i,k+1} = \sum_{m=k+1}^{n-1} \bar{A}_{im} A_{mn} \qquad \text{if} \quad 0 < i \leqslant k$$

$$C_{ij} = A_{ij} \quad \text{if} \quad i \geqslant l; \qquad C_{l-1,j} = \sum_{m=1}^{l-1} \bar{A}_{0m} A_{mj} \qquad \text{if} \quad l \leqslant j \leqslant n.$$

In particular, $B_{k,k+1} = \tilde{A}_{kn}$ and $C_{l-1,l} = \tilde{A}_{0l}$, and it is easy to verify that B and C are defining systems for $\langle \bar{V}_1, ..., \bar{V}_k, \{\tilde{A}_{kn}\} \rangle$ and for $\langle \{\tilde{A}_{0l}\}, V_{l+1}, ..., V_n \rangle$. If $l = k+1$, then $\tilde{B} + \tilde{C} = 0$. If $l > k+1$, then a straightforward calculation shows that $\tilde{B} + \tilde{C} = d(\sum_{m=k+1}^{l-1} \bar{A}_{0m} A_{mn})$. This proves (iii). To prove (i), we merely observe that we could start the above argument using any defining system for $\langle V_2, ..., V_n \rangle$ and that $\tilde{B}$ would then be precisely $A_{01}\tilde{A}_{1n}$. This shows that each element of $V_1\langle V_2, ..., V_n \rangle$ belongs to $-\langle \{\tilde{A}_{0l}\}, V_{l+1}, ..., V_n \rangle$ for some $\{\tilde{A}_{0l}\} \in \langle V_1, ..., V_l \rangle$. The modifications needed to prove (ii) are equally simple.

We single out certain frequently used special cases in the following corollary.

COROLLARY 3.2. *Massey products and ordinary products are related by the following formulas.*

(i) *If* $\langle V_2, ..., V_n \rangle$ *is defined, then* $\langle \bar{V}_1 V_2, V_3, ..., V_n \rangle$ *is defined and*

$$V_1 \langle V_2, ..., V_n \rangle \subset -\langle \bar{V}_1 V_2, V_3, ..., V_n \rangle.$$

(ii) *If* $\langle V_1, ..., V_{n-1} \rangle$ *is defined, then* $\langle V_1, ..., V_{n-2}, V_{n-1}V_n \rangle$ *is defined and*

$$\langle V_1, ..., V_{n-1} \rangle V_n \subset \langle V_1, ..., V_{n-2}, V_{n-1}V_n \rangle.$$

(iii) *If* $\langle V_1, ..., V_{n-1} \rangle$ *and* $\langle V_2, ..., V_n \rangle$ *are strictly defined, then*

$$V_1 \langle V_2, ..., V_n \rangle = \langle \bar{V}_1, ..., \bar{V}_{n-1} \rangle V_n.$$

Proof. (i) and (ii) follow from the cases $l = 2$ and $k = n-2$ of (i) and (ii) of the theorem where, for (ii), V_i is replaced by $\bar{V}_i$ if $i < n$. Part (iii) follows from the cases $l = n-1$ of (i) and $k = 1$ of (ii) of the theorem, since these are simultaneously valid under the present hypotheses.

The last part of the corollary contains the simplest special case, $l = n-2$, of the following general system of relations between $(l+1)$-tuple matric Massey products and $(n-l)$-tuple matrix Massey products.

THEOREM 3.3. *Let l be given such that $1 \leqslant l \leqslant n-2$ and, if $l < \frac{1}{2}(n-1)$, let k be given such that $1 \leqslant k \leqslant n-2l$. Assume that $\langle V_q, ..., V_{q+l}\rangle$ is strictly defined and consist of the single matrix Y_q for each q such that $1 \leqslant q \leqslant n-l$ if $l \geqslant \frac{1}{2}(n-1)$ or such that $k \leqslant q \leqslant k+l$ if $l < \frac{1}{2}(n-1)$. Assume further that $\langle \bar{V}_1, ..., \bar{V}_{q-1}, Y_q, V_{q+l+1}, ..., V_n\rangle$ is strictly defined for each such q. Then $0 \in \sum_q \langle \bar{V}_1, ..., \bar{V}_{q-1}, Y_q, V_{q+l+1}, ..., V_n\rangle$.*

Proof. We first prove the more complicated case $l < \frac{1}{2}(n-1)$ and then indicate the modifications necessary in the case $l \geqslant \frac{1}{2}(n-1)$. We have $0 \in \langle V_1, ..., V_{k+l-1}\rangle$ and $0 \in \langle V_{k+l+1}, ..., V_n\rangle$ since

$$\langle \bar{V}_1, ..., \bar{V}_{q-1}, Y_q, V_{q+l+1}, ..., V_n\rangle$$

is strictly defined for $q = k+l$ and $q = k$. Choose matrices A_{ij}, $j \leqslant k+l-1$ or $i \geqslant k+l$, such that $\{A_{i-1,i}\} = V_i$ and $dA_{ij} = \tilde{A}_{ij}$ for $j > i+1$. Since $\langle V_q, ..., V_{q+l}\rangle$ is strictly defined, $k \leqslant q \leqslant k+l$, we can choose further matrices

$$A_{ij}, k-1 \leqslant i < k+l \leqslant j \leqslant k+2l \qquad \text{and} \qquad j-i \leqslant l,$$

such that $\{A_{k+l-1,k+l}\} = V_{k+l}$ and $dA_{ij} = \tilde{A}_{ij}$ for $j > i+1$. Since each $\langle V_q, ..., V_{q+l}\rangle$ has zero indeterminacy, we necessarily have that $\{\tilde{A}_{q-1,q+l}\} = Y_q$. By induction on q, we can obtain defining systems A^q for the $\langle \bar{V}_1, ..., \bar{V}_{q-1}, Y_q, V_{q+l+1}, ..., V_n\rangle$ which satisfy

(a) $\quad A^q_{ij} = \bar{A}_{ij} \quad$ if $\; j < q; \qquad A^q_{ij} = A_{i+l,j+l} \quad$ if $\; i \geqslant q;$

(b) $$A^q_{iq} = \sum_{m=q}^{\max(i+l,k+l-1)} \bar{A}_{im}A_{m,q+l} - \sum_{r=\max(k,i+1)}^{k+l-1} A^r_{iq}$$

if $\; i < q < k+l;$

(c) $$A^{k+l}_{ij} = \sum_{m=k+l}^{i+l} \bar{A}_{im}A_{m,j+l} - \sum_{r=\max(k,i+1)}^{k+l-1} A^r_{ij} \qquad \text{if } \; i < k+l \leqslant j.$$

Here $A^q_{q-1,q} = \tilde{A}_{q-1,q+l}$, and a quite tedious calculation demonstrates that (b) and (c) are consistent with $dA^q_{ij} = \tilde{A}^q_{ij}$. The A^q_{ij} for $i < q < j$ and $q < k+l$ are chosen arbitrarily so that $dA^q_{ij} = \tilde{A}^q_{ij}$. Another tedious calculation shows that $\sum_{q=k}^{k+l} \tilde{A}^q = 0$, and this proves the result for $l < \frac{1}{2}(n-1)$. Suppose that $l \geqslant \frac{1}{2}(n-1)$. Since $\langle V_q, ..., V_{q+l}\rangle$ is strictly defined with zero indeterminacy for $1 \leqslant q \leqslant n-l$, we can choose matrices A_{ij}, $j-i \leqslant l$, such that $\{A_{ij} \mid q-1 \leqslant i < j \leqslant q+l\}$ is a defining system for $\langle V_q, ..., V_{q+l}\rangle$ and $\{\tilde{A}_{q-1,q+l}\} = Y_q$. By induction on q, we can find defining systems A^q for $\langle \bar{V}_1, ..., \bar{V}_{q-1}, Y_q, V_{q+l+1}, ..., V_n\rangle$ which satisfy (a) above and

(b′) $$A^q_{iq} = \sum_{m=q}^{i+l} \bar{A}_{im}A_{m,q+l} - \sum_{r=i+1}^{q-1} A^r_{iq}, \qquad i < q \leqslant n-l.$$

The A_{ij}^q, $i < q < j$, are arbitrary such that $dA_{ij}^q = \tilde{A}_{ij}^q$. Now if $l = \frac{1}{2}(n-1)$, then $\sum_{q=1}^{n-1} \tilde{A}^q = 0$, while if $l > \frac{1}{2}(n-1)$, then $d(\sum_{m=n-l}^{l} \bar{A}_{0m}A_{mn}) = \sum_{q+1}^{n-l} \tilde{A}^q$, and this completes the proof.

The following corollary complements Corollary 3.2.

COROLLARY 3.4. *Massey products and ordinary products are related by the following formulas.*

(i) *If* $\langle V_1V_2, V_3, ..., V_n \rangle$ *is defined, then* $\langle V_1, \bar{V}_2V_3, V_4, ..., V_n \rangle$ *is defined and*

$$\langle V_1V_2, V_3, ..., V_n \rangle \subset -\langle V_1, \bar{V}_2V_3, V_4, ..., V_n \rangle.$$

(ii) *If* $\langle V_1, ..., V_{n-2}, \bar{V}_{n-1}V_n \rangle$ *is defined, then* $\langle V_1, ..., V_{n-3}, V_{n-2}V_{n-1}, V_n \rangle$ *is defined and*

$$\langle V_1, ..., V_{n-2}, \bar{V}_{n-1}V_n \rangle \subset -\langle V_1, ..., V_{n-3}, V_{n-2}V_{n-1}, V_n \rangle.$$

(iii) *If* $\langle V_1, ..., V_kV_{k+1}, ..., V_n \rangle$ *and* $\langle V_1, ..., \bar{V}_{k+1}V_{k+2}, ..., V_n \rangle$ *are strictly defined, then*

$$\langle V_1, ..., V_{k-1}, V_kV_{k+1}, V_{k+2}, ..., V_n \rangle$$
$$\cap -\langle V_1, ..., V_k, \bar{V}_{k+1}V_{k+2}, V_{k+3}, ..., V_n \rangle \neq \varnothing.$$

Proof. (iii) is precisely the case $l = 1$ of the theorem, with V_i replaced by $\bar{V}_i$ for $i \leqslant k$. (i) and (ii) are easily obtained sharpenings of (iii) in the cases $k = 1$ and $k = n - 2$.

We now turn to commutativity relations. We first define an involution of the category $\mathcal{O}\!\ell_n$ and then relate it to matric Massey products.

DEFINITION 3.5. Let $R = \{R_{ij}, \mu_{ijk}\}$ be an object in $\mathcal{O}\!\ell_n$. Define the opposite object $R^0 = \{R_{ij}^0, \mu_{ijk}^0\}$ in $\mathcal{O}\!\ell_n$ by

$$\text{(i)} \quad R_{ij}^0 = R_{n-j,n-i}; \qquad \mu_{ijk}^0 = \mu_{n-k,n-j,n-i}T : R_{ij}^0 \otimes R_{jk}^0 \to R_{ik}^0,$$

where $T : R_{n-j,n-i} \otimes R_{n-k,n-j} \to R_{n-k,n-j} \otimes R_{n-j,n-i}$ is the standard twisting map, $T(x \otimes y) = (-1)^{\deg x \deg y} y \otimes x$. A simple diagram chase proves that $R^0 \in \mathcal{O}\!\ell_n$. Observe that $R_{i-1,i}^0 = R_{n-i,n-i+1}$ and that $R_{0n}^0 = R_{0n}$.

If $(V_1, ..., V_n)$ is a multipliable system, $V_i \in MH(R_{i-1,i})$, and if V_i' denotes the matrix transpose to V_i, then clearly $(V_n', ..., V_1')$ is a multipliable system such that $D(V_1, ..., V_n)' = D(V_n', ..., V_1')$. It therefore makes sense to compare $\langle V_1, ..., V_n \rangle'$, computed in R, to $\langle V_n', ..., V_1' \rangle$, computed in R^0. This can always be done, but assumptions as to the degrees of the entries of the V_i must be made in order to obtain a uniform statement (if $2 \neq 0$ in Λ). Thus suppose that each entry of V_i has the same parity ϵ_i (where $\epsilon_i = 0$ if the degrees are all even, $\epsilon_i = 1$ if the degrees are all odd) and define

$$s(i,j) = j - i + \sum_{i \leqslant k < l \leqslant j} (\epsilon_k + 1)(\epsilon_l + 1).$$

With these notations, we have the following result.

PROPOSITION 3.6. *Let* $\langle V_1, \ldots, V_n \rangle$ *be defined in* R, $V_i \in MH(R_{i-1,i})$. *If* $2 \neq 0$ *in* Λ, *assume that each entry of* V_i *has the same parity* ϵ_i. *Then* $\langle V_n', \ldots, V_1' \rangle$ *is defined in* R^0, *and*

$$\langle V_1, \ldots, V_n \rangle' = (-1)^{s(1,n)} \langle V_n', \ldots, V_1' \rangle$$

as a subset of $MH(R_{0n})$.

Proof. Observe first that if $X \in MR_{ij}$ and $Y \in MR_{jk}$, and if the degrees of the entries of X all have parity ϵ and the degrees of the entries of Y all have parity ϕ, then

$$(XY)' = (-1)^{\epsilon\phi} Y'X' \in MR_{ik} = MR^0_{n-k,n-i},$$

where XY is computed in R and $Y'X'$ is computed in R^0. Now let A be a defining system for $\langle V_1, \ldots, V_n \rangle$ and define $B_{n-j,n-i} = (-1)^{s(i+1,j)} A'_{ij}$. Since $s(i, i) = 0$, $B_{n-i,n-i+1} = A'_{i-1,i}$, which represents V_i'. In view of our first observation, it requires only a tedious verification of signs to see that B is a defining system in R^0 for $\langle V_n', \ldots, V_1' \rangle$ such that $(-1)^{s(1,n)} \bar{B} = \bar{A}'$. This proves that

$$\langle V_1, \ldots, V_n \rangle' \subset (-1)^{s(1,n)} \langle V_n', \ldots, V_1' \rangle,$$

and the opposite inclusion follows from symmetry.

For the remainder of this section, we shall assume given a DGA Λ-algebra U, regarded as an object of $\mathfrak{A}_n$. Observe that U^0, the opposite algebra of U, is precisely the opposite object of U in the category $\mathfrak{A}_n$. Moreover, writing elements of U^0 in the form u^0, the map $f: U \to U^0$ defined by $f(u) = u^0$ is a homology isomorphism of differential Λ-modules. Of course, f is a morphism of algebras if and only if U is commutative. We shall say that U is n-homotopy commutative if $f: U \to U^0$ is n-homotopy multiplicative in the sense of Definition 1.4. This is clearly a condition as to the existence of certain homotopies $U^i \to U$, $2 \leqslant i \leqslant n$, and is satisfied in each of the following cases:

(1) $U = C^*(X; \Lambda)$, where X is a topological space

(2) $U = C_*(\Omega X; \Lambda)$, where X is an H-space with unit

(Moore loops are understood, so as to have associativity)

(3) $U = B(\Lambda)^*$, the dual of the bar construction of a cocommutative Hopf algebra Λ.

Indeed, the requisite homotopies can be obtained by the method of acyclic models in case (1), by passing to chains from geometric homotopies in case (2) (see Clark [*1*]), and by use of the contracting homotopy in $B(\Lambda)$ in case (3) (see [*10*]).

We can now use our naturality result to convert Proposition 3.6 into an internal statement about matric Massey products in $H(U)$.

Corollary 3.7. *Let U be an n-homotopy commutative* DGA *Λ-algebra. Let $\langle V_1, ..., V_n\rangle$ be defined, $V_i \in MH(U)$. If $2 \neq 0$ in Λ, assume that each entry of V_i has the same parity ϵ_i. Then $\langle V_n', ..., V_1'\rangle$ is also defined, and*

$$\langle V_1, ..., V_n\rangle' = (-1)^{s(1,n)}\langle V_n', ..., V_1'\rangle,$$

where $s(1, n) = n - 1 + \sum_{1 \leq k < l \leq n} (\epsilon_k + 1)(\epsilon_l + 1)$.

Proof. Consider $f : U \to U^0$. By Theorem 1.5, we have that

$$f_*\langle V_1, ..., V_n\rangle = \langle f_*(V_1), ..., f_*(V_n)\rangle.$$

On the other hand, the previous proposition gives

$$f_*\langle V_1, ..., V_n\rangle' = (-1)^{s(1,n)} \langle f_*(V_n)', ..., f_*(V_1)'\rangle.$$

Therefore $\langle f_*(V_1), ..., f_*(V_n)\rangle' = (-1)^{s(1,n)} \langle f_*(V_n)', ..., f_*(V_1)'\rangle$, with both sides computed in U^0. Applying f_*^{-1} to both sides, we obtain the desired relation by use of Theorem 1.5.

The corollary is due to Kraines [5] in the case of ordinary Massey products. We complete this section by giving two different generalizations to n-tuple products of the well-known permutativity relation

$$0 \in \langle a, b, c\rangle \pm \langle b, c, a\rangle \pm \langle c, a, b\rangle,$$

which is valid for ordinary triple products in the cohomology of spaces. Since permutations of multipliable systems are usually not multipliable systems, these results will only make sense for ordinary Massey products. To simplify the statement of the first relation, let us say that U is n-homotopy permutative if there exist morphisms of Λ-modules $h_l : U^l \to U$, of degree $1 - l$, such that h_1 is the identity map and, for $2 \leqslant l \leqslant n$,

$$dh_l(u_1 \otimes \cdots \otimes u_l) = \sum_{k=1}^{l} h_l(\bar{u}_1 \otimes \cdots \otimes \bar{u}_{k-1} \otimes du_k \otimes u_{k+1} \otimes \cdots \otimes u_l)$$

$$- \sum_{k=1}^{l-1} h_{l-1}(\bar{u}_1 \otimes \cdots \otimes \bar{u}_{k-1} \otimes \bar{u}_k u_{k+1} \otimes u_{k+2} \otimes \cdots \otimes u_l)$$

$$- (-1)^{(\deg u_1 + 1) \cdot \sum_{k=2}^{n} (\deg u_k + 1)} h_{l-1}(\bar{u}_2 \otimes \cdots \otimes \bar{u}_{n-1} \otimes \bar{u}_n \bar{u}_1).$$

(A calculation is required to show that this notion makes sense, that is, is consistent with $d^2 = 0$.) This condition is satisfied in the first and third examples listed above. We can now obtain the following result, which is also due to Kraines [5]. We sketch a proof since our notations and sign conventions differ from his.

PROPOSITION 3.8. *Let U be an n-homotopy permutative* DGA *Λ-algebra. Let $v_i \in H(U)$ and suppose that $\langle v_1, \ldots, v_n \rangle$ is defined and that*

$$\langle v_{l+1}, \ldots, v_n, v_1, v_2, \ldots, v_l \rangle$$

is strictly defined for $1 \leqslant l < n$. Then

$$(-1)^{s(1,n)} \langle v_1, \ldots, v_n \rangle \subset \sum_{l=1}^{n-1} (-1)^{s(1,l)+s(l+1,n)} \langle v_{l+1}, \ldots, v_n, v_1, \ldots, v_l \rangle,$$

where $s(i,j) = j - i + \sum_{i \leqslant k < l \leqslant j} (\deg v_k + 1)(\deg v_l + 1)$.

Proof. Let $a = (a_{ij})$ be a defining system for $\langle v_1, \ldots, v_n \rangle$. Since the $\langle v_{l+1}, \ldots, v_n, v_1, \ldots, v_l \rangle$ are strictly defined, we can choose elements a_{ji}, $0 < i < j < n$, by induction on $j - i$, such that

$$da_{ji} = \sum_{m=j+1}^{n-1} \bar{a}_{jm} a_{mi} + \bar{a}_{jn} a_{0i} + \sum_{m=1}^{i-1} \bar{a}_{jm} a_{mi} .$$

Then defining systems a^l for the $\langle v_{l+1}, \ldots, v_n, v_1, \ldots, v_l \rangle$ are given by

$$a_{ij}^l = \begin{cases} a_{i-n+l, j-n+l} & \text{if } \ n - l \leqslant i \\ a_{i+l, j-n+l} & \text{if } \ i < n - l < j \\ a_{i+l, j+l} & \text{if } \ j \leqslant n - l. \end{cases}$$

Write $u(k) = s(1, k) + s(k + 1, n)$, $0 < k < n$. Then a tedious verification of signs demonstrates the formula

$$\sum_{l=1}^{n-1} (-1)^{u(l)} \tilde{a}^l = \sum_{l=1}^{n-1} (-1)^{u(l)} \bar{a}_{ln} a_{0l} + \sum_{0<k<l<n} (-1)^{u(k)} [\bar{a}_{kl}, a_{lk}].$$

Adding and subtracting $(-1)^{s(1,n)} \tilde{a}$ on the right, we obtain

$$\sum_{l=1}^{n-1} (-1)^{u(l)} \tilde{a}^l = (-1)^{s(1,n)} \tilde{a} + b, \qquad b = \sum_{0\leqslant k<l<n} (-1)^{u(k)} [\bar{a}_{kl}, a_{lk}],$$

where $u(0) = 1 + s(1, n)$ and a_{i0} is defined to be a_{in}. Let homotopies h_l give the n-homotopy permutativity of U. Define

$$c = \sum_{l=2}^{n} \sum_{I} (-1)^{u(i_0)} h_l(a_{i_0 i_1} \otimes \cdots \otimes a_{i_{l-1} i_0}),$$

where the second sum is taken over all sequences $I = (i_0, ..., i_{l-1})$ such that $0 \leqslant i_0 < i_1 < \cdots < i_{l-1} < n$. Then a tedious calculation shows that $d(c) = -b$, and this proves the result.

For our final commutativity result, we require that U be homotopy commutative via a $\cup_1$-product which satisfies the Hirsch formula. This means that we are given $\cup_1 : U \otimes U \to U$, of degree -1, satisfying

(i) $d(x \cup_1 y) = [x, y] - d(x) \cup_1 y + \bar{x} \cup_1 d(y)$, and

(ii) $(xy) \cup_1 z = (-1)^{\deg y \deg z} (x \cup_1 z) y - \bar{x}(y \cup_1 z)$.

This hypothesis is satisfies if $U = C^*(X; A)$ [3, 10] or if $U = B(A)^*$, A a cocommutative Hopf algebra [*10*]. It should be observed that the existence of a $\cup_1$-product in U satisfying (i) is equivalent to either 2-homotopy commutativity or permutativity: in both cases, the required homotopy h_2 is given by $h_2(x \otimes y) = -\bar{x} \cup_1 y$.

PROPOSITION 3.9. *Let U be a* DGA *Λ-algebra equipped with a $\cup_1$-product which satisfies the Hirsch formula. Let $v_i \in H(U)$ and suppose that $\langle v_1, ..., v_n\rangle$ is defined and that $\langle v_2, ..., v_l, v_1, v_{l+1}, ..., v_n\rangle$ is strictly defined, $2 \leqslant l \leqslant n$. Then*

$$\langle v_1, ..., v_n\rangle \subset - \sum_{l-2}^{n} (-1)^{t(l)} \langle v_2, ..., v_l, v_1, v_{l+1}, ..., v_n\rangle,$$

where $t(l) = (\deg v_1 + 1). \sum_{k=2}^{l} (\deg v_k + 1)$.

Proof. Let $a^1 = (a^1_{ij})$ be a defining system for $\langle v_1, ..., v_n \rangle$. Define $t(i,j) = (\deg v_1 + 1) \cdot \sum_{k=i}^{j} (\deg v_k + 1), 2 \leqslant i \leqslant j \leqslant n$, so that $t(l) = t(2, l)$. Since each $\langle v_1, ..., v_l, v_1, v_{l+1}, ..., v_n \rangle$ is strictly defined, we can choose defining systems for these products, by induction on l, such that

(a) $a^l_{ij} = a^1_{i+1,j+1}$ if $j < l$; $a^l_{ij} = a_{ij}$ if $i \geqslant l$; an

(b) $a^l_{l-1,l} = a^1_{01}$; $a^l_{il} = - \sum_{k=i+1}^{l-1} (-1)^{t(k+1,l)} a^k_{il} + \bar{a}^1_{i+1,l} \cup_1 a^1_{01}$ if $i < l - 1$.

A lengthy computation, using (i) and (ii) above, shows that (b) is consistent with the formula $da^l_{il} = \tilde{a}^l_{il}$. The a^l_{ij}, $i < l < j$, are chosen arbitrarily such that $da^l_{ij} = \tilde{a}^l_{ij}$. Another lengthy computation shows that

$$d((-1)^{t(n)} \bar{a}^1_{1n} \cup_1 a^1_{01}) = \sum_{l=1}^{n} (-1)^{t(l)} \tilde{a}^l.$$

Since a^1 was an arbitrary defining system for $\langle v_1, ..., v_n \rangle$, this completes the proof.

4. Matric Massey Products in Spectral Sequences

Let $R = \{R_{ij}, \mu_{ijk}\}$ be an object in $\mathcal{O}\!\iota_n$. We say that R is filtered if each R_{ij} is a decreasingly filtered differential Λ-module and each μ_{ijk} is filtration preserving. Here the filtration on the tensor product is defined, as usual, by

$$F^p(R_{ij} \otimes R_{jk}) = \sum_{r+s=p} F^r R_{ij} \otimes F^s R_{ij}.$$

We say that the filtration is complete if, for each R_{ij},

$$\lim_{p \to -\infty} F^p R_{ij} = R_{ij} = \lim_{\infty \leftarrow p} R_{ij}/F^p R_{ij}.$$

$H(R_{ij})$ is filtered by letting $F^p H(R_{ij})$ be the image of $H(F^p R_{ij})$ under the map induced by inclusion. We say that a complete filtered object R is convergent if, for each R_{ij},

$$\lim_{p \to -\infty} F^p H(R_{ij}) = H(R_{ij}) = \lim_{\infty \leftarrow p} H(R_{ij})/F^p H(R_{ij}).$$

We shall assume throughout this section that we are given a fixed convergent complete filtered object $R \in \mathcal{O}\!\iota_n$.

By the usual procedures, we can obtain a spectral sequence $\{E_r R_{ij}\}$ for each R_{ij}. By results of Eilenberg and Moore ([2] and unpublished material), our hypotheses are sufficient to guarantee that $\{E_r R_{ij}\}$ converges to $H(R_{ij})$. The pairings μ_{ijk} induce morphisms of spectral sequences $\{E_r(R_{ij} \otimes R_{jk})\} \to \{E_r R_{ik}\}$, and we have the standard homology products $\{E_r R_{ij} \otimes E_r R_{jk}\} \to \{E_r(R_{ij} \otimes R_{jk})\}$. By composition, therefore, we have the well-defined pairings

$$E_r\mu_{ijk} : E_r R_{ij} \otimes E_r R_{jk} \to E_r R_{ik} .$$

Let $E_r R = \{E_r R_{ij}, E_r\mu_{ijk}\}$. Then $E_r R$ is an object in a category $\mathscr{B}\mathfrak{A}_n(r)$, which is defined precisely as was $\mathfrak{A}_n$, except that the component differential Λ-modules are bigraded and have differentials of bidegree $(r, 1 - r)$. After making the appropriate modifications in Notations 1.1, specified below, we can define matric Massey products for objects in $\mathscr{B}\mathfrak{A}_n(r)$ by Definition 1.2, and all of our preceding results go through unchanged.

We shall here investigate the convergence of matric Massey products contained in $E_{r+1}R_{0n}$ to matric Massey products contained in $H(R_{0n})$ and shall study the relationship between the operations in $E_{r+1}R_{0n}$ and the higher differentials. The statements of our results are somewhat technical, but the essential ideas are not difficult. One may regard a matric Massey product $\langle V_1, ..., V_n\rangle \subset E_{r+1}R_{0n}$ as an approximation to a matric Massey product in $H(R_{0n})$. How good the approximation is depends upon how much interference there is from higher differentials. Under appropriate hypotheses, there is no interference and we obtain our convergence theorem. Under other hypotheses, the interference is forced to take the form of specific higher differentials defined on $\langle V_1, ..., V_n\rangle$ and we then obtain precise expressions for such differentials.

We must first specify the necessary modifications in Notations 1.1. Let $D = (d_{ij}, d'_{ij})$ and $E = (e_{ij}, e'_{ij})$ be $p \times q$ and $p' \times q'$ matrices with entries in $Z \times Z$. We say that (D, E) is a compatible pair if $q = p'$ and if both $d_{ik} + e_{kj}$ and $d'_{ik} + e'_{kj}$ are independent of k for each i and j. We then define the $p \times q'$ matrix $D * E = (f_{ij})$ by $f_{ij} = (d_{ik} + e_{kj}, d'_{ik} + e'_{kj})$. If $D_1, ..., D_n$ are matrices with entries in $Z \times Z$, we say that $(D_1, ..., D_n)$ is a compatible system if each (D_i, D_{i+1}) is a compatible pair. If $D = (d_{ij}, d'_{ij})$ and $(s, t) \in Z \times Z$, we define $D^{(s,t)} = (d_{ij} + s, d'_{ij} + t)$. Let $(D_1, ..., D_n)$ be a compatible system. Define D_{ij}, $0 \leqslant i < j \leqslant n$, by

$$D_{ij} = (D_{i+1} * \cdots * D_j)^{(-rt,(r-1)t)}, \quad t = j - i - 1, \quad 0 \leqslant i < j \leqslant n. \quad (1)$$

Inductively, each (D_{ik}, D_{kj}) is a compatible pair, and

$$D_{ij}^{(r,1-r)} = D_{ik} * D_{kj} \quad \text{for each } k \text{ such that} \quad 0 \leqslant i < k < j \leqslant n. \quad (2)$$

Now the notion of a multipliable system of matrices can be defined just as in

Notations 1.1 (b), and the sign conventions of Notations 1.1 (c) remain in force with total degrees understood.

We shall work throughout this section with a fixed given multipliable system of matrices $(V_1, ..., V_n)$, where $V_i \in ME_{r+1}R_{i-1,i}$. We shall assume that V_1 is a row matrix and V_n is a column matrix. We let $D_i = D(V_i)$, the matrix of bidegrees of V_i, and we define D_{ij} by (1) above. If $A = (A_{ij})$ is a defining system for $\langle V_1, ..., V_n\rangle \subset E_{r+1}R_{0n}$, then $A_{ij} \in ME_rR_{ij}$ satisfies $D(A_{ij}) = D_{ij}$. Observe that we have

$$D < V_1, ..., V_n\rangle = D(V_1, ..., V_n)^{(s,t)}, \tag{3}$$

where

$$D(V_1, ..., V_n) = D_1 * \cdots * D_n \quad \text{and} \quad (s, t) = (-r(n-2), (r-1)(n-2)).$$

We shall find it convenient to adopt the notation $(p, q) \in D_{ij}$ to indicate the fact that (p, q) occurs as an entry of the matrix $D_{ij} = D(A_{ij})$.

As a final preliminary, we shall need some terminology to allow concise statements about elements and matrices of elements in spectral sequences. Let G be a filtered differential Λ-module, with convergent spectral sequence. $E_{r,\infty}G$ will denote the Λ-submodule of E_rG consisting of all permanent cycles, and $ME_{r,\infty}G$ will denote the set of matrices with entries in $E_{r,\infty}G$. Let $i : E_{r,\infty}G \to E_\infty G$ denote the natural epimorphism (for any r). If $x \in E_{r,\infty}^{p,q}G$ and $y \in F^pH(G)$ projects to $i(x)$ in $E_\infty G$, then we say that x converges to y. If $X = (x_{ij}) \in ME_{r,\infty}G$ and x_{ij} converges to y_{ij}, then we say that X converges to $Y = (y_{ij})$. We let π denote the natural epimorphism $F^pG \to E_0^{p,*}G$. If $B = (b_{ij}) \in MG$ and if the b_{ij} are of known filtration, then $\pi(B)$ is a well-defined element of ME_0G. If, moreover, the entries of $\pi(B)$ are known to survive to E_rG, then we shall indicate this fact by writing $\pi(B) \in ME_rG$.

With these notations, we can first prove a convergence theorem and then a generalized Leibnitz formula for matric Massey products contained in $E_{r+1}R_{0n}$.

THEOREM 4.1. *Let $\langle V_1, ..., V_n\rangle$ be defined in $E_{r+1}R_{0n}$. Assume that $V_i \in ME_{r+1,\infty}R_{i-1,i}$ and that V_i converges to W_i, where $\langle W_1, ..., W_n\rangle$ is strictly defined in $H(R_{0n})$. Assume further that the following condition (*) is satisfied*

$$(*) \quad \text{If } (p, q) \in D_{ij}, \quad 1 < j - i < n, \quad \text{then} \quad E_{r+u+1}^{p-u,q+u}R_{ij} \subset E_{r+u+1,\infty}R_{ij} \quad \text{for} \quad u \geqslant 0.$$

Let A be any defining system for $\langle V_1, ..., V_n\rangle$. Then $\bar{A}$ is a permanent cycle which converges to an element of $\langle W_1, ..., W_n\rangle$.

Proof. We shall construct a defining system B for $\langle W_1, ..., W_n\rangle$ such that

$\pi(B_{ij}) = A_{ij} \in ME_r R_{ij}$. $\tilde{B}$ will then represent an element of $\langle V_1, ..., W_n \rangle$ to which $\tilde{A}$ converges. We proceed by induction on $j - i$. For $j - i = 1$, we simply choose appropriate matrices of representative cycles for the W_i; we can do this since V_i converges to W_i. Suppose the B_{ij} have been found for $j - i < k$, where $1 < k < n$. Fix i and j such that $j - i = k$. Let $a \in E_r^{pq} R_{ij}$ be any entry of A_{ij}, say the (y, z)th. Let e and f denote the (y, z) entries of $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$, respectively. Then $d_r(a) = e$ and, by the induction hypothesis, $\pi(f) = e \in E_r R_{ij}$. We may therefore choose $c \in F^p R_{ij}$ such that $\pi(c) = a \in E_r^{pq} R_{ij}$ and $d(c) \equiv f \bmod F^{p+r+1} R_{ij}$. It suffices to show that by adding a suitable chain in $F^{p+1} R_{ij}$ to c we can obtain a chain b such that $d(b) = f$. Indeed, we can then take b to be the (y, z) entry of B_{ij}; the desired condition $\pi(b) = a \in E_r^{pq} R_{ij}$ will follow since we will have that $\pi(b) = \pi(c)$. Suppose that no such chain can be found. Then $g = f - d(c)$ must not be a boundary in $F^{p+1} R_{ij}$. However, since $\langle W_1, ..., W_n \rangle$ is strictly defined, f and therefore g must be a boundary in R_{ij}. Let t be that integer such that g is homologous in $F^{p+1} R_{ij}$ to an element, say h, of $F^{p+t} R_{ij}$ but not to an element of $F^{p+t+1} R_{ij}$. Then $t > r$ and $\pi(h)$ is nonzero in $E_t^{p+t, q-t+1} R_{ij}$. Since g and therefore h is a boundary in R_{ij}, there exists $u \geqslant 0$ such that $\pi(h)$ is killed in $E_{t+u} R_{ij}$. Then $E_{t+u}^{p-u, q+u} R_{ij} \not\subset E_{t+u,\infty} R_{ij}$. Since this contradicts (*), the B_{ij} can be constructed and the proof is complete.

The result above was previously stated by Ivanovskii [*4*] for ordinary Massey products in a certain spectral sequence, with $A = Z_2$. It should be noticed that there is nothing in the statement or proof to prevent $\tilde{A}$ from being killed by higher differentials. If this occurs and if $\tilde{A} \in E_r^{pq} R_{0n}$, then the cycle $\tilde{B} \in F^p R_{0n}$ found in the proof is necessarily homologous to an element (possibly zero) of $F^{p+1} R_{0n}$.

Remarks 4.2. Suppose that all of the hypotheses of the theorem are satisfied, except that it is not known that $\langle V_1, ..., V_n \rangle$ is defined in $E_{r+1} R_{0n}$. Suppose, in addition to (*), that

$$(*') \quad \text{If } (p, q) \in D_{ij}, \quad 1 < j - i < n, \quad \text{then} \quad E_{r+u}^{p-u, q+u} R_{ij} \subset E_{r+u,\infty} R_{ij} \quad \text{for} \quad u \geqslant 1.$$

Then $\langle V_1, ..., V_n \rangle$ is strictly defined, and the conclusion of the theorem is therefore valid. To see this, suppose given any partial defining system $\{A_{ij} \mid j - i < k\}$, where $1 < k < n$. Fix i and j such that $j - i = k$. Then $\tilde{A}_{ij} \in ME_{r,\infty} R_{ij}$ since, by the proof of the theorem, $\tilde{A}_{ij} = \pi(\tilde{B}_{ij})$ where $\tilde{B}_{ij}$ represents an element of $\langle W_i, ..., W_j \rangle$. Let e be the (y, z) entry of $\tilde{A}_{ij}$. If $e = 0$, let 0 be the (y, z) entry of A_{ij}. Suppose $e \neq 0$. Since $\langle W_i, ..., W_j \rangle = \{0\}$ and $\tilde{A}_{ij} = \pi(\tilde{B}_{ij})$, e cannot survive to a non-zero element of $E_\infty R_{ij}$, hence e must be killed in some $E_{r+u} R_{ij}$, $u \geqslant 0$. By (*') we must have $u = 0$. Thus there exists $a \in E_r R_{ij}$ such that $d_r(a) = e$ and the given partial defining system can be completed to a defining system A. By Lemma 1.3, it follows that $\langle V_1, \ldots, V_n \rangle$ is strictly defined.

We next turn to the problem of computing $d_s \langle V_1, \ldots, V_n \rangle$ when the $d_s V_i$ are known. The most useful form of the result is given in Corollary 4.4.

THEOREM 4.3. *Let $\langle V_1, \ldots, V_n \rangle$ be defined in $E_{r+1} R_{0n}$. Let $s > r$ be given such that $d_t V_i = 0$ for $t < s$ and $1 \leqslant i \leqslant n$ and such that the following condition (*) is satisfied.*

(*) *If $(p, q) \in D_{ij}$, $1 < j - i < n$, then, for each t such that $r < t < s$,*

$$E_t^{p+t, q-t+1} R_{ij} = 0 \quad \textit{and} \quad E_{r+s-t}^{p+t, q-t+1} = 0.$$

Let $\alpha \in \langle V_1, \ldots, V_n \rangle \subset E_{r+1} R_{0n}$. Then $d_t(\alpha) = 0$ for $t < s$ and there exist $Y_k \in ME_{r+1,k}$ such that Y_k survives to $d_s V_k$ and such that $\langle X_1, \ldots, X_n \rangle$ is defined in $E_{r+1} R_{0n}$ and contains an element γ which survives to $-d_s(\alpha)$, where

$$X_1 = (Y_1, \bar{V}_1), \qquad X_i = \begin{pmatrix} V_i & 0 \\ Y_i & \bar{V}_i \end{pmatrix} \quad \textit{if} \quad 1 < i < n, \qquad X_n = \begin{pmatrix} V_n \\ Y_n \end{pmatrix}.$$

Proof. Let A be a defining system for $\langle V_1, \ldots, V_n \rangle$ such that $\{\tilde{A}\} = \alpha$. Suppose that we have found matrices $B_{ij} \in MR_{ij}$ which satisfy

(i) $\pi(B_{ij}) = A_{ij} \in ME_r R_{ij}$ and

(ii) If $b \in F^p R_{ij}$ is the (y, z) entry of B_{ij} and $f \in F^{p+r} R_{ij}$ is the (y, z) entry of $\tilde{B}_{ij}$ if $j > i + 1$, or zero if $j = i + 1$, then $db \equiv f \bmod F^{p+s} R_{ij}$.

Then define $G_{ij} \in ME_r R_{ij}$ to be the matrix whose (y, z) entry is $\pi(db - f) \in E_s^{p+s,*} R_{ij}$ (b and f as in (ii)). Let $Y_i = \{G_{i-1,i}\} \in ME_{r+1} R_{i-1,i}$; clearly Y_i survives to $d_s V_i$. Define

$$C_{0j} = (G_{0j}, \bar{A}_{0j}), \quad C_{ij} = \begin{pmatrix} A_{ij} & 0 \\ G_{ij} & \bar{A}_{ij} \end{pmatrix} \text{ if } 0 < i < j < n, \quad C_{in} = \begin{pmatrix} A_{in} \\ G_{in} \end{pmatrix}.$$

We claim that C is a defining system for $\langle X_1, \ldots, X_n \rangle$ such that $-\tilde{C}$ survives to $d_s \alpha$. $\tilde{C}$ will therefore represent the desired element γ. To justify our claim, observe that $d_r G_{ij} = -\pi d\tilde{B}_{ij}$ if $j > i + 1$. By the usual trick of adding and subtracting the same elements, we find

$$\text{(a)} \qquad -d\tilde{B}_{ij} = \sum_{k=i+1}^{j-1} \left[\left(-d\bar{B}_{ik} + \sum_{m=i+1}^{k-1} B_{im} \bar{B}_{mk} \right) B_{kj} + B_{ik} \left(dB_{kj} - \sum_{m=k+1}^{j-1} \bar{B}_{km} B_{mj} \right) \right].$$

Clearly π applied to the right side gives $\sum_{k=i+1}^{j-1} (\bar{G}_{ik}A_{kj} + A_{ik}G_{kj})$. This shows that C is indeed a defining system for $\langle X_1, ..., X_n \rangle$. Applied to $\tilde{B}_{0n}$, (a) also shows that $\tilde{C}_{0n}$ survives to $-d_s(\alpha)$. (Observe the change in filtration between $\tilde{B}_{ij}$ and G_{ij}; this is what permits (a) to give a computation of d_r if $(i, j) \neq (0, n)$ and of d_s if $(i, j) = (0, n)$.) Now that we have justified our claim, it remains only to construct the matrices B_{ij}. Since $d_t V_i = 0$ for $t < s$, $B_{i-1,i}$ can surely be found as desired. Suppose that B_{ij} have been found for $j - i < k$, where $1 < k < n$, and fix i and j such that $j - i = k$. Let $a \in E_r^{pq} R_{ij}$ and $e \in E_r^{p+r,q-r+1} R_{ij}$ denote the (y, z) entries of A_{ij} and $\tilde{A}_{ij}$, respectively. Then $d_r(a) = e$ and, with f as in (ii), $\pi(f) = e$. We may therefore choose $c \in F^p R_{ij}$ such that $\pi(c) = a$ and $d(c) \equiv f \bmod F^{p+r+1} R_{ij}$. It suffices to show that by adding a suitable chain in $F^{p+1} R_{ij}$ to c we can obtain a chain b such that $d(b) \equiv f \bmod F^{p+s} R_{ij}$, since $\pi(b) = a$ will then follow from $\pi(b) = \pi(c)$. Suppose that no such chain can be found. Let t be that integer such that $g = f - d(c)$ is homologous in $F^{p+1} R_{ij}$ to an element, say h, of $F^{p+t} R_{ij}$ but not to an element of $F^{p+t+1} R_{ij}$. Then $r < t < s$ and $\pi(h)$ is nonzero in $E_u^{p+t,q-t+1} R_{ij}$, where $u = \min(t, r + s - t)$. Here $\pi(h)$ survives to $E_{r+s-t} R_{ij}$ since $d(h) = d(f)$ and $d(f) \in F^{p+r+s} R_{ij}$ by the induction hypothesis, and $\pi(h)$ does not bound before stage t by the choice of t. This contradiction to (*) establishes the theorem.

It should be observed that the technical hypotheses of the theorem are vacuous in the case $s = r + 1$.

COROLLARY 4.4. *Assume in addition to the hypotheses of the theorem that for* $1 \leq k \leq n$ *there is just one matrix* $Y_k \in ME_{r+1} R_{k-1,k}$ *which survives to* $d_s V_k$. *By abuse, write* $Y_k = d_s V_k$, *and suppose that each*

$$\langle \bar{V}_1, ..., \bar{V}_{k-1}, d_s V_k, V_{k+1}, ..., V_n \rangle$$

is strictly defined in $E_{r+1} R_{0n}$. *Assume further that all matric Massey products in sight have zero indeterminacy. Then*

$$d_s \langle V_1, ..., V_n \rangle = - \sum_{k=1}^{n} \langle \bar{V}_1, ..., \bar{V}_{k-1}, d_s V_k, V_{k+1}, ..., V_n \rangle.$$

Proof. This follows immediately from the case $l = 0$ of Proposition 2.10.

It is perfectly possible, and quite common in the applications, to have an element $\alpha \in \langle V_1, ..., V_n \rangle \subset E_{r+1} R_{0n}$ such that $d_s(\alpha) \neq 0$ for some $s > r$ even though each V_i is a matrix of permanent cycles. For example, such behavior must occur in any spectral sequence $\{E_r\}$ such that $E_2^{pq} = \mathrm{Ext}_A^{pq}(A, A)$ for some connected A-algebra A, $E_2^{1,*} = E_{2,\infty}^{1,*}$, and $E_2 \neq E_\infty$ (because, by [*10*], all elements of $E_2^{p,*}$ for $p > 1$ are built up via matric Massey products from the elements of $E_2^{1,*}$). The following result relates such differentials to lower order matric Massey products in the various $MH(R_{ij})$. The most useful form of the result is given in Corollary 4.6. (In our general context, the remarks preceding Proposition 2.3 are needed to interpret the statement of the theorem.)

THEOREM 4.5. *Let $\langle V_1, ..., V_n \rangle$ be defined in $E_{r+1}R_{0n}$. Assume that $V_i \in ME_{r+1,\infty}R_{i-1,i}$ and that V_i converges to W_i. Let l, $1 \leqslant l \leqslant n-2$, be given such that $\langle W_k, ..., W_{k+l} \rangle$ is strictly defined in $MH(R_{k-1,k+l})$ for $1 \leqslant k \leqslant n-l$ and such that the following condition (*) is satisfied.*

(*) If $(p, q) \in D_{ij}$, $1 < j - i \leqslant l$, then $E_{r+u+1}^{p-u,q+u}R_{ij} \subset E_{r+u+1,\infty}R_{ij}$ for $u \geqslant 0$.

Further, let $s > r$ be given such that the following condition (**) is satisfied.

(**) *If $(p, q) \in D_{ij}$, $l < j - i < n$, then, for each t such that $r < t < s$,*

$$E_t^{p+t,q-t+1}R_{ij} = 0 \qquad \textit{and, if} \qquad j - i > l + 1, \qquad E_{r+s-t}^{p+t,q-t+1}R_{ij} = 0.$$

Let $\alpha \in \langle V_1, ..., V_n \rangle \subset E_{r+1}R_{0n}$. Then $d_t(\alpha) = 0$ for $t < s$ and there exist $Y_k \in ME_{r+1}R_{k-1,k+l}$ such that Y_k converges to an element of $\langle W_k, ..., W_{k+l} \rangle$ and such that $\langle X_1, ..., X_{n-l} \rangle$ is defined in $E_{r+1}R_{0n}$ and contains an element γ which survives to $d_s(\alpha)$, where

$$X_1 = (Y_1, V_1), \quad X_i = \begin{pmatrix} V_{i+l} & 0 \\ Y_i & V_i \end{pmatrix} \text{ if } 1 < i < n - l, \quad X_{n-l} = \begin{pmatrix} V_n \\ Y_{n-l} \end{pmatrix}.$$

Proof. The hypotheses and proof both borrow elements from Theorems 4.1 and 4.3. Let A be a defining system for $\langle V_1, ..., V_n \rangle$ such that $\{\tilde{A}\} = \alpha$. We claim that there exist matrices $B_{ij} \in MR_{ij}$, $j - i < n$, which satisfy

(i) $\pi(B_{ij}) = A_{ij} \in ME_rR_{ij}$

(ii) $\{B_{ij} \mid k-1 \leqslant i < j \leqslant k + l, j - i \leqslant l\}$ is a defining system for $\langle W_k, ..., W_{k+l} \rangle$,

(iii) If $i < j \leqslant n - l$ and if $b \in F^pR_{i,j+l}$ is the (y, z) entry of $B_{i,j+l}$ and $f \in F^{p+r}R_{i,j+l}$ is the (y, z) entry of $\tilde{B}_{i,j+l}$, then $d(b) \equiv f \bmod F^{p+s}R_{i,j+l}$.

To prove this, we first construct the B_{ij} for $j - i \leqslant l$ by the proof of Theorem 4.1 and then construct the B_{ij} for $j - i > l$ by the proof of Theorem 4.3. (Here, in (**), we do not require $E_{r+s-t}^{p+t,q-t+1}R_{ij} = 0$ for $j - i = l + 1$ since the $\tilde{B}_{k-1,k+l}$ are known to be matrices of cycles.) This proves the claim. Now define $G_{ij} \in ME_rR_{i,j+l}$, $j \leqslant n - l$, to be the matrix whose (y, z) entry is $\pi(f - db) \in E^{p+s,*}R_{i,j+l}$ (b and f as in (iii)). Let

$Y_k = \{G_{k-1,k}\} \in ME_{r+1}R_{k-1,k+l}$. Since $Y_k = \pi(\tilde{B}_{k-1,k+l} - dB_{k-1,k+l})$, Y_k converges to an element of $\langle W_k ,..., W_{k+l}\rangle$. Next, define

$$C_{0j} = (G_{0j}, \bar{A}_{0j}), \qquad C_{ij} = \begin{pmatrix} A_{i+l,j+l} & 0 \\ G_{ij} & \bar{A}_{ij} \end{pmatrix} \quad \text{if} \quad 0 < i < j < n - l,$$

$$C_{i,n-l} = \begin{pmatrix} A_{i+l,n} \\ G_{i,n-l} \end{pmatrix}.$$

Here $d_r G_{ij} = \pi d\tilde{B}_{i,j+l}$. A straightforward calculation, using (ii), yields the following formula

$$\text{(a)} \qquad d\tilde{B}_{i,j+l} = \sum_{k=i+1}^{j-1} \Big[\Big(-\sum_{m=i+1}^{k+l-1} B_{im}\bar{B}_{m,k+l} + d\bar{B}_{i,k+l}\Big) B_{k+l,j+l}$$

$$+ B_{ik}\Big(\sum_{m=k+1}^{j+l-1} \bar{B}_{km}B_{m,j+l} - dB_{k,j+l}\Big)\Big].$$

Clearly π applied to the right side gives $\sum_{k=i+1}^{j-1} (\bar{G}_{ik}A_{k+l,j+l} + A_{ik}G_{kj})$. This shows that C is a defining system for $\langle X_1 ,..., X_n\rangle$, and (a) applied in the case $(i, j + l) = (0, n)$ shows that $\gamma = \{\tilde{C}\}$ survives to $d_s(\alpha)$. (As in the proof of Theorem 4.3, the filtration shift between $\tilde{B}_{i,j+l}$ and G_{ij} permit (a) to give a computation of d_r if $(i, j + l) \neq (0, n)$ and of d_s if $(i, j + l) = (0, n)$.) This completes the proof.

COROLLARY 4.6. *Assume in addition to the hypotheses of the theorem that for $1 \leqslant k \leqslant n - l$ there is just one matrix $Y_k \in ME_{r+1}R_{k-1,k+l}$ such that Y_k converges to an element of $\langle W_k ,..., W_{k+l}\rangle$, and suppose that each $\langle \bar{V}_1 ,..., \bar{V}_{k-1} , Y_k , V_{k+l+1} ,..., V_n\rangle$ is strictly defined in $E_{r+1}R_{0n}$. Assume further that all matric Massey products in sight, except possibly the $\langle W_k ,..., W_{k+l}\rangle$, have zero indeterminacy. Then*

$$d_s\langle V_1 ,..., V_n\rangle = \sum_{k=1}^{n-l} \langle \bar{V}_1 ,..., \bar{V}_{k-1} , Y_k , V_{k+l+1} ,..., V_n\rangle.$$

Proof. This follows immediately from Proposition 2.10.

It should be observed that hypothesis (*) of the theorem is vacuous if $l = 1$ and hypothesis (**) is vacuous if $s = r + 1$. The uniqueness hypothesis on Y_k in the corollary can often be verified in the applications by showing that there is only one element of the required degree in $E_{r+1}R_{k-1,k+l}$ for each entry of Y_k. We remark that, as in Theorem 4.1, there is nothing in the statement or proof of the theorem to prevent Y_k from being killed by higher differentials.

Applications of this theorem go in two directions. One can use it either to compute higher differentials from knowledge of matric Massey products in the $H(R_{ij})$ or to compute matric Massey products in the $H(R_{ij})$ from knowledge of the higher differentials. The principle behind the latter application is as follows. Suppose, under the hypotheses of the theorem, that $d_s(\alpha)$ is known and that there are unique matrices Y_k for which $d_s(\alpha)$ can be written in the form $\langle X_1, ..., X_n \rangle$. Then we are guaranteed that Y_k converges to an element of $\langle W_k, ..., W_{k+l} \rangle$. A matric Massey product, or ordinary product if $l = 1$, so detected necessarily takes values in too high a filtration degree to be detected by Theorem 4.1, since Y_k has higher filtration than does $\langle V_k, ..., V_{k+l} \rangle$, to which Theorem 4.1 applies. In other words, the theorem gives formal precision to the commonly observed phenomenon that nontrivial higher differentials are related to nontrivial extensions in the limit terms of spectral sequences. Examples of applications of the theorem in both directions will appear in [*12*], where the results of this section will be employed in studying the cohomology of the Steenrod algebra.

Acknowledgment

Some of the material of this paper is folklore, but little of it has appeared in the literature. Correspondence with D. S. Kahn, M. Mahowald and R. M. F. Moss about triple products in the Adams spectral sequence aided in the formulation of the results of Section 4, and B. Gray suggested some of the new relations given in Section 3. Parts of Sections 2 and 3 owe much to the work of D. Kraines. Considerable simplification of notation resulted from suggestions of S. MacLane.

References

1. Clark, A. Homotopy commutativity and the Moore spectral sequence. *Pac. J. Math.* **15** (1965), 65–74.
2. Eilenberg, S. and Moore, J. C. Limits and spectral sequences. *Topology* **1** (1961), 1–23.
3. Hirsch, G. Quelques propriétés des produits de Steenrod. *C. R. Acad. Sci. Paris* **241** (1955), 923–925.
4. Ivanovskii, N. Cohomology of the Steenrod algebra. *Doklady* **157** (1964), 1284–1287.
5. Kraines, D. Massey higher products. *Trans. Am. Math. Soc.* **124** (1966), 431–449.
6. Lawrence, R. Ph.D. Dissertation. University of Chicago, 1968.
7. Massey, W. S. Some higher order cohomology operations. Symposium internacional de topologia algebraica, pp. 145–154. Mexico City, 1958.
8. May, J. P. The cohomology of restricted Lie algebras and of Hopf algebras. *J. Algebra* **3** (1966), 123–146.
9. May, J. P. The cohomology of principal bundles, homogeneous spaces, and two-stage Postnikov systems. *Bull. Am. Math. Soc.* **74** (1968), 334–339.

10. May, J. P. The structure and applications of the Eilenberg-Moore spectral sequences. To appear.
11. May, J. P. The cohomology of B Spin(n). To appear.
12. May, J. P. The cohomology of the Steenrod algebra. To appear.
13. Moss, R. M. F. Secondary compositions and the Adams spectral sequence. To appear.
14. Stasheff, J. D. Homotopy associativity of H-spaces I, II. *Trans. Am. Math. Soc.* **108** (1963), 275–293 and 293–312.