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RING SPECTRA WHICH ARE THOM COMPLEXES

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For our purposes a ring spectrum E is a spectrum with a map $i : E \wedge E \to E$ and a unit $i : S^0 \to E$ such that the following diagrams commute up to homotopy:

$$\begin{array}{ccc} E \wedge E \wedge E & \xrightarrow{\mu \wedge 1} & E \wedge E \\ \downarrow {\scriptstyle 1 \wedge \mu} & & \downarrow {\scriptstyle \mu} \\ E \wedge E & \xrightarrow{\mu} & E \end{array} \qquad \begin{array}{ccccc} S^0 \wedge E & \xrightarrow{i \wedge 1} & E \wedge E & \xleftarrow{1 \wedge i} & E \wedge S^0 \\ & {\scriptstyle l} \searrow & \downarrow {\scriptstyle \mu} & \swarrow {\scriptstyle r} & \\ & & E & & \end{array}$$

The ring spectrum is abelian if

$$\begin{array}{ccc} E \wedge E & \xrightarrow{T} & E \wedge E \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & E & \end{array}$$

commutes up to homotopy where T is the map that exchanges factors.

Let L be a space and let ξ be a fibration over L classified by a map $f : L \to BF$ (the classifying space of stable spherical fibrations). We can form the Thom spectrum $T(f)$ of f as a suspension spectrum by letting $(T(f))_n$ be the Thom complex of $L^n \to BF_n$ where L^n is the n-skeleton of L. This makes $T(f) = \{(T(f))_n\}$ into a suspension spectrum.

Spectra which arise in this fashion have a unit which is the inclusion of the fiber on the Thom class.

Natural examples of maps $f : L \to BF$ give a plethora of interesting spectra: among them are $K(\mathbf{Z}_2, 0)$, $K(\mathbf{Z}, 0)$, the Brown-Gitler spectrum, and a spectrum for which the secondary operation of Adams $\varphi_{j,j}$ [1] is defined and non-zero on the Thom class.

Frequently, the Thom spectra which we obtain in this manner are commutative ring spectra. A useful feature of these Thom spectra is that they admit particularly nice resolutions. Consequently, these spectra give rise to

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spectral sequences converging to $\pi_*^S(S^0)$ which are quite manageable for certain applications such as the immersion conjecture for RP^{8k+7} [4].

In this paper we describe the above point of view together with a number of related results. In outline we proceed as follows: In Section 1 we give 2 theorems on the properties of Thom spectra $T(f)$. These theorems should be regarded as technical tools which are used to identify $T(f)$ as a ring spectrum and to describe nice resolutions of $T(f)$; these theorems are needed to obtain our main results in Sections 2 and 3. In Section 2 we catalogue examples of Thom spectra obtained from H-maps $f: L \to BF$. In particular, we study the spectra mentioned in the previous two paragraphs. In Section 3 we study resolutions of $T(f)$ where f is now assumed to be a loop map. Theorem 1.2 is used to identify $T(f) \wedge T(f)$ in our resolutions. The resolutions of Section 3 give rise to certain spectral sequences which are described in Section 4. The techniques are applied to the May spectral sequence, the Adams spectral sequence, and the immersion conjecture for RP^{8k+7}.

1. General theorems on Thom spectra. The following simple theorem is basic.

THEOREM 1.1. *Suppose L is an H-space with multiplication μ and $f: L \to BF$ is an H-map. Then the Thom spectrum $T(f)$ is a ring spectrum. If L is a homotopy commutative H-space and f is a morphism of homotopy commutative H-spaces then $T(f)$ is a commutative ring spectrum.*

Proof. The hypothesis gives a commutative diagram

$$\begin{array}{ccc} L \times L & \xrightarrow{f \times f} & BF \times BF \\ \downarrow{\scriptstyle \mu_L} & & \downarrow{\scriptstyle \mu_{BF}} \\ L & \xrightarrow[f]{} & BF \end{array}$$

Taking Thom complexes we have $T(\mu_L): T(f) \wedge T(f) \to T(f)$. The Thom class multiplies and so the spectrum has a unit. The commutative conclusion is also immediate from an appropriate diagram at the space level.

The ring of operations on spectra which arise when f is a loop map is often tractible.

THEOREM 1.2. *If $T(f)$ is a ring spectrum which is the Thom complex of a bundle over a loop space L classified by a loop map $f: L \to BF$, then $T(f) \wedge T(f) = L_+ \wedge T(f)$. (+ denotes a disjoint basepoint.)*

Proof. Let $\Delta: L \to L \times L$ be the map defined by $\Delta(x) = (x, x^{-1})$. Let $g: L \times L \to L \times L$ be the composite

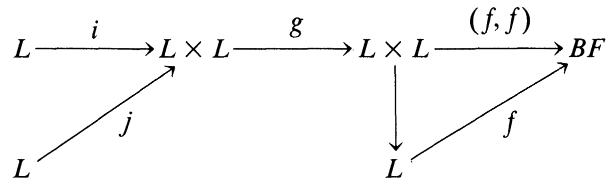
$$L \times L \xrightarrow{\Delta \times \mathrm{id}} L \times L \times L \xrightarrow{(d, \mu)} L \times L$$

where μ is the multiplication in L. Then, clearly, g is a homotopy equivalence.

Consider the bundle over $L \times L$ given by

$$L \times L \xrightarrow{g} L \times L \xrightarrow{(f,f)} BF.$$

The bundle induced by (f, f) is equivalent to the bundle induced by $(f, f) \circ g$. Consider



where $i : L \to L \times L$ is the left hand inclusion, and j is the right hand inclusion.

The Thom complex of $f\mu gi$ is homotopy equivalent to $T(f)$ while $T(f\mu gj)$ is trivial. Thus, as spectra $L_+ \wedge T(f) \cong T(f) \wedge T(f)$.

2. Some examples I. In this section all spaces should be considered as localized at the prime 2 unless otherwise noted as in example 2.8 where odd primes are considered. Many results are valid more generally.

Some very useful spectra are given by taking $L_i = \Omega S^i$ for $i = 2, 3, 5, 9$ and letting f_i be $\Omega\omega$ where $\omega : S^i \to B^2O$ is a generator. We will use these spectra frequently and so let $X_i = T(f_i)$, $i = 2, 3, 5$, and 9. By a different procedure Barratt described similar spectra in 1967. His approach was quite different but he obtained some of these properties. Theorems 1.1 and 1.2 give a much more direct path to these properties. We note several of them.

2.1. The ring spectrum X_3 is abelian.

Proof. The map $S^3 \to B^2O$ is equivalent to the loop of $\mathrm{H}P^\infty \xrightarrow{\bar\omega} B^3U$ where $\bar\omega$ is a generator of π_3 and is extended by standard obstruction theory. Then the realification of $\Omega\bar\omega$ is ω.

2.2. These spectra have some tractible homotopy properties. The following result is illustrative.

$\mathrm{Ext}_A^{s,t}(H^*(X_2), \mathrm{Z}_2)$ contains $\mathrm{Z}_2(v_1, w_5, v_2)$ where v_1, w_5, v_2 have filtration (1, 2), (1, 6), (1, 7) respectively and v_i are related to the BP generators of the same name.

Sketch proof. It is not hard to calculate by hand $\mathrm{Ext}_{A_2}(H^*(X_2), \mathrm{Z}_2)$ and show that it equals $\mathrm{Z}_2(a, v_1, w_5, v_2)$ where a has filtration (0, 8). Next one calculates, by hand again, to show that v_1, w_5, v_2 all exist in $\mathrm{Ext}_A(H^*(X_2), \mathrm{Z}_2)$. The ring map and the map $\mathrm{Ext}_A(\tilde H^*(X_2), \mathrm{Z}_2) \to \mathrm{Ext}_{A_2}(\tilde H^*(X_2), \mathrm{Z}_2)$ complete the proof.

2.3. From 1.2 we have maps $k_j : X_i \to \Sigma^{(i-1)j} X_i$ which have degree 1 in dimension $(i-1)j$. The evaluation of these maps in all other dimensions will be important later on. To do so we will describe k_j more explicitly. Let g_i^{-1} be the homotopy inverse of the map g described in 1.2 as applied to ΩS^i. Then k_j is the composite

$$X_i \xrightarrow{\text{id} \wedge S^0} X_i \wedge X_i \xrightarrow{T(g_i^{-1})} \Omega S^i_+ \wedge X_i = \bigvee_{j=0}^{\infty} \Sigma^{(i-1)j} X_i \to \Sigma^{(i-1)j} X_i$$

The first three maps are the maps induced in Thom complexes by the following space maps

$$\Omega S^i \xrightarrow{\text{id},} \Omega S^i \times \Omega S^i \xrightarrow{\Delta' \times 1} \Omega S^i \times \Omega S^i \times \Omega S^i \xrightarrow{\text{id},} \Omega S^i \times \Omega S^i$$

where $\Delta'(x) = (x, x)$. Let a_j be a class in $H_{(i-1)j}(\Omega S^i)$. Then

$$a_l \to (a_l \otimes 1) \to \sum_{l+k=l} \binom{l}{j} a_j \otimes a_k \otimes 1 \to \sum_{l+k=l} \binom{l}{j} a_j \otimes a_k .$$

Thus

2.4. $k_{j*}(a_l) = \binom{l}{j} a_{l-j}$.

If $i = 2$, then everything is with $\mathbf{Z}_2$ for coefficients and this formula is less interesting.

2.5. (Brayton Gray and M. G. Barratt). If $\alpha \in \pi_j(S^0)$ let M_α be the stable complex $S^0 U_\alpha e^{j+1}$. Then $X_5 \wedge M_\eta = X_3$ and $X_3 \wedge M_{2\iota} = X_2$.

Neither of these will follow from H maps but note that up to homotopy equivalence $\Omega S^2 = S^1 \times \Omega S^3$. Note that $X_5 \neq X_9 \wedge M_\nu$. First to see that $X_2 = X_3 \wedge M_{2\iota}$, note that $S^3 \to S^2 \to B^2 O$ gives a generator. Thus there is a map $X_3 \to X_2$ of degree 1 on the Thom class. Now it is easy to verify that $M_{2\iota} \wedge X_3 = X_2$. (Note that in X_3, $Sq^{2^i} U \neq 0$ for every i).

It is a little harder to verify $M_\nu \wedge X_5 = X_3$. The starting place is the observation that there is a map $M_\nu \to X_3$ with degree 1 on the Thom class. Using the multiplication we have $M_\nu \wedge M_\nu \to X_3$. Using the homotopy commutativity of X_3 we see that $S^4 \to M_\nu \wedge M_\nu \to X_3$ is null homotopic and the cofiber of $S^4 \to M_\nu \wedge M_\nu$ is the 2-skeleton on X_5. Now suppose we have a commutative diagram

$$\begin{array}{ccc} M_\nu \wedge X_5^{4l} & \longrightarrow & X_5^{4l+4} \\ & \searrow \quad \swarrow & \\ & X_3 & \end{array}$$

Then we have $M_\nu \wedge M_\nu \wedge X_5^{4l} \to X_5^{4l+4} \to X_3$ and the composite $S^4 \wedge X_5^{4l} \to M_\nu \wedge M_\nu \wedge X_5^{4l} \to M_\nu \wedge X_5^{4l+4}$ has X_5^{4l+8} as the cofiber. But as above the composite $S^4 \to M_\nu \wedge M_\nu \to X_3$ is zero and so $M_\nu \wedge X_5^{4l+4}$ extends to X_5^{4l+8}. Hence $X_5 \to X_3$. Now $X_5 \wedge M_\nu = X_3$ by again checking the Steenrod operations. (Everything is still localized at the prime 2.)

2.6. Let $L = \Omega^2 S^3$ and let $w : S^3 \to B^3 O$ be a generator. Let $f = \Omega^2 w$. Then $T(f) = K(\mathbf{Z}_2, 0)$. This case has received a lot of attention in recent literature [6], [5], [8], and [12].

2.7. If F_n is the filtration of $\Omega^2 S^3$ (see [7]) and $f_n = f/F_n$ (with f as in 2.6), then in [6] it was shown that $H^*(T(f_n)) = M(n) = A/\{A\chi Sq^k \mid k > n\}$. ($A$ is the mod 2 Steenrod algebra.) Brown and Peterson have shown that a lot of the spaces which arise in this fashion are actual Brown-Gitler spectra [2].

Let $W(1)$ be the fiber of the degree 1 map of $\Omega^2 S^3 \to S^1$. Then f induces a map $\bar{f} : W(1) \to BSO$ and $T(\bar{f}) = K(\mathbf{Z}, 0)$ at the prime 2. Snaith [9] has shown that $\Omega^n \Sigma^n X$ splits stably into a certain wedge if X is path-connected. Specializing to $\Omega^2 S^3$, Snaith gives a map

$$h_p : \Omega^2 S^3 \to QD_p$$

where QD_p localized at an odd prime p is $QS^{2p-2} \cup_p e^{2p-1}$. The first element of order p in the stable stems, α_1, induces a map $S^{2p-2} \cup_p e^{2p-1} \to BF$. Since BF is an infinite loop space, we may extend this last map to QD_p and consider the composite

$$\Omega^2 S^3 \xrightarrow{h_p} QD_p \to BF.$$

We multiply together the maps h_p, one for each odd prime, together with the composite

$$W(1) \xrightarrow{\bar{f}} BSO \to BF$$

to obtain the map

$$g : W(1) \to BF.$$

PROPOSITION 2.8. $T(g) = K(\mathbf{Z}, 0)$.

Proof. We will outline the proof since the result is really one dealing with primes other than 2. The proof follows closely that given in [6] for 2.6. First note that $\Omega^2 S^3 \to Q\Sigma^{2p-2}M_p$ is part of a commutative diagram

$$\begin{array}{ccccc} \Omega^2 S^3 & \longrightarrow & Q\Sigma^{2p-2}M_p & \longrightarrow & BF \\ \uparrow & & & \nearrow f_p & \\ \Omega S^2 & \longrightarrow & \Omega S^{2p-1} & & \end{array}$$

(We will do one prime, they all work the same way.) By using the Cartan formula and $\Pi S^{2p-2} \to \Omega S^{2p-1} \to BF$ we see that in $T(f_p)$ $P^i U \neq 0$ and $\chi P^i U \neq 0$ for all i. (χ is the anti-isomorphism.)

Next we filter A_p, mod p Steenrod algebra, by letting $\mathfrak{F}_n A_p$ = vector space generated by $\{\chi P^I \parallel I = (s_1, \epsilon_1, \ldots, s_k, \epsilon_k, 0 \ldots),\ \epsilon_i = 0, 1,\ s_i = 1, 2, 3, \ldots,$ and $x_i \geqslant p s_{i+1} + \epsilon_i$ for each i. In addition $s_1 \geqslant n\}$. Then $\mathfrak{F}_1 A_p = H^*(K(Z, 0), Z_p)$ and $\mathfrak{F}_i A_p \supset \mathfrak{F}_{i+1} A_p$. Clearly $\mathfrak{F}_n(A_p)/\mathfrak{F}_{n+1}(A_p) = \Sigma^{n(2p-2)} A_p / A_p\{\chi(\Delta^\epsilon P^i) \mid i > n, \epsilon = 0, 1\}$.

Let $Y_n(p) = i^{-1}\mathcal{F}_{pn}$. Using the product methods of [6] it is easily shown that $H^*(Y_n(p)/Y_{n-1}(p)) \cong \mathcal{F}_n A_p/\mathcal{F}_{n+1}A_p$ as A_p modules. Combining this filtered A_p action with the generator given by 2.9 completes the proof. (Recently we have received a copy of the thesis of Ralph Cohen [3]. The modules $\mathcal{F}_n A_p/\mathcal{F}_{n+1}(A_p)$ are discussed there in some detail.)

The referee offered the following somewhat easier proof of 2.8.

Proof. It suffices to show that the map

$$\theta : A/\beta A \to H^*(T(g) ; \mathbf{Z}/p\mathbf{Z})$$

given by $\theta(a) = aU$ is an isomorphism at each prime p. Assume that p is odd. Since $A/\beta A$ and $H^*(T(g) ; \mathbf{Z}/p\mathbf{Z})$ have the same finite rank in any fixed degree and g is an H-map, it suffices to check that θ is an injection on primitives. We check this result for the Milnor Q_i's here. Let w_i denote the ith Wu class in $H^*(BSF ; \mathbf{Z}/p\mathbf{Z})$. By [14],

$$Q_i U = U \cup (\beta w_{1+p+p^2\cdots+p^{i-1}} + \Delta)$$

where Δ is decomposable. Let $x_{2p^k-2+\epsilon}$, $\epsilon = 0, 1$, denote the unique primitive in $H_{2p^k-2+\epsilon}(W(1) ; \mathbf{Z}/p\mathbf{Z})$. It is well-known that (1) $P^1_* x_{2p^k-2} = -(x_{2p^{k-1}-2})^p$ if $k > 1$, (2) $P^i_* x_{2p^k-2} = 0$ if $i > 1$, and (3) $P^1 P^p \ldots P^{p^{i-1}} w_{1+p+\cdots+p^{i-1}} = w_{1+p+\cdots+p^i} + d$ where d is decomposable. We then have by the evident Kronecker pairings,

$$\begin{aligned}\langle g^*(\beta w_{1+p+\cdots+p^{i-1}} + \Delta), x_{2p^{i+1}-1}\rangle &= \langle g^*\beta w_{1+p+\cdots+p^{i-1}}, x_{2p^{i+1}-1}\rangle \\ &= \langle g^* w_{1+p+\cdots+p^{i-1}}, x_{2p^{i+1}-2}\rangle \\ &= \langle g^* P^1 P^p \ldots P^{p^{i-2}} w_{1+p+\cdots+p^{i-2}}, x_{2p^{i+1}-2}\rangle \\ &= \langle g^* w_{1+p+\cdots+p^{i-2}}, P^{p^{i-1}}_* \ldots P^p_* P^1_* x_{2p^{i+1}-2}\rangle \\ &= \langle g^* w_{1+p+\cdots+p^{i-2}}, \pm (x^{p^i}_{2p-2})\rangle \\ &= \pm 1.\end{aligned}$$

Hence $Q_i g^* U \neq 0$. The other primitives are checked similarly.

Let $W(1) = Y$. Then we have that filtration induces a filtration on Y so that $Y_n = i^{-1}(F_{2^n})$ where $i : Y \to \Omega^2 S^3$. Let $\bar{B}(n) = T(\bar{f}/Y_n)$. Note that $\bar{B}(n) \wedge M_{2\iota} = B(2n+1)$.

PROPOSITION 2.10. *$H^*(\bar{B}(n))$ is isomorphic to $M(2n) \otimes_{A_0} \mathbf{Z}_2$.*

Proof. Recall that $\bar{B}(n)$ is given as a Thom complex. The right action of Sq^1 is obtained by looking at the classes $Sq^I Sq^1$. Since $Sq^1 U = U \cup \chi_1$ and $Sq^I \chi_1 = 0$ for all I we see that under the map $\bar{B}(n) \xrightarrow{i} B(2n)$, i^* is just the projection $M(2n) \to M(2n) \otimes_{A_0} \mathbf{Z}_2$.

2.11. Another collection of interesting spectra results from restricting f of 2.6 to $\Omega J_{2^i-1}(S^2) \subset \Omega^2 S^3$ where J_k is the James construction. The homology of

$\Omega J_{2^i-1}(S^2)$ is $\mathbf{Z}_2[x_1, \ldots, x_{i-1}]$ and $T(f/\Omega J_{2^i-1}(S^2))$ is a ring spectrum realizing the part of A^* which is $\mathbf{Z}_2(\xi_1, \ldots, \xi_{i-1})$. An evident modification of the second proof of 2.8 yields this result.

2.12. As a last example of an interesting spectrum which arises this way we give the following. Consider $S^5 \to B^3F$ which represents a generator. Let $f: \Omega^2 S^5 \to BF$ be the double loop map. Then $T(f)$ has the property $\varphi_{j,j} U \neq 0$ for every j where $\varphi_{j,j}$ is the secondary operation described by Adams [1].

Sketch proof. It is easily verified that $f^*(w_i) = 0$ for all w_i where w_i in $H^*(BF)$ are defined by the Thom isomorphism and Sq^i on the Thom class. Next standard formulae in BF show that if x_3 generates $H_3(\Omega^2 S^5)$ then $Q_1 \cdots Q_1 x_3 = \sigma e_{2^s-1} e_{2^s-1}$ modulo decomposables where e_i generate $H_*(SO) \subset H_*(SF)$ and σ is the suspension homomorphism. Formulae such as this are proved in [13] for example. Standard arguments show that if $\varphi_{j,j}$ is defined on U then $\varphi_{j,j} U \neq 0$ if and only if there is a homology class x such that $f_* x = \sigma e_{2^j-1} e_{2^j-1}$ modulo decomposables.

2.13. Finally, having constructed lots of examples of spectra, we conjecture that BP, bo and bu cannot be gotten in this fashion.

3. Resolutions with respect to ring spectra. The ring spectra which arise from 1.1 yield particularly nice resolutions. Before describing these resolution we fix some notation. Let Ω be a loop space and let X be the Thom spectrum of a bundle over Ω given by a loop map. Let $\Delta : \Omega_+ \times X \cong X \wedge X$ be given by the proof of 1.2. By the geometric bar resolution with respect to a spectrum X with unit we mean the tower of fibrations in the stable category

$$\begin{array}{ccc}
\vdots & & \\
\big\downarrow p_3 & & \\
X_2 & \xrightarrow{1 \wedge S^0} & X_2 \wedge X \\
\big\downarrow p_2 & & \\
X_1 & \xrightarrow{1 \wedge S^0} & X_1 \wedge X \\
\big\downarrow p_1 & & \\
S^0 & \xrightarrow{p_0} & X
\end{array} \tag{3.1}$$

where $S^0 \to X$ is the inclusion of the unit. X_1 is the fiber of p_0. In general X_n is the fiber of $X_{n-1} \xrightarrow{1 \wedge S^0} X_{n-1} \wedge X$. Associated to this resolution is the cofiber sequence

$$\begin{array}{ccccccccc}
 & X & \longrightarrow & & IX \wedge X & \longrightarrow & & I^2X \wedge X \to & \\
\nearrow & & \searrow p_0' & \nearrow i_1 = 1 \wedge S^0 & & \searrow p_1' & \nearrow i_2 = 1 \wedge S^0 & & \searrow p_2' \\
S^0 & & & IX & & & I^2X & & & I^3X
\end{array} \tag{3.2}$$

Here IX is the cofiber of the inclusion $S^0 \xrightarrow{i_0} X$, $i_1 : IX \to IX \wedge X$ is $1 \wedge S^0$. Inductively we define $I^j(X)$ to be the cofiber of $i_{j-1} : I^{j-1}X \to I^{j-1}X \wedge X$. (The notation I^jX is suggestive of the augmentation ideal analogue.) Note that $\Sigma^i X_i = I^i X$.

If we apply π_* to 3.1 or 3.2 we get a spectral sequence $\{E_r^{s,t}(S^0, X, \pi), \delta_r\}$. The E_1 term is $E_t^{s,t} = \pi_{t-s}(X_s \wedge X)$. Under reasonable hypothesis this spectral sequence converges to $E_0\pi_*(S^0)$. The chain complex $\{E_1, \delta_1\}$ is most easily handled by the chain complex 3.2.

Associated to 3.1 or 3.2 is the sequence

$$X \xrightarrow{p_0 \wedge S^0} IX \wedge X \xrightarrow{p_1 \wedge S^0} I^2X \wedge X \to \cdots \xrightarrow{p_\sigma \wedge S^0} IX^{\sigma+1} \wedge X \to \cdots \tag{3.3}$$

Let $d_i = p_{i-1} \wedge S^0$. Clearly $d_{i+1} \circ d_i$ is null homotopic.

Consider the sequence

$$X \xrightarrow{\bar{d}_1} X \wedge X \xrightarrow{\bar{d}_2} X \wedge X \wedge X \to \cdots \xrightarrow{\bar{d}_{\sigma-1}} X^\sigma \tag{3.4}$$

where X^σ is $X \wedge \cdots \wedge X$ σ times and $\bar{d}_\sigma = \Sigma_{i=1}^{\sigma+1}(-1)^i d_\sigma^i$ for $d_\sigma^i : X^\sigma \to X^{\sigma+1}$ defined by $1 \wedge \cdots \wedge S^0 \wedge \cdots \wedge 1$ and S^0 occurs in the ith place. By standard nonsense we see that $\bar{d}_{\sigma+1}\bar{d}_\sigma$ is null homotopic. The sequence 3.4 maps, in an obvious way, to 3.3. Indeed, it seems easiest to consider the following diagram displaying these maps

$$\begin{array}{ccccccc}
 & & I^2X \wedge X & \xrightarrow{1 \wedge \bar{d}_1} & I^2X \wedge X & \longrightarrow & \cdots \\
 & \nearrow & \uparrow{\scriptstyle p_1 \wedge 1} & & \uparrow{\scriptstyle p_1 \wedge 1} & & \\
 & IX \wedge X & \xrightarrow{1 \wedge \bar{d}_1} IX \wedge X \wedge X & \xrightarrow{1 \wedge \bar{d}_2} & IX \wedge X^3 & \longrightarrow & \cdots \\
{\scriptstyle p_0 \wedge S^0} \nearrow & \uparrow{\scriptstyle p_0 \wedge 1} & \uparrow{\scriptstyle p_0 \wedge 1} & & \uparrow{\scriptstyle p_0 \wedge 1} & & \\
X \xrightarrow{\bar{d}_1} & X \wedge X & \longrightarrow X \wedge X \wedge X & \longrightarrow & X \wedge X \wedge X \wedge X & \to & \cdots
\end{array} \tag{3.5}$$

Continuing this process yields the desired maps from $X^{\sigma+1} \to I^\sigma X \wedge X$.

For notational purposes we write it again as

$$\begin{array}{ccccccccc}
X & \longrightarrow & IX \wedge X & \longrightarrow & I^2X \wedge X & \longrightarrow \cdots \longrightarrow & I^\sigma X \wedge X & \to \\
\uparrow{\scriptstyle f_1} & & \uparrow{\scriptstyle f_2} & & \uparrow{\scriptstyle f_3} & & \uparrow{\scriptstyle f_{\sigma+1}} & \\
X & \xrightarrow{\bar{d}_1} & X \wedge X & \longrightarrow & X^3 & \longrightarrow \cdots \xrightarrow{\bar{d}_\sigma} & X^{\sigma+1} & \to
\end{array} \tag{3.5'}$$

Next we wish to compare 3.4 with what we have using the structure maps of 1.2 in the case X is a Thom complex over some loop space classified by a loop map. We have the following diagram

$$\begin{array}{ccccccccc}
X & \xrightarrow{\bar{d}_1} & X^2 & \xrightarrow{\bar{d}_2} & X^3 & \longrightarrow \cdots \xrightarrow{\bar{d}_\sigma} & X^{\sigma+1} & \longrightarrow \cdots \\
\uparrow g_1 & & \uparrow g_2 & & \uparrow g_3 & & \uparrow g_{\sigma+1} & \\
X & \xrightarrow{\delta_1} & \Omega_+ \wedge X & \xrightarrow{\delta_2} & \Omega_+ \wedge \Omega_+ \wedge X & \longrightarrow \cdots \xrightarrow{\delta_\sigma} & (\Omega_+)^\sigma \wedge X & \to \cdots
\end{array} \tag{3.6}$$

where the g_i are homotopy equivalences by $g : \Omega \wedge \Omega \to \Omega \wedge \Omega$ (1.2) where $\delta_1 = \bar{\Delta} + S^0 \wedge 1$, $\delta_2 = \bar{\Delta} \wedge 1 - 1 \wedge \bar{\Delta} + S^0 \wedge 1$, $\delta_3 = \bar{\Delta} \wedge 1 \wedge 1 - 1 \wedge \bar{\Delta} \wedge 1 + 1 \wedge 1 \wedge \bar{\Delta} - S^0 \wedge 1 \wedge 1$, etc., and $\bar{\Delta}$ is the map induced by the usual diagonal.

PROPOSITION 3.7. *Diagram* 3.6 *commutes.*

Proof. It is sufficient to look at the space level. The first square becomes

$$\begin{array}{ccc}
\Omega & \xrightarrow{(1,0)-(0,1)} & \Omega \times \Omega \\
\uparrow \mathrm{id} & & \uparrow g \\
\Omega & \xrightarrow[\bar{\Delta}-(0,1)]{} & \Omega \times \Omega
\end{array}$$

Now $\Delta \circ \bar{\Delta} = (1, 0)$ and $\Delta(0, 1) = (0, 1)$. (Recall g is the composite $\Omega \times \Omega \xrightarrow{\Delta \times 1} \Omega \times \Omega \times \Omega \xrightarrow{1 \times \mu} \Omega \times \Omega$ and Δ' is $(1, -1)$.) The general case represents a sequence of similar steps.

Also note that the sequence of maps in 3.5 which eliminate the various axes amount to removing the basepoint in 3.6. This gives

PROPOSITION 3.8 *We have the following commutative diagram*

$$\begin{array}{ccccccc}
X & \xrightarrow{p_0 \wedge S^0} & IX \wedge X & \longrightarrow \cdots \xrightarrow{p_{\sigma-1} \wedge S^0} & I^\sigma X \wedge X & \longrightarrow \cdots \\
\uparrow \mathrm{id} & & \uparrow \bar{g}_2 & & \uparrow \bar{g}_{\sigma+1} & \\
X & \xrightarrow[\delta_1]{} & \Omega \wedge X & \xrightarrow[\delta_2]{} \cdots \xrightarrow[\delta_\sigma]{} & \Omega^\sigma \wedge X & \longrightarrow \cdots
\end{array}$$

4. Some examples II. In this section we apply the ideas of §3 to a few of the spectra described in §2.

4.1. The theory gives a particularly nice situation when applied to ΩS^i and X_i of 2. For each i we have spectral sequences coming from the exact couple of the resolutions whose $E_1^{s,t} = \pi_t((\Omega S^i)^s \wedge X_i) = [\tilde{H}_*(\Omega S^i \wedge \cdots \wedge \Omega^i S\,;\, \mathbf{Z}) \otimes \pi_*(X_i)]_t$. The d_1 is induced by δ_s above.

4.2. When we apply the theory to Ω^2S^3 and $K(\mathbf{Z}_2)$ we get the classical bar resolution from 3.1. The resolution 3.1 looks slightly different than the bar resolution since it appears to make each of the exterior algebra generators in $H^*(\Omega^2S^3)$ primitive in the resolution. These generators can be identified with $\xi_i^{2^j}\epsilon A^*$ and $\xi_i^{2^j}$ is not primitive. This apparent discrepancy is cleared up when one recalls the fact that Ω^2S^3, as a stable complex, breaks up into parts each of which has a nontrivial Steenrod algebra action. The action is given by $x_i \to \Sigma_{j+k-i} x_j^{2^k} \otimes \xi_k$. When this additional term is added to the primitive term we have the usual bar resolution.

The May spectral sequence is obtained this way also. We look at the resolution

$$\begin{aligned} &\mathbf{Z}_2 \to K(\mathbf{Z}_2, 0) \to \Omega^2S^3 \wedge K(\mathbf{Z}_2, 0) \\ &\quad \to (\Omega^2S^3)^2 \wedge K(\mathbf{Z}_2, 0) \to \cdots \to (\Omega^2S^3)^\sigma \wedge K(\mathbf{Z}_2, 0) \to \cdots \end{aligned}$$

Now $\mathrm{Hom}_A(C_s, \mathbf{Z}_2) \cong (\Omega^2S^3)^s$. The differential in the associated chain complex has two parts, one is the differential in

$$\begin{aligned} &\Omega^2S^3 \xrightarrow{\bar\Delta} (\Omega^2S^3)^2 \xrightarrow{1\wedge\bar\Delta + \bar\Delta\wedge 1} (\Omega^2S^3)^3 \\ &\xrightarrow{1\wedge 1\wedge\bar\Delta + 1\wedge\bar\Delta\wedge 1 + \bar\Delta\wedge 1\wedge 1} (\Omega^2S^3)^4 \to \cdots \end{aligned}$$

and the second part interprets the action of the Steenrod algebra in Ω^2S^3. Using the Koszul resolution we see that $H_*(C_1) = \mathbf{Z}_2(R_{i,j})$ $i \geqslant 0, j \geqslant 1$ where $R_{i,j}$ is represented by $x_j^{2^i}$ and $H_*(\Omega^2S^3) = \mathbf{Z}_2(x_i)$. This is the E_1 term of the May spectral sequence. The d_1 results from identifying $x_j^{2^i}$ with $\alpha\epsilon A$ and asking how $\alpha_{i,j}$ acts on $x_l^{2^k}$.

We have $x_j^{2^i} = \alpha_{i,k} X_{j-k}^{2^{i+k}}$ for $k = 1, \ldots, j-1$. This follows easily from the Brown-Gitler decomposition description of A (see [6]). It probably is easily read from the Nishida relation. Anyway, when dualized this yields $dR_{ij} = \Sigma_{k=1}^{j-1} R_{i,k}R_{i+k,j-k}$. The higher differentials reflect more complicated squaring operations. The evaluation of differentials seems to be easier in this setting. In particular in Tangora [11], 4.9, the proposition $d_4(b_{03})^2 = h_2b_{12}^2 + h_4b_{02}^2$ is proved. It is apparently not easy to verify that the term $h_2b_{12}^2$ is present. From this approach it is rather easy. It seems likely that 1.3 of [11] could be proved in this manner.

4.3. An interesting description of the E_2 term for the Novikov spectral sequence results when one applied the theory of 3 to BU and MU. The resulting chain complex is

$$MU \xrightarrow{f_1} BU \wedge MU \xrightarrow{\delta} BU \wedge BU \wedge MU \to \cdots$$

where δ_1 is the map of Thom complexes given by $BU \xrightarrow{\Delta} BU \times BU \xrightarrow{0,1} BU$, $\delta_2 = \Delta \wedge 1 - 1 \wedge \delta_1$, $\Delta_3 = \Delta \wedge 1 \wedge 1 - 1 \wedge \Delta \wedge 1 + 1 \wedge 1 \wedge \delta$ and so forth. Many standard formulae result.

4.4. BO [8, ...] and MO [8, ...] yield an interesting spectral sequence and recent work of Davis and Mahowald [4] have applied it.

4.5. The space $\Omega(J_{2^i-1}S^2)$ where J_k is the James construction yields interesting spectra when one uses the composite $\Omega(J_{2^i-1}S^2) \subset \Omega^2 S^3 \xrightarrow[f]{} BO$. The homology of $\Omega J_{2^i-1}S^2$ is equal to $P(x_1, \ldots, x_{i-1})$. The resulting resolution seems to give a geometric realization of the various spectral sequence of Adams [1], Chapter 2.

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