The Kervaire invariant in homotopy theory

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Abstract

In this note we discuss how the first author came upon the Kervaire invariant question while analyzing the image of the J-homomorphism in the EHP sequence.

One of the central projects of algebraic topology is to calculate the homotopy classes of maps between two finite CW complexes. Even in the case of spheres – the smallest non-trivial CW complexes – this project has a long and rich history.

Let S^n denote the n-sphere. If $k < n$, then all continuous maps $S^k \to S^n$ are null-homotopic, and if $k = n$, the homotopy class of a map $S^n \to S^n$ is detected by its degree. Even these basic facts require relatively deep results: if $k = n = 1$, we need covering space theory, and if $n > 1$, we need the Hurewicz theorem, which says that the first non-trivial homotopy group of a simply-connected space is isomorphic to the first non-vanishing homology group of positive degree. The classical proof of the Hurewicz theorem as found in, for example, [28] is quite delicate; more conceptual proofs use the Serre spectral sequence.

Let us write $\pi_i S^n$ for the ith homotopy group of the n-sphere; we may also write $\pi_{k+n} S^n$ to emphasize that the complexity of the problem grows with k. Thus we have $\pi_{n+k} S^n = 0$ if $k < 0$ and $\pi_n S^n \cong \mathbb{Z}$. Given the Hurewicz theorem and knowledge of the homology of Eilenberg-MacLane spaces it is relatively simple to compute that

$$\pi_{n+1} S^n \cong \begin{cases} 0, & n = 1; \\ \mathbb{Z}, & n = 2; \\ \mathbb{Z}/2\mathbb{Z}, & n \geq 3. \end{cases}$$

The generator of $\pi_3 S^2$ is the Hopf map; the generator in $\pi_{n+1} S^n$, $n > 2$ is the *suspension* of the Hopf map. If X has a basepoint y, the suspension ΣX is given by

$$\Sigma X = S^1 \times X / (S^1 \times y \cup 1 \times X)$$

where $1 \in S^1 \subseteq \mathbb{C}$. Then $\Sigma S^n \cong S^{n+1}$ and we get a suspension homomorphism

$$E : \pi_{n+k} S^n \to \pi_{(n+1)+k} S^{n+1}.$$

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By the Freudenthal Suspension Theorem, this map is onto if $k \leq n-1$, and if $k < n-1$ it is an isomorphism. The common value of this group for large n is the kth stable homotopy group of spheres, written $\pi_k^s S^0$. For short, we may write

$$\mathrm{colim}_k \pi_{n+k} S^n = \pi_k^s S^0.$$

Note that this formula makes sense even if $k < 0$.

There has been a great deal of computation in the stable homotopy groups of spheres; see, for example, Appendix 3 of [26]. The answer is fairly complete for k up to about 60; if we divide out by 2- and 3-torsion, this can be improved to about $k = 1000$. However, we are a long way from any sort of complete calculation. Research since the mid-1970s has shifted to the investigation of large-scale phenomena, especially after the paper by Miller, Ravenel, and Wilson [24] on periodic phenomena and the proofs of Ravenel's nilpotence conjectures by Devinatz, Hopkins, and Smith [10, 13].

Historically, the Kervaire invariant arose in Pontryagin's calculation of $\pi_2^s S^0$. He noted that $\pi_k^s S^0$ is isomorphic to the group of cobordism classes of *framed* k-manifolds; that is, differentiable manifolds with a chosen trivialization of the stable normal bundle. Let $\mathbb{F}_2$ be the field with two elements. and let M be a connected framed manifold of dimension $k = 4m-2$. By collapsing all but the top cell of M we obtain a map

$$M \longrightarrow S^{4m-2}$$

which is an isomorphism of $H^{4m-2}(-, \mathbb{F}_2)$. Using surgery [6, 29] we can try to to build a cobordism from M to the sphere. This may not be possible, but we do find that the non-singular pairing

$$\lambda : H^{2m-1}(M, \mathbb{F}_2) \times H^{2m-1}(M, \mathbb{F}_2) \to H^{4m-2}(M, \mathbb{F}_2) \cong \mathbb{F}_2$$

given by Poincaré duality has a quadratic refinement μ; that is, there is a function $\mu : H^{2m-1}(M, \mathbb{F}_2) \to \mathbb{F}_2$ so that

$$\mu(x+y) + \mu(x) + \mu(y) = \lambda(x, y).$$

Up to isomorphism, the pair $(H^{2m-1}(M, \mathbb{F}_2), \mu)$ is completely determined by the *Arf Invariant.* This invariant is 1 if $\mu(x) = 1$ for the majority of the elements in $H^{2m-1}(M, \mathbb{F}_2)$; otherwise it is 0.[1] The *Kervaire invariant* of M is the Arf invariant of this quadratic refinement.

After first getting the computation wrong, Pontryagin [25] noted that for a particular framing of $S^1 \times S^1$, the Kervaire invariant was non-zero, giving a non-trivial cobordism class. Then $\pi_2^s S^0 \cong \mathbb{Z}/2\mathbb{Z}$ generated by this element.

To study the higher homotopy groups of spheres, we must consider more sophisticated methods. One such is the Adams spectral sequence

$$\mathrm{Ext}_A^s(\mathbb{F}_2, \Sigma^t \mathbb{F}_2) \Longrightarrow \pi_{t-s}^s S^0 \otimes \mathbb{Z}_2.$$

[1] For this reason, Browder has called the Arf invariant the "democratic invariant".

Here A is the Steenrod algebra, $\mathbb{Z}_2$ is the 2-adic integers, and $\Sigma^t\mathbb{F}_2 = \tilde{H}^*(S^t, \mathbb{F}_2)$. The Kervaire invariant elements are then classes

$$h_j^2 \in \mathrm{Ext}_A^2(\mathbb{F}_2, \Sigma^{2^{j+1}}\mathbb{F}_2)$$

which could detect elements in $\pi^s_{2^{j+1}-2}S^0$. If $j = 1$, this element detects Pontryagin's class.

In his work in smoothing theory, Kervaire [16] constructed a topological manifold of dimension $4m-2$ for $m \neq 1,\ 2,\ 4$ which had Kervaire invariant one and which was smooth if a point was removed. The question then became "Is the boundary sphere smoothable?" Browder [5], proved that it was smoothable if and only if $m = 2^{j-1}$, $j \geq 4$ and if the elements h_j^2 detected a homotopy class. The homotopy class was constructed for $j = 4$ before Browder's work by Peter May in his thesis [22]; the finer properties of this element were uncovered in [4]. The class in dimension 62 (that is, $j = 5$) was constructed later in [3].

Hill, Hopkins, and Ravenel [11] have shown that for $j \geq 7$ the class h_j^2 is not a permanent cycle in the Adams spectral sequence and cannot detect a stable homotopy class. This settles Browder's question in all but one case. Their proof is a precise and elegant application of equivariant stable homotopy theory. It is also very economical: they develop the minimum amount needed to settle exactly the question at hand. The very economy of this solution leaves behind numerous questions for students of $\pi^s_*S^0$. One immediate problem is to find the differential on h_j^2 in the Adams spectral sequence. The target would be an important element which we as yet have no name for.

The Kervaire invariant and Arf invariant have appeared in other places and guises in geometry and topology. For example, it is possible to formulate the Kervaire invariant question not for framed manifolds, but for oriented manifolds whose structure group reduces to $SO(1) = S^1$. In this formulation, Ralph Cohen, John Jones, and the first author showed that the problem had a positive solution [8]. The relevant homotopy classes are in the stable homotopy of the Thom spectrum $MSO(1)$; they are constructed using a variant of the methods of [19], which certainly don't extend to the sphere. Note that $MSO(1) = \Sigma^{-2}\mathbb{C}\mathrm{P}^\infty$ is an infinite CW spectrum, but still relatively small. This may the best of all positive worlds for this problem.

In [7], Brown found a way to extend the Kervaire invariant to another more general class of manifolds. And, by contemplating work of Witten, Hopkins and Singer found an application of the Arf invariant in dimension 6 to some problems in mathematical physics. See [12].

Parallel to this geometric story, the Kervaire invariant problem also arose in an entirely different line of research in homotopy theory, and here the negative solution of [11] leaves as many questions as it answers. This line of inquiry, long studied by the first author, asks just how the stable homotopy groups of spheres are born. To make this question precise, we must introduce the EHP sequence. This was discovered by James [15] in the mid-1950s and related techniques were exploited by Toda to great effect in his landmark book [27].

If X is a based space, let ΩX denote the space of based loops in X. In his work on loop spaces [14], James produced a small CW complex with the homotopy type of ΩS^{n+1} and later he noticed that this gave a splitting

$$\Sigma\Omega S^{n+1} \simeq \bigvee_{t>0} S^{nt+1}.$$

Here $\bigvee$ is the one-point union or *wedge*. By collapsing all factors of the wedge except for $t = 2$ and then taking the adjoint, we obtain the first *Hopf invariant*

$$H : \Omega S^{n+1} \to \Omega S^{2n+1}$$

There is also the map $E : S^n \to \Omega S^{n+1}$ adjoint to the identity; it induces the suspension homomorphism on homotopy groups. A calculation with the Serre spectral sequence shows that

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$$

is a fiber sequence when localized at 2. As a consequence there is a long exact sequence in homotopy groups, once we divide out by the odd torsion:

$$(1) \qquad \cdots \to \pi_{i+2}S^{2n+1} \xrightarrow{P} \pi_i S^n \xrightarrow{E} \pi_{i+1}S^{n+1} \xrightarrow{H} \pi_{i+1}S^{2n+1} \to \cdots .$$

This is the EHP sequence. As mentioned, E is the suspension map and H is the Hopf invariant. The map P is more difficult to describe; however we do have that if $\alpha \in \pi_* S^{n-1}$, then (up to sign)

$$P(E^{n+2}\alpha) = [\iota_n, E\alpha]$$

where $[-,-]$ is the Whitehead product and $\iota_n \in \pi_n S^n$ is the identity. Thus, for example, $P(\iota_{2n+1}) = [\iota_n, \iota_n]$.

From this point forward in this note, we will implicitly localize all groups at the prime 2.

The EHP sequence gives an inductive method for calculating the homotopy groups of spheres; the key is to do double induction on n and k in $\pi_{n+k}S^n$. To this end we reindex the subscripts in Equation (1) and write a triangle

$$(2) \qquad \begin{array}{ccc} \pi_{n+k}S^n & \xrightarrow{\quad E \quad} & \pi_{(n+1)+k}S^{n+1} \\ & {\scriptstyle P}\nwarrow \quad \swarrow{\scriptstyle H} & \\ & \pi_{(2n+1)+(k-n)}S^{2n+1} & \end{array}$$

for the EHP sequence. The dotted arrow indicates a map of degree -1. Then, assuming we know $\pi_{m+i}S^m$ for all $m \leq n$ and for all $i < k$, we can try to calculate $\pi_{(n+1)+k}S^{n+1}$. Coupled with the unstable Adams Spectral Sequence, it is possible to do low dimensional calculations very quickly – but, as with all algebraic approximations to the homotopy groups of spheres, it gets difficult fairly soon.[2] Tables for this computation can be found in a number of places; see, for example, [23] or §I.5 of [26].

Question 1.1. Suppose $\alpha \in \pi_k^s S^0$ is a stable element.

1. What is the smallest n so that α is in the image of $\pi_{n+k} S^n \to \pi_k^s S^0$? Then S^n is the *sphere of origin*.
2. Suppose S^n is the sphere of origin of α and a is a class in $\pi_{n+k} S^n$ which suspends to α. What is $H(a)$? This is "the" *Hopf invariant* of α.

Technical Warning 1.2. As phrased, the second question is not precise, as there maybe more than one a which suspends to α. There are several ways out of this difficulty. One is to ignore it. In practice, this works well. Another is to note that the EHP sequences, as in Equation (2) assemble into an exact couple which gives a spectral sequence

$$E^1_{k,n} = \pi_{n+k} S^{2n-1} \Longrightarrow \pi_k^s S^0.$$

Then questions (1) and (2) can be rephrased by asking for the non-zero element in E^∞ which detects α.

It is a feature of this spectral sequence that the E^2-page is an $\mathbb{F}_2$-vector space. This means, for example, that elements of high order must have high sphere of origin. Charts for this spectral sequence can be developed from [17] and can be found in explicit form in [23], which is based on work of the first author.

Example 1.3. As a simple test case, the sphere of origin the generator $\eta \in \pi_1^s S^0 \cong \mathbb{Z}/2\mathbb{Z}$ is S^2 and a can be taken to the Hopf map $S^3 \to S^2$. The Hopf invariant of this map is (up to sign) the identity $\iota_3 \in \pi_3 S^3$. We can ask when $\iota_{2n-1} \in \pi_{2n-1} S^{2n-1}$ can be the Hopf invariant of some stable class. This is the *Hopf invariant one* problem, settled by Adams in [1]: it only happens when n is 2, 4, or 8; the resulting stable classes are $\eta \in \pi_1^s S^0$, $\nu \in \pi_3^s S^0$, and $\sigma \in \pi_7^s S^0$.

There are instructive reformulations of the Hopf invariant one problem. First, by the EHP sequence, ι_{2n-1} is the Hopf invariant of a stable class if and only if

$$[\iota_{n-1}, \iota_{n-1}] = 0 \in \pi_{2n-1} S^{n-1}.$$

Thus we are asking about the behavior of the Whitehead product.

Second, an argument with Steenrod operations shows there is an element of Hopf invariant one if and only if the element

$$h_j \in \mathrm{Ext}^1_A(\mathbb{F}_2, \Sigma^{2^j} \mathbb{F}_2)$$

survives to E_∞ in the Adams spectral sequence. It is this last question that Adams settled showing

$$d_2 h_j = h_0 h_{j-1}^2$$

[2] Doug Ravenel has dubbed this general observation "The Mahowald Uncertainty Principle".

if $j \geq 4$.

This example, while now part of our basic tool kit, remains very instructive for the interplay of stable and unstable information, and the role of the Adams spectral sequence. Notice also that we changed our question in the middle of the discussion.

Question 1.4. Let $\alpha \in \pi_k^s S^0$ be a stable element, then α desuspends uniquely to $\pi_{n+k} S^n$ if $n > k+1$. Suppose $2n-1 > k+1$. Is

$$\alpha \in \pi_{2n-1+k} S^{2n-1}$$

the Hopf invariant of a stable element in $\pi_{2n-1+k} S^n$?

The solution of the Hopf invariant one problem, completely answers this question for a generator of $\pi_0^s S^0$. We will have other examples below.

It is exactly in thinking about Question (1.4) that the first author came to the Kervaire invariant problem.

In the middle 1960s, Adams [2] (with an addendum by the first author at the prime 2 [18]) wrote down an infinite family of non-zero elements in the homotopy groups of spheres. These elements we now call the *image of J*, and they were the first example of "periodic" families. They are easy to define, although less easy to show they are non-trivial.

Let $SO(n)$ be the special orthogonal group. Then $SO(n)$ acts on S^n by regarding S^n as the one-point compactification of $\mathbb{R}^n$. This action defines a map

$$SO(n) \to \mathrm{map}_*(S^n, S^n)$$

from $SO(n)$ to the space of pointed maps. Taking the adjoint, assembling all n, and applying homotopy yields map

$$J : \pi_k SO \to \pi_k^s S^0. \tag{3}$$

By Bott periodicity, we know the homotopy groups of SO. What Adams did was compute the image. To state the result, let $k > 0$ be an integer, let $\nu_2(-)$ denote 2-adic valuation, and define

$$\lambda(k) = \nu_2(k+1) + 1.$$

Thus $\lambda(7) = 4$ and $\lambda(11) = 3$. The image of the J-homomorphism lies in a split summand $\mathrm{Im}(J)_* \subseteq \pi_*^2 S^0$ with $\mathrm{Im}(J)_1 \cong \mathbb{Z}/2\mathbb{Z}$ generated by η and for $k \geq 2$,

$$\mathrm{Im}(J)_k = \begin{cases} \mathbb{Z}/2^{\lambda(k)}\mathbb{Z} & k = 8t-1, 8t+3; \\ \mathbb{Z}/2\mathbb{Z} & k = 8t, 8t+2; \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & k = 8t+1; \\ 0 & k = 8t+4, 8t+5, 8t+6. \end{cases}$$

Let's write ρ_{8t-1} and ζ_{8t+3} for the generators of the groups in degrees $8t-1$ and $8t+3$ respectively. Some of these elements are familiar; for example, $\nu = \zeta_3$ and $\sigma = \rho_7$. The elements $\eta\rho_{8t-1}$ and $\eta^2\rho_{8t-1}$ are non-zero in $\mathrm{Im}(J)_*$. There is another generator μ_{8t+1} in degree $8t+1$; $\eta\mu_{8t+1} \neq 0$ and $\eta^2\mu_{8t+1} = 4\zeta_{8t+3}$. Despite the name, the J-homomorphism of Equation (3) is not onto $\mathrm{Im}(J)_*$, as the elements μ_{8t} and $\eta\mu_{8t}$ are not in the image, although we see that they are intimately connected to that image. In fact, we can think of μ_1 as η; then

$$\eta^2\mu_1 = \eta^3 = 4\nu = \zeta_3$$

and the equation $\eta^2\mu_{8t+1} = 4\zeta_{8t+3}$ is then forced by the periodic behavior of these elements.

There were several revealing new features to this family. One was that it was infinite: this was the first systematic collection of elements produced in the stable homotopy groups of spheres and took us beyond the era of stem-by-stem calculations. Another feature was that this was the first of what we now call periodic families of stable homotopy classes. The attempt to understand stable homotopy theory in terms of periodic families led to a reorganization of the field, including the work of Miller, Ravenel, and Wilson [24], and the Ravenel's nilpotence conjectures, proved by Devinatz, Hopkins, and Smith [10, 13].

We now say that $\mathrm{Im}(J)_*$ is the v_1-periodic homotopy groups of spheres. We won't dwell on this point, but in modern language (a language not available in the 1960s), we say the the composite

$$\mathrm{Im}(J)_* \xrightarrow{\subseteq} \pi_*^s S^0 \longrightarrow \pi_* L_{K(1)} S^0$$

is an isomorphism in degrees greater than 1 and an injection in degree 1. Here $L_{K(1)}S^0$ is the localization of the sphere spectrum at the K-theory with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

In the mid-1960s, the first author began an extensive study of the image of J in the EHP sequence; the first results appeared in [17] and there was followup paper [20] almost fifteen years later.

The sphere of origin and the Hopf invariants of the elements in the image of J are all known. For example, the sphere of origin of ζ_{8k+3} is S^5 and its Hopf invariant is an element on S^9 which suspends to $2^{\lambda(8k-1)-1}\rho_{8k-1}$; in particular, the Hopf invariant of ζ_{8k+3} is another element in the image of J. The remarkable fact is that this is (almost) true in general; the exception is $\eta\rho_{8k-1}$ which has sphere of origin S^3 and Hopf invariant $\nu\zeta_{8k-5}$ on the 5-sphere.[3] In this sense the image J is very nearly a closed family. The detailed answers are nicely laid out in [23].

But there are exceptions. Once the the sphere-of-origin and Hopf invariant calculations have been answered for the elements in the image of J, there are still a few elements left that could be Hopf invariants of new elements in the stable homotopy groups of spheres. There are some sporadic examples (see the table below) and there are also two infinite families. The first is

$$\nu = \zeta_3 \in \pi_{2^{j+1}-2} S^{2^{j+1}-5}, \quad j \geq 4.$$

[3]Since $\nu = \zeta_3$, the element $\nu\zeta_{8k-5}$ could be regarded as an honorary element of the image of J, failing to attain full membership because it is unstable.

This turns out to be the Hopf invariant of an element

$$\eta_j \in \pi^s_{2^j} S^0$$

detected by the element

$$h_1 h_j \in \mathrm{Ext}^2_A(\mathbb{F}_2, \Sigma^{2^j+2}\mathbb{F}_2).$$

This element was constructed by the first author in [19].

The Kervaire invariant elements arose as part of conjectural solution to what happens for the second infinite family. To describe this conjecture we need some notation.

Let $j \geq 2$ and define integers a and b by the equation $j = 4a + b$ for $0 \leq b \leq 3$. Define $\phi(j) = 8a + 2^b$. Notice that if $i \geq 2$, then $\pi_i SO \neq 0$ if and only if $i = \phi(j) - 1$ for some j. Let β_j be a generator of the image of J in degree $\phi(j) - 1$; thus, for example, we have

$$\beta_2 = \nu \quad \beta_3 = \sigma \quad \beta_4 = \eta\sigma \quad \beta_5 = \eta^2\sigma \quad \beta_6 = \zeta_{11}.$$

Notice we are excluding the generators μ_{8k+1} and $\eta\mu_{8k+1}$.

The remaining classes available to be Hopf invariants were the infinite family

$$\beta_j \in \pi_{2n+\phi(j)} S^{2n+1}, \quad n + \phi(j) + 1 = 2^{j+1}.$$

The first author made the following conjecture in 1967 [17].

Conjecture 1.5. Let $n + \phi(j) + 1 = 2^{j+1}$. The Whitehead product $[\iota_n, \beta_j] = 0$ if and only if h_j^2 detects a non-zero homotopy class.

To paraphrase the conjecture we have: if h_j^2 detects a non-zero homotopy class Θ_j, then Θ_j has sphere of origin $S^{2^{j+1}-\phi(j)}$ and Hopf invariant β_j.

This conjecture has been proved in all aspects by Crabb and Knapp [9], but, of course, the negative solution of [11] leaves only the case $j = 6$ of interest. Indeed, we now see that for $j > 6$

$$[\iota_n, \beta_j] \neq 0 \in \pi_* S^{2^{j+1}-\phi(j)-1}.$$

So, somewhat surprisingly, the image of J has led us into unknown territory. What else can we say about this class?

Open Problem 1.6. There is another, richer, question left as well. If the Kervaire invariant class Θ_j had existed, it would have had Hopf invariant β_j. A likely consequence of this was that for all odd k, the element

$$P(\beta_j) \in \pi_* S^{k2^{j+1}-\phi(j)-1}$$

would have had Θ_j as its Hopf invariant. Now we have no idea what the Hopf invariant of this family of elements could be, but they are presumably a new and very interesting collection of elements in the stable homotopy groups of spheres. For example, they should play a key role in the iterated root invariant [21] of $2\iota \in \pi^s_0 S^0$. The elements in this family should depend only on m, and not on k.

Here is a table showing the generators in the stable homotopy groups of spheres which are not in the image of J, yet which have Hopf invariants in the image of J. Listed also are their spheres of origin, and their Hopf invariants. There are five (or six) sporadic elements and one infinite family. The element Θ_6 is the unsettled case of the Kervaire invariant problem. It may or may not exist. The element ν^* is the Toda bracket $\langle \sigma, 2\sigma, \nu \rangle$ in $\pi^s_{18} S^0$. It is detected by $h_2 h_4$ in the Adams Spectral Sequence. In this context the η_j family looks quite curious. Why does it have this privileged role?

Element	Sphere of Origin	Hopf Invariant
ν^2	4	ν
σ^2	8	σ
ν^*	12	σ
Θ_4	23	$\eta\sigma = \beta_4$
Θ_5	54	$\eta^2\sigma = \beta_5$
$\Theta_6(?)$	116	$\zeta_{11} = \beta_6$
η_j	$2^j - 2$	ν

References

[1] Adams, J. F., "On the non-existence of elements of Hopf invariant one", *Ann. of Math. (2)*, 72 (1960), 20–104.

[2] Adams, J. F., "On the groups $J(X)$, IV", *Topology* 5 (1966), 21-71.

[3] Barratt M., Jones, J., and Mahowald, M., "Relations amongst Toda brackets and the Kervaire invariant in dimension 62," *J. London Math. Soc. (2)*, 30 (1984), 533-550.

[4] Barratt, M. G. and Mahowald, M. E. and Tangora, M. C., "Some differentials in the Adams spectral sequence. II", *Topology*, 9 (1970), 309–316.

[5] Browder, W.,"The Kervaire invariant of framed manifolds and its generalization", *Ann. of Math. (2)* 90 (1969), 157–186.

[6] Browder, W., *Surgery on simply-connected manifolds*, Springer-Verlag, Berlin 1972.

[7] Brown, Jr., Edgar H., "Generalizations of the Kervaire invariant", *Ann. of Math. (2)*, 95 (1972), 368–383.

[8] Cohen, R. L. and Jones, J. D. S. and Mahowald, M. E., "The Kervaire invariant of immersions", *Invent. Math.*, 79 (1985) no. 1, 95–123.

[9] Crabb, M. C. and Knapp, K., "Applications of nonconnective Im(J)-theory", *Handbook of algebraic topology*, 463–503, North-Holland, Amsterdam, 1995.

[10] Devinatz, Ethan S. and Hopkins, Michael J. and Smith, Jeffrey H., "Nilpotence and stable homotopy theory. I", *Ann. of Math. (2)*, 128 (1988), no. 2, 207–241.

[11] Hill, M.A, Hopkins, M.J., and Ravenel, D.C, "On the non-existence of elements of Kervaire invariant one," `arxiv.org/abs/0908.3724`.

[12] Hopkins, M. and Singer, I., "Quadratic functions in geometry, topology, and M-theory," *J. Differential Geom.* 70 (2005), no. 3, 329–452.

[13] Hopkins, Michael J. and Smith, Jeffrey H., "Nilpotence and stable homotopy theory. II", *Ann. of Math. (2)*, 148 (1998), no. 1, 1–49.

[14] James, I. M., "Reduced product spaces", *Ann. of Math. (2)*, 62 (1955), 170–197.

[15] James, I. M., "The suspension triad of a sphere", *Ann. of Math. (2)*, 63 (1956), 407–429.

[16] Kervaire, M, "A manifold which does not admit any differential structure," *Comment. Math. Helv.*, 34 (1960), 256-270.

[17] Mahowald, M., *The metastable homotopy of* S^n, Memoirs of AMS 72 (1967).

[18] Mahowald, M, "On the order of the image of J", *Topology* 6 (1967), 371-378.

[19] Mahowald, Mark, "A new infinite family in ${}_2\pi_*{}^s$", *Topology*, 16 (1977), no. 3, 249–256.

[20] Mahowald, Mark, "The image of J in the EHP sequence", *Ann. of Math. (2)*, 116 (1982) no.1, 65–112.

[21] Mahowald, Mark E. and Ravenel, Douglas C., "The root invariant in homotopy theory", *Topology*, 32 (1993) no. 4, 865–898.

[22] May, J.P., "The cohomology of restricted Lie algebras and of Hopf algebras; application to the Steenrod algebra", Thesis, Princeton University, 1964.

[23] Miller, H. R. and Ravenel, D. C., "Mark Mahowald's work on the homotopy groups of spheres", *Algebraic topology (Oaxtepec, 1991)*, Contemp. Math., 146, 1–30, Amer. Math. Soc., Providence, RI 1993.

[24] Miller, Haynes R. and Ravenel, Douglas C. and Wilson, W. Stephen, "Periodic phenomena in the Adams-Novikov spectral sequence", *Ann. Math. (2)*, 106 (1977), no. 3, 469–516.

[25] Pontrjagin, C.R.. *Acad. Sci. URSS* 19 (1938) 147-149, 361-363

[26] Ravenel, Douglas C., *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, 121, Academic Press Inc., Orlando, FL, 1986.

[27] Toda, Hirosi, *Composition methods in homotopy groups of spheres*, *Annals of Mathematics Studies, No. 49*, Princeton University Press, Princeton, N.J., 1962.

[28] Spanier, E.H., *Algebraic Topology*, McGraw-Hill, New York, 1966.

[29] Wall, C. T. C., *Surgery on compact manifolds*, Mathematical Surveys and Monographs, 69 (2nd ed.), A.M.S., Providence, R.I., 1999.

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