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Abstract

We discuss some general properties of TR and its $K(1)$-localization. We prove that after $K(1)$-localization, TR of $H\mathbb{Z}$-algebras is a truncating invariant in the Land–Tamme sense, and deduce h-descent results. We show that for regular rings in mixed characteristic, TR is asymptotically $K(1)$-local, extending results of Hesselholt and Madsen. As an application of these methods and recent advances in the theory of cyclotomic spectra, we construct an analog of Thomason's spectral sequence relating $K(1)$-local K-theory and étale cohomology for $K(1)$-local TR.

1. Introduction

The topological cyclic homology, $\mathrm{TC}(R)$, of a ring (or ring spectrum) R is a basic invariant introduced by Bökstedt, Hsiang, and Madsen [BHM93] (see also Dundas, Goodwillie, and McCarthy [DGM13] and Nikolaus and Scholze [NS18]) with many applications in algebraic K-theory. Its p-adic completion $\mathrm{TC}(R;\mathbb{Z}_p)$ arises as the fixed points of an operator called Frobenius on another invariant $\mathrm{TR}(R;\mathbb{Z}_p)$, which plays a central role in the approach to TC via equivariant stable homotopy theory. The construction $\mathrm{TR}(-;\mathbb{Z}_p)$ is often of arithmetic significance; for instance, the foundational calculations [HM03, HM04, GH06] of the p-adic K-theory of local fields F are based on the relationship between $\mathrm{TR}(\mathcal{O}_F;\mathbb{Z}_p)$ and the de Rham–Witt complex with log poles of $\mathcal{O}_F$.

In this paper, we prove some structural results about TR and how it relates to its $K(1)$-localization, throughout at an implicit prime number p. We will only consider p-typical TR and after p-adic completion. The operation of $K(1)$-localization when applied to algebraic K-theory is dramatically simplifying, as shown by Thomason [Tho85, TT90]; in particular, $K(1)$-local K-theory satisfies étale descent and admits a descent spectral sequence from étale cohomology under mild hypotheses; see [CM19] for a modern account. Here we study analogs of some of these properties for $L_{K(1)}\mathrm{TR}(-)$. Our starting point is the following theorem.

Theorem 1.1. *As a functor on connective $H\mathbb{Z}$-algebras, $L_{K(1)}\mathrm{TR}(-)$ is a truncating invariant in the Land–Tamme sense [LT19]. In other words, if R is a connective $H\mathbb{Z}$-algebra, then $L_{K(1)}\mathrm{TR}(R) \xrightarrow{\sim} L_{K(1)}\mathrm{TR}(\pi_0 R)$.*

Theorem 1.1 refines results of [LMMT20, BCM20], where it is shown that $L_{K(1)}K(-)$ and (equivalently) $L_{K(1)}\mathrm{TC}(-)$ are truncating invariants of connective $H\mathbb{Z}$-algebras, which ultimately

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follows from the claim

$$L_{K(1)}K(\mathbb{Z}/p^n) = 0, \quad n \geq 1, \tag{1}$$

and more generally (and consequently) for any $H\mathbb{Z}$-algebra R,

$$L_{K(1)}K(R) \xrightarrow{\sim} L_{K(1)}K(R[1/p]). \tag{2}$$

In [BCM20], (1) is proved via a calculation in prismatic cohomology; in [LMMT20], (1) is proved using some unstable chromatic homotopy theory. Our proof of Theorem 1.1 (which also gives a new proof of (1)) is based on a direct TC-theoretic argument via estimation of exponents of nilpotence of the Bott element; in fact, it yields a slightly stronger result (Theorem 3.1 below).

The property of being truncating yields many pleasant features of the construction $L_{K(1)}\mathrm{TR}(-)$: by [LT19], one obtains *cdh*-descent and excision. Since we are working $K(1)$-locally, we can combine this with results of [CMNN20] to obtain h-descent.

Theorem 1.2. *Any $K(1)$-local localizing invariant which is truncating, such as $L_{K(1)}\mathrm{TR}(-)$, satisfies h-descent on quasi-compact and quasi-separated (qcqs) schemes.*

In particular, $L_{K(1)}\mathrm{TR}(-)$ satisfies étale descent. This is not so surprising, since $\mathrm{TR}(-;\mathbb{Z}_p)$ itself (like all Hochschild-theoretic invariants) actually satisfies flat descent; see [BMS19, § 3]. However, Theorem 1.2 (together with (1)) leads to étale descent in the generic fiber. Since $\mathrm{TR}(R;\mathbb{Z}_p)$ of a ring R depends only on the (derived) p-adic completion of R, we can informally view $L_{K(1)}\mathrm{TR}(R)$ as an invariant of the 'rigid space' associated to $R[1/p]$.

Example 1.3 (Galois descent in the generic fiber). Let $R \to S$ be a finite, finitely presented map of rings. Suppose we have a finite group G acting on S such that $R[1/p] \to S[1/p]$ is G-Galois. Then, for any $K(1)$-local localizing invariant E which is truncating, we have $E(R) \xrightarrow{\sim} E(S)^{hG}$.

Recall that the Lichtenbaum–Quillen conjecture, refined by the Beilinson–Lichtenbaum conjecture proved by Voevodsky and Rost (see [HW19] for an account), predicts that for $\mathbb{Z}[1/p]$-algebras A satisfying mild finiteness conditions, the p-adic K-theory $K(A;\mathbb{Z}_p)$ is 'asymptotically $K(1)$-local'; that is, the map $K(A;\mathbb{Z}_p) \to L_{K(1)}K(A;\mathbb{Z}_p)$ is an equivalence in high enough degrees. We next discuss analogs of such statements for $\mathrm{TR}(R;\mathbb{Z}_p)$ for p-adic rings R. Indeed, in [HM03, HM04] it is shown that if R is smooth of relative dimension d over a discrete valuation ring $\mathcal{O}_K$ of mixed characteristic with perfect residue field of characteristic $p > 2$, then $\mathrm{TR}(R;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R;\mathbb{F}_p)$ is d-truncated; more precisely, this is a consequence of the relationship shown in [HM03, HM04] with the absolute de Rham–Witt complex. We prove this asymptotic $K(1)$-locality more generally for regular rings satisfying F-finiteness hypotheses from the Beilinson–Lichtenbaum conjecture applied to the generic fiber as well as the connection between TR and p-typical curves [Hes96]. We expect that there should be a purely p-adic proof of this result (as in [HM03, HM04] in the smooth case).

Theorem 1.4. *Let R be a p-torsion-free excellent regular noetherian ring. Suppose that R/p is finitely generated as a module over its subring of pth powers. Suppose, furthermore, that for all $\mathfrak{p} \in \mathrm{Spec}(R)$ containing (p), we have $\dim R_{\mathfrak{p}} + \log_p[\kappa(\mathfrak{p}) : \kappa(\mathfrak{p})^p] \leq d$ for some $d \geq 0$. Then the map $\mathrm{TR}(R;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R;\mathbb{F}_p)$ is $(d-1)$-truncated.*

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Using the Antieau–Nikolaus theory of topological Cartier modules [AN21], we relate the property of $\mathrm{TR}(R;\mathbb{F}_p)$ being 'asymptotically $K(1)$-local' to the extensively studied Segal conjecture for $\mathrm{THH}(R)$, that is, the condition that the cyclotomic Frobenius $\varphi\colon \mathrm{THH}(R;\mathbb{F}_p) \to \mathrm{THH}(R;\mathbb{F}_p)^{tC_p}$ should be an equivalence in high degrees. In particular, we obtain a version of the Segal conjecture for $\mathrm{THH}(R)$ when R is regular. We expect that there should be a filtered version of this statement, using the motivic filtrations of Bhatt, Morrow, and Scholze [BMS19].

COROLLARY 1.5. *Let R be as in Theorem 1.4. Then the cyclotomic Frobenius* $\varphi\colon \mathrm{THH}(R;\mathbb{F}_p) \to \mathrm{THH}(R;\mathbb{F}_p)^{tC_p}$ *is $(d-1)$-truncated.*

Finally, we study the analog of Thomason's spectral sequence [Tho85, TT90] from étale cohomology to $K(1)$-local algebraic K-theory. For a scheme X over $\mathbb{Z}[1/p]$ satisfying mild finiteness conditions (to wit: X should be qcqs of finite Krull dimension, with a uniform bound on the mod p virtual cohomological dimensions of the residue fields [RØ06, CM19]), one has a convergent spectral sequence

$$E_2^{s,t} = H^s(X, \mathbb{Z}_p(t)) \implies \pi_{2t-s} L_{K(1)} K(X).$$

We can construct a similar spectral sequence for $L_{K(1)}\mathrm{TR}$ under significantly stronger finiteness and regularity conditions, arising from a natural filtration. To formulate the E_2-term (or the graded pieces of this filtration), we use the arc_p-topology of [BM18].

DEFINITION 1.6 (The arc_p-topology and arc_p-cohomology). We say that a map of derived p-complete rings $R \to R'$ is an *arc_p-cover* if, for every map $R \to V$ for V a rank 1 valuation ring which is p-complete and such that $p \neq 0$, there exist an extension of rank 1 valuation rings $V \to W$ and the following commutative diagram.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \\ V \arrow[r] & W \end{tikzcd}$$

The arc_p-topology is the finitary Grothendieck topology on the opposite of the category of derived p-complete rings defined such that a family $\{R \to R'_\alpha\}_{\alpha \in A}$ is a covering family if and only if there exists a finite subset $A' \subset A$ such that $R \to \prod_{\alpha \in A'} R'_\alpha$ is an arc_p-cover.

Given any functor F from derived p-complete rings to abelian groups, we let $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), F(-))$ denote the arc_p-cohomology of $F(-)$ on a derived p-complete ring R.[1]

Example 1.7. (i) We can consider the arc_p-cohomology of the structure presheaf $\mathcal{O}$, $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), \mathcal{O})$. This is closely related to the perfectoidizations considered in [BS19, §§ 7 and 8], which work with the p-complete arc-topology rather than the arc_p-topology. For $R = \mathbb{Z}_p$, it is not difficult to see that this is the continuous $\mathrm{Gal}(\mathbb{Q}_p)$-homotopy invariants of the derived saturation $(\mathcal{O}_C)_*$ for $C = \widehat{\overline{\mathbb{Q}_p}}$ (in the sense of almost ring theory [GR03]).

(ii) We consider the arc_p-cohomology of the Witt vector presheaf $W(\mathcal{O})$, denoted $R \mapsto R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O}))$, as well as its p-adic Tate twists $W(\mathcal{O})(i)$ for $i \in \mathbb{Z}$.

[1] For set-theoretic reasons, to define arc_p-cohomology we should fix a cutoff cardinal. We will only consider situations where the choice of cutoff cardinal does not affect the result.

Theorem 1.8. *Let K be a complete nonarchimedean field of mixed characteristic $(0,p)$ with ring of integers $\mathcal{O}_K$ and residue field k with $[k:k^p]<\infty$. Suppose either K is discretely valued or K is perfectoid. Let R be a formally smooth $\mathcal{O}_K$-algebra. Then there exists a natural convergent, exhaustive $\mathbb{Z}$-indexed descending filtration $\mathrm{Fil}^{\geq *}L_{K(1)}\mathrm{TR}(R)$ on $L_{K(1)}\mathrm{TR}(R)$ such that*

$$\mathrm{gr}^i L_{K(1)}\mathrm{TR}(R) \simeq R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O})(i))[2i]. \tag{3}$$

Theorem 1.8 is effectively an étale hyperdescent (in the generic fiber) result together with the calculation for perfectoids. For illustration, we specialize to the case where $R=\mathcal{O}_K$, for K discretely valued with perfect residue field. Then the filtration (3) arises via a type of pro-Galois descent in the generic fiber. If L/K is G-Galois, Example 1.3 and Theorem 1.4 imply that $\mathrm{TR}(\mathcal{O}_K;\mathbb{F}_p)\to\mathrm{TR}(\mathcal{O}_L;\mathbb{F}_p)^{hG}$ is a 0-truncated map. However, this does not help with passage to $\bar{K}$ since TR does not commute with filtered colimits; note that TR has a simple form for $\mathcal{O}_{\bar{K}}$; see [Hes06]. Using again the theory of topological Cartier modules as a 'decompletion' of the theory of cyclotomic spectra (see [AN21]), we prove the following pro-Galois result (see Example 6.9).

Theorem 1.9. *Let K be a complete, discretely valued field of characteristic 0 with perfect residue field k of odd characteristic p. Let $\mathrm{TR}(\mathcal{O}_K|K)$ denote the cofiber of the transfer map $\mathrm{TR}(k)\to\mathrm{TR}(\mathcal{O}_K)$. Then the natural map induces an equivalence*

$$\mathrm{TR}(\mathcal{O}_K|K;\mathbb{F}_p)\to\tau_{\geq 0}\mathrm{Tot}\Big(\mathrm{TR}(\mathcal{O}_{\bar{K}};\mathbb{F}_p)\rightrightarrows\mathrm{TR}(\mathcal{O}_{\bar{K}\otimes_K\bar{K}};\mathbb{F}_p)\overset{\to}{\underset{\to}{\to}}\dots\Big).$$

The idea that TR should satisfy this type of pro-Galois descent in the generic fiber is expressed in [Hes02]; in particular, [Hes02, Conjecture 5.1] predicts a related (but stronger) conclusion at the level of homotopy groups (in particular, the vanishing of higher Galois cohomology groups in the associated descent spectral sequence with $\mathbb{Q}_p/\mathbb{Z}_p$ coefficients).

Conventions

We write Sp for the ∞-category of spectra and $\mathbb{S}$ for the sphere spectrum. We use the theory of cyclotomic spectra in the form developed in [NS18], as well as the theory of topological Cartier modules developed in [AN21]; we write CycSp for the ∞-category of cyclotomic spectra. We write TR for p-typical TR. Given an E_∞-ring B, a homogeneous element $x\in\pi_*(B)$, and a B-module M, we often write M/x for the cofiber of multiplication by x on M. In the case $x=p$, we will often write this as '$;\mathbb{F}_p$'; for example, $\mathrm{THH}(B;\mathbb{F}_p)$ refers to the cofiber of p on $\mathrm{THH}(B)$. A B-algebra always refers, unless otherwise specified, to an E_1-algebra in B-modules.

2. Generalities on $K(1)$-local truncating invariants

Let B be a base connective E_∞-ring. In this section we work with localizing invariants on small B-linear idempotent-complete stable ∞-categories. Unlike in [BGT13], we do not assume compatibility with filtered colimits, so for us a localizing invariant is simply a functor from (small, idempotent-complete) B-linear stable ∞-categories to spectra which carries Verdier quotient sequences to cofiber sequences. Following [LT19], we say that such a localizing invariant E is *truncating* if for every connective B-algebra A, we have $E(A)\xrightarrow{\sim}E(H\pi_0A)$. This implies [LT19, Theorem B] that E satisfies excision.

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Example 2.1. The constructions $L_{K(1)}K(-), L_{K(1)}\mathrm{TC}(-)$ are truncating on connective $H\mathbb{Z}$-algebras, as verified in [LMMT20]. For commutative p-complete rings, the two invariants actually agree (we do not know if this is true for noncommutative p-complete rings; see [BCM20, Question 2.20]). Below we will show that $L_{K(1)}\mathrm{TR}(-)$ is truncating.

In the rest of the section we will assume for simplicity of notation that B is discrete; by the assumption of truncatedness, this does not affect any of the results.

Proposition 2.2. *Let E be a $K(1)$-local localizing invariant of B-linear ∞-categories which is truncating. Then, on the category of discrete B-algebras, we have the following assertions.*

(i) *E is nilinvariant.*
(ii) *E annihilates any B-algebra C which is annihilated by a power of p.*
(iii) *Let $A \to A'$ be a map of B-algebras which is a p-isogeny. Then $E(A) \to E(A')$ is an equivalence.*

Proof. For (i), the fact that E is nilinvariant follows from [LT19, Theorem B]. For (ii), since E is nilinvariant, we may assume C is an $\mathbb{F}_p$-algebra, so that $E(C)$ is by the theory of noncommutative motives [BGT13] a $K(\mathbb{F}_p;\mathbb{Z}_p) = H\mathbb{Z}_p$-module (the last identification by [Qui72]); since E is $K(1)$-local we get $E(C) = 0$.

For (iii), the kernel of $A \to A'$ is annihilated by a power of p, so by (ii) (and excision) we can assume that $A \subset A'$. Let $n \gg 0$, so $p^n A' \subset A$. Then the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ A/(p^n A' \cap A) \arrow[r] & A'/p^n A' \end{tikzcd}$$

is a Milnor square of rings. Applying the localizing invariant E and using excision and (ii) again, we conclude (iii). $\square$

In the next result, we use Voevodsky's h-topology for possibly non-noetherian schemes; in other words, the topology generated by finitely presented v-covers (also called universally subtrusive morphisms); see [Ryd10] or [BS17, § 2].

Theorem 2.3 (h-descent for truncating $K(1)$-local invariants). *Let E be a $K(1)$-local localizing invariant on B-linear ∞-categories which is truncating. Then E satisfies h-descent on qcqs B-schemes.*

Proof. By the results of [LT19, Appendix A], E satisfies *cdh*-descent, and in particular satisfies excision for abstract blow-up squares. The results of [CMNN20] imply that E satisfies finite locally free descent. Since E also satisfies Nisnevich descent (as does any localizing invariant [TT90]), we obtain that E satisfies fppf descent thanks to [Sta20, Tag 05WN]. By [BS17, Theorem 2.9], h-descent (for any sheaf) is implied by fppf descent and excision for abstract blow-up squares. Combining these facts, we conclude. $\square$

Example 2.4. Let E be as above. Let $\pi\colon X' \to X$ be a finitely presented proper morphism (e.g. a finitely presented closed immersion) of qcqs B-schemes such that $X'[1/p] \xrightarrow{\sim} X[1/p]$.

Then $E(X) \xrightarrow{\sim} E(X')$. In fact, by *cdh*-descent, we have a pullback square

$$\begin{tikzcd} E(X) \arrow[r] \arrow[d] & E(X') \arrow[d] \\ E(X \otimes \mathbb{F}_p) \arrow[r] & E(X' \otimes \mathbb{F}_p) \end{tikzcd}$$

and the terms on the bottom vanish by Proposition 2.2.

In the next result we use the notion of nilpotence of a group action; see [Mat18, § 4.1] or [CM19, Definition 2.17] for accounts, or [MNN17] for the general setup in equivariant stable homotopy theory. Let G be a finite group. The collection of *nilpotent* objects of the ∞-category $\mathrm{Fun}(BG, \mathrm{Sp})$ is the thick subcategory generated by the objects which are induced from the trivial subgroup. For an algebra object of $\mathrm{Fun}(BG, \mathrm{Sp})$, nilpotence holds if and only if the Tate construction vanishes. A module over a nilpotent algebra object in $\mathrm{Fun}(BG, \mathrm{Sp})$ is itself nilpotent.

Corollary 2.5 (Galois descent in the generic fiber). *Let E be a $K(1)$-local localizing invariant of B-linear ∞-categories which is truncating. Let $R \to S$ be a finite and finitely presented map of (commutative, discrete) B-algebras. Let G be a finite group acting on S via R-algebra maps. Suppose that $R[1/p] \to S[1/p]$ is G-Galois. Then the natural map induces an equivalence $E(R) \xrightarrow{\sim} E(S)^{hG}$. Moreover, the G-action on $E(S)$ is nilpotent.*

Proof. Replacing S with $S \times R/p$, we may assume without loss of generality that $R \to S$ is an h-cover. Then, by Theorem 2.3, we have

$$E(R) \simeq \mathrm{Tot}(E(C(R \to S)^\bullet)) = \mathrm{Tot}\Big(E(S) \rightrightarrows E(S \otimes_R S) \substack{\to\\\to\\\to} \dots\Big). \tag{4}$$

Since the group G acts on S, we have a natural map of cosimplicial rings from the Čech nerve $C(R \to S)^\bullet$ to the standard resolution $S \rightrightarrows \prod_G S \substack{\to\\\to\\\to}$ for G acting on S (which calculates S^{hG}). This map of cosimplicial rings is an isogeny in each degree; for example, in degree 1, $S \otimes_R S \to \prod_G S$ is an isogeny because it is a map of finitely presented R-modules (see [Sta20, Tag 0564]) which induces an isomorphism after inverting p thanks to the Galois hypothesis. Therefore, by Proposition 2.2, we find that the map induces an equivalence after applying E, and we find from (4) that

$$E(R) \xrightarrow{\sim} \mathrm{Tot}\left(E(S) \rightrightarrows E\left(\prod_G S\right) \substack{\to\\\to\\\to} \dots\right) = E(S)^{hG},$$

which is the desired claim. Finally, to see that the G-action on $E(S)$ is nilpotent, we use that $E(S)$ is a module G-equivariantly over $L_{K(1)}K(S) \xrightarrow{\sim} L_{K(1)}K(S[1/p])$ (via (2)), and the G-action on $L_{K(1)}K(S[1/p])$ is nilpotent by [CMNN20, Theorem 5.6] (see also [CM19, Lemma 4.20]). $\square$

3. The truncating property of $L_{K(1)}\mathrm{TR}(-)$

In this section we prove the following basic result. Throughout, we fix a connective, $K(1)$-acyclic E_∞-ring B (e.g. $H\mathbb{Z}$).

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Theorem 3.1. *The construction $L_{K(1)}\mathrm{TR}(-)$ is truncating on connective E_1-B-algebras. More generally, for any set Q,*[2] *the construction $L_{K(1)}(\prod_Q \mathrm{TR}(-))$ is truncating on connective E_1-B-algebras.*

The proof of Theorem 3.1 will rely on a $K(1)$-acyclicity criterion for cyclotomic spectra (Proposition 3.6), which will use some elementary estimates for exponents of nilpotence with respect to the Bott element β.

In the following, we let ku denote the connective topological K-theory spectrum, so $\pi_*(ku) = \mathbb{Z}[\beta]$ with $|\beta| = 2$. Since B is $K(1)$-acyclic, the associative ring spectrum $B \otimes ku/p$ is annihilated by a power of β. Our strategy is roughly based on bounding the exponents of nilpotence for β in the fixed points $\mathrm{THH}(B)^{C_{p^n}} \otimes ku/p$ and, in particular, showing that they are $O(p^n)$.

Recall that ku is complex-oriented, leading to the following result.

Proposition 3.2. *Let M be a ku-module equipped with an S^1-action. Then, for each $n \geq 1$, the natural map induces an equivalence*

$$M^{hS^1} \otimes_{ku^{BS^1}} ku^{BC_{p^n}} \xrightarrow{\sim} M^{hC_{p^n}}. \tag{5}$$

Proof. Compare [MNN17, § 7.4]. In fact, via the projection formula, $M^{hC_{p^n}} \simeq (M \otimes_{ku} ku^{S^1/C_{p^n}})^{hS^1}$. Here $ku^{S^1/C_{p^n}}$ denotes the ku-valued function spectrum of S^1/C_{p^n} with the corresponding S^1-action, and the tensor product is taken in $\mathrm{Fun}(BS^1, \mathrm{Mod}(ku))$. Let V_n denote the one-dimensional complex representation of S^1 where $z \in S^1$ acts by multiplication by z^{p^n}, and let $S(V_n)$ denote the unit circle in V_n as an S^1-space, so $S(V_n) \simeq S^1/C_{p^n}$. The Spanier–Whitehead dual of the Euler sequence in $\mathrm{Fun}(BS^1, \mathrm{Sp})$, $S(V_n)_+ \to S^0 \to S^{V_n}$ and the complex orientability of ku together show that $ku^{S^1/C_{p^n}} \in \mathrm{Fun}(BS^1, \mathrm{Mod}(ku))$ belongs to the thick subcategory generated by the unit. Therefore, applying the right adjoint $(-)^{hS^1}\colon \mathrm{Fun}(BS^1, \mathrm{Mod}(ku)) \to \mathrm{Mod}(ku^{BS^1})$, we find that the natural map

$$M^{hS^1} \otimes_{ku^{BS^1}} (ku^{S^1/C_{p^n}})^{hS^1} \to (M \otimes_{ku} ku^{S^1/C_{p^n}})^{hS^1} = M^{hC_{p^n}}$$

is an equivalence, whence the result. □

By complex orientability, we have

$$\pi_*(ku^{BS^1}) = \mathbb{Z}[\beta][[x]], \quad |\beta| = 2, |x| = -2.$$

Consider the formal group law over $\mathbb{Z}[\beta]$ given by $f(u, v) = u + v + \beta uv$; this is the formal group law associated to the complex-oriented ring spectrum ku, and is homogeneous of degree -2 if $|u|, |v| = -2$. We have an equivalence $ku^{BS^1}/x \simeq ku$. More generally, let $[p^n](x) \in \pi_*(ku^{BS^1})$ denote the p-series of the formal group law f; modulo p, we have $[p^n](x) \equiv \beta^{p^n-1}x^{p^n}$ since $[p^n](x)$ is homogeneous of degree -2 and recovers the multiplicative formal group law under the specialization $\beta \mapsto 1$. By the Eilenberg–Moore spectral sequence or Gysin sequence (of which (5) is a form), we have

$$ku^{BC_{p^n}} = ku^{BS^1}/([p^n](x)). \tag{6}$$

[2] This states informally that if A is a connective B-algebra, then the fiber of the map $\mathrm{TR}(A; \mathbb{F}_p) \to \mathrm{TR}(H\pi_0(A); \mathbb{F}_p)$ has the property that each degree is annihilated by a power of v_1 (depending on the degree; note that this condition is slightly stronger than the fiber simply being $K(1)$-acyclic).

Proposition 3.3. *Let A be a connective E_∞-ring spectrum with S^1-action. Suppose that $\beta^r = 0$ in $\pi_*(A \otimes ku/p)$. Then in $\pi_*((A \otimes ku)^{hC_{p^n}}/p)$ we have $\beta^{p^n-1+rp^n} = 0$.*

Proof. Let $R = A \otimes ku/p$, so R is an associative ku-algebra spectrum with S^1-action. Consider the S^1-homotopy fixed points R^{hS^1}, which is a ku^{BS^1}-algebra. Since $p = 0$ in $\pi_0 R$, we have by Proposition 3.2 and (6),

$$R^{hC_{p^n}} = R^{hS^1} \otimes_{ku^{BS^1}} ku^{BC_{p^n}} = R^{hS^1}/[p^n](x) = R^{hS^1}/\big(\beta^{p^n-1}x^{p^n}\big).$$

Our assumption is that $\beta^r = 0$ in $\pi_*(R)$, which means that we can write $\beta^r = xv \in \pi_*(R^{hS^1})$ for some v since $R = R^{hS^1}/x$. It follows that in $\pi_*(R^{hC_{p^n}})$, we have $\beta^{p^n-1+rp^n} = \beta^{p^n-1}(xv)^{p^n} = 0$. ∎

Now we apply the above to the E_∞-ring $\mathrm{THH}(B)$ equipped with its S^1-action; recall that B is assumed connective and $K(1)$-acyclic.

Proposition 3.4. *There exists a constant $\kappa > 0$ such that for each $i > 0$ we have the following assertions.*

(i) *We have that $\beta^{\kappa i} = 0$ in $(ku/p \otimes \mathrm{THH}(B))^{hC_i}$.*
(ii) *Let N be a $\mathrm{THH}(B) \otimes ku$-module in $\mathrm{Fun}(BS^1, \mathrm{Sp})$. Suppose that N (as a $\mathrm{THH}(B) \otimes ku$-module in $\mathrm{Fun}(BS^1, \mathrm{Sp})$) is induced from the cyclic group $C_i \subset S^1$. Then, for any t, N_{hC_t}/p is annihilated by $\beta^{\kappa i}$.*

Proof. To prove (i), using the transfer and restricting to a p-Sylow subgroup, we may reduce to the case where i is a power of p, say $i = p^n$. Then the claim follows from Proposition 3.3, since β is nilpotent in $ku/p \otimes \mathrm{THH}(B)$ since B is $K(1)$-acyclic. It follows that we can find a κ_1 such that $\beta^{\kappa_1 i} = 0$ in $(ku/p \otimes \mathrm{THH}(B))^{hC_i}$ for all $i > 0$.

For (ii), consider a $\mathrm{THH}(B) \otimes ku$-module N in $\mathrm{Fun}(BS^1, \mathrm{Sp})$ which is the induction of a $\mathrm{THH}(B) \otimes ku$-module N' in $\mathrm{Fun}(BC_i, \mathrm{Sp})$. It follows that $N_{hS^1}/p = (N'/p)_{hC_i}$ is a module over $(ku/p \otimes \mathrm{THH}(B))^{hC_i}$, writing homotopy orbits as a module over homotopy fixed points; this is therefore annihilated by $\beta^{\kappa_1 i}$ by the previous part of the result. For any t, let W_t be the one-dimensional complex representation of S^1 where z acts by multiplication by z^t, so the unit circle $S(W_t)_+ = (S^1/C_t)_+$ as S^1-spaces. We have $N_{hC_t} = (N \otimes S(W_t)_+)_{hS^1}$ by the projection formula for induction and restriction along $C_t \subset S^1$. Then the Euler sequence $S(W_t)_+ \to S^0 \to S^{W_t}$ (as in the proof of Proposition 3.2) and the complex orientability of ku yield a fiber sequence $N_{hC_t} \to N_{hS^1} \to \Sigma^2 N_{hS^1}$. Therefore, N_{hC_t}/p is annihilated by $\beta^{2\kappa_1 i}$; taking $\kappa = 2\kappa_1$ we conclude. ∎

For the next result, we use the notion of a (nonnegatively) *graded cyclotomic spectrum*' see [AMMN20, § 3 and Appendix A] or [Bru01]. A graded cyclotomic spectrum consists of a graded spectrum $X = \bigoplus_{i \geq 0} X_i$ equipped with an S^1-action together with a graded S^1-equivariant cyclotomic Frobenius $\varphi_i \colon X_i \to X_{pi}^{tC_p}$ for each i. Given a nonnegatively graded E_1-ring R, the topological Hochschild homology $\mathrm{THH}(R)$ acquires the structure of a graded cyclotomic spectrum. Given a graded cyclotomic spectrum X, we can consider a graded cyclotomic spectrum $X_{\geq i}$ where we only consider the graded summands in degrees i or higher; this gives any graded cyclotomic spectrum a natural descending filtration. The filtration quotients $X_{\geq i}/X_{\geq pi}$ have trivialized Frobenius because of the grading, and their TR can be thus described explicitly.

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Construction 3.5 (TR of cyclotomic spectra with zero Frobenius). Suppose $X \in \mathrm{CycSp}_{\geq 0}$ is p-complete with trivialized Frobenius. Then, as in [AN21, Remark 2.5] and [NS18, Corollary II.4.7], we obtain a natural equivalence

$$\mathrm{TR}(X; \mathbb{Z}_p) \simeq \prod_{i \geq 0} X_{hC_{p^i}}. \tag{7}$$

For this, we also use the identification for $n \geq 1$, $(X^{tC_p})^{hC_{p^{n-1}}} \simeq X^{tC_{p^n}}$, see [NS18, Lemma II.4.1].

Proposition 3.6 ($K(1)$-acyclicity criterion). *Let X be a positively graded cyclotomic spectrum with the structure of a $\mathrm{THH}(B)$-module. Suppose, for each $i > 0$, that the following assertions hold.*

(i) *X_i is $(2p+1)\kappa i$-connected (where κ is as in Proposition 3.4).*
(ii) *As a $\mathrm{THH}(B)$-module in $\mathrm{Fun}(BS^1, \mathrm{Sp})$, X_i is induced from the cyclic group $C_i \subset S^1$.*

Then, for any set Q, $\prod_Q \mathrm{TR}(X)$ is $K(1)$-acyclic.

Proof. For simplicity of notation, we write $\mathrm{TR}_Q(-) = \prod_Q \mathrm{TR}(-)$. The construction $\mathrm{TR}_Q(-)$ is exact and commutes with geometric realizations on $\mathrm{CycSp}_{\geq 0}$; therefore, it commutes with tensoring with ku. Without loss of generality, we can therefore assume that X is a $\mathrm{THH}(B) \otimes ku$-module in graded cyclotomic spectra. Here we regard ku as a trivial cyclotomic spectrum, that is, via the image of the unique symmetric monoidal functor $\mathrm{Sp} \to \mathrm{CycSp}$.

We consider the descending filtration $\{\mathrm{Fil}^{\geq n} X = X_{\geq p^n}\}_{n \geq 0}$ on the cyclotomic spectrum X; note that the associated graded terms have trivialized Frobenii for degree reasons. This yields a filtration on the spectrum $\mathrm{TR}_Q(X)$, with $\mathrm{Fil}^{\geq n}\mathrm{TR}_Q(X) = \mathrm{TR}_Q(X_{\geq p^n})$. Since $\mathrm{TR}(-)$ preserves connectivity, our assumptions imply that $\mathrm{Fil}^{\geq n}\mathrm{TR}_Q(X)$ is $(2p+1)\kappa p^n$-connective.

We have by (7) that $\mathrm{gr}^n \mathrm{TR}_Q(X) = \bigoplus_{p^n \leq i < p^{n+1}} \prod_Q \prod_{t \geq 0} (X_i)_{hC_{p^t}}$, since the cyclotomic Frobenius is trivial on $\mathrm{gr}^n X$ for grading reasons. Since X_i is induced from $C_i \subset S^1$, it follows from Proposition 3.4 that the ku-module $\mathrm{gr}^n \mathrm{TR}_Q(X)/p$ is annihilated by $\beta^{\kappa p^{n+1}}$.

Now $|\beta| = 2$, and $2\sum_{i=0}^{n-1} \kappa p^{i+1} < (2p+1)\kappa p^n$ with the difference tending to ∞ as $n \to \infty$. Applying Lemma 3.7 below, we conclude that inverting β annihilates $\mathrm{TR}_Q(X)/p$, whence $\mathrm{TR}_Q(X)$ is $K(1)$-acyclic as desired. □

Lemma 3.7. *Let R be a connective E_1-ring spectrum, and let $x \in \pi_t(R)$ be an element. Let $\{\mathrm{Fil}^{\geq n} Y\}_{n \geq 0}$ be a filtered R-module spectrum. Suppose that there exist functions $f, g\colon \mathbb{N} \to \mathbb{N}$ such that:*

(i) *$\mathrm{gr}^n(Y)$ is annihilated by $x^{f(n)}$;*
(ii) *$\mathrm{Fil}^{\geq n}(Y)$ is $g(n)$-connective;*
(iii) *$g(n) - t\sum_{i=0}^{n-1} f(i) \to \infty$ for $n \to \infty$.*

Then $Y[1/x] = 0$.

Proof. Let $y \in \pi_s(Y)$. For each $n > 0$, the class $x^{f(0)+f(1)+\cdots+f(n-1)} y \in \pi_*(Y)$ naturally lifts to $\pi_{s+t(f(0)+\cdots+f(n-1))}(\mathrm{Fil}^{\geq n} Y)$. But for $n \gg 0$, the connectivity of $\mathrm{Fil}^{\geq n} Y$ forces this last group to vanish. Therefore, the image of y in $\pi_s(Y[1/x])$ vanishes as desired. □

We now use the following basic calculation of THH of a free associative algebra, as a spectrum equipped with S^1-action. Versions of this are classical in ordinary Hochschild homology; see [Lod98, § 3.1]. In the language of factorization homology, this result is a special case of the calculation of the factorization homology of a free algebra ([AFT17, Proposition 4.13] and [AF15, Proposition 5.5]).

Theorem 3.8 (THH of a free associative algebra). *Let M be a spectrum, and let $T(M) = \bigoplus_{n\geq 0} M^{\otimes n}$ be the free E_1-algebra spectrum generated by M. Then there is a natural equivalence in* $\mathrm{Fun}(BS^1, \mathrm{Sp})$,

$$\mathrm{THH}(T(M)) \simeq \bigoplus_{n\geq 0} \mathrm{Ind}_{C_n}^{S^1}(M^{\otimes n}), \tag{8}$$

where we use the natural C_n-action on $M^{\otimes n}$ by permuting the factors.

Proof. The results of [AFT17, AF15] (applied to the framed manifold S^1) imply that $\mathrm{THH}(T(M)) \simeq \int_{S^1} T(M) = \bigoplus_{n\geq 0}(\mathrm{Conf}_n(S^1)_+ \otimes M^{\otimes n})_{h\Sigma_n}$, for $\mathrm{Conf}_n(S^1)$ the configuration space of n ordered points on the circle. One checks now (see [CJ98, Example II.14.4]) that $\mathrm{Conf}_n(S^1)$, as a space with $S^1 \times \Sigma_n$-action, is homotopy equivalent to $(S^1 \times \Sigma_n)/C_n$ (with C_n embedded diagonally), whence the claim. □

Proposition 3.9. *Let M, N be connective spectra. Then the map of cyclotomic spectra*

$$\mathrm{THH}(T(M \oplus N)) \otimes \mathrm{THH}(B) \to \mathrm{THH}(T(M)) \otimes \mathrm{THH}(B)$$

induces an equivalence on $L_{K(1)}(\prod_Q \mathrm{TR}(-))$ for any set Q if N is at least $\kappa(2p+1)$-connective.

Proof. We can consider the tensor algebra $T(M \oplus N)$ as a graded E_1-ring spectrum where M is placed in degree 0 and N is placed in degree 1. In this case, if we collect the terms in (8), we find that the ith graded piece of $\mathrm{THH}(T(M \oplus N))$ is the component which is i-homogeneous. Explicitly, for any subset $I \subset \langle n\rangle = \{1, 2, \ldots, n\}$, we write $(M, N)^{\otimes(\langle n\rangle\setminus I, I)}$ for the ordered tensor product of n factors, where the jth factor is M if $j \notin I$ and N if $j \in I$. Expanding (8) gives

$$\mathrm{THH}(T(M \oplus N))_i = \bigoplus_{n\geq 0} \mathrm{Ind}_{C_n}^{S^1}\Bigg(\bigoplus_{I\subset\langle n\rangle, |I|=i} (M, N)^{\otimes(\langle n\rangle\setminus I, I)}\Bigg). \tag{9}$$

Here the C_n-action on the parenthesized term in (9) permutes the various summands. Note in particular that the stabilizer of $I \subset \langle n\rangle$ in the nth summand is a subgroup of a cyclic group $C_i \subset C_n$ since $|I| = i$. In particular, it follows that the ith graded piece of $\mathrm{THH}(T(M \oplus N))$ is induced from $C_i \subset S^1$. Furthermore, since N is at least $\kappa(2p+1)$-connective, it follows that the ith graded piece of $\mathrm{THH}(T(M \oplus N))$ is at least $\kappa(2p+1)$-connective. By Proposition 3.6, it follows that the positively graded part of $\mathrm{THH}(T(M \oplus N)) \otimes \mathrm{THH}(B)$ has vanishing $K(1)$-local $\prod_Q \mathrm{TR}(-)$ for any set Q, whence the result. □

For the next result, if R is an E_1-ring spectrum and N an (R, R)-bimodule, we let $T_R(N) = \bigoplus_{n\geq 0} N \otimes_R N \otimes_R \cdots \otimes_R N$ be the free E_1-algebra under R generated by N.

Proposition 3.10. *Let R be a connective B-algebra. Let N be an at least $\kappa(2p+1)$-connective (R, R)-bimodule. Then the map $\mathrm{THH}(T_R(N)) \to \mathrm{THH}(R)$ induces an equivalence on $L_{K(1)}(\prod_Q \mathrm{TR}(-))$ for any set Q.*

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Proof. Simplicially resolving N by free (R,R)-bimodules (since $\prod_Q \mathrm{TR}(-)$ commutes with geometric realizations), we may assume that N is free on generators (possibly infinitely many) in degrees $\kappa(2p+1)$ or higher. Simplicially resolving R by free B-algebras, we may assume that R is free as well, on some classes in degrees at least 0. In this case, the result follows from Proposition 3.9. □

Lemma 3.11. *Let E be a localizing invariant of B-linear ∞-categories. Suppose that there exists $k \geq 0$ such that for every connective B-algebra R, we have $E(R) \xrightarrow{\sim} E(\tau_{\leq k} R)$. Then E is truncating.*

Proof. We show by descending induction[3] that for any $i \geq 0$ and for any connective B-algebra R, the map $R \to \tau_{\leq i} R$ induces an equivalence on E; taking $i = 0$ gives the theorem. For $i \geq k$, we already know the claim by assumption. Suppose we know the claim for $i+1$; to prove the claim for i, we need to see that $\tau_{\leq i+1} R \to \tau_{\leq i} R$ induces an equivalence on E. Now $\tau_{\leq i+1} R \to \tau_{\leq i} R$ is a square-zero extension, that is, we have the following pullback diagram of B-algebras.

$$\begin{tikzcd}
\tau_{\leq i+1} R \arrow[r] \arrow[d] & H(\pi_0 R) \arrow[d] \\
\tau_{\leq i} R \arrow[r] & H(\pi_0 R) \oplus H(\pi_{i+1} R)[i+2]
\end{tikzcd}$$

Since E is a localizing invariant, the main result of Land and Tamme [LT19] yields a B-algebra $\tilde{R}$ with underlying spectrum $H(\pi_0 R) \otimes_{\tau_{\leq i+1} R} \tau_{\leq i} R$ fitting into a commutative diagram of B-algebras

$$\begin{tikzcd}
\tau_{\leq i+1} R \arrow[r] \arrow[d] & H(\pi_0 R) \arrow[d] \\
\tau_{\leq i} R \arrow[r] & \tilde{R}
\end{tikzcd}$$

which is carried to a pullback by E. But the map $H(\pi_0 R) \to \tilde{R}$ is an equivalence in degrees up to $i+1$ and therefore induces an equivalence on E by the inductive hypothesis. Therefore, $E(\tau_{\leq i+1} R) \to E(\tau_{\leq i} R)$ is an equivalence, whence the inductive step and the result. □

Proof of Theorem 3.1. As before, we write $\mathrm{TR}_Q(-) = \prod_Q \mathrm{TR}(-)$ for a set Q. For any connective B-algebra R, we claim that $R \to \tau_{\leq \kappa(2p+1)} R$ induces an equivalence on $L_{K(1)} \mathrm{TR}_Q(-)$. Indeed, this follows from Proposition 3.10 because we can simplicially resolve $\tau_{\leq \kappa(2p+1)} R$ using free R-algebras over free (R,R)-bimodules on classes in degrees $\kappa(2p+1)$ or higher and since $\mathrm{TR}_Q(-)$ commutes with geometric realizations. The result now follows from Lemma 3.11. □

Corollary 3.12 (cf. [LMMT20]). *The invariants $L_{K(1)} K(-), L_{K(1)} \mathrm{TC}(-)$ are truncating on connective B-algebras.*

Proof. The result for $L_{K(1)} \mathrm{TC}(-)$ follows from Theorem 3.1 by taking Frobenius fixed points. The result for $L_{K(1)} K(-)$ is a formal consequence since the fiber of the trace $K(-) \to \mathrm{TC}(-)$ is truncating by the Dundas–Goodwillie–McCarthy theorem [DGM13]. □

[3] This type of argument is also used in [LMMT20, § 3.2].

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Corollary 3.13 [LMMT20, BCM20]. *For any n, we have $L_{K(1)}K(\mathbb{Z}/p^n) = 0$.*

Proof. Take $B = H\mathbb{Z}$ in Theorem 3.1, so $L_{K(1)}\mathrm{TC}(-)$ is truncating and therefore nilinvariant on connective $H\mathbb{Z}$-algebras. Then the result follows because the Dundas–Goodwillie–McCarthy theorem and comparison with $\mathbb{F}_p$ yield $L_{K(1)}K(\mathbb{Z}/p^n) = L_{K(1)}\mathrm{TC}(\mathbb{Z}/p^n)$, but the above shows that this equals $L_{K(1)}\mathrm{TC}(\mathbb{F}_p) = 0$. □

4. Asymptotic $K(1)$-locality

In this section we show (Theorem 4.8) that TR is asymptotically $K(1)$-local for a regular ring satisfying mild hypotheses, using the Beilinson–Lichtenbaum conjecture. This result is due to Hesselholt and Madsen in the case of smooth algebras over a DVR with perfect residue field of characteristic greater than 2, which we begin by reviewing.

Theorem 4.1 (Hesselholt–Madsen [HM03, HM04]). *Let K be a complete, discretely valued field of characteristic 0 with ring of integers $\mathcal{O}_K \subset K$ and perfect residue field k of characteristic $p > 2$. Let R be a smooth $\mathcal{O}_K$-algebra of relative dimension d. Then the map*

$$\mathrm{TR}(R; \mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R; \mathbb{F}_p)$$

is d-truncated.

Proof. Without loss of generality, we can assume that $\mu_p \subset K$, since otherwise $\mathrm{TR}(R; \mathbb{F}_p)$ is a retract of $\mathrm{TR}(R[\zeta_p]; \mathbb{F}_p)$ via the transfer. In this case, the result follows from [HM04, Theorem E]. Indeed, [HM04, Theorem E] gives a calculation of the cofiber $\mathrm{TR}(R|R_K; \mathbb{F}_p)$ of the transfer map $\mathrm{TR}(R \otimes_{\mathcal{O}_K} k; \mathbb{F}_p) \to \mathrm{TR}(R; \mathbb{F}_p)$. Since $\mathrm{TR}(R \otimes_{\mathcal{O}_K} k; \mathbb{F}_p)$ is $K(1)$-acyclic and d-truncated in view of the identification [Hes96] with the de Rham–Witt complex of $R \otimes_{\mathcal{O}_K} k$, it suffices to verify the (stronger) claim that $\mathrm{TR}(R|R_K; \mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R|R_K; \mathbb{F}_p)$ is $(d-2)$-truncated. Equivalently (see Lemma 5.8 below), it suffices to show that the cofiber of the Bott element on $\mathrm{TR}(R|R_K; \mathbb{F}_p)$ is $(d+1)$-truncated. In fact, this follows from the calculation in [Hes96], once we know that the absolute de Rham–Witt complex $W\Omega^*_{(R,M_R)}$ is p-divisible in degrees $d+2$ or higher. This in turn follows from the case where $R = \mathcal{O}_K$ (see [HM03, Corollary 3.2.7]; the case of polynomial rings via the functor P (see [HM04, Lemma 7.1.4] and its proof); and finally, étale base-change [HM04, Lemma 7.1.1]. □

Construction 4.2 (TR as p-typical curves). Let R be an animated ring.[4] We let

$$C(R) = \varprojlim_n \Omega K(R[x]/x^n, (x)) \tag{10}$$

denote the spectrum of *curves* on the K-theory of R, defining a functor from animated rings to spectra. By [Hes96, Theorem 3.1.9], if R is a discrete $\mathbb{Z}/p^j$-algebra for some j, we have a natural expression of $\mathrm{TR}(R; \mathbb{Z}_p)$ as a summand of $C(R; \mathbb{Z}_p)$ (note that we are considering p-typical TR, while in [Hes96, Theorem 3.1.9] global TR is considered). Left Kan extending both sides to animated rings (since both $\mathrm{TR}(-)$ and $C(-)$ commute with geometric realizations),[5] we obtain

[4] Also called a simplicial commutative ring; see [ČS19] for a discussion of this terminology.

[5] For $C(-)$, this follows from the expression $K(R[x]/x^n, (x)) = \mathrm{TC}(R[x]/x^n, (x))$, using that TC of connective ring spectra commutes with geometric realizations, and passing to the limit along n.

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that $\mathrm{TR}(R;\mathbb{Z}_p)$ is naturally a summand of $C(R)$ for R an animated $\mathbb{Z}/p^j$-algebra. Passing to the limit over j and using the p-adic continuity of K-theory [CMM21, Theorem 5.21], we find that for any p-henselian animated ring R (i.e. $\pi_0(R)$ is p-henselian), $\mathrm{TR}(R;\mathbb{Z}_p)$ is a summand of $C(R;\mathbb{Z}_p)$.

Next, we discuss the comparison between the p-adic K-theory of R and $R[1/p]$; see [Niz08, Lemma 3.5] for this argument.

Proposition 4.3. *Let R be a regular noetherian ring of finite Krull dimension. Suppose, for every $x \in \mathrm{Spec}(R/p)$, that we have $[\kappa(x) : \kappa(x)^p] \leq p^d$. Then the map $K(R;\mathbb{F}_p) \to K(R[1/p];\mathbb{F}_p)$ is d-truncated.*

Proof. The homotopy fiber of the map in question is, by Quillen's dévissage theorem, the mod p K-theory of the abelian category of finitely generated R/p-modules, that is, the mod p G-theory of R/p, $G(R/p;\mathbb{F}_p)$. So it suffices to show that $G(R/p;\mathbb{F}_p)$ is concentrated in degrees up to d. Using the filtration by codimension of support and dévissage (see [Qui73, Theorem 5.4]), we find that $G(R/p;\mathbb{F}_p)$ has a filtration whose associated graded terms are direct sums of the $K(\kappa(x);\mathbb{F}_p)$ for $x \in \mathrm{Spec}(R/p)$; it therefore suffices to show that these terms are d-truncated. But now the Geisser–Levine theorem [GL00] implies that for any field E of characteristic p, there is an embedding $K_*(E;\mathbb{F}_p) \hookrightarrow \Omega^*_{E/\mathbb{F}_p}$. This implies that $K(\kappa(x);\mathbb{F}_p)$ is $\dim_{\kappa(x)} \Omega^1_{\kappa(x)/\mathbb{F}_p}$-truncated. But $\dim_{\kappa(x)} \Omega^1_{\kappa(x)/\mathbb{F}_p} = \log_p[\kappa(x) : \kappa(x)^p] \leq d$ by the theory of p-bases [Mat86, Theorem 26.5]. Combining these facts, the result follows. □

Proposition 4.4 (Rosenschon–Østvær [RØ06]). *Let R be a regular noetherian $\mathbb{Z}[1/p]$-algebra of finite Krull dimension. Suppose that $\mathrm{vcd}_p(\kappa(x)) \leq d$ for all $x \in \mathrm{Spec}(R[1/p])$. Then the map $K(R;\mathbb{F}_p) \to L_{K(1)}K(R;\mathbb{F}_p)$ has $(d-3)$-truncated homotopy fiber.*

Proof sketch. Using Nisnevich descent [TT90], we reduce to the case where R is henselian local, and even a field of characteristic not equal to p by Gabber–Suslin rigidity [Gab92]; note that we do not have to worry about the distinction between connective and nonconnective K-theory by regularity. Then the result follows from the Beilinson–Lichtenbaum conjecture (proved by Voevodsky and Rost; see [HW19]) describing the associated graded terms of the motivic filtration on $K(R;\mathbb{F}_p)$ (see, for example, [CM19, § 6.2] for an account). □

Proposition 4.5. *Let R be an excellent normal domain which is henselian along an ideal $I \subset R$ containing (p). Suppose that, for all $\mathfrak{p} \in \mathrm{Spec}(R)$ containing I, we have $\dim R_{\mathfrak{p}} + \log_p[\kappa(\mathfrak{p}) : \kappa(\mathfrak{p})^p] \leq d$. Then, for all $\mathfrak{q} \in \mathrm{Spec}(R[1/p])$, the residue field $\kappa(\mathfrak{q})$ has p-cohomological dimension at most $d+1$.*

Proof. By standard continuity arguments, it suffices to show that for any affine open $U = \mathrm{Spec}(R[1/f]) \subset \mathrm{Spec}(R[1/p])$ and any constructible p-torsion sheaf $\mathcal{F}$ on U, we have $H^n(U,\mathcal{F}) = 0$ for $n > d+1$. Denote by $j\colon U \hookrightarrow \mathrm{Spec}(R)$ the open inclusion, and denote by $i_0\colon \mathrm{Spec}(R/I) \hookrightarrow \mathrm{Spec}(R)$ the closed embedding. Using that $\mathrm{Spec}(R/I)$ has p-cohomological dimension at most 1 and the affine analog of proper base change [Gab94, Hub93] applied to the henselian ideal $I \subset R$, we see that it suffices to show that $i_0^* R^n j_* \mathcal{F} = 0$ for $n > d$.

Working stalkwise on R and using the compatibility of étale cohomology with filtered colimits, we can now reduce to the case where R is an excellent, strictly henselian normal local domain with residue field k (and $I \subset R$ is contained in the maximal ideal), since excellence and normality are preserved by strict henselization (see [Gre76] for the former). The statement then becomes that $H^n(U, \mathcal{F}) = 0$ for $n > d$, which follows from the Gabber–Orgogozo bound [GO08, Theorem 6.1] (noting that the p-dimension of the residue field k is $\log_p[k : k^p]$ since k is separably closed). $\square$

Remark 4.6. Suppose that R_0 is a ring such that R_0/p is finite type over a perfect field k of characteristic p. Then, for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(R_0)$ containing (p), we have $\dim(R_0)_{\mathfrak{p}} + \log_p[\kappa(\mathfrak{p}) : \kappa(\mathfrak{p})^p] \leq \dim(R_0)$. Indeed, we have $\dim(R_0)_{\mathfrak{p}} + \dim(R_0/\mathfrak{p}) \leq \dim(R_0)$. Therefore, it suffices to prove $\log_p[\kappa(\mathfrak{p}) : \kappa(\mathfrak{p})^p] = \dim(R_0/\mathfrak{p})$. But in this case, both are the transcendence degree of the field $\kappa(\mathfrak{p})$ over k; see [Mat80, Theorem 27.B].

Lemma 4.7. *Let R_0 be an $\mathbb{F}_p$-algebra. Suppose that for all residue fields κ of R_0, we have $[\kappa : \kappa^p] \leq p^d$ for some $d \geq 0$. Then, for all residue fields κ' of $R_0[[x]]$, we have $[\kappa' : \kappa'^p] \leq p^{d+1}$.*

Proof. We have that $R_0[[x]]$ is generated by p elements over the subring generated by its pth powers and by R_0. For every residue field κ of R_0, it follows that $R_0[[x]] \otimes_{R_0} \kappa$ is generated by p elements over the subring generated by pth powers together with κ. In particular, it is generated by p^{d+1} elements as a module over its pth powers. It follows that the same holds for any residue field of $R_0[[x]] \otimes_{R_0} \kappa$ and, varying κ, we obtain the conclusion for every residue field of R_0. $\square$

Theorem 4.8. *Let R be an excellent, p-torsion-free regular noetherian ring. Suppose R/p is finitely generated as a module over its subring of pth powers. Suppose, furthermore, that for all $\mathfrak{p} \in \operatorname{Spec}(R)$ containing (p), we have $\dim R_{\mathfrak{p}} + \log_p[\kappa(\mathfrak{p}) : \kappa(\mathfrak{p})^p] \leq d$ for some $d \geq 0$. Then the map $\mathrm{TR}(R; \mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R; \mathbb{F}_p)$ is $(d-1)$-truncated.*

Proof. Without loss of generality, we can assume that R is p-henselian (since henselization preserves excellence; see [Gre76]), so $\dim R \leq d$. By Construction 4.2, it suffices to show that $C(R)/p \to L_{K(1)}C(R)/p$ is $(d-1)$-truncated. Now $C(R)$ is the desuspension of the fiber of the map $\varprojlim_n K(R[x]/x^n) \to K(R)$. With p-adic coefficients, the fact that R/p is finitely generated as a module over its pth powers allows us to pass to the limit [CMM21, Theorem F] and we obtain

$$C(R; \mathbb{F}_p) = \Omega K(R[[x]], (x); \mathbb{F}_p).$$

Since $K(R)$ is a retract of $K(R[[x]])$, it suffices to show that the fiber of $K(R[[x]]; \mathbb{F}_p) \to L_{K(1)}K(R[[x]]; \mathbb{F}_p)$ is d-truncated. We will verify this by comparing both sides with the intermediate term $K(R[[x]][1/p]; \mathbb{F}_p)$.

First, for every characteristic p residue field κ of R, corresponding to a prime ideal $\mathfrak{p} \subset R$ containing (p), we have $[\kappa : \kappa^p] \leq p^{d-1}$ by our assumption, since $\dim R_{\mathfrak{p}} \geq 1$. Now $R[[x]]$ is also a p-torsion-free regular ring of dimension $\dim(R) + 1$. For every characteristic p residue field κ' of $R[[x]]$, we have $[\kappa' : \kappa'^p] \leq p^d$ by Lemma 4.7. By Proposition 4.3, it follows that $K(R[[x]]; \mathbb{F}_p) \to K(R[[x]][1/p]; \mathbb{F}_p)$ is d-truncated.

Second, we apply Proposition 4.5 to the ring $R[[x]]$ and the ideal $I = (p, x)$. The power series ring $R[[x]]$ remains excellent (and regular) since R is excellent, thanks to [KS16]. For any prime ideal $\mathfrak{p} \in \operatorname{Spec}(R[[x]])$ containing I, we let $\mathfrak{p}_0 = \mathfrak{p} \cap R \subset R$, so that $R[[x]]_{\mathfrak{p}}/(x) = R_{\mathfrak{p}_0}$. Then $\dim(R[[x]]_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}_0}) + 1$ and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}_0)$. Thus, Proposition 4.5 applies to $R[[x]]$ (with d replaced by $d+1$) and we find that the characteristic 0 residue fields of $R[[x]]$ have

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p-cohomological dimension at most $d+2$. Therefore, by Proposition 4.4, $K(R[[x]][1/p];\mathbb{F}_p) \to L_{K(1)}K(R[[x]][1/p])$ is $(d-1)$-truncated.

Combining the above, we find that the composite map $K(R[[x]];\mathbb{F}_p) \to L_{K(1)}K(R[[x]];\mathbb{F}_p) \simeq L_{K(1)}K(R[[x]][1/p];\mathbb{F}_p)$ (the last identification by (2)) is d-truncated, whence the result. □

This recovers in particular Theorem 4.1. More generally, we have the following example.

Example 4.9. Let R be a d-dimensional regular, excellent p-torsion-free noetherian ring with R/p finitely generated over its pth powers. Suppose that $(R/p)_{\mathrm{red}}$ is finite type over a perfect field (and necessarily of dimension $d-1$). Then Theorem 4.8 (in view of Remark 4.6) applies to show that $\mathrm{TR}(R;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R;\mathbb{F}_p)$ is $(d-1)$-truncated.

5. The Segal conjecture

In this section we discuss the relationship between the following two properties of a cyclotomic spectrum X:

(i) $\mathrm{TR}(X)$ agrees with its $K(1)$-localization in high enough degrees;
(ii) the cyclotomic Frobenius $\varphi_X\colon X \to X^{tC_p}$ is an equivalence in high enough degrees.

Property (i) is the Lichtenbaum–Quillen style statement discussed in the previous section, and verified for $\mathrm{THH}(R)$ under regularity and finiteness hypotheses. Property (ii) is often referred to as the 'Segal conjecture' since for $X = \mathrm{THH}(\mathbb{S})$, the Frobenius $\mathbb{S} \to \mathbb{S}^{tC_p}$ is a p-adic equivalence by the Segal conjecture for C_p, proved in [Lin80, Gun80]. The Segal conjecture has been studied extensively for $\mathrm{THH}(R)$ for R a ring (or ring spectrum).

We first show the implication (i) $\implies$ (ii). We use the Antieau–Nikolaus theory of topological Cartier modules [AN21], which we begin by briefly reviewing.

Definition 5.1. A *topological Cartier module* M is an object of $\mathrm{Fun}(BS^1, \mathrm{Sp})$ together with maps $V\colon M_{hC_p} \to M$ and $F\colon M \to M^{hC_p}$ in $\mathrm{Fun}(BS^1, \mathrm{Sp})$ together with a homotopy between the composite and the norm map $M_{hC_p} \to M^{hC_p}$ (considered $S^1 \simeq S^1/C_p$-equivariantly). The collection of topological Cartier modules is naturally organized into a presentable stable ∞-category.

Given a bounded-below, p-typical cyclotomic spectrum X, we can consider $\mathrm{TR}(X)$ as a topological Cartier module, and we have an identification $X \simeq \mathrm{cofib}(V)$. Under these identifications, the cyclotomic Frobenius $X \to X^{tC_p}$ is obtained from $F\colon \mathrm{TR}(X) \to \mathrm{TR}(X)^{hC_p}$ by taking cofibers by V on both domain and codomain and identifying $\mathrm{TR}(X)^{tC_p} \simeq X^{tC_p}$ as $(\mathrm{TR}(X)_{hC_p})^{tC_p} = 0$ [NS18, Lemma I.2.1]. On bounded-below objects, this construction establishes a fully faithful embedding from cyclotomic spectra into topological Cartier modules, with image given by the V-complete objects [AN21, Theorem 3.21].

Proposition 5.2. *Let X be a connective, p-complete cyclotomic spectrum whose underlying spectrum is $K(1)$-acyclic. Suppose the map $\mathrm{TR}(X) \to L_{K(1)}\mathrm{TR}(X)$ is d-truncated. Then the Frobenius $\varphi\colon X \to X^{tC_p}$ is d-truncated.*

In the case $L_{K(1)}\mathrm{TR}(X) = 0$, the result is [AN21, Proposition 2.25].

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Proof. Since X is $K(1)$-acyclic, it follows that $V\colon \mathrm{TR}(X)_{hC_p} \to \mathrm{TR}(X)$ is $K(1)$-locally an equivalence. The $K(1)$-localization $L_{K(1)}\mathrm{TR}(X)$ acquires the structure of a topological Cartier module as well by $K(1)$-localizing F, V and using the comparison map $L_{K(1)}(\mathrm{TR}(X)^{hC_p}) \to (L_{K(1)}\mathrm{TR}(X))^{hC_p}$. The composite map $(L_{K(1)}\mathrm{TR}(X))_{hC_p} \to (L_{K(1)}\mathrm{TR}(X))^{hC_p}$ is an equivalence after p-completion since Tate constructions vanish in the $K(1)$-local category. Since we saw that V is an equivalence on $L_{K(1)}\mathrm{TR}(X)$ after p-completion, it follows that the Frobenius on $L_{K(1)}\mathrm{TR}(X)$ induces an equivalence

$$L_{K(1)}\mathrm{TR}(X) \xrightarrow{\sim} (L_{K(1)}\mathrm{TR}(X))^{hC_p}.$$

For a topological Cartier module Y, we consider the fiber of $F = F_Y\colon Y \to Y^{hC_p}$, which we denote $\mathrm{fib}(F)$. As we saw above, $\mathrm{fib}(F)$ is contractible for the topological Cartier module $L_{K(1)}\mathrm{TR}(X)$. Moreover, $\mathrm{fib}(F)$ is d-truncated for the topological Cartier module $\mathrm{fib}(\mathrm{TR}(X) \to L_{K(1)}\mathrm{TR}(X))$ because this topological Cartier module is itself d-truncated. In particular, we find that $\mathrm{fib}(F\colon \mathrm{TR}(X) \to \mathrm{TR}(X)^{hC_p})$ is d-truncated. Taking the cofiber on both the domain and codomain of the Verschiebung, we find that $\mathrm{fib}(F\colon \mathrm{TR}(X) \to \mathrm{TR}(X)^{hC_p}) = \mathrm{fib}(\varphi\colon X \to X^{tC_p})$ which is therefore d-truncated as desired. □

Combining Proposition 5.2 and Theorem 4.8, we obtain the following result. Versions of the Segal conjecture have been studied by many authors. For instance, it is known that $\mathrm{THH}(\mathbb{Z}_p; \mathbb{F}_p) \to \mathrm{THH}(\mathbb{Z}_p; \mathbb{F}_p)^{tC_p}$ is an equivalence on connective covers [BM94, Lemma 6.5]. Compare [HM03] for the more general case of a DVR $\mathcal{O}_K$ of mixed characteristic and perfect residue field of characteristic $p > 2$. The Segal conjecture for smooth algebras in characteristic p appears as [Hes18, Proposition 6.6] and (at the filtered level) [BMS19, Corollary 8.18]. The Segal conjecture has also been verified for certain ring spectra as well; see [AR02, LR11, AQ21].

Corollary 5.3 (The Segal conjecture for regular rings). *Let R be a p-torsion-free excellent regular noetherian ring with R/p finitely generated over its pth powers. Suppose that for all $\mathfrak{p} \in \mathrm{Spec}(R)$ containing (p), we have $\dim R_{\mathfrak{p}} + \log_p[\kappa(\mathfrak{p}) : \kappa(\mathfrak{p})^p] \leq d$. Then the map $\mathrm{THH}(R; \mathbb{F}_p) \to \mathrm{THH}(R; \mathbb{F}_p)^{tC_p}$ is $(d-1)$-truncated.*

Question 5.4. Is it possible to prove a filtered version of this result, with respect to the motivic filtrations [BMS19] on both sides?

We next discuss the converse direction. Here we only prove the result under a more restrictive hypothesis, namely for cyclotomic spectra which are ku-modules.

Example 5.5. Let $C = \widehat{\overline{\mathbb{Q}_p}}$. If R is a $\mathcal{O}_C$-algebra, then the cyclotomic trace makes $\mathrm{THH}(R; \mathbb{Z}_p)$ into a ku-module in view of the equivalence $ku_{\hat{p}} \simeq K(\mathcal{O}_C; \mathbb{Z}_p)$; see [Niz98, Lemma 3.1], [Sus83], and [Hes06]. One can improve this slightly: $K(\widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}; \mathbb{Z}_p)$ has the structure of an E_∞-algebra under ku, so the same applies to any $\widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$-algebra R. This follows from three facts. First, $K(\widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}; \mathbb{Z}_p) \simeq K(\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}; \mathbb{Z}_p)$; see [HN19, Lemma 1.3.7] for this argument (which only uses that the field $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ is perfectoid). Second, since $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ is perfectoid, it has p-cohomological dimension at most 1 by the tilting equivalence, whence $K(\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}; \mathbb{Z}_p) \to L_{K(1)}K(\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}; \mathbb{Z}_p)$ is the connective cover map; see Proposition 4.4. Third, $L_{K(1)}K(\mathbb{Z}[\zeta_{p^\infty}])$ has the structure of an E_∞-algebra under KU and hence ku; see [BCM20, Construction 3.7].

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Lemma 5.6. *Let $M \in \mathrm{Fun}(BS^1, \mathrm{Mod}(H\mathbb{F}_p))$ be an r-connected object. Suppose that there exists a map $f\colon M \to M'$ in $\mathrm{Fun}(BS^1, \mathrm{Mod}(H\mathbb{F}_p))$ such that M' is induced (as an object with S^1-action) and such that f is an equivalence on $\tau_{\geq r}$. Then there exists a map $\tilde{M} \to M$ of r-connected objects in $\mathrm{Fun}(BS^1, \mathrm{Mod}(H\mathbb{F}_p))$ which induces an equivalence on $\tau_{\geq r+1}$ and such that $\tilde{M}$ is induced.*

Proof. The homotopy groups $\pi_*(M), \pi_*(M')$ form graded modules over the ring $\pi_*(\mathbb{F}_p[S^1]) = \mathbb{F}_p[\epsilon]/\epsilon^2$, $|\epsilon| = 1$. By assumption, $\pi_*(M')$ is a free graded $\mathbb{F}_p[\epsilon]/\epsilon^2$-module and $\pi_*(M)$ is the submodule of those elements in degree r or higher. Choosing an $\mathbb{F}_p$-subspace V of $\pi_r(M) = \pi_r(M')$ which is complementary to $\epsilon\pi_{r-1}(M') \subset \pi_r(M')$, we can modify M to form $\tilde{M}$ with $\pi_r(\tilde{M}) = V$; it is now easy to see that $\pi_*(\tilde{M})$ is a free graded $\mathbb{F}_p[\epsilon]/\epsilon^2$-module, so we can conclude. □

Proposition 5.7. *Let X be a connective, p-complete ku-module in* CycSp*. Suppose that $\varphi\colon X \to X^{tC_p}$ is d-truncated. Then the fiber of $\mathrm{TR}(X)/p \to L_{K(1)}\mathrm{TR}(X)/p$ is $(d+3)$-truncated.*

Proof. It follows that the cyclotomic spectrum $Y = X/(p,\beta) = X \otimes_{ku} H\mathbb{F}_p \in \mathrm{CycSp}_{\geq 0}$ has the property that $\varphi\colon Y \to Y^{tC_p}$ is $(d+4)$-truncated. It follows that the comparison map $Y^{C_p} \to Y^{hC_p}$ is $(d+4)$-truncated as well, via the fiber square

$$\begin{tikzcd} Y^{C_p} \arrow[r] \arrow[d] & Y^{hC_p} \arrow[d] \\ Y \arrow[r, "\varphi"] & Y^{tC_p} \end{tikzcd}$$

Now, for each $n \geq 1$, we have the following pullback diagram [NS18, Lemma II.4.5].

$$\begin{tikzcd} Y^{C_{p^n}} \arrow[r] \arrow[d, "R"] & Y^{hC_{p^n}} \arrow[d] \\ Y^{C_{p^{n-1}}} \arrow[r] & Y^{tC_{p^n}} \end{tikzcd} \tag{11}$$

The bottom horizontal map is $Y^{C_{p^{n-1}}} \to Y^{hC_{p^{n-1}}} \xrightarrow{\varphi^{hC_{p^{n-1}}}} (Y^{tC_p})^{hC_{p^{n-1}}}$, and the last term is identified with $Y^{tC_{p^n}}$ using the Tate orbit lemma; see [NS18, Lemma II.4.1].

Now $Y^{C_p} \to Y^{hC_p}$ is an equivalence in degrees at least $d+6$, hence $Y^{C_{p^{n-1}}} \to Y^{hC_{p^{n-1}}}$ is an equivalence in degrees at least $d+6$ by [NS18, Corollary II.4.9] (a generalization of results of Tsalidis [Tsa98] and Bökstedt, Bruner, Lunø, Nielsen, and Rognes [BBLR14]). Since $\varphi\colon Y \to Y^{tC_p}$ is an equivalence in degrees at least $d+6$, it follows that the bottom horizontal map in (11) is an equivalence in degrees at least $d+6$. Note also that Y^{tC_p} is a module over $\mathbb{F}_p^{tC_p}$ in $\mathrm{Fun}(BS^1, \mathrm{Mod}(H\mathbb{F}_p))$; since the latter has induced S^1-action, it follows that the former does too.

By Lemma 5.6, we can find an at least $(d+6)$-connected Y' with an S^1-equivariant map $Y' \to Y$ which induces an equivalence in degrees at least $d+7$ and such that Y' is induced as an object of $\mathrm{Fun}(BS^1, \mathrm{Mod}(H\mathbb{F}_p))$. It follows that the map $Y'^{hC_{p^n}} \to Y^{hC_{p^n}}$ is an equivalence in

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degrees at least $d+7$. But the commutative square

$$\begin{tikzcd} Y'^{hC_{p^n}} \arrow[r] \arrow[d] & Y^{hC_{p^n}} \arrow[d] \\ Y'^{tC_{p^n}} = 0 \arrow[r] & Y^{tC_{p^n}} \end{tikzcd}$$

now shows that any $\alpha \in \pi_r(Y^{hC_{p^n}})$ for $r \geq d+7$ has vanishing image in $\pi_r(Y^{tC_{p^n}})$. Using the commutative square (11) again (since the horizontal maps are equivalences in degrees at least $d+6$), it follows that the restriction map $Y^{C_{p^n}} \to Y^{C_{p^{n-1}}}$ induces the zero map in degrees at least $d+7$. Consequently, taking the inverse limit over R, we find that $\mathrm{TR}(Y) \in \mathrm{Sp}_{\leq d+6}$, whence the fiber of $\mathrm{TR}(X)/p \to L_{K(1)}\mathrm{TR}(X)/p$ belongs to $\mathrm{Sp}_{\leq d+3}$ by Lemma 5.8. □

Lemma 5.8. *Let M be a ku-module spectrum. Then the following assertions are equivalent.*

(i) *$M/(p,\beta)$ is concentrated in degrees at most $d+3$.*
(ii) *The fiber of $M/p \to L_{K(1)}M/p$ is concentrated in degrees at most d.*

Proof. Given a ku-module N, it suffices to show that the fiber of $N \to N[1/\beta]$ (which after p-completion is $L_{K(1)}N$) is concentrated in degrees at most d if and only if N/β is concentrated in degrees at most $d+3$; then the result will follow by taking $N = M/p$. To prove the claim, first replace N by $\mathrm{fib}(N \to N[1/\beta])$; thus we may assume that N is actually β-power torsion.

Then the claim follows from the observation that a β-power torsion ku-module N is concentrated in degrees at most d if and only if N/β is concentrated in degrees at most $d+3$. The 'only if' direction is evident, so we verify the 'if' direction. In fact, if there exists a nonzero $x \in \pi_i(N)$ for $i > d$, then by multiplying by a power of β we may assume $\beta x = 0$, whence there exists a nonzero class in $\pi_{i+3}(N/\beta)$ from the long exact sequence. □

Remark 5.9. Let $X = \mathrm{THH}(\mathbb{F}_p)$; in this case, we have that $\mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(\mathbb{F}_p)^{tC_p}$ is (-3)-truncated. This computation is due to Hesselholt and Madsen [HM97, § 5], and is refined in [NS18, Appendix IV-4]. Meanwhile $\mathrm{TR}(\mathbb{F}_p) = H\mathbb{Z}_p \to L_{K(1)}\mathrm{TR}(\mathbb{F}_p) = 0$ is 0-truncated modulo p. Thus, the bound of Proposition 5.7 is the best possible.

Proposition 5.10. *Let R_0 be a p-torsion-free perfectoid ring. Let R be a formally smooth R_0-algebra (with respect to the p-adic topology) of relative dimension d. Then the map $\varphi\colon \mathrm{THH}(R;\mathbb{F}_p) \to \mathrm{THH}(R;\mathbb{F}_p)^{tC_p}$ is $(d-3)$-truncated, and the map $\mathrm{TR}(R;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R;\mathbb{F}_p)$ is $(d-1)$-truncated.*

Proof. Let $X_i, i \in \mathcal{C}$ be a diagram of spectra. Suppose that for each $i \in \mathcal{C}$, the map $X_i \to L_{K(1)}X_i$ is m-truncated for some m. Then the map $\varprojlim_i X_i \to L_{K(1)}(\varprojlim_i X_i)$ is m-truncated, and $L_{K(1)}(\varprojlim_i X_i) \to \varprojlim_i L_{K(1)}X_i$ is an equivalence. This follows because $L_{K(1)}(-)$ annihilates bounded-above spectra. Applying the above observation, and using flat descent of $\mathrm{TR}(-;\mathbb{Z}_p)$ (which follows from [BMS19, § 3]) and $\mathrm{THH}(-;\mathbb{Z}_p), \mathrm{THH}(-;\mathbb{Z}_p)^{tC_p}$, we reduce to the case where R is a $\widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$-algebra (e.g. using André's lemma; see [BS19, Theorem 7.12]), so $\mathrm{THH}(R;\mathbb{Z}_p)$ is a $ku_{\hat{p}}$-module in $\mathrm{CycSp}_{\geq 0}$ (Example 5.5). With this reduction in mind, the first claim implies the second in view of Proposition 5.7 (applied to the cyclotomic spectrum $\mathrm{THH}(R;\mathbb{F}_p)$).

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Thus, we prove the first claim (i.e. the Segal conjecture for $\mathrm{THH}(R;\mathbb{F}_p)$); this result appears as [BMS19, Corollary 9.12] in the case where R_0 is the ring of integers in a complete, algebraically closed nonarchimedean field of mixed characteristic, and in [BMS19, § 6] when $R = R_0$.

Let (A, I) be the perfect prism associated to the perfectoid ring R_0, and let $\tilde{\xi} \in I$ be a generator. We use the prismatic cohomology $\Delta_{R/A}$ of [BS19] and its Nygaard completion $\hat{\Delta}_{R/A}$. For a quasisyntomic R_0-algebra S, there are [BMS19] complete, exhaustive descending $\mathbb{Z}$-indexed filtrations on $\mathrm{TC}^-(S;\mathbb{Z}_p), \mathrm{TP}(S;\mathbb{Z}_p)$ with

$$\mathrm{gr}^i \mathrm{TC}^-(S;\mathbb{Z}_p) \simeq \mathcal{N}^{\geq i}\hat{\Delta}_{S/A}[2i], \quad \mathrm{gr}^i \mathrm{TP}(S;\mathbb{Z}_p) \simeq \hat{\Delta}_{S/A}[2i],$$

and the map $\varphi\colon \mathrm{TC}^-(S;\mathbb{Z}_p) \to \mathrm{TP}(S;\mathbb{Z}_p)$ on graded pieces is given by the prismatic divided Frobenius

$$\varphi/\tilde{\xi}^i \colon \mathcal{N}^{\geq i}\hat{\Delta}_{S/A}[2i] \to \hat{\Delta}_{S/A}[2i]; \tag{12}$$

see [BS19, § 13] for the comparison between prismatic cohomology and the objects of [BMS19]. Here we have trivialized the Breuil–Kisin twists involved since we are over the base perfectoid ring R_0, and we can compute the Nygaard-completed absolute prismatic cohomology as the Nygaard completed relative prismatic cohomology over R_0.

Now the map $\varphi\colon \mathrm{TC}^-(S;\mathbb{Z}_p) \to \mathrm{TP}(S;\mathbb{Z}_p)$ arises by taking S^1-invariants on the cyclotomic Frobenius $\varphi\colon \mathrm{THH}(S;\mathbb{Z}_p) \to \mathrm{THH}(S;\mathbb{Z}_p)^{tC_p}$. Both sides of this map are filtered (again, as in [BMS19]) and the associated graded pieces are

$$\mathrm{gr}^i \mathrm{THH}(S;\mathbb{Z}_p) \simeq \mathcal{N}^i\hat{\Delta}_{S/A}[2i], \quad \mathrm{gr}^i \mathrm{THH}(S;\mathbb{Z}_p)^{tC_p} = (\hat{\Delta}_{S/A})/\tilde{\xi}[2i],$$

and the cyclotomic Frobenius is induced from (12). But since R is formally smooth over R_0, the Nygaard filtration is complete and $\varphi/\tilde{\xi}^i$ induces an equivalence $\mathcal{N}^i\Delta_{R/A} \xrightarrow{\sim} \tau^{\leq i}(\Delta_{R/A}/\tilde{\xi})$ (see [BS19, § 12.4]), where the right-hand side is the ith stage of the conjugate filtration on $\Delta_{R/A}/\tilde{\xi}$. Using the Hodge–Tate comparison [BS19, Theorem 4.10], it follows that the cohomology groups of $\Delta_{R/A}/\tilde{\xi}$ are p-torsion-free. It follows that

$$\mathrm{gr}^i\varphi\colon \mathrm{gr}^i \mathrm{THH}(R;\mathbb{F}_p) \to \mathrm{gr}^i \mathrm{THH}(R;\mathbb{F}_p)^{tC_p}$$

exhibits the domain as the i-connective cover of the codomain, and therefore has $(i-2)$-truncated homotopy fiber for all i. The codomain is $(2i-d)$-connective by the Hodge–Tate comparison, since R/R_0 is formally smooth of relative dimension d; therefore, $\mathrm{gr}^i\varphi$ is an equivalence for $i \geq d$. Therefore, the fiber of $\varphi\colon \mathrm{THH}(R;\mathbb{F}_p) \to \mathrm{THH}(R;\mathbb{F}_p)^{tC_p}$ has a complete filtration such that gr^i is $(i-2)$-truncated for all i and contractible for $i \geq d$. This proves that $\varphi\colon \mathrm{THH}(R;\mathbb{F}_p) \to \mathrm{THH}(R;\mathbb{F}_p)^{tC_p}$ is $(d-3)$-truncated, whence the result. $\square$

6. Pro-Galois descent

In this section, we prove a type of pro-Galois descent for TR in the generic fiber, which is related to the conjecture in [Hes02]. The basic example is when K is a characteristic 0 local field, and one tries to relate $\mathrm{TR}(\mathcal{O}_K;\mathbb{Z}_p)$ (computed by [HM03]) with the 'continuous' homotopy fixed points for the Galois group $\mathrm{Gal}(\bar{K}/K)$ on $\mathrm{TR}(\mathcal{O}_{\bar{K}};\mathbb{Z}_p)$, before or after $K(1)$-localization. The advantage is that the latter is much more tractable; see [Hes06] for the calculation of $\mathrm{TR}(\mathcal{O}_{\bar{K}};\mathbb{Z}_p)$. For finite Galois extensions in the generic fiber, these claims follow from § 2. However, there are

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some additional subtleties to extend to pro-Galois descent because TR fails to commute with filtered colimits.

6.1 An auxiliary construction

Let B be a base ring. Let E be a $K(1)$-local, localizing invariant on B-linear ∞-categories which is truncating. Let R_0 be a p-adically complete B-algebra. Given a $K(1)$-local truncating invariant E, we now describe a construction of a sheaf on the finite étale site of $R_0[1/p]$. For every finite étale $R_0[1/p]$-algebra S, we can choose a 'ring of integers' S_0 which is finite and finitely presented over R_0 with $S_0[1/p] = S$ and consider $E(S_0)$. It is not difficult to see that this only depends on S and that it defines the desired sheaf; to make the functoriality precise, we use left Kan extension.

Construction 6.1 (E as a sheaf on the finite étale site of the generic fiber). Let R_0 be a p-adically complete B-algebra and let $R = R_0[1/p]$. Using the $K(1)$-local, localizing invariant E which is assumed to be truncating, we define a sheaf $\mathcal{F}_E$ of spectra on the finite étale site of $\mathrm{Spec}(R)$ as follows. Let $\mathcal{C}_0$ denote the category of finite, finitely presented R_0-algebras S_0 with $S_0[1/p]$ étale over R, and let $\mathcal{C}$ denote the category of finite étale R-algebras. Consider the functor $F\colon \mathcal{C}_0 \to \mathcal{C}$ given by inverting p. Note the following observations.

(i) F is essentially surjective. That is, given any finite étale R-algebra S, there exists a finite, finitely presented R_0-algebra S_0 with $S = S_0[1/p]$. In fact, consider any finite R_0-subalgebra $S_0' \subset S$ with $S_0'[1/p] = S$. The algebra S_0' is not necessarily finitely presented, but it is a directed colimit of finite, finitely presented R_0-algebras $S_0^{(\alpha)}$ under surjective maps; one of them will have $S_0^{(\alpha)}[1/p] = S$, and can be taken for S_0.

(ii) If $S \in \mathcal{C}$, then $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/S}$ is a filtered category. In fact, the subcategory of $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/S}$ spanned by those S_0 such that the structure map $S_0[1/p] \to S$ is an isomorphism is itself filtered and cofinal. This follows similarly by comparing S_0 with its image in S.

We consider the functor $S_0 \mapsto E(S_0)$ on the category $\mathcal{C}_0$ of finite, finitely presented R_0-algebras S_0 with $S_0[1/p]$ finite étale over R. Then, to define $\mathcal{F}_E$ on finite étale R-algebras, we left Kan extend $E(-)$ along the functor $\mathcal{C}_0 \to \mathcal{C}$. Explicitly, it follows from Example 2.4 that if S is a finite étale R-algebra and S_0 is a finite, finitely presented R_0-algebra with $S_0[1/p] \simeq S$, then we have a canonical equivalence $\mathcal{F}_E(S) \simeq E(S_0)$. It also follows from h-descent that $\mathcal{F}_E$ is indeed a sheaf of $K(1)$-local spectra on the finite étale site of $\mathrm{Spec}(R)$. Note that it is a sheaf of modules over the sheaf $S \mapsto L_{K(1)}K(S)$, which is also the sheaf $\mathcal{F}_{L_{K(1)}K}$ because $L_{K(1)}K(-)$ is insensitive to inverting p on $H\mathbb{Z}$-algebras as in (2).

Proposition 6.2 (Hypercompleteness of $\mathcal{F}_E$). *With notation as above, suppose there is a uniform bound on the mod p cohomological dimensions of the residue fields of R and R has finite Krull dimension. Then $\mathcal{F}_E$ defines a hypercomplete sheaf on the finite étale site of* $\mathrm{Spec}(R)$.

Proof. Our hypotheses imply that $\mathrm{Spec}(R)$ has finite mod p étale cohomological dimension; see [CM19, Corollary 3.29].

Now we use the $K(\pi, 1)$ property, which states that for any p-henselian ring A and any $\mathbb{F}_p$-local system $\mathcal{L}$ on $\mathrm{Spec}(A[1/p])$, the cohomology of $\mathcal{L}$ on the étale sites and the finite étale sites of $\mathrm{Spec}(A[1/p])$ coincide. For A noetherian and p-complete, this is proved in [Sch13, Theorem 4.9]. The case of A noetherian and p-henselian then follows using the Fujiwara–Gabber theorem (see [BM18, Theorem 6.11] for an account), and then one can pass to the limit to obtain the arbitrary p-henselian case using the commutation of cohomology (on either the étale or finite étale sites) and filtered colimits.

From the above two paragraphs, it follows that the étale fundamental group $\pi_1^{\mathrm{et}}(\mathrm{Spec}(R))$ has finite mod p cohomological dimension, which implies that the notion of hypercompleteness for sheaves of p-complete spectra on the finite étale site of $\mathrm{Spec}(R)$ can be made explicit in terms of exponents of nilpotence [CM19, § 4.1]. Now $\mathcal{F}_E$ is a module over the hypercomplete sheaf $S \mapsto L_{K(1)}K(S)$ (see [CM19, Theorem 7.14] for hypercompleteness), hence a hypercomplete sheaf itself, thanks to [CM19, Corollary 4.26]. $\square$

We also need the following variant of the above with respect to a fixed profinite group.

Construction 6.3 ($\mathcal{F}_E$ relative to a profinite group). Let S_0 be an R_0-algebra equipped with the action of a profinite group Γ which is continuous (with respect to the discrete topology on S_0), and such that $R \to S := S_0[1/p]$ is Γ-Galois. Suppose that there exist a cofinal collection of open normal subgroups $N_i \subset \Gamma$, $i \in I$, for which the fixed points $S_0^{N_i}$ form a finite, finitely presented R_0-algebra. It follows from the above that we obtain a sheaf on the category of finite continuous Γ-sets which carries $\Gamma/N_i \mapsto E(S_0^{N_i})$. This is just the restriction of $\mathcal{F}_E$ to the site of finite continuous Γ-sets (which maps to the finite étale site of R).

When is the sheaf of Construction 6.3 hypercomplete? When Γ has finite cohomological dimension, hypercompleteness is smashing [CM19, § 4.1], so hypercompleteness holds if the sheaf of spectra which sends $\Gamma/H \mapsto L_{K(1)}K(S^H)$ (for any cofinal collection of open normal subgroups H) is hypercomplete.

Example 6.4. Suppose $\Gamma = \mathbb{Z}_p^n$ and the Γ-extension of R_0 is obtained by adding compatible systems of p-power roots. Explicitly, suppose R_0 is a $\mathbb{Z}[\zeta_{p^\infty}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$-algebra and $S_0 = R_0 \otimes_{\mathbb{Z}[\zeta_{p^\infty}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]} \mathbb{Z}[\zeta_{p^\infty}, t_1^{\pm 1/p^\infty}, \ldots, t_n^{\pm 1/p^\infty}]$, with the evident Γ-action. In this case, the sheaf of Construction 6.3 is hypercomplete. Again since hypercompleteness is smashing, this follows because any $K(1)$-local localizing invariant yields a hypercomplete étale sheaf on the site of finite étale (or even all étale) $\mathbb{Z}[1/p, \zeta_{p^\infty}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$-algebras; see [CM19, Theorem 7.14]. In particular, we take as the localizing invariant $A \mapsto L_{K(1)}K(A \otimes_{\mathbb{Z}[1/p, \zeta_{p^\infty}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]} R)$.

6.2 Completion of topological Cartier modules

Here again we use the 'decompletion' of the theory of cyclotomic spectra given by the topological Cartier modules of Antieau and Nikolaus [AN21] and an amplitude property of the completion. We recall that $\mathrm{TR}(-)$ gives a fully faithful right adjoint embedding from bounded-below cyclotomic spectra into bounded-below topological Cartier modules, with image the V-complete objects; see [AN21, Theorem 3.21].

Proposition 6.5. *Let M be a topological Cartier module which is d-truncated and bounded below. Then the V-completion of M is $(d+3)$-truncated. If M is p-complete, then the V-completion of M is $(d+2)$-truncated.*

Proof. We reduce by dévissage to the case where M is concentrated in degree 0, and $d = 0$. The completion of M is given by $\mathrm{cofib}(\varprojlim M_{hC_{p^n}} \to M)$ where the maps $M_{hC_{p^n}} \to M_{hC_{p^{n-1}}}$ are given by $V_{hC_{p^{n-1}}}$; see [AN21, Proposition 3.22].

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Now since M is discrete, the Verschiebung is simply given by a map $V\colon M \to M$ of abelian groups, and the S^1-action is trivial. We can (as in the proof of [AN21, Lemma 3.25]) form a $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-indexed diagram $Y_{i,j} = M \otimes BC_{p^j+}$ such that the transition maps in the i-direction are given by V and in the j-direction are given by the canonical projections. By the above, the completion of M is given by the cofiber of the map $\varprojlim_{i,j} Y_{i,j} \to M$.

Now a simple computation shows that for any abelian group A, $\varprojlim_j HA \otimes BC_{p^j+}$ are concentrated in degrees at most 2, and degrees at most 1 if A is derived p-complete. This claim will imply the result. Indeed, for the first part, it suffices to show that $\varprojlim_j HA \otimes BC_{p^j+}$ is concentrated in degrees at most 1 when A is torsion-free; this follows because the pro abelian group $\{H_*(BC_{p^j};\mathbb{Z})\}_{j\geq 0}$ is pro-zero for $* \geq 2$. For the second part, we also observe that the pro-abelian group $\{H_1(BC_{p^j};\mathbb{Z})\}_{j\geq 0}$ is simply the tower $\cdots \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$, and our assumption that A is derived p-complete gives that $A \simeq \varprojlim A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^j$ is in particular discrete. $\square$

Remark 6.6. We can give another proof of Proposition 6.5 using the results from the previous section in the case where M is annihilated by a power of p. By dévissage, we can reduce to the case where M is an $H\mathbb{F}_p$-module (in topological Cartier modules). Let $X = M/M_{hC_p} = \mathrm{cofib}(V)$ be the associated cyclotomic spectrum, so $\mathrm{TR}(X)$ is the derived V-completion of M by the correspondence between bounded-below V-complete Cartier modules and bounded-below cyclotomic spectra; see [AN21, Theorem 3.21]. Then the proof of Proposition 5.2 shows that the cyclotomic Frobenius $\varphi\colon X \to X^{tC_p}$ is d-truncated, since $\mathrm{fib}(\varphi) = \mathrm{fib}(F\colon M \to M^{hC_p})$. Using Proposition 5.7, we find that $\mathrm{TR}(X)/p$ is $(d+3)$-truncated, whence $\mathrm{TR}(X)$ is $(d+2)$-truncated.

Corollary 6.7. *Let $R_i, i \in I$ be a filtered system of rings and let $R = \varinjlim R_i$. Suppose that the map $\mathrm{TR}(R_i;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R_i;\mathbb{F}_p)$ is d-truncated for all $i \in I$. Then $\mathrm{TR}(R;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R;\mathbb{F}_p)$ is $(d+2)$-truncated.*

Proof. Let $v\colon \Sigma^u(S^0/p) \to (S^0/p)$ be a v_1-self map (so we can take $u = 2p-2$ for p odd and $u = 8$ for $p = 2$). Let X be any spectrum. Then we observe that the following assertions are equivalent.

(i) The map $X/p \to L_{K(1)}X/p$ is d-truncated.
(ii) The spectrum $X \otimes S^0/(p,v)$ is $(d+u+1)$-truncated.

The equivalence is proved analogously to Lemma 5.8, noting that $L_{K(1)}(X/p) = X \otimes (S^0/p)[v^{-1}]$ by the telescope conjecture at height 1 [Mah81, Mil81].

Therefore, it suffices to show that $\mathrm{TR}(R;\mathbb{Z}_p) \otimes S^0/(p,v)$ is $(d+u+3)$-truncated. To this end, let $M = \varinjlim \mathrm{TR}(R_i;\mathbb{Z}_p)$, so M is a connective topological Cartier module which is not necessarily derived V-complete; the V-completion of its p-completion is $\mathrm{TR}(R;\mathbb{Z}_p)$ since $\mathrm{THH}(-)$ commutes with filtered colimits (as a functor from rings to $\mathrm{CycSp}_{\geq 0}$). Now the topological Cartier module $M \otimes S^0/(p,v)$ is $(d+u+1)$-truncated as a filtered colimit of $(d+u+1)$-truncated objects. Taking the V-completion and using Proposition 6.5, we find that $\mathrm{TR}(R;\mathbb{Z}_p)/(p,v)$ is $(d+u+3)$-truncated. Therefore, $\mathrm{TR}(R;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R;\mathbb{F}_p)$ is $(d+2)$-truncated. $\square$

6.3 The main pro-Galois result

In this subsection we prove the following pro-Galois descent result.

Theorem 6.8 (Pro-Galois descent in the generic fiber). *Fix $d \geq 0$. Let R be a p-complete ring such that $\mathrm{TR}(R;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R;\mathbb{F}_p)$ is d-truncated. Let S be an R-algebra.*

Let G be a profinite group of finite p-cohomological dimension which acts continuously on the R-algebra S (given the discrete topology). Suppose that the following assertions hold.

(i) *$R[1/p] \to S[1/p]$ is a G-Galois extension.*
(ii) *There is a cofinal collection of open normal subgroups $N_i \subset G$, $i \in I$, such that $S_i := S^{N_i}$ is a finite, finitely presented R-algebra and such that $\mathrm{TR}(S_i;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(S_i;\mathbb{F}_p)$ is d-truncated.*
(iii) *The induced sheaf of spectra on finite continuous G-sets given by*

$$T \mapsto L_{K(1)}(K(\mathrm{Fun}_G(T, S[1/p])))$$

(for $\mathrm{Fun}_G(T, S[1/p])$ denoting G-equivariant functions $T \to S$) is hypercomplete. For example, this holds if $R[1/p]$ has finite Krull dimension and there is a uniform bound on the mod p cohomological dimensions of the residue fields, but also in cases such as Example 6.4.

Then the map

$$L_{K(1)}\mathrm{TR}(R) \to (L_{K(1)}\mathrm{TR}(S))^{hG_{\mathrm{cts}}} := \mathrm{Tot}\Big(L_{K(1)}\mathrm{TR}(S) \rightrightarrows L_{K(1)}\mathrm{TR}(\mathrm{Fun}_{\mathrm{cts}}(G,S)) \substack{\to\\ \to\\ \to} \dots\Big) \quad (13)$$

is an equivalence and the map

$$\mathrm{TR}(R;\mathbb{F}_p) \to \mathrm{Tot}\Big(\mathrm{TR}(S;\mathbb{F}_p) \rightrightarrows \mathrm{TR}(\mathrm{Fun}_{\mathrm{cts}}(G,S);\mathbb{F}_p) \substack{\to\\ \to\\ \to} \dots\Big)$$

is $(d+2)$-truncated.

Proof. Let $\mathcal{F}$ be any product-preserving presheaf from finite continuous G-sets to p-torsion spectra. We let $\mathcal{F}^{\mathrm{disc}}$ denote the left Kan extension to profinite G-sets, so if S is a profinite G-set which can be written $S \simeq \varprojlim S_j$ for some finite continuous G-sets S_j, then $\mathcal{F}^{\mathrm{disc}}(S) \simeq \varinjlim \mathcal{F}(S_j)$. We let

$$R\Gamma(*, \mathcal{F}) = \mathrm{Tot}\Big(\mathcal{F}^{\mathrm{disc}}(G) \rightrightarrows \mathcal{F}^{\mathrm{disc}}(G \times G) \substack{\to\\ \to\\ \to} \dots\Big).$$

As in [CM19, § 4.1], $R\Gamma(*, \mathcal{F})$ is the value of the hypersheafification or Postnikov sheafification of $\mathcal{F}$ (with respect to the topology on finite continuous G-sets where covering families are jointly surjective ones) at $*$. When we work with p-torsion spectra, our assumption that G has finite cohomological dimension implies that $R\Gamma(*, -)$ commutes with all colimits.

Now we take $\mathcal{F}$ to be the presheaf which sends a finite continuous G-set T to $\mathrm{TR}(\mathrm{Fun}_G(T, S);\mathbb{F}_p)$, where $\mathrm{Fun}_G(T, S)$ is (as in the statement) the ring of G-equivariant functions $T \to S$. Unwinding the definitions and hypotheses, we find from Corollary 2.5 that if $T = G/N_j$ for one of the distinguished normal subgroups N_j, we have that

$$L_{K(1)}\mathcal{F}(*) \xrightarrow{\sim} \mathrm{Tot}\Big(L_{K(1)}\mathcal{F}(T) \rightrightarrows L_{K(1)}\mathcal{F}(T \times T) \substack{\to\\ \to\\ \to} \dots\Big)$$

is an equivalence. Note $L_{K(1)}\mathcal{F}$ is an example of Construction 6.3 for the profinite group G (though we denote by R the p-complete ring). In particular, our assumptions imply hypercompleteness of $L_{K(1)}\mathcal{F}$, so, passing to the limit, we find that

$$L_{K(1)}\mathcal{F}(*) \simeq R\Gamma(*, L_{K(1)}\mathcal{F}). \quad (14)$$

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Note here that $\mathcal{F}^{\mathrm{disc}}$ is not given by $\mathrm{TR}(-;\mathbb{F}_p)$ because TR does not commute with filtered colimits; instead, for instance, $\mathcal{F}^{\mathrm{disc}}(G) = \varinjlim_{i\in I} \mathrm{TR}(S_i;\mathbb{F}_p)$. By assumption (ii), $\mathcal{F}^{\mathrm{disc}}(G), \mathcal{F}^{\mathrm{disc}}(G\times G), \dots$ are spectra with the property that the map to their $K(1)$-localization is d-truncated; therefore, $R\Gamma(*,\mathcal{F}) \to L_{K(1)}R\Gamma(*,\mathcal{F}) = R\Gamma(*, L_{K(1)}\mathcal{F})$ is d-truncated.

Now we apply two-out-of-three to the sequence of maps $\mathcal{F}(*) \to R\Gamma(*,\mathcal{F}) \to R\Gamma(*, L_{K(1)}\mathcal{F})$. We just showed that the second map is d-truncated, while the composite map is by (14) identified with $\mathrm{TR}(R;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(R;\mathbb{F}_p)$, which is d-truncated by assumption. Therefore, by two-out-of-three, we find that $\mathcal{F}(*) \to R\Gamma(*,\mathcal{F})$ is d-truncated. In other words,

$$\mathcal{F}(*) \to \mathrm{Tot}\Big(\mathcal{F}^{\mathrm{disc}}(G) \rightrightarrows \mathcal{F}^{\mathrm{disc}}(G\times G) \substack{\to\\ \to\\ \to} \dots\Big) \tag{15}$$

is d-truncated. Here both sides of (15) have the structure of topological Cartier modules since the forgetful functor from topological Cartier modules to spectra commutes with limits and colimits [AN21, Proposition 3.11].

Now we take the V-completion of both sides in (15) (considered as topological Cartier modules). The left-hand side of (15) is already V-complete. To analyze the right-hand-side, observe that the totalization in (15) commutes with $(-)_{hC_{p^{n-1}}}$ because $R\Gamma(*,-)$ commutes with colimits. The limit over n in computing the V-completion clearly commutes with the totalization. Thus, the V-completion of the right-hand side of (15) is given by

$$\mathrm{Tot}\Big(\mathrm{TR}(S;\mathbb{F}_p) \rightrightarrows \mathrm{TR}(\mathrm{Fun}_{\mathrm{cts}}(G,S);\mathbb{F}_p) \substack{\to\\ \to\\ \to} \dots\Big).$$

Note finally that the right-hand side of (15) is bounded below by the finiteness of the p-cohomological dimension. Taking the V-completion in (15), we find from Proposition 6.5 that

$$\mathrm{TR}(R;\mathbb{F}_p) \to \mathrm{Tot}\Big(\mathrm{TR}(S;\mathbb{F}_p) \rightrightarrows \mathrm{TR}(\mathrm{Fun}_{\mathrm{cts}}(G,S);\mathbb{F}_p) \substack{\to\\ \to\\ \to} \dots\Big)$$

is $(d+2)$-truncated, whence the last claim of the theorem. Finally, $\mathrm{TR}(S;\mathbb{F}_p), \mathrm{TR}(\mathrm{Fun}_{\mathrm{cts}}(G,S);\mathbb{F}_p)$ map via $(d+2)$-truncated maps to their $K(1)$-localizations, via Corollary 6.7, so the $K(1)$-local descent statement follows. $\square$

Example 6.9 (Discrete valuation rings). Let K be a complete, discretely valued field of characteristic 0 whose residue field k is of characteristic p with $[k:k^p] \leq p^d$. It follows that if k'/k is any finite extension, then $[k':k'^p] \leq p^d$; see [GO08, Lemma 2.1.1].

By the main result of [GO08] (which is due to [Kat82] in this case), it follows that the Galois group $\mathrm{Gal}(\bar{K}/K)$ has p-cohomological dimension at most $d+2$. Moreover, by Theorem 4.8, it follows that if L is any finite extension of K and $\mathcal{O}_L \subset L$ the ring of integers (which is excellent as a complete local ring), then $\mathrm{TR}(\mathcal{O}_L;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(\mathcal{O}_L;\mathbb{F}_p)$ is d-truncated.

We can now apply Theorem 6.8 to conclude that

$$\mathrm{TR}(\mathcal{O}_K;\mathbb{F}_p) \to \mathrm{TR}(\mathcal{O}_{\bar{K}};\mathbb{F}_p)^{h\mathrm{Gal}_K^{\mathrm{cts}}} := \mathrm{Tot}\Big(\mathrm{TR}(\mathcal{O}_{\bar{K}};\mathbb{F}_p) \rightrightarrows \mathrm{TR}(\mathcal{O}_{\bar{K}\otimes_K \bar{K}};\mathbb{F}_p) \substack{\to\\ \to\\ \to} \dots\Big)$$

is d-truncated. Note that the theorem gives a priori that the map is $(d+2)$-truncated, but we can upgrade the conclusion to d-truncated as follows: the analogous comparison map with $L_{K(1)}\mathrm{TR}(-)$ everywhere is an equivalence; the maps $\mathrm{TR}(\mathcal{O}_{\bar{K}\otimes_K\cdots\otimes_K\bar{K}};\mathbb{F}_p) \to \mathrm{TR}(\mathcal{O}_{\bar{K}\otimes_K\cdots\otimes_K\bar{K}};\mathbb{F}_p)$ are (-1)-truncated by Proposition 5.10 since these are p-torsion-free rings whose completions are perfectoid; and the map $\mathrm{TR}(\mathcal{O}_K;\mathbb{F}_p) \to L_{K(1)}\mathrm{TR}(\mathcal{O}_K;\mathbb{F}_p)$ is d-truncated (Theorem 4.8).

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Suppose, in particular, that k is perfect and of characteristic greater than 2.[6] In this case, the results of [HM03] (recalled in the proof of Theorem 4.1) show that $\mathrm{TR}(\mathcal{O}_K|K;\mathbb{F}_p)$ is the connective cover of its $K(1)$-localization, which is $L_{K(1)}\mathrm{TR}(\mathcal{O}_K;\mathbb{F}_p)$. Again, $\mathcal{O}_{\bar{K}}, \mathcal{O}_{\bar{K}\otimes_K\bar{K}}, \dots$ have p-completions which are perfectoid rings, so for them $\mathrm{TR}(-;\mathbb{F}_p)$ agrees with the connective cover of its $K(1)$-localization (Proposition 5.10). It follows that $\mathrm{TR}(\mathcal{O}_K|K;\mathbb{F}_p) \simeq \tau_{\geq 0}\mathrm{TR}(\mathcal{O}_{\bar{K}};\mathbb{F}_p)^{h\mathrm{Gal}_K^{\mathrm{cts}}}$.

Example 6.10. Let R_0 be a p-torsion-free perfectoid ring containing a system of primitive p-power roots of unity. Let R be a formally smooth R_0-algebra, which is formally étale over the formal torus (i.e. p-completed Laurent polynomial algebra) $R_0\langle t_1^{\pm 1}, \dots, t_n^{\pm 1}\rangle$.

Let $G = \mathbb{Z}_p(1)^n$ and let $S = R \otimes_{R_0\langle t_1^{\pm 1},\dots,t_n^{\pm 1}\rangle} \varinjlim_r R_0\langle t_1^{\pm 1/p^r}, \dots, t_n^{\pm 1/p^r}\rangle$ with the evident G-action by roots of unity. Using Proposition 5.10, one sees that the hypothesis of Theorem 6.8(ii) applies. The hypothesis of Theorem 6.8(iii) applies thanks to Example 6.4. It follows that the comparison map (13) is an equivalence. In particular, the $K(1)$-local TR of R is expressed as the inverse limit of a diagram of the $K(1)$-local TR of various rings whose p-completion is perfectoid.

7. The analog of Thomason's spectral sequence

In this section, we construct in certain cases an analog of Thomason's étale descent spectral sequence for $L_{K(1)}K(-)$ in terms of étale cohomology (see [Tho85, TT90] for $L_{K(1)}\mathrm{TR}(-)$). Our construction splits into two parts. First, we give a formula for TR and its $K(1)$-localization on the category of perfectoid rings. Second, we hypersheafify $L_{K(1)}\mathrm{TR}(-)$ on all rings in the arc_p-topology and compare $L_{K(1)}\mathrm{TR}(-)$ to this hypersheafification.

7.1 arc_p-cohomology

In this section we discuss the cohomology with respect to the arc_p-topology, mentioned briefly in the introduction (Definition 1.6). This is a variant of the following topology (see [ČS19, § 2.2.1]).[7]

DEFINITION 7.1 (The p-complete arc-topology). The *p-complete* arc-*topology* or $\mathrm{arc}_{\hat{p}}$-*topology* is the finitary Grothendieck topology (see Remark A.1) on the opposite of the category of derived p-complete commutative rings such that a map $R \to R'$ is a cover if, for every p-complete valuation ring V of rank at most 1 and map $R \to V$, there is an extension $V \to W$ of p-complete valuation rings of rank at most 1 and a map $R' \to W$ fitting into the following commutative diagram.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \\ V \arrow[r] & W \end{tikzcd}$$

We will also consider restrictions of the $\mathrm{arc}_{\hat{p}}$-topology to appropriate subcategories of the category of derived p-complete rings, such as the category Perfd of perfectoid rings [BMS18, § 3].

[6] We expect that this works when k has characteristic 2 as well.

[7] Throughout, for set-theoretic reasons one should impose cardinality bounds, that is, assume all the rings one allows into the site to be of cardinality less than some uncountable strong limit cardinal κ, so that the sites will be small. However, the choice of κ will not matter in all the constructions we consider and we will consequently suppress it.

The arc_p-*topology* is defined similarly, but we only consider rank 1 p-complete valuation rings where $p \neq 0$.

The $\mathrm{arc}_{\hat{p}}$-topology behaves well with respect to perfectoid rings; one knows that the structure presheaf is a sheaf of rings with respect to this topology, and one even has no higher cohomology [BS19, § 8]. The arc_p-topology is a variant where in some sense we also impose derived saturatedness conditions. Note that a functor is an arc_p-sheaf if and only if it is an $\mathrm{arc}_{\hat{p}}$-sheaf and annihilates any $\mathbb{F}_p$-algebra. Any derived p-complete ring can be covered in the $\mathrm{arc}_{\hat{p}}$-topology (respectively, the arc_p-topology) by a product of p-complete absolutely integrally closed rank 1 valuation rings (respectively, p-complete absolutely integrally closed rank 1 valuation rings where $p \neq 0$).

We now give some examples of arc_p-cohomology. To begin with, we consider the simplest case of constant sheaves (or p-adically constant ones); the result is that one essentially recovers the p-adic étale cohomology of the generic fiber.

Construction 7.2 (Constant sheaves in the arc_p-topology). Given an abelian group M, we can consider the associated constant sheaf of abelian groups in the arc_p-topology; its value on a derived p-complete ring R is given by $H^0_{\text{ét}}(\mathrm{Spec}(R[1/p]), M)$. To see this, we observe that the presheaf of abelian groups $R \mapsto H^0_{\text{ét}}(\mathrm{Spec}(R[1/p]), M)$ is a sheaf in the arc_p-topology on derived p-complete rings [BM18, Corollary 6.17] (which assumes M torsion; however, this is not necessary since we are only working with H^0). It admits a map from the constant presheaf M, and the kernel and cokernel vanish locally in the arc_p-topology.

Construction 7.3 (p-adically constant sheaves and Tate twists in the arc_p-topology). Consider the sheaf of rings $\mathbb{Z}_p$ in the arc_p-topology, defined as the p-completion of the constant sheaf associated to $\mathbb{Z}_p$, or equivalently as the inverse limit of the constant sheaves $\mathbb{Z}/p^n\mathbb{Z}$; explicitly, $\mathbb{Z}_p(R) = H^0_{\mathrm{arc}_p}(\mathrm{Spec}(R), \mathbb{Z}_p) = H^0_{\text{proét}}(\mathrm{Spec}(R[1/p]), \mathbb{Z}_p)$. We construct an invertible $\mathbb{Z}_p$-module $\mathbb{Z}_p(1)$ as the arc_p-sheafification of the presheaf $R \mapsto \mathbb{Z}_p(1)^{\mathrm{pre}}(R) := T_p(R^\times)$. To check that this is an invertible module, we observe (see [BM18, Proposition 3.30]) that $\mathbb{Z}_p(1)^{\mathrm{pre}}(R)$ is an invertible $\mathbb{Z}_p(R)$-module whenever R is a product of absolutely integrally closed valuation rings of mixed characteristic $(0, p)$. Note that $\mathbb{Z}_p(1)/p$ can equally be described as the arc_p-sheaf associated to the presheaf $R \mapsto \mu_p(R) = R^\times[p]$, since arc_p-locally all elements have pth roots.

Proposition 7.4 (arc_p-cohomology as p-adic vanishing cycles). *Let R be any derived p-complete ring. Then there is a natural equivalence*

$$R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), \mathbb{Z}_p(i)) \simeq R\Gamma_{\text{proét}}(\mathrm{Spec}(R[1/p]), \mathbb{Z}_p(i)), \tag{16}$$

where $\mathbb{Z}_p(i)$ on the right-hand side refers to the usual Tate twist on the pro-étale site of $\mathrm{Spec}(R[1/p])$.

Proof. We know that the right-hand side is a $D(\mathbb{Z}_p)^{\geq 0}$-valued sheaf for the arc_p-topology by [BM18, Corollary 6.17], and we obtain a map from the presheaf $R \mapsto T_p(R^\times)^{\otimes i}$ to the right-hand side. Sheafifying in the arc_p-topology and p-completing, we obtain a map from the left-hand side to the right-hand side as in (16). To see that (16) is an equivalence of arc_p-sheaves, it suffices to check on products of rank 1 absolutely integrally closed, p-complete valuation rings of mixed characteristic as these form a basis (see [BM18, Proposition 3.29]), which is handled by the next lemma. ☐

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LEMMA 7.5. *Let R be a product $\prod_{t\in T} V_t$ of absolutely integrally closed, p-complete valuation rings. Then:*

(i) $H^j_{\text{proét}}(\mathrm{Spec}(R[1/p]), \mathbb{Z}_p(i)) = 0$ *for* $j > 0$*;*
(ii) *the map* $T_p(R^\times) \to H^0_{\text{proét}}(\mathrm{Spec}(R[1/p]), \mathbb{Z}_p(1))$ *is an isomorphism.*

Proof. It suffices to prove both claims after reducing modulo p. Consider the functor $A \mapsto F_j(A) := H^j(\mathrm{Spec}(\hat{A}_p[1/p]; \mathbb{F}_p(i))$. This functor commutes with finite products and filtered colimits by the Gabber–Fujiwara theorem; see [BM18, Theorem 6.11] for an account. To prove that $F_j(R) = 0$ for $j > 0$ which implies (i), it suffices as in [BM18, Corollary 3.17] to show that F vanishes on every ultraproduct of the $\{V_t\}$ for each ultrafilter on T. But these ultraproducts are all absolutely integrally closed, p-henselian valuation rings, whence (i). Claim (ii) is proved similarly using the map $A^\times[p] \to H^0_{\text{proét}}(\mathrm{Spec}(A[1/p]), \mathbb{F}_p(1))$. □

Next we consider the cohomology of the structure presheaf.

Construction 7.6 (arc$_p$-cohomology of the structure presheaf). For a derived p-complete commutative ring R, we let $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), \mathcal{O})$ denote the arc$_p$-cohomology of the structure presheaf.

The arc$_p$-topology restricts to a topology on the opposite of Perfd. Our first goal is to identify concretely arc$_p$-cohomology on Perfd.

Construction 7.7 (Saturation). Let R be a perfectoid ring. We have the natural quotient $R \to R' := R/\mathrm{rad}((p))$, which is the universal map from R to a perfect $\mathbb{F}_p$-algebra. Note that $R' \otimes^L_R R'$ is discrete (and equivalent to R') since relative tensor products of perfectoid rings are p-completely discrete (and perfectoid).

Let $J = \mathrm{rad}(p)$. We have $J \otimes^L_R J \xrightarrow{\sim} J$. It follows that one is in the setup of almost mathematics, and we have a derived almostification functor $(-)_*\colon D(R) \to D(R)$, which is also given by $R\mathrm{Hom}_R(J, -)$. We claim that in fact J can be made explicit, and in particular that it has projective dimension at most 1, so that for any discrete R-module M, $M_* \in D(R)^{[0,1]}$. Indeed, there exists an element $\omega \in R$ such that ω is a unit multiple of p and such that ω admits a compatible system of p-power roots $\{\omega^{1/p^n}\}_{n\geq 0}$; see [BMS18, Lemma 3.9]. We claim that as an R-module, there is an equivalence

$$J \simeq \varinjlim \Big(R \overset{\omega^{1-1/p}}{\to} R \overset{\omega^{1/p-1/p^2}}{\to} \dots \Big). \tag{17}$$

To see this, we observe first that the filtered colimit on the right-hand side is a submodule of R given by the ideal $J' := \bigcup_n (\omega^{1/p^n})$ (i.e. given by multiplication by ω^{1/p^n} on the nth term); we can see this by arc$_{\hat{p}}$-descent to reduce to the case where R is a product of valuation rings of rank at most 1, in which case the claim is clear. Now clearly $J' \subset J$ and R/J' is a ring of characteristic p on which the Frobenius is surjective. To obtain $J' = J$, we use [BMS18, Lemma 3.10] to see that the Frobenius induces an isomorphism $R/\omega^{1/p^n} \xrightarrow{\sim} R/\omega^{1/p^{n-1}}$ for $n \geq 1$. This implies that R/J' is perfect, whence $J = J'$ as desired.

DEFINITION 7.8 (Spherically complete fields). We recall (see [vR78, Chapter 4] for an account) that a nonarchimedean field is called *spherically complete* if every decreasing sequence of closed

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disks has nonempty intersection; in particular, any such field is complete. Any nonarchimedean field admits an extension which is spherically complete and algebraically closed.

Lemma 7.9. *Let C be spherically complete with ring of integers $\mathcal{O}_C$. Given any tower $\{M_i\}_{i\geq 1}$ of cyclic $\mathcal{O}_C$-modules, we have $\varprojlim^1 M_i = 0$.*

Proof. Writing $\{M_i\}$ as the cokernel of an injective map $\{N_i\}_{i\geq 1} \to \{N_i'\}_{i\geq 1}$ of inverse systems with the N_i' levelwise isomorphic to $\mathcal{O}_C$ and using the long exact sequence, we reduce to the case that the M_i are individually isomorphic to $\mathcal{O}_C$ (in particular, torsion-free). Moreover, upon passing to a cofinal subfamily, we may assume the $\{M_i\}_{i\geq 1}$ form a descending sequence of cyclic ideals $\{I_i\}_{i\geq 1} \subset \mathcal{O}_C$; using the short exact sequence of inverse systems $0 \to \{I_i\}_{i\geq 1} \to \{\mathcal{O}_C\}_{i\geq 1} \to \{\mathcal{O}_C/I_i\}_{i\geq 1} \to 0$ (with the middle sequence constant), we see that it suffices to show that $\mathcal{O}_C \to \varprojlim_i \mathcal{O}_C/I_i$ is surjective. But this is precisely the definition of spherical completeness. $\square$

Proposition 7.10 (arc_p-cohomology of perfectoids). *The functor $R \mapsto R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), \mathcal{O})$ restricted on* Perfd *agrees with $R \mapsto R_*$. In particular, for $R \in \mathrm{Perfd}$, $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), \mathcal{O}) \in D(R)^{[0,1]}$.*

Proof. Let R be a perfectoid ring. First, we claim that arc_p-cohomology of R with $\mathcal{O}$-coefficients can be calculated on either all derived p-complete rings or the subcategory of perfectoid rings (endowed with the arc_p-cohomology). To see this, we use that perfectoid rings form a basis for the arc_p-topology. In fact, any ring admits an arc_p-cover by a semiperfectoid ring S, and then one can take the perfectoidization [BS19, § 8] of S, which is an arc_p-cover of S (e.g. maps of S or its perfectoidization into p-complete absolutely integrally closed valuation rings are the same). The rest of the claim is a general argument following from the fact that perfectoid rings form a basis for the arc_p-topology; see Proposition A.6 below.

Now it suffices to prove that $R \mapsto R_*$ is a $D(\mathbb{Z})^{\geq 0}$-valued arc_p-sheaf on Perfd and that the map $R \to R_*$ is locally an equivalence. The fact that it is an arc_p-sheaf follows because the structure presheaf is a $D(\mathbb{Z})^{\geq 0}$-valued sheaf on the $\mathrm{arc}_{\hat{p}}$-topology on Perfd and $R \mapsto R_*$ annihilates perfect $\mathbb{F}_p$-algebras. The fact that $R \to R_*$ is locally an equivalence follows by taking an arc_p-cover by a product of rings of integers in various spherically complete, algebraically closed nonarchimedean fields of mixed characteristic; for such rings, $R = R_*$. $\square$

Let R_0 be a perfectoid base ring, and let R be a derived p-complete R_0-algebra. One has the construction of the *perfectoidization* R_{perfd} of R, a coconnective E_∞-algebra under R which has the property that if R is semiperfectoid, then R_{perfd} is discrete and is the universal perfectoid ring that R maps to. As shown in [BS19, § 8], $R \mapsto R_{\mathrm{perfd}}$ is the $\mathrm{arc}_{\hat{p}}$-cohomology of the structure presheaf on the category of derived p-complete R_0-algebras.

Proposition 7.11 (arc_p-cohomology as saturated perfectoidization). *For R an R_0-algebra which is derived p-complete, we have a natural equivalence of E_∞-algebras in $D(R_0)$, $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), \mathcal{O}) \simeq (R_{\mathrm{perfd}})_*$ (where almost mathematics is taken relative to R_0 and the ideal* $\mathrm{rad}(p)$*).*

Proof. This is proved similarly to Proposition 7.10. We have a natural (in R) map $R \to (R_{\mathrm{perfd}})_*$. To show that it is the arc_p-sheafification, it suffices to show that the codomain is an arc_p-sheaf, and that the map $R \to (R_{\mathrm{perfd}})_*$ has cofiber which vanishes locally in the arc_p-topology;

note also that we can compute the arc_p-cohomology either over all derived p-complete rings or over R_0-algebras (Example A.7). The codomain is an $\mathrm{arc}_{\hat{p}}$-sheaf because of the identification of perfectoidization with $\mathrm{arc}_{\hat{p}}$-cohomology [BS19, § 8], and hence is an arc_p-sheaf since it annihilates $\mathbb{F}_p$-algebras. The cofiber of $R \to (R_{\mathrm{perfd}})_*$ vanishes locally in the arc_p-topology, as one sees by working with perfectoid R and using the argument of Proposition 7.10. □

Example 7.12. Suppose R is the p-completion of a ring which is integral over the perfectoid ring R_0. Then the perfectoidization R_{perfd} is discrete [BS19, Theorem 10.11]. It thus follows that $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), \mathcal{O}) \in D(R)^{[0,1]}$.

Construction 7.13 (Witt vector cohomology in the arc_p-topology). We consider the presheaf $W(\mathcal{O})$ given by $R \mapsto W(R)$ and its arc_p-cohomology (with Tate twists) $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O})(i))$. Since the Witt vector functor is endowed with Frobenius and Verschiebung operators, so is the construction $R \mapsto R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O})(i))$.

In our setting, we can think of the Witt vector cohomology considered above as a one-parameter (along V) deformation of the structure presheaf cohomology, especially in light of the next result.

PROPOSITION 7.14. *For any ring R, $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O})(i))$ is p-complete and complete with respect to the Verschiebung.*

Proof. By base-changing to $\mathbb{Z}_p[\zeta_{p^\infty}]$, we may assume without loss of generality that $i = 0$. Then this follows from Proposition A.10 below, since the presheaf $W(\mathcal{O})$ commutes with finite products and is complete with respect to (p, V). □

COROLLARY 7.15. *Let R be finite and finitely presented over a perfectoid ring. Then $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O})(i)) \in D(\mathbb{Z}_p)^{[0,1]}$ for any i.*

Proof. By p-completeness and V-completeness (Proposition 7.14), it suffices to prove the analogous statement for $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), \mathcal{O}(i)/p)$. Replacing R with $R[\zeta_p]$ and using $(\mathbb{Z}/p)^\times$-Galois descent along this extension, we may assume that R contains a primitive pth root of unity. This lets us reduce to the case $i = 0$, whence the result follows from Example 7.12. □

We note, finally, that one can recover the p-adic nearby cycles (see Proposition 7.4) as the fixed points of Frobenius on Witt vector arc_p-cohomology. This will not be used in the sequel.

PROPOSITION 7.16. *For any ring R, there is a natural equivalence for each i,*

$$R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), \mathbb{Z}_p(i)) \simeq R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O})(i))^{F=1}. \tag{18}$$

Proof. We have a natural map $\mathbb{Z}_p \to W(\mathcal{O})^{F=1}$ of presheaves. Twisting and sheafifying in the arc_p-topology, we obtain the map from left to right in (18). To see that it is an equivalence, it suffices to check that it is an equivalence on products (over some indexing set T) of rings of the form $\mathcal{O}_C$, for C spherically complete and algebraically closed of mixed characteristic $(0, p)$; now we can trivialize the Tate twists, so can assume $i = 0$. Using Lemma 7.5, we find that the left-hand side is discrete and simply given by $\prod_T \mathbb{Z}_p$. The right-hand side is also given by $\prod_T \mathbb{Z}_p$ by Lemma 7.17 below. □

Lemma 7.17. *Let C be an algebraically closed complete nonarchimedean field. Then the natural map $\mathbb{Z}_p \to W(\mathcal{O}_C)^{F=1}$ is an equivalence.*

Proof. It suffices to work modulo p, that is, to show that the natural map $\mathbb{F}_p \to (W(\mathcal{O}_C)/p)^{F=1}$ is an equivalence. But the Witt vector Frobenius reduces modulo p to the ordinary Frobenius on the ring $W(\mathcal{O}_C)/p$. By the Artin–Schreier sequence, $(W(\mathcal{O}_C)/p)^{F=1}$ is the mod p étale cohomology of $W(\mathcal{O}_C)/p$. This is unchanged by taking the quotient by the ideal $VW(\mathcal{O}_C)/p$, which squares to zero; therefore, it is also the étale cohomology of $\mathcal{O}_C/p$, or equivalently of the residue field k; this in turn is $\mathbb{F}_p$ as C is algebraically closed and hence so is k. $\square$

7.2 $\mathrm{THH}^{C_{p^r}}$ of perfectoid rings

Here we calculate the fixed points of THH of perfectoid rings, generalizing the main result of [Hes06]; these results are known to experts. Recall that for any p-complete ring R, we have $\pi_0\mathrm{TR}(R;\mathbb{Z}_p) \simeq W(R)$; see [HM97, Theorem F].

Our strategy is to treat the case of a p-torsion-free perfectoid ring by a direct spectral sequence argument and appeal to known results for a perfect $\mathbb{F}_p$-algebra; we will glue both cases together using an excision argument. To begin with, we consider the characteristic p case. For perfect fields, this result appears in [HM97, § 5].

Lemma 7.18. *Let R be a perfect $\mathbb{F}_p$-algebra. Then:*

(i) $\pi_*(\mathrm{THH}(R;\mathbb{Z}_p)^{C_{p^r}}) \simeq W_{r+1}(R)[\sigma_r]$ *for* $|\sigma_r| = 2$;
(ii) $\mathrm{TR}(R;\mathbb{Z}_p) \simeq HW(R)$.

Proof. The Segal conjecture holds for R: in fact, $\varphi\colon \mathrm{THH}(R;\mathbb{Z}_p) \to \mathrm{THH}(R;\mathbb{Z}_p)^{tC_p}$ is (-3)-truncated; see [BMS19, § 6]. Therefore, we have that $\mathrm{THH}(R;\mathbb{Z}_p)^{C_{p^r}} \to \mathrm{THH}(R;\mathbb{Z}_p)^{hC_{p^r}} \xrightarrow{\varphi^{hC_{p^r}}} \mathrm{THH}(R;\mathbb{Z}_p)^{tC_{p^{r+1}}}$ are equivalences on connective covers; see [NS18, Corollary II.4.9]. Using the homotopy fixed point spectral sequence (or [BMS19, § 6]), we find $\pi_*(\mathrm{THH}(R;\mathbb{Z}_p)^{tS^1}) \simeq W(R)[\sigma^{\pm 1}]$. Now $\mathrm{THH}(R;\mathbb{Z}_p)^{tC_{p^{r+1}}} \simeq \mathrm{THH}(R;\mathbb{Z}_p)^{tS^1}/p^{r+1}$ by [NS18, Lemma IV.4.12]. Combining these facts, we see that claim (i) follows as $W_{r+1}(R) = W(R)/p^{r+1}$.

For claim (ii), note that $\mathrm{THH}(R;\mathbb{Z}_p)$ is a module in CycSp over $K(\mathbb{F}_p;\mathbb{Z}_p) \simeq H\mathbb{Z}_p$. By Proposition 5.7, we have that $\mathrm{TR}(R;\mathbb{Z}_p) \to L_{K(1)}\mathrm{TR}(R;\mathbb{Z}_p)$ is 0-truncated; since the target vanishes, it follows that $\mathrm{TR}(R;\mathbb{Z}_p)$ is 0-truncated, whence the result as we know $\pi_0\mathrm{TR}(R;\mathbb{Z}_p)$. $\square$

In the following, we will say that an E_∞-ring A is *weakly even periodic* if the odd homotopy groups of A vanish, and if for all $m, n \in \mathbb{Z}$, the map $\pi_{2m}(A) \otimes_{\pi_0(A)} \pi_{2n}(A) \to \pi_{2(n+m)}(A)$ is an isomorphism; in particular, $\pi_2(A)$ is an invertible $\pi_0(A)$-module, and Zariski locally $\pi_*(A)$ is a Laurent polynomial algebra over $\pi_0(A)$ on a degree 2 class.

Lemma 7.19. *Let A, B, C be weakly even periodic E_∞-rings and fix maps $A \to C, B \to C$ such that $\pi_0(A) \to \pi_0(C)$ is surjective. Then $A \times_C B$ is weakly even periodic.*

Proof. It follows that $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(C) \xrightarrow{\sim} \pi_*(C)$ and, similarly, $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \xrightarrow{\sim} \pi_*(B)$. Now since $\pi_0(A \times_C B), \pi_0(A), \pi_0(B), \pi_0(C)$ form a Milnor square, the category of finitely generated projective $\pi_0(A \times_C B)$-modules is the homotopy pullback of the categories of finitely generated projective $\pi_0(A)$-modules, $\pi_0(B)$-modules, and $\pi_0(C)$-modules; see [Mil71, § 2]. The result now follows. $\square$

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Construction 7.20 (p-torsion-free quotients of perfectoid rings). Let R be a perfectoid ring, and let $I \subset R$ be the ideal of p-power torsion. Then $pI = 0$; see [BMS19, Proposition 4.19].

We have in fact that R/I is perfectoid as well. To see this, we use the theory of perfectoidizations [BS19, §§ 7 and 8]. Let $R' = (R/p)_{\mathrm{perf}}$. For every p-complete valuation ring V with a map $R \to V$ which does not annihilate $I \subset R$, we find that V has characteristic p and we obtain a unique extension $R' \to V$. Therefore, applying [BS19, Corollary 8.11], we obtain a pullback square of perfectoid rings

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \\ (R/I)_{\mathrm{perfd}} \arrow[r] & (R'/I)_{\mathrm{perfd}} \end{tikzcd} \tag{19}$$

Since the vertical arrows are surjective (see [BS19, Theorem 7.4]), this is a Milnor square of rings. Since the terms on the right-hand side are both $\mathbb{F}_p$-algebras, it follows that the kernel of $R \to (R/I)_{\mathrm{perfd}}$ is annihilated by p; since it contains I, it must be equal to I and we have $(R/I)_{\mathrm{perfd}} = R/I$.

In particular, this shows that any perfectoid ring naturally fits into a Milnor square involving a p-torsion-free perfectoid ring and a map of perfect $\mathbb{F}_p$-algebras. The diagram (19) is in addition a homotopy pushout square of E_∞-rings, that is, there are no higher Tor terms, since everything involved is perfectoid. It follows by [LT19] that (19) induces a pullback after applying any localizing invariant.

PROPOSITION 7.21. *Let R be a perfectoid ring. Then, for each $r \geq 1$, we have the following assertions.*

(i) $\pi_*\mathrm{THH}(R;\mathbb{Z}_p)^{C_{p^r}}$ *is concentrated in even degrees.*
(ii) $\pi_2\mathrm{THH}(R;\mathbb{Z}_p)^{C_{p^r}}$ *is an invertible module over* $\pi_0\mathrm{THH}(R;\mathbb{Z}_p)^{C_{p^r}} \simeq W_{r+1}(R)$.
(iii) *The multiplication map* $\mathrm{Sym}^t(\pi_2\mathrm{THH}(R;\mathbb{Z}_p)^{C_{p^r}}) \to \pi_{2t}\mathrm{THH}(R;\mathbb{Z}_p)^{C_{p^r}}$ *is an isomorphism for all* $t \geq 0$.

Proof. Recall [BMS19, § 6] that we have a noncanonical isomorphism $\pi_*\mathrm{THH}(R;\mathbb{Z}_p)^{tC_p} \simeq R[\sigma^{\pm 1}]$ and that $\varphi\colon \mathrm{THH}(R;\mathbb{Z}_p) \to \mathrm{THH}(R;\mathbb{Z}_p)^{tC_p}$ exhibits the source as the connective cover of the target. Therefore, we have that $\mathrm{THH}(R;\mathbb{Z}_p)^{C_{p^r}} \to \mathrm{THH}(R;\mathbb{Z}_p)^{hC_{p^r}} \xrightarrow{\varphi^{hC_{p^r}}} \mathrm{THH}(R;\mathbb{Z}_p)^{tC_{p^{r+1}}}$ are equivalences on connective covers; see [NS18, Corollary II.4.9].

Our next claim is that $\mathrm{THH}(R;\mathbb{Z}_p)^{tC_{p^{r+1}}}$ is a weakly even periodic E_∞-ring for any $r \geq 0$. If R is p-torsion-free, this follows from the (degenerate) Tate spectral sequence applied to $\mathrm{THH}(R;\mathbb{Z}_p)$, whose homotopy groups form a polynomial algebra over R on a class in degree 2. If R is an $\mathbb{F}_p$-algebra, the claim also follows from Lemma 7.18. We now treat the case of a general perfectoid R. In view of the Milnor square (19) which is also a homotopy pushout of E_∞-rings and the main result of [LT19], we find that $\mathrm{THH}(-;\mathbb{Z}_p)^{tC_{p^{r+1}}}$ carries (19) to a pullback of E_∞-ring spectra. Moreover, on $\pi_0(-)$ this square yields a Milnor square since $\pi_0\mathrm{THH}(-;\mathbb{Z}_p)^{tC_{p^{r+1}}} \simeq W_{r+1}(-)$ for perfectoids via the previous paragraph and [HM97, Theorem F]. Via Lemma 7.19, the claim of weak even periodicity follows.

Now the above paragraphs show that $\mathrm{THH}(R;\mathbb{Z}_p)^{C_{p^r}}$ is the connective cover of a weakly even periodic E_∞-ring, and we already know its π_0 is given by $W_{r+1}(R)$, whence the result. $\square$

7.3 Description of TR of perfectoids

Let C be spherically complete and algebraically closed of mixed characteristic $(0,p)$. Then we recall some of the additional rings attached to C. We have the tilt $C^\flat$, the ring of integers $\mathcal{O}_{C^\flat} \subset C^\flat$, and the Fontaine ring $A_{\mathrm{inf}} = A_{\mathrm{inf}}(\mathcal{O}_C) = W(\mathcal{O}_{C^\flat})$. Choosing a compatible system $(1, \zeta_p, \zeta_{p^2}, \ldots)$ of primitive p^nth roots in C, we obtain an element $\epsilon \in \mathcal{O}_{C^\flat}$ and let $[\epsilon]$ denote the corresponding element of A_{inf}. We have the map $\theta\colon A_{\mathrm{inf}} \to \mathcal{O}_C$, which exhibits the source as the universal p-adically pro-nilpotent thickening of the latter; in particular, this leads to a map $A_{\mathrm{inf}} \to W(\mathcal{O}_C)$, which is a surjection in this case [BMS18, Lemma 3.23].

Lemma 7.22 (TR in the spherically complete case). *Let C be a spherically complete, algebraically closed nonarchimedean field of mixed characteristic $(0,p)$ and let $\mathcal{O}_C \subset C$ be the ring of integers. Let $\beta \in \pi_2\mathrm{TR}(\mathcal{O}_C;\mathbb{Z}_p)$ be the Bott element (arising from the image of the cyclotomic trace). Then $\mathrm{TR}_*(\mathcal{O}_C;\mathbb{Z}_p) \simeq W(\mathcal{O}_C)[\beta]$.*

Proof. Since everything is p-complete, it suffices to see that $\mathrm{TR}(\mathcal{O}_C;\mathbb{F}_p)/\beta$ is 0-truncated. Now for each n, each homotopy group in $\pi_*(\mathrm{THH}^{C_{p^n}}(\mathcal{O}_C;\mathbb{F}_p))$ is a cyclic module over $W(\mathcal{O}_C)/p$ (Proposition 7.21), which in turn is a quotient of $A_{\mathrm{inf}}(\mathcal{O}_C)/p = \mathcal{O}_{C^\flat}$. Moreover, the homotopy groups are concentrated in even degrees.

Now $C^\flat$ is spherically complete; see the proof of [BMS18, Lemma 3.23]. Using the previous paragraph, Lemma 7.9, and the Milnor exact sequence, it follows that $\pi_*(\mathrm{TR}(\mathcal{O}_C;\mathbb{F}_p))$ is concentrated in even degrees. By Lemma 5.8 and Proposition 5.10, we also find that the cofiber of β is 2-truncated, so $\pi_*(\mathrm{TR}(\mathcal{O}_C;\mathbb{F}_p)/\beta)$ is concentrated in degrees 0 and 2, and is given by $W(\mathcal{O}_C)/p$ in degree 0. It thus suffices to show that the zeroth Postnikov section $\mathrm{TR}(\mathcal{O}_C;\mathbb{Z}_p)/\beta \to HW(\mathcal{O}_C)$ induces an isomorphism on $\pi_0(-)^{tS^1}$. But we have by [AN21, Corollary 10] that $\mathrm{TR}(\mathcal{O}_C;\mathbb{Z}_p)^{tS^1} = \mathrm{THH}(\mathcal{O}_C;\mathbb{Z}_p)^{tS^1}$, and on homotopy groups this is $A_{\mathrm{inf}}[\sigma^{\pm 1}]$ (see [BMS19, §6]). Moreover, with respect to the above identifications, we have that β is a $\mathbb{Z}_p^\times$-multiple of $([\epsilon]-1)\sigma$; see [HN19, Theorem 1.3.6]. Therefore, we have $\pi_0(\mathrm{TR}(\mathcal{O}_C;\mathbb{Z}_p)/\beta)^{tS^1} \simeq A_{\mathrm{inf}}/([\epsilon]-1)[\sigma^{\pm 1}]$, whence the result since $A_{\mathrm{inf}}/([\epsilon]-1) \xrightarrow{\sim} W(\mathcal{O}_C)$ by [BMS18, Lemma 3.23]. □

Proposition 7.23 (TR of perfectoid rings). *For $R \in \mathrm{Perfd}$ and for $i > 0$, we have natural equivalences*

$$\tau_{[2i-1,2i]}\mathrm{TR}(R;\mathbb{Z}_p) \simeq R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O})(i))[2i]. \tag{20}$$

For $R \in \mathrm{Perfd}$ we have, for $i \in \mathbb{Z}$,

$$\tau_{[2i-1,2i]}\left(L_{K(1)}\mathrm{TR}(R;\mathbb{Z}_p)\right) \simeq R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O})(i))[2i]. \tag{21}$$

Proof. The structure presheaf on Perfd defines an $\mathrm{arc}_{\hat{p}}$-sheaf [BS19, Proposition 8.9]. Taking the inverse limit in Proposition 7.21 over the restriction maps (and accounting for possible $\lim^1$ terms), we find in particular that $R \mapsto \tau_{[2i-1,2i]}\mathrm{TR}(R;\mathbb{Z}_p)$ defines an $\mathrm{arc}_{\hat{p}}$-sheaf of spectra on Perfd. Since, for $i > 0$, this functor annihilates perfect $\mathbb{F}_p$-algebras, it in fact defines an arc_p-sheaf on Perfd.

The cyclotomic trace gives a map of $\mathbb{Z}_p$-modules $T_p(R^\times) \to \pi_2\mathrm{TR}(R;\mathbb{Z}_p)$. Since $\pi_0(\mathrm{TR}(R;\mathbb{Z}_p)) \simeq W(R)$, we obtain natural maps $f_i\colon W(R)(i) \to \pi_{2i}(\mathrm{TR}(R;\mathbb{Z}_p))$ for $i > 0$. To complete the proof, it suffices to show that:

(a) f_i is an isomorphism for a cofinal (in the arc_p-topology) collection of $R \in \mathrm{Perfd}$;
(b) $\pi_{2i-1}(\mathrm{TR}(R;\mathbb{Z}_p)) = 0$ for a cofinal (in the arc_p-topology) collection of $R \in \mathrm{Perfd}$.

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We will take this collection to be the set of products of perfectoid rings of the form $\mathcal{O}_C$, for C a spherically complete and algebraically closed nonarchimedean field of mixed characteristic $(0, p)$. First, Proposition 7.21 shows that the construction $R \mapsto \mathrm{THH}(R; \mathbb{Z}_p)$ commutes with products for $R \in \mathrm{Perfd}$; inductively and taking the limit, we find that $R \mapsto \mathrm{TR}(R; \mathbb{Z}_p)$ commutes with products for $R \in \mathrm{Perfd}$. Thus, it suffices to verify the above claims for $R = \mathcal{O}_C$ itself, and that follows from Lemma 7.22.

For the claim about $L_{K(1)}\mathrm{TR}(R; \mathbb{Z}_p)$, note first that the first part of the proof and Proposition 5.10 (and (19) to reduce to the p-torsion-free case) show easily that $R \mapsto L_{K(1)}\mathrm{TR}(R; \mathbb{Z}_p)$ defines an arc_p-sheaf on Perfd (see also the remarks at the beginning of the proof of Proposition 5.10, concerning when $L_{K(1)}(-)$ commutes with homotopy limits). It then follows from the above (by inverting the Bott element over perfectoid rings containing p-power roots of unity) that the homotopy groups of the arc_p-sheaf $L_{K(1)}\mathrm{TR}(R; \mathbb{Z}_p)$ on Perfd are given by $\pi_{2i} \simeq W(\mathcal{O})(i)$, and we obtain a filtration with associated graded terms the right-hand side of (21) via the Postnikov filtration as arc_p-sheaves. Note that the Postnikov filtration is always exhaustive, and it is complete because it is complete on the arc_p-sheaf $R \mapsto \tau_{\geq 1} L_{K(1)}\mathrm{TR}(R) \simeq \tau_{\geq 1}\mathrm{TR}(R; \mathbb{Z}_p)$. Since the relevant associated graded pieces are in homological degrees $[2i-1, 2i]$ by Corollary 7.15, the result follows. $\square$

7.4 The main results

In this subsection we prove Theorem 1.8 from the introduction. Our strategy is to compare $L_{K(1)}\mathrm{TR}(-)$ with its arc_p-hypersheafification.

Construction 7.24 (The invariant $\left(L_{K(1)}\mathrm{TR}(-)\right)^\sharp$). We define the functor $(L_{K(1)}\mathrm{TR}(-))^\sharp$ on derived p-complete rings as the arc_p-hypersheafification of the functor $L_{K(1)}\mathrm{TR}(-)$ on derived p-complete rings. We have a comparison map

$$L_{K(1)}\mathrm{TR}(-) \to \left(L_{K(1)}\mathrm{TR}(-)\right)^\sharp. \tag{22}$$

For a ring which is not necessarily derived p-complete, we define $(L_{K(1)}\mathrm{TR}(-))^\sharp$ as that of its derived p-completion.

To analyze this construction, we will use the results about $L_{K(1)}\mathrm{TR}$ of perfectoids proved in the previous section, as well as some general tools from the appendix. On Perfd, Proposition 7.23 implies that $L_{K(1)}\mathrm{TR}(-)$ defines an arc_p-hypersheaf. Using Proposition A.6, it follows that the restriction of $(L_{K(1)}\mathrm{TR}(-))^\sharp$ to Perfd is just $L_{K(1)}\mathrm{TR}(-)$ again, that is, (22) is an equivalence.[8]

Proposition 7.25. *For any ring R, there is a natural, complete exhaustive $\mathbb{Z}$-indexed filtration on $(L_{K(1)}\mathrm{TR}(R))^\sharp$, denoted $\mathrm{Fil}^{\geq *}(L_{K(1)}\mathrm{TR}(R))^\sharp$, with associated graded terms $\mathrm{gr}^i(L_{K(1)}\mathrm{TR}(R))^\sharp \simeq R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O})(i))[2i]$.*

Proof. The filtration in question is the arc_p-Postnikov filtration (on the derived p-completion of R). Note that Postnikov towers converge for hypercomplete arc_p-sheaves (Proposition A.10). For the identification of the graded pieces (or equivalently, the sheafified homotopy groups), it suffices by descent (see Example A.9) to work with the subcategory Perfd, where the result follows from (21). $\square$

[8] Alternatively, one could build $(L_{K(1)}\mathrm{TR}(-))^\sharp$ using the unfolding construction of Example A.9, by starting with the arc_p-hypersheaf $L_{K(1)}\mathrm{TR}(-)$ on Perfd.

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We do not know in general for which R the comparison map (22) is an equivalence; for such R, one obtains a 'motivic' filtration on $L_{K(1)}\mathrm{TR}(R)$ with associated graded terms the arc$_p$-cohomology complexes $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O})(i))[2i]$. Here we will show that the comparison map is an equivalence in certain formally smooth cases, using the pro-Galois descent result (Theorem 6.8).

THEOREM 7.26. *Let R_0 be a p-torsion-free perfectoid ring. Let R be a formally smooth p-complete R_0-algebra. Then $L_{K(1)}\mathrm{TR}(R) \xrightarrow{\sim} (L_{K(1)}\mathrm{TR}(R))^\sharp$.*

Proof. First, we reduce to the case where R_0 admits a compatible system of p-power roots of unity. By André's lemma [BS19, Theorem 7.12], we know that there exists a p-completely faithfully flat perfectoid R_0-algebra R_0' which has this property (e.g. is absolutely integrally closed). Now we have by descent

$$\mathrm{TR}(R;\mathbb{Z}_p) \simeq \mathrm{Tot}\Big(\mathrm{TR}(R\otimes_{R_0} R_0';\mathbb{Z}_p) \rightrightarrows \mathrm{TR}(R\otimes_{R_0} R_0' \otimes_{R_0} R_0';\mathbb{Z}_p) \substack{\to\\ \to\\ \to} \dots\Big). \tag{23}$$

Since in high degrees all of the terms in the above totalization agree with their $K(1)$-localization by Proposition 5.10, it follows that the descent property (23) holds for $L_{K(1)}\mathrm{TR}(-)$ as well (see the beginning of the proof of Proposition 5.10). Moreover, $(L_{K(1)}\mathrm{TR}(-))^\sharp$ satisfies descent for the map $R \to R\otimes_{R_0} R_0'$ by construction. Therefore, we reduce to the case where R_0 contains $\widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$.

Working locally on $\mathrm{Spf}(R)$, we can assume that R receives a map from $R_0\langle t_1^{\pm 1}, \dots, t_n^{\pm 1}\rangle$ which is étale mod p. We consider the extension $R_\infty = R \otimes_{\mathbb{Z}[\zeta_{p^\infty}, t_1^{\pm 1}, \dots, t_n^{\pm 1}]} \mathbb{Z}[\zeta_{p^\infty}, t_1^{\pm 1/p^\infty}, \dots, t_n^{\pm 1/p^\infty}]$ and the evident $\mathbb{Z}_p(1)^n$-action on R_∞. As in Example 6.10, we find from Theorem 6.8 that the natural map induces an equivalence

$$L_{K(1)}\mathrm{TR}(R) \xrightarrow{\sim} \mathrm{Tot}\Big(L_{K(1)}\mathrm{TR}(R_\infty) \rightrightarrows L_{K(1)}\mathrm{TR}(\mathrm{Fun}_{\mathrm{cts}}(\mathbb{Z}_p(1)^n, R_\infty)) \substack{\to\\ \to\\ \to} \dots\Big).$$

Now this is also true for $(L_{K(1)}\mathrm{TR}(-))^\sharp$, because the above augmented cosimplicial ring is an arc$_p$-hypercover (strictly speaking, for that we replace all rings by their derived p-completions). But now this gives $L_{K(1)}\mathrm{TR}(R) \xrightarrow{\sim} (L_{K(1)}\mathrm{TR}(R))^\sharp$, since they agree on perfectoids and every term in the above cosimplicial resolution has perfectoid p-completion. $\square$

THEOREM 7.27. *Suppose R is a formally smooth $\mathcal{O}_K$-algebra, where K is a complete discretely valued field of mixed characteristic $(0,p)$ whose residue field k satisfies $[k : k^p] < \infty$. Then $L_{K(1)}\mathrm{TR}(R) \xrightarrow{\sim} (L_{K(1)}\mathrm{TR}(R))^\sharp$.*

Proof. We let $G = \mathrm{Gal}(\bar{K}/K)$ and consider the G-action on $S = R\otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}$. As in Example 6.9, our assumptions imply that G has finite cohomological dimension and, moreover, that the map (13) is an equivalence. Therefore, it suffices to show that $L_{K(1)}\mathrm{TR}(A) \xrightarrow{\sim} (L_{K(1)}\mathrm{TR}(A))^\sharp$ when A is one of $S, \mathrm{Fun}_{\mathrm{cts}}(G, S), \dots$. But these are all (up to p-completions) formally smooth over perfectoids, whence the claim by Theorem 7.26 and descent. $\square$

Remark 7.28. Suppose $L_{K(1)}\mathrm{TR}(R) \xrightarrow{\sim} (L_{K(1)}\mathrm{TR}(R))^\sharp$. On Frobenius fixed points, one recovers the Thomason [Tho85, TT90] filtration (i.e. the pro-étale Postnikov filtration) on $L_{K(1)}\mathrm{TC}(R) \simeq L_{K(1)}K(R[1/p])$ (see [BCM20] for this identification), whose associated graded terms are given by $\mathrm{gr}^i \simeq R\Gamma_{\text{proét}}(\mathrm{Spec}(R[1/p]), \mathbb{Z}_p(i))[2i]$; see Proposition 7.16.

Remark 7.29. The above filtration on $L_{K(1)}\mathrm{TR}(-)$ and the calculations of TR of smooth algebras over a DVR in [HM03, HM04, GH06] suggest that the cohomology $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O})(i))$ should be related to the absolute de Rham–Witt complex. The work [Mor18] also suggests that for smooth algebras over a perfectoid base containing all p-power roots of unity, $R\Gamma_{\mathrm{arc}_p}(\mathrm{Spec}(R), W(\mathcal{O}))$ should be related to the relative de Rham–Witt complex (over the perfectoid base).

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Appendix. Topological preliminaries

In this appendix, we record some basic topological preliminaries about (hyper)sheaves of spectra.

Remark A.1 (Conventions for sites). We will for simplicity work only with sites of the following nice form (see [Lur18, § A.3.2]). Let $\mathcal{C}$ be a category with pullbacks and finite coproducts, such that coproducts distribute over pullbacks and are disjoint. Suppose $\mathcal{C}$ is equipped with a class of morphisms $S = S_{\mathcal{C}}$ which contains all isomorphisms and is stable under composition and pullback. We equip $\mathcal{C}$ with the Grothendieck topology where a collection $\{X_i \to X\}_{i \in I}$ is a covering if there exists a finite subset $I' \subset I$ such that $\bigsqcup_{i \in I'} X_i \to X$ can be refined by a map belonging to S. In this case, a presheaf on $\mathcal{C}$ with values in an ∞-category $\mathcal{D}$ with all small limits is a sheaf if and only if it carries finite coproducts in $\mathcal{C}$ to finite products in $\mathcal{D}$ and if it satisfies Čech descent for maps in S; see [Lur18, § A.3.3].

Example A.2.

(i) The small or big étale site of a qcqs scheme (where we only allow qcqs schemes) is an example, with S the class of étale surjections.
(ii) The arc_p-topology or $\mathrm{arc}_{\hat{p}}$-topology on the opposite of the category of derived p-complete rings is an example, with S the class of arc_p or $\mathrm{arc}_{\hat{p}}$-covers.
(iii) The arc_p-topology or $\mathrm{arc}_{\hat{p}}$-topology on the opposite of the category of perfectoid rings is an example, with S the class of arc_p or $\mathrm{arc}_{\hat{p}}$-covers.

Remark A.3 (Examples of continuous functors). Let $\mathcal{C}, \mathcal{C}'$ be sites as in Remark A.1. Let $u\colon \mathcal{C}' \to \mathcal{C}$ be a functor which preserves finite coproducts and pullbacks, as well as morphisms in the respective classes $S_{\mathcal{C}}, S_{\mathcal{C}'}$. It follows that if $\mathcal{F}$ is a sheaf (with values in any ∞-category $\mathcal{D}$ with all small limits) on $\mathcal{C}$, then $\mathcal{F} \circ u$ is a sheaf on $\mathcal{C}'$. These are examples of continuous functors; the notion can be defined for more general sites; see [SGA4, Exp. III.1].

Construction A.4 (Sheaves of spectra). Given a site $\mathcal{C}$ (as in Remark A.1), we let $\mathrm{PSh}(\mathcal{C}, \mathrm{Sp})$ denote the ∞-category of presheaves of spectra on $\mathcal{C}$ and $\mathrm{Shv}(\mathcal{C}, \mathrm{Sp}) \subset \mathrm{PSh}(\mathcal{C}, \mathrm{Sp})$ denote the subcategory of sheaves of spectra [Lur18, § 1.3]. We equip these both with their canonical t-structures; see [Lur18, § 1.3.2].

Given $u\colon \mathcal{C}' \to \mathcal{C}$ as in Remark A.3, we obtain a right adjoint and left t-exact functor $(-) \circ u\colon \mathrm{Shv}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Shv}(\mathcal{C}', \mathrm{Sp})$. It has a left adjoint $u_!\colon \mathrm{Shv}(\mathcal{C}', \mathrm{Sp}) \to \mathrm{Shv}(\mathcal{C}, \mathrm{Sp})$ given by left Kan extension along u followed by sheafification. By adjunction, necessarily $u_!$ is right t-exact.

Construction A.5 (Hypersheaves of spectra). Let $\mathcal{C}$ be a site as in Remark A.1. Any presheaf $\mathcal{F}$ of spectra on $\mathcal{C}$ fits into a unique fiber sequence of spectra

$$\mathcal{F}_{\mathrm{null}} \to \mathcal{F} \to \mathcal{F}^{\sharp}, \tag{A.1}$$

where $\mathcal{F}_{\mathrm{null}}$ has trivial sheafified homotopy groups and $\mathcal{F}^{\sharp}$ is a hypercomplete sheaf of spectra, that is, for every presheaf $\mathcal{G}$ with trivial sheafified homotopy groups, we have $\mathrm{Hom}(\mathcal{G}, \mathcal{F}^{\sharp}) = 0$. If $\mathcal{F}$ is a sheaf of spectra, then the cofiber sequence shows that $\mathcal{F}_{\mathrm{null}}$ is also a sheaf of spectra; it is then ∞-connective with respect to the t-structure on $\mathrm{Shv}(\mathcal{C}, \mathrm{Sp})$. We let $\mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C}, \mathrm{Sp}) \subset \mathrm{Shv}(\mathcal{C}, \mathrm{Sp})$ denote the full subcategory of hypercomplete sheaves. We refer to [CM19, § 2] for an exposition of some of these constructions, which go back to [Jar87, DHI04].

Proposition A.6. *Let $u\colon \mathcal{C}' \to \mathcal{C}$ be a morphism of sites as in Remark A.1 preserving finite coproducts and pullbacks and carrying the class of arrows $S_{\mathcal{C}'}$ into $S_{\mathcal{C}}$. Suppose that for any object $X' \in \mathcal{C}'$ and a finite covering family $\{Y_i \to u(X')\}_{i\in I}$ in $\mathcal{C}$, there is a refinement which is the image under u of a finite covering family $\{Y_i' \to X'\}_{i\in I}$ in $\mathcal{C}'$.*

Then the restriction functor $(-) \circ u\colon \mathrm{PSh}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{PSh}(\mathcal{C}', \mathrm{Sp})$ commutes with hypersheafification (and, in particular, preserves hypercomplete sheaves).

Proof. Using the cofiber sequence (A.1), we see that it suffices to prove that $(-) \circ u\colon \mathrm{Shv}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Shv}(\mathcal{C}', \mathrm{Sp})$ preserves both the subclasses of objects with trivial homotopy groups and hypercomplete objects.

Our hypotheses imply that if a presheaf of abelian groups on $\mathcal{C}$ has trivial sheafification, then its pullback to $\mathcal{C}'$ has trivial sheafification. Therefore, $(-) \circ u$ preserves objects with trivial homotopy groups. It thus suffices to show that if $\mathcal{F} \in \mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C}, \mathrm{Sp})$, then the sheaf $\mathcal{F} \circ u \in \mathrm{Shv}(\mathcal{C}', \mathrm{Sp})$ is also hypercomplete; this will not use the assumption in the second sentence of the statement. Equivalently, given $\mathcal{G} \in \mathrm{Shv}(\mathcal{C}', \mathrm{Sp})$ which is ∞-connective, it suffices to show that $\mathrm{Hom}_{\mathrm{Shv}(\mathcal{C}', \mathrm{Sp})}(\mathcal{G}, \mathcal{F} \circ u) = 0$. By adjointness, this is $\mathrm{Hom}_{\mathrm{Shv}(\mathcal{C}, \mathrm{Sp})}(u_!\mathcal{G}, \mathcal{F})$; since $u_!\colon \mathrm{Shv}(\mathcal{C}', \mathrm{Sp}) \to \mathrm{Shv}(\mathcal{C}, \mathrm{Sp})$ is right t-exact and therefore preserves ∞-connective objects and $\mathcal{F}$ is hypercomplete, this is contractible. ☐

Example A.7 (Overcategories). Suppose $a \in \mathcal{C}$ and $\mathcal{C}' = \mathcal{C}_{/a}$ is the overcategory of a, with the induced topology. Then the natural forgetful functor $\mathcal{C}' \to \mathcal{C}$ clearly satisfies the conditions of Proposition A.6. In particular, the hypersheafification of a presheaf on $\mathcal{C}$ when restricted to $\mathcal{C}_{/a}$ is the hypersheafification of the restriction to $\mathcal{C}_{/a}$.

A direct consequence is that given an appropriate site with a 'basis', hypersheaves of spectra can be entirely recovered from their values on the basis. The result is a direct analog of [SGA4, Theorem 4.1, Exp. III], and appears (for sheaves of spaces) in [Aok20, Appendix A].

Proposition A.8. *Let $\mathcal{C}$ be a site as in Remark A.1. Let $\mathcal{C}' \subset \mathcal{C}$ be a full subcategory closed under finite coproducts and fiber products, and define $S_{\mathcal{C}'}$ to be the intersection of $S_{\mathcal{C}}$ with $\mathcal{C}'$. Suppose every object $X \in \mathcal{C}$ admits a map $Y \to X$ in $S_{\mathcal{C}}$ with $Y \in \mathcal{C}'$. Then the restriction functor $\mathrm{Shv}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Shv}(\mathcal{C}', \mathrm{Sp})$ restricts to an equivalence on hypercomplete objects.*

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Proof. Let $u\colon \mathcal{C}' \subset \mathcal{C}$ be the inclusion, so we have a restriction functor $(-) \circ u\colon \mathrm{PSh}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{PSh}(\mathcal{C}', \mathrm{Sp})$. By Proposition A.6, it restricts to a functor on hypercomplete sheaves, so we have $(-) \circ u\colon \mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C}', \mathrm{Sp})$. This last functor has a left adjoint $L\colon \mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C}', \mathrm{Sp}) \to \mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C}, \mathrm{Sp})$, given by $\mathcal{F} \mapsto L\mathcal{F} := (\mathrm{Lan}_u \mathcal{F})^\sharp$, that is, L is obtained by applying the left Kan extension $\mathrm{Lan}_u\colon \mathrm{PSh}(\mathcal{C}', \mathrm{Sp}) \to \mathrm{PSh}(\mathcal{C}, \mathrm{Sp})$ followed by hypersheafification $(-)^\sharp$. We now show that L is fully faithful. Indeed, we have, for $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C}', \mathrm{Sp})$,

$$\mathrm{Hom}_{\mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C},\mathrm{Sp})}(L\mathcal{F}, L\mathcal{G}) = \mathrm{Hom}_{\mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C}',\mathrm{Sp})}\big(\mathcal{F}, (\mathrm{Lan}_u \mathcal{G})^\sharp \circ u\big). \tag{A.2}$$

Now since u is fully faithful, since hypersheafification commutes with $(-) \circ u$ by Proposition A.6, and since $\mathcal{G}$ is already hypercomplete, we have $(\mathrm{Lan}_u \mathcal{G})^\sharp \circ u = \mathcal{G}$. Therefore, the right-hand side of (A.2) simplifies to $\mathrm{Hom}_{\mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C}',\mathrm{Sp})}(\mathcal{F}, \mathcal{G})$ as desired.

Our hypotheses imply that the restriction functor $(-) \circ u$ is conservative on hypercomplete sheaves, because it is conservative on sheaves of abelian groups. Since the restriction functor is conservative and has a fully faithful left adjoint, the result follows. $\square$

Example A.9 (Unfolding in the arc_p-topology). The natural forgetful functor establishes an equivalence of ∞-categories between arc_p-hypersheaves of spectra on all derived p-complete rings and arc_p-hypersheaves of spectra on perfectoid rings.

Finally, we include a basic observation about commuting hypersheafification and certain products when the site $\mathcal{C}$ is sufficiently large (e.g. the arc_p-site). This fact is closely related to the theory of replete topoi; see [BS15, § 3]. Compare [BS15, Proposition 3.3.3] for the second part of the next result for the derived category of abelian sheaves, or equivalently hypercomplete $H\mathbb{Z}$-module sheaves of spectra. Note, in particular, that it applies to the arc_p-site. This follows because arc-covers in the sense of [BM18] are closed under filtered colimits of rings [BM18, Corollary 2.20] and because a map of derived p-complete rings $R \to R'$ is an arc_p-cover if and only if $R \to R' \times R/p \times R[1/p]$ is an arc-cover.

In the following, we write $\mathrm{PSh}_\sqcup(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$ for the subcategory of presheaves which carry finite coproducts to finite products.

Proposition A.10. *Let $\mathcal{C}$ be a site as in Remark A.1. Suppose $\mathcal{C}$ admits countable filtered limits. Suppose, moreover, that if $\cdots \to X_i \to X_{i-1} \to \cdots \to X_0$ is a sequence of arrows in S, then $\varprojlim_i X_i \to X_0$ belongs to S. Then the following assertions hold.*

(i) *The hypersheafification functor, $(-)^\sharp\colon \mathrm{PSh}_\sqcup(\mathcal{C}) \to \mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C})$ commutes with countable products and limits along $\mathbb{Z}_{\geq 0}$-indexed towers.*

(ii) *For any $\mathcal{G} \in \mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C}) \subset \mathrm{Shv}(\mathcal{C})$, the Postnikov tower of $\mathcal{G}$ (as a sheaf of spectra) converges.*

Proof. For (i), since hypersheaves are always closed (inside presheaves) under arbitrary limits and since $\mathbb{Z}_{\geq 0}$-indexed limits can be built from countable products, it suffices to show (via the unique cofiber sequence (A.1)) that if $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$ is a countable family of presheaves in $\mathrm{PSh}_\sqcup(\mathcal{C})$ with trivial hypersheafification (i.e. trivial sheafified homotopy groups), then the product presheaf $\prod_{i \in \mathbb{N}} \mathcal{F}_i$ has trivial hypersheafification. But this is just a claim about the presheaves of abelian groups $\{\pi_j(\mathcal{F}_i)\}_{i \in \mathbb{N}}$ for each j. Explicitly, given a class $\alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} \pi_j(\mathcal{F}_i(X))$ for some $X \in \mathcal{C}$, we

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find for each i a cover $X_i \to X$ in S which annihilates[9] $\alpha_i \in \pi_j(\mathcal{F}_i(X))$ and then form the cover $X_1 \times_X X_2 \times_X \ldots$ of X, which annihilates α.

Now let $\mathcal{G}$ be a hypercomplete sheaf of spectra on $\mathcal{C}$. The Postnikov tower of $\mathcal{G} \in \mathrm{Shv}(\mathcal{C})$ is obtained by taking the presheaf truncations $\tau_{\leq n}^{\mathrm{pre}}\mathcal{G}$ and applying the hypersheafification (or sheafification, since these objects are truncated), that is, one forms $\{(\tau_{\leq n}^{\mathrm{pre}}\mathcal{G})^\sharp\}$, which is a tower in $\mathrm{PSh}_\sqcup(\mathcal{C})$. Since Postnikov towers converge for presheaves of spectra, that is, $\mathcal{G} \simeq \varprojlim_n \tau_{\leq n}^{\mathrm{pre}}\mathcal{G}$, and we have just seen that $(-)^\sharp \colon \mathrm{PSh}_\sqcup(\mathcal{C}) \to \mathrm{Shv}_{\mathrm{hyp}}(\mathcal{C})$ commutes with limits along $\mathbb{Z}_{\geq 0}$-indexed towers, we find $\mathcal{G} \simeq \varprojlim_n (\tau_{\leq n}^{\mathrm{pre}}\mathcal{G})^\sharp$ as desired. □

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