ELSEVIER Journal of Pure and Applied Algebra 121 (1997) 137–159

JOURNAL OF PURE AND APPLIED ALGEBRA

$THH(R) \cong R \otimes S^1$ for E_∞ ring spectra[1]

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Communicated by C.A. Weibel; received 7 March 1994; received in revised form 28 June 1994

Abstract

We prove that the topological Hochschild homology spectrum $THH(R)$ of an E_∞ spectrum R is the S^1-indexed sum of copies R in the category of E_∞ ring spectra. As a consequence, we obtain a natural S^1-action and compatible power operations on $THH(R)$. In addition, $THH(R)$ admits an A_∞-comultiplication making it an A_∞ Hopf algebra spectrum over R. © 1997 Elsevier Science B.V.

AMS Classification: 55P42; 19D10; 16E40

1. Introduction and main results

Around 1985 Bökstedt introduced the notion of Topological Hochschild Homology THH of a functor with smash products [2] (for a published version see [4]). The category of such functors is topologically enriched. In a conversation Bökstedt pointed out that if this category were tensored $THH(R)$ ought to be the tensor $R \otimes S^1$ in the commutative case.

Recall that a category is *topologically enriched* if the morphism sets are topologized such that composition is continuous. In a topologically enriched category we have the notions of indexed limits and colimits.

1.1. Definition. Let $\mathscr{K}$ and $\mathscr{B}$ be topologically enriched categories and let $F : \mathscr{K} \to \mathscr{T}op$, $G : \mathscr{K}^{op} \to \mathscr{T}op$ and $X : \mathscr{K} \to \mathscr{B}$ be continuous functors. The *limit of X indexed*

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[1] Supported by the NSF.

PII S0022-4049(97)00118-7

by F is an object $\lim_F X$ in $\mathcal{B}$ together with a natural homeomorphism

$$\mathcal{B}(B, \lim_F X) \cong Funct(\mathcal{K}, \mathcal{T}op)(F, \mathcal{B}(B, X(-))),$$

where $Funct(\mathcal{K}, \mathcal{T}op)$ denotes the category of continuous functors $\mathcal{K} \to \mathcal{T}op$. The *colimit of X indexed by G* is an object $\mathrm{colim}_G X$ in $\mathcal{B}$ together with a natural homeomorphism

$$\mathcal{B}(\mathrm{colim}_G X, B) \cong Funct(\mathcal{K}^{op}, \mathcal{T}op)(G, \mathcal{B}(X(-), B)).$$

If F and G are the constant functors to a point we get the usual definitions of limits and colimits with the additional requirement that the natural bijections be homeomorphisms. If $\mathcal{K}$ consists of one object and its identity and F and G take the space K as value while X takes $B \in ob\mathcal{B}$ as value, we denote $\lim_F X$ by B^K and $\mathrm{colim}_G X$ by $B \otimes K$.

1.2. Definition. If $B \otimes K$ exist for all $B \in \mathcal{B}$ and $K \in \mathcal{T}op$, B is called *tensored.* If all B^K exist it is called *cotensored.*

In view of Bökstedt's remark and its implications tensored and cotensored structures are our central concern. They have the following universal property.

1.3. Let $\mathcal{B}$ be a topologically enriched tensored and cotensored category. Then for $B_1, B_2 \in \mathcal{B}$ and $K \in \mathcal{T}op$ we have natural homeomorphisms

$$\mathcal{B}(B_1 \otimes K, B_2) \cong \mathcal{T}op(K, \mathcal{B}(B_1, B_2)) \cong \mathcal{B}(B_1, B_2^K).$$

This shows that $B \otimes K$ is the K-indexed sum of copies B if K is discrete and B^K the K-indexed product. Hence, $B \otimes K$ and B^K are topologically parametrized versions of K-indexed sums and products.

From (1.3) we immediately deduce

1.4. For $B \in \mathcal{B}$ and $K, L \in \mathcal{T}op$ we have natural isomorphisms

$$(B \otimes K) \otimes L \cong B \otimes (K \times L),$$
$$(B^K)^L \cong B^{K \times L}.$$

The central idea of Bökstedt's definition of *THH* is to take the classical definition of Hochschild homology, replace ring by a suitable notion of ring spectrum and the tensor product by the smash product. At that time there was no known category of spectra with an associative, commutative and unital smash product. So Bökstedt introduced unstably defined functors with smash products whose structures allow a stabilization procedure, and a fairly small coherence machinery took care of a coherently homotopy associative, commutative and unital smash product of such functors.

Recently Elmendorf et al. [6, 7] discovered a category of spectra which admits a strictly (up to natural isomorphisms) associative, commutative and unital smash product making it into a symmetric monoidal category. Moreover, it has the pleasant property that the A_∞ ring spectra and E_∞ ring spectra, which are structured by the linear isometry operad $\mathscr{L}$, are exactly the monoids, respectively, commutative monoids in this category. Hence, it is very simple to define THH in terms of this smash product $\star$ (see Section 4), and therefore we will work in this setting. It has not been established yet that this definition of THH agrees with the one of Bökstedt, but this is very likely to be true [15], at least for CW ring spectra [7, Ch. I].

The restriction to $\mathscr{L}$-structured ring spectra is not substantial: recall from [13] that an *E_∞ operad* is a Σ-free operad $\mathscr{C}$ such that each $\mathscr{C}(n)$ is contractible. If $\mathscr{C}$ is an operad without an action of Σ_n on $\mathscr{C}(n)$ and each $\mathscr{C}(n)$ is contractible, we call $\mathscr{C}$ an *A_∞ operad.* Our canonical example is the *linear isometries operad* $\mathscr{L}$: let $\mathscr{I}$ denote the category of real inner product spaces and linear isometries. Let $\mathscr{U} \cong \mathbb{R}^\infty$ be an object in $\mathscr{I}$, then $\mathscr{L}(n) = \mathscr{I}(\mathscr{U}^{\oplus n}, \mathscr{U})$ defines an E_∞ operad whose structure maps are given by composition. If we forget the action of Σ_n on $\mathscr{U}^{\oplus n}$, $\mathscr{L}$ reduces to an A_∞ operad.

1.5. Definition. An A_∞ *ring spectrum* consists of a spectrum R, an A_∞ operad $\mathscr{C}$ augmented over $\mathscr{L}$ and structure maps

$$\zeta_n : \mathscr{C}(n) \ltimes R^n \to R,$$

$n \geq 0$, defining an action of $\mathscr{C}$ on R. Here R^n is the n-fold external smash product (see Section 2 for a recollection of the basic definitions). If $\mathscr{C}$ is an E_∞ operad augmented over $\mathscr{L}$ (as E_∞ operad) and the ζ_n are Σ_n-equivariant, R is an E_∞ *ring spectrum.*

Let R be an A_∞ or E_∞ ring spectrum structured by an operad $\mathscr{C}$. Let C denote its associated monad [10, VII.3]. Since $\mathscr{C}$ augments over $\mathscr{L}$ the monad C acts on the monad L associated with the operad $\mathscr{L}$. Applying the functorial two-sided barconstruction we obtain maps of ring spectra

$$B(L, C, R) \leftarrow B(C, C, R) \to R,$$

which are weak equivalences and homomorphisms with respect to the C-structure. Moreover, the left ring spectrum is structured by the linear isometry operad (for the two-sided barconstruction on space level and its properties see [13, Section 9]. The spectrum level construction is similar, details will appear in [7]).

Hence, there is a functorial way to replace each A_∞ or E_∞ ring spectrum by a weakly equivalent one structured by the operad $\mathscr{L}$. This allows the following:

1.6. Convention. A_∞ or E_∞ ring spectrum will always mean a ring spectrum structured by the linear isometry operad.

Let $\mathscr{E}_\infty$ denote the category of E_∞ ring spectra and homomorphisms. Our main results are

Theorem A. *$\mathscr{E}_\infty$ is topologically enriched in a canonical way and contains all indexed limits and colimits. In particular, $\mathscr{E}_\infty$ is tensored and cotensored.*

A different proof that $\mathscr{E}_\infty$ is tensored and cotensored will appear in a joint paper by Mike Hopkins and the first author.

Theorem B. *For E_∞ ring spectra R there is a natural isomorphism in $\mathscr{E}_\infty$*

$$THH(R) \cong R \otimes S^1.$$

In particular, $THH(R)$ is again an E_∞ ring spectrum and its multiplication ν: $THH(R) \star THH(R) \to THH(R)$ is induced by the folding map $S^1 \sqcup S^1 \to S^1$.

This result has a number of interesting consequences. We include just the most straightforward ones; others will be studied in a separate paper.

Since $R \otimes S^1$ is a continuous functor in both variables we have a homomorphism of topological monoids

$$\lambda : \mathscr{T}op(S^1, S^1) \to \mathscr{E}_\infty(THH(R), THH(R))$$

natural in the $\mathscr{E}_\infty$ ring spectrum R. The adjoint of the multiplication $S^1 \times S^1 \to S^1$ also defines a homomorphism of topological monoids

$$\rho : S^1 \to \mathscr{T}op(S^1, S^1).$$

The composite $\lambda \circ \rho = \hat{\alpha}$ defines an S^1-action on $THH(R)$ and we obtain

Theorem C. *For an E_∞ ring spectrum R there is an S^1-action on $THH(R)$ through homomorphisms, i.e. a homomorphism of topological monoids*

$$\hat{\alpha} : S^1 \to \mathscr{E}_\infty(THH(R), THH(R)).$$

There are also obvious power operations

$$\Phi^r : THH(R) \to THH(R)$$

of the types considered by Loday [11] and McCarthy [14] defined on S^1 by

$$\varphi^r(e^{2\pi i t}) = e^{2\pi i r t}, \quad r \in \mathbb{Z}.$$

Since the product of $THH(R)$ is given by the folding map $S^1 \sqcup S^1 \to S^1$ the following result can easily be checked by considering the S^1 factor.

Theorem D. *For each E_∞ ring spectrum R there exist natural power operations*

$$\Phi^r : THH(R) \to THH(R)$$

one for each $r \in \mathbb{Z}$ satisfying

(i) *Φ^0 factors through the natural monomorphism*

$$\iota_R : R = R \otimes \{1\} \to THH(R)$$

induced by the inclusion $\{1\} \subset S^1$. It defines a retraction of ι_R.

(ii) *$\Phi^1 = id$.*

(iii) *$\Phi^r \circ \Phi^s = \Phi^{r \cdot s}$.*

(iv) *Each Φ^r is multiplicative, i.e. a homomorphism of E_∞ ring spectra.*

(v) *Compatability with the S^1-action. The following diagram commutes*:

$$\begin{array}{ccc} THH(R) \otimes S^1 & \xrightarrow{\alpha} & THH(R) \\ \Big\downarrow{\scriptstyle \Phi^r \otimes \varphi^r} & & \Big\downarrow{\scriptstyle \Phi^r} \\ THH(R) \otimes S^1 & \xrightarrow{\alpha} & THH(R) \end{array}$$

Here α denotes the adjoint of $\hat{\alpha}$.

$THH(R)$ with its S^1-action has an obvious universal property:

Theorem E. *The natural monomorphism $\iota_R : R = R \otimes \{1\} \to THH(R)$ has the following universal property: given an E_∞ ring spectrum R' with an S^1-action through homomorphisms and a homomorphism $f : R \to R'$ then there exists a unique S^1-equivariant homomorphism $\overline{f} : THH(R) \to R'$ such that $\overline{f} \circ i_R = f$.*

By (1.3) $- \otimes S^1 : \mathscr{E}_\infty \to \mathscr{E}_\infty$ is left adjoint to $(-)^{S^1} : \mathscr{E}_\infty \to \mathscr{E}_\infty$. We deduce

Theorem F. *$THH(-) : \mathscr{E}_\infty \to \mathscr{E}_\infty$ preserves colimits. In particular, since $*$ is the coproduct in $\mathscr{E}_\infty$,*

$$THH(R_1 * R_2) \cong THH(R_1) * THH(R_2).$$

In [2] Bökstedt defined a map

$$\lambda : R \wedge S^1_+ \to THH(R),$$

which plays an important role in the calculations of [3]. The existence of λ is obvious in our set-up. Let $\underline{x} : \{*\} \to S^1$ denote the map sending $*$ to $x \in S^1$.

Theorem G. *For any E_∞ ring spectrum R there is a natural spectrum level Bökstedt map*

$$\lambda : R \wedge S^1_+ \to THH(R)$$

with the following properties:

(i) λ *is* S^1*-equivariant,*

(ii) *if* T *is an* E_∞ *ring spectrum and* $f : R \wedge S^1_+ \to T$ *a map of spectra such that*

$$f \circ (R \wedge \underline{x}_+) : R = R \wedge \{*\}_+ \to R \wedge S^1_+ \to T$$

is an E_∞ *homomorphism for each* $x \in S^1$, *there is a unique* E_∞ *homomorphism* $\hat{f} : THH(R) \to T$ *such that* $f = \hat{f} \circ \lambda$.

Theorem H. *For each* $x \in S^1$ *the* E_∞ *homomorphism*

$$i_x = id \otimes \underline{x} : R = R \otimes \{*\} \to THH(R)$$

defines an E_∞ *R-algebra structure on* $THH(R)$. *Hence, if R is an Eilenberg–MacLane spectrum* $THH(R)$ *is a product of Eilenberg–MacLane spectra.*

Let $*_R$ be the smash product over R as defined in Definition 6.1 below. If A and B are E_∞ R-algebras $A *_R B$ is the pushout in $\mathscr{E}_\infty$ of

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A *_R B \end{array}$$

(the proof is the same as the usual one). Give $THH(R)$ the R-algebra structure from ι_R. Then the projection

$$THH(R) * THH(R) \to THH(R) *_R THH(R)$$

is induced by the map $S^1 \sqcup S^1 \to S^1 \vee S^1$ with $1 \in S^1$ as base point. The algebra multiplication is given by the based folding map $S^1 \vee S^1 \to S^1$.
The pinch map $S^1 \to S^1 \vee S^1$ now defines an A_∞ comultiplication

$$THH(R) \to THH(R) *_R THH(R)$$

with (homotopy) counit induced by $S^1 \to *$, and we obtain

Theorem I. *If* R *is an* E_∞ *ring spectrum* $THH(R)$ *has an* A_∞ *Hopf algebra structure over* R.

1.7. Remark. It might be worth noticing that there is a simplicial E_∞ ring spectrum $E_\bullet$ with $E_0 = R$ and E_n the n-fold application of THH to R. Its realization is $R \otimes \mathbb{C}P^\infty$.

The paper is organized as follows. In Section 2 we recall the basic definitions and results from [6, 7, 10] which we will use in our constructions and add a few facts left out in these papers. Section 3 provides the proof of Theorem A. Topological Hochschild homology will be defined and studied in Section 4 which includes the proof of Theorem B. Section 5 deals with the Bökstedt map in a more general context. The paper ends with a section on the extension of the definition of topological Hochschild homology to E_∞ algebras over E_∞ ring spectra and some remarks on possible variations due to changes of the frame work provided by the work of Elmendorf et al. [7]

2. Spectra and (unital) S-modules

Throughout this paper $\mathcal{T}op$ denotes the category of compactly generated weak Hausdorff spaces and $\mathcal{T}op_*$ its based version. Limits, colimits and function spaces are taken in these categories.

We will work with coordinate-free spectra in the sense of [10]. Given a *universe*, i.e. real inner product space $\mathcal{U} \cong \mathbb{R}^\infty$, a $\mathcal{U}$-indexed *prespectrum* D assigns to each finite-dimensional subspace V of $\mathcal{U}$ a based space DV and to each orthogonal pair V, W a structure map

$$\sigma_{V,W} : DV \to \Omega^W D(V \oplus W)$$

satisfying the obvious associativity condition. D is a *spectrum* if the $\sigma_{V,W}$ are homeomorphisms. A map of $\mathcal{U}$-indexed (pre)-spectra $f : D \to D'$ is a family $fV : DV \to D'V$ of based maps preserving the structure. We denote the resulting categories of prespectra and spectra by $\mathcal{P}\mathcal{U}$ and $\mathcal{S}\mathcal{U}$, respectively. Both categories are topologically enriched by giving their morphism sets from D to D' the subspace topology of $\Pi_V\, \mathcal{T}op_*(DV, D'V)$.

For a prespectrum D and a based space K the assignments $V \mapsto DV \wedge K$ and $V \mapsto \mathcal{T}op_*(K, DV)$ define the *small smash product* and the *small function spectrum* functors

$$\mathcal{P}\mathcal{U} \times \mathcal{T}op_* \to \mathcal{P}\mathcal{U}, \quad (D,K) \mapsto D \wedge K,$$
$$\mathcal{P}\mathcal{U} \times \mathcal{T}op_*^{op} \to \mathcal{P}\mathcal{U}, \quad (D,K) \mapsto F(K,D).$$

The small function spectrum functor restricts to a functor

$$\mathcal{S}\mathcal{U} \times \mathcal{T}op_*^{op} \to \mathcal{S}\mathcal{U}.$$

For the small smash product we use the composite

$$\mathcal{S}\mathcal{U} \times \mathcal{T}op_* \to \mathcal{P}\mathcal{U} \to \mathcal{S}\mathcal{U}, \quad (E,K) \mapsto L(E \wedge K),$$

where L is the spectrification [10, p. 13]. In abuse of notation we again write $E \wedge K$ for $L(E \wedge K)$.

2.1. Proposition (Lewis et al. [10, I.3.3; I.3.4]). *Let E, $E' \in \mathcal{S}\mathcal{U}$, $K, L \in \mathcal{T}op_*$.*

(1) *There are natural homeomorphisms*

$$\mathcal{S}\mathcal{U}(E \wedge K, E') \cong \mathcal{T}op_*(K, \mathcal{S}\mathcal{U}(E,E')) \cong \mathcal{S}\mathcal{U}(E, F(K,E')).$$

(2) *There are natural isomorphisms*

$$E \wedge S^0 \cong E, \quad (E \wedge K) \wedge L \cong E \wedge (K \wedge L),$$
$$F(S^0, E) \cong E, \quad F(K \wedge L, E) \cong F(K, F(L, E)).$$

To obtain smash products between spectra we note the existence of an associative, commutative and unital *external smash product functor*

$$\mathscr{S}\mathscr{U} \times \mathscr{S}\mathscr{U}' \to \mathscr{S}(\mathscr{U} \oplus \mathscr{U}'), \quad (E, E') \mapsto E \wedge E'$$

induced by the spectrification of the functor defined by the formula $(E \wedge E')(V \oplus V') = EV \wedge E'V'$. To obtain an internal smash product in $\mathscr{S}\mathscr{U}$ we apply the twisted half smash product of [10]: let $\mathscr{I}$ be the topologically enriched category of universes and $\mathscr{T}op/\mathscr{I}(\mathscr{U}, \mathscr{U}')$ the category of spaces over $\mathscr{I}(\mathscr{U}, \mathscr{U}')$, then

2.2. Proposition (Lewis et al. [10, VI.1.1; VI.1.5; VI.3.1]). *There are functors*

$$\mathscr{T}op/\mathscr{I}(\mathscr{U}, \mathscr{U}') \times \mathscr{S}\mathscr{U} \to \mathscr{S}\mathscr{U}', \quad (A, E) \mapsto A \ltimes E,$$
$$\mathscr{T}op/\mathscr{I}(\mathscr{U}, \mathscr{U}')^{op} \times \mathscr{S}\mathscr{U}' \to \mathscr{S}\mathscr{U}, \quad (A, E) \mapsto F[A, E),$$

such that, for $A \in \mathscr{T}op/\mathscr{I}(\mathscr{U}, \mathscr{U}')$, $B \in \mathscr{T}op/\mathscr{I}(\mathscr{U}', \mathscr{U}'')$, $E \in \mathscr{S}\mathscr{U}$, $E' \in \mathscr{S}\mathscr{U}'$ *and* $K \in \mathscr{T}op_*$,

(1) *there is a natural homeomorphism*

$$\mathscr{S}\mathscr{U}'(A \ltimes E, E') \cong \mathscr{S}\mathscr{U}(E, F[A, E')),$$

(2) $A \ltimes E$ *preserves colimits in both variables,* $F[A, E')$ *preserves limits in* E' *and converts colimits in* A *to limits,*

(3) *there are isomorphisms*

$$A \ltimes (E \wedge K) \cong (A \ltimes E) \wedge K \quad \text{and} \quad F(K, F[A, E')) \cong F[A, F(K, E')),$$

(4) *for* $B \times A \to \mathscr{I}(\mathscr{U}', \mathscr{U}'') \times \mathscr{I}(\mathscr{U}, \mathscr{U}') \xrightarrow{\circ} \mathscr{I}(\mathscr{U}, \mathscr{U}'')$ *we have a natural isomorphism*

$$B \ltimes (A \ltimes E) \cong (B \times A) \ltimes E.$$

We could now define an internal smash product in $\mathscr{S}\mathscr{U}$ by the correspondence

$$(E, E') \mapsto \mathscr{I}(\mathscr{U} \oplus \mathscr{U}, \mathscr{U}) \ltimes (E \wedge E').$$

By the geometry of the spaces $\mathscr{I}(\mathscr{U}, \mathscr{U}')$ this internal smash product is coherently homotopy associative, homotopy commutative and homotopy unital. The coherence theory makes any construction involving this smash product cumbersome.

Recently, Elmendorf et al. [6, 7] managed to incorporate the coherence directly into the definition to come up with a smash product on a sufficiently large category of spectra with much better properties:

2.3. Let $\mathscr{L}(n) = \mathscr{I}(\mathscr{U}^n, \mathscr{U})$ denote the linear isometry operad on $\mathscr{U}$. An *S-module* is a spectrum M with an $\mathscr{L}(1)$-action, i.e. M comes equipped with a map $\xi : \mathscr{L}(1) \ltimes M \to M$ such that

$$\begin{array}{ccc} \{id\} \ltimes M & \longrightarrow & \mathscr{L}(1) \ltimes M \\ & {\scriptstyle \cong} \searrow \quad \swarrow {\scriptstyle \xi} & \\ & M & \end{array}$$

$$\begin{array}{ccccc} \mathscr{L}(1) \ltimes (\mathscr{L}(1) \ltimes M) & \xrightarrow{\cong} & (\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes M & \xrightarrow{\gamma \ltimes M} & \mathscr{L}(1) \ltimes M \\ \downarrow {\scriptstyle \mathscr{L}(1) \ltimes \xi} & & & & \downarrow {\scriptstyle \xi} \\ \mathscr{L}(1) \ltimes M & & \xrightarrow{\qquad \xi \qquad} & & M \end{array}$$

commute (γ is composition in the operad $\mathscr{L}$). A map of S-modules is a map $f : M \to N$ of spectra respecting the $\mathscr{L}(1)$-action.

Recall that the stable categories are obtained by formally inverting weak equivalences, i.e. maps of spectra $f : E \to E'$ for which each $f(V) : E(V) \to E'(V)$ is a weak equivalence.

2.4. Proposition (Elmendorf et al. [6, Theorem 1; 7, Ch. I]). *The category S-$\mathscr{M}od$ of S-modules is complete and cocomplete, with both limits and colimits created in $\mathscr{S}\mathscr{U}$. The forgetful functor S-$\mathscr{M}od \to \mathscr{S}\mathscr{U}$ induces an equivalence of the associated stable categories.*

We need a slightly stronger version of the first part of (2.4) which is implicit in [10]. Let $F : \mathscr{C} \to \mathscr{S}\mathscr{U}$ be a diagram of spectra. The spectrum $\lim F$ is defined by $(\lim F)(V) = \lim F(-)(V)$. The same procedure for colimits produces a prespectrum which we have to spectrify. By [10] spectrification is a continuous functor. Since both functors $\mathscr{T}op_\star(-, Y)$ and $\mathscr{T}op_\star(X, -)$ are continuous, we have

2.5. Proposition. *Let $F : \mathscr{C} \to \mathscr{S}\mathscr{U}$ be a diagram of spectra. Then there are natural homeomorphisms*

$$\mathscr{S}\mathscr{U}(\operatorname{colim} F, E) \cong \lim \mathscr{S}\mathscr{U}(F, E),$$

$$\mathscr{S}\mathscr{U}(E, \lim F) \cong \lim \mathscr{S}\mathscr{U}(E, F).$$

And, by restricting the morphism spaces to their subspaces of maps of S-modules,

2.6. Addendum to (2.4). Let $F : \mathscr{C} \to S\text{-}\mathscr{M}od$ be a diagram of S-modules. Then there are natural homeomorphisms

$$S\text{-}\mathscr{M}od(\operatorname{colim} F, M) \cong \lim S\text{-}\mathscr{M}od(F, M),$$

$$S\text{-}\mathscr{M}od(M, \lim F) \cong \lim S\text{-}\mathscr{M}od(M, F).$$

As an immediate consequence we have

2.7. Proposition. *As a topologically enriched category $\mathscr{S}\mathscr{U}$ contains all indexed limits and colimits.*

Proof. By (2.1) $\mathscr{S}\mathscr{U}$ is tensored by $(E,K) \mapsto E \wedge (K_+)$ and cotensored by $(E,K) \mapsto F(K_+, E)$. Together with (2.5) this proves the claim [9, (3.70)]. □

To obtain the same result for $S\text{-}\mathscr{M}od$ we need to know that it is tensored and cotensored. This is a consequence of the following variant of a result of Linton.

2.8. Lemma. *Let $T : \mathscr{C} \to \mathscr{C}$ be a continuous monad on a topologically enriched category $\mathscr{C}$ and let $\mathscr{C}^T$ be its category of algebras. Then*

(1) the forgetful functor $U : \mathscr{C}^T \to \mathscr{C}$ creates all indexed limits,

(2) if $\mathscr{C}^T$ has continuous coequalizers (in the sense of (2.5), (2.6)) and $\mathscr{C}$ is tensored, then $\mathscr{C}^T$ contains all indexed colimits.

Proof. (1) is the topologically enriched version of [12, VI.2, Excercise 2]. If we denote the free functor $\mathscr{C} \to \mathscr{C}^T$ also by T, then T is left adjoint to U, hence preserves indexed colimits. In particular, $T(C \otimes K) = (TC) \otimes K$ for $K \in \mathscr{T}op$. Let $\mu : T \circ T \to T$ denote the multiplication of T. Then for $X \in \mathscr{C}^T$ with structure map $\xi : TX \to X$

$$TTX \underset{T\xi}{\overset{\mu X}{\rightrightarrows}} TX \xrightarrow{\xi} X$$

is a coequalizer in $\mathscr{C}^T$ by Beck's tripleability theorem [12, VI.7]. Hence the coequalizer of

$$TTX \otimes K \underset{T\xi \otimes K}{\overset{\mu X \otimes K}{\rightrightarrows}} TX \otimes K$$

satisfies the universal property of $X \otimes K$. The result (2) now follows [9, (3.70)]. □

By (2.8) we can apply the proof of the corresponding result from [7, Ch. I] to obtain

2.9. Proposition. *As topologically enriched category $S\text{-}\mathscr{M}od$ contains all indexed limits and colimits, and both are created in $\mathscr{S}\mathscr{U}$.*

For S-modules M and N we define the *smash product over S* to be the coequalizer in $\mathscr{SU}$

$$(\mathscr{L}(2)\times\mathscr{L}(1)\times\mathscr{L}(1))\ltimes(M\wedge N)\ \underset{\mathscr{L}(2)\times\xi_M\wedge\xi_N}{\overset{\gamma\ltimes M\wedge N}{\rightrightarrows}}\ \mathscr{L}(2)\ltimes(M\wedge N)\to M\wedge_S N.$$

2.10. Proposition (Elmendorf et al. [6, Theorem 1; 7, Ch. I]). *The smash product over S is an associative and commutative bifunctor, and there is a natural map $S\wedge_S M\to M$ which is a weak equivalence if M is a CW spectrum and an isomorphism if $M=S$. There is also a functorial function spectrum $F_S(M,P)$ of S-modules and a natural homeomorphism*

$$S\text{-}\mathscr{M}od(M\wedge_S N,P)\cong S\text{-}\mathscr{M}od(N,F_S(M,P)).$$

To obtain a smash product which is also unital Elmendorf et al. consider the category S-$\mathscr{M}od_\star$ of S-modules under S. Its objects are S-modules M with a map of S-modules $\eta:S\to M$. Such an object is called *unital S-module.*

2.11. Proposition. *As a topologically enriched category S-$\mathscr{M}od_\star$ contains all indexed limits and colimits. Ordinary colimits of connected diagrams and all indexed limits are created in $\mathscr{SU}$.*

Proof. $M\mapsto M\vee S$ is a monad on S-$\mathscr{M}od$ whose algebras are precisely the unital S-modules: if M is an algebra with structure map $\xi:M\vee S\to M$ then M is unital via $S\to M\vee S\to M$; conversely, if M is unital, $\eta:S\to M$, an algebra structure map is defined by

$$M\vee S\xrightarrow{id\vee\eta}M\vee M\xrightarrow{fold}M.$$

Hence, the forgetful S-$\mathscr{M}od_\star\to S$-$\mathscr{M}od$ creates all indexed limits. Now let $F:\mathscr{C}\to S$-$\mathscr{M}od_\star$ be a diagram in the usual sense, $\operatorname{col}F$ its colimit in S-$\mathscr{M}od$ and $\operatorname{col}S$ the colimit in S-$\mathscr{M}od$ of the constant $\mathscr{C}$-diagram on S. Then $\operatorname{colim}F$ in S-$\mathscr{M}od_\star$ is the pushout in S-$\mathscr{M}od$ of

$$\begin{array}{ccc}\operatorname{col}S & \longrightarrow & S\\ \downarrow & & \\ \operatorname{col}F & & \end{array}$$

Hence, S-$\mathscr{M}od$ has arbitrary indexed colimits by (2.8). If $\mathscr{C}$ is connected $S=\operatorname{col}S$ and hence $\operatorname{col}F=\operatorname{colim}F$. $\square$

2.12. Definition (*Elmendorf et al.* [6, *Section* 2; 7, *Chap.* 1]). Let M and N be unital S-modules. The *reduced smash product* $M\star N$ is the pushout in S-$\mathscr{M}od$

$$\begin{array}{ccc}
(M \wedge_S S) \vee (S \wedge_S N) & \longrightarrow & M \wedge_S N \\
\Big\downarrow & & \Big\downarrow \\
M \vee N & \longrightarrow & M \star N
\end{array}$$

with the structure map $S \cong S \wedge_S S \to M \wedge_S N \to M \star N$.

2.13. Proposition (Elmendorf et al. [6, Section 2]). *S-$\mathcal{M}od_\star$ is a symmetric monoidal category with tensor product $\star$ and unit S. Its monoids and commutative monoids are precisely the A_∞ and E_∞ ring spectra, respectively.*

Although $M\star -$ does not preserve sums in the category S-$\mathcal{M}od_\star$ we note for later use.

2.14. Lemma. *$M\star -$ preserves colimits of connected diagrams in S-$\mathcal{M}od_\star$.*

Proof. By (2.9) and (2.11) colimits of connected diagrams are created in S-$\mathcal{M}od$. For an S-module K the functor $K \vee - : S\text{-}\mathcal{M}od \to S\text{-}\mathcal{M}od$ preserves colimits of connected diagrams while $K \wedge_S - : S\text{-}\mathcal{M}od \to S\text{-}\mathcal{M}od$ preserves all colimits because it is a left adjoint (2.10). The result follows since colimits commute with pushouts. □

3. Ring spectra and R-modules

Before we start with the proof of Theorem A we need to consider R-modules.

3.1. Definition. Let R be an A_∞ or E_∞ ring spectrum. A *unital* (*left*) *R-module* is a unital S-module M with a structure map $\xi : R \star M \to M$ in S-$Mod_\star$ such that

$$\begin{array}{ccc}
R \star R \star M & \xrightarrow{\mu\star M} & R \star M \\
\Big\downarrow{\scriptstyle R\star\xi} & & \Big\downarrow{\scriptstyle \xi} \\
R \star M & \xrightarrow{\xi} & M
\end{array}
\qquad
\begin{array}{ccc}
S \star M & \xrightarrow{\eta\star M} & R \star M \\
& \searrow{\scriptstyle \cong} & \Big\downarrow{\scriptstyle \xi} \\
& & M
\end{array}$$

commute, where $\mu : R \star R \to R$ is the multiplication and $\eta : S \to R$ the unit. A map of unital R-modules is a map of unital S-modules respecting the R-structure.

3.2. Proposition. *The category R-$\mathcal{M}od_\star$ of unital R-modules is topologically enriched and contains all indexed limits and colimits. Colimits of connected diagrams and arbitrary indexed limits are created in S-$\mathcal{M}od_\star$.*

Proof. $M \mapsto R \star M$ is a continuous monad on S-$\mathcal{M}od_\star$ with unit $M \cong S \star M \to R \star M$ whose algebras are by definition precisely the unital R-modules (the module structure maps are the algebra structure maps of the monad and vice versa). Hence, the forgetful $R\text{-}\mathcal{M}od_\star \to S\text{-}\mathcal{M}od_\star$ creates all indexed limits (Lemma 2.8(1)). Now let $F:\mathcal{C}\to R\text{-}\mathcal{M}od_\star$ be a connected diagram and $Q\in S\text{-}\mathcal{M}od_\star$ its colimit in $S\text{-}\mathcal{M}od_\star$. Since $R\star$ — preserves colimits of connected diagrams in $S\text{-}\mathcal{M}od_\star$ (Lemma 2.14) Q is a unital R-module with structure map

$$\xi_Q : R\star Q = \operatorname{colim}(R\star F)\to \operatorname{colim} F = Q.$$

It is easy to check that Q has the topologized universal property of a colimit in $R\text{-}\mathcal{M}od_\star$. Since coequalizers are colimits of connected diagrams the result follows from Propositions 2.8 and 2.11. □

Let $\mathscr{E}_\infty$ be the category of E_∞ ring spectra and their homomorphisms. As always before we topologically enrich $\mathscr{E}_\infty$ by giving $\mathscr{E}_\infty(E,E')$ the subspace topology from $\mathscr{S}\mathscr{U}(E,E')$.

3.3. Proposition (Elmendorf et al. [6, Proposition 2]). *$E\star E'$ is the sum of E and E' in $\mathscr{E}_\infty$, and the folding map $E\star E\to E$ is the multiplication of E.*

3.4. Proof of Theorem A. Let $L:\mathscr{S}\mathscr{U}\to\mathscr{S}\mathscr{U}$ be the monad associated with the linear isometry operad $\mathscr{L}$. Then L is continuous and $\mathscr{E}_\infty$ its category of algebras [10, VIII. 3]. Hence, the forgetful $\mathscr{E}_\infty\to\mathscr{S}\mathscr{U}$ creates all indexed limits (Lemma 2.8(1)). To show the existence of indexed colimits we have to estabish the existence of continuous coequalizers (Lemma 2.8(2)). So given

$$Q \underset{g}{\overset{f}{\rightrightarrows}} R$$

in $\mathscr{E}_\infty$. Let f_1 denote the composite

$$f_1 : R\star Q \xrightarrow{R\star f} R\star R \xrightarrow{\mu_R} R,$$

where μ_R is the multiplication of R. Then f_1 is a homomorphism of R-modules. Take the coequalizer T in $R\text{-}\mathcal{M}od_\star$

$$(3.5)\quad R\star Q \underset{g_1}{\overset{f_1}{\rightrightarrows}} R \xrightarrow{t} T.$$

We show that T is the coequalizer of f and g in $\mathscr{E}_\infty$. Since $T\star$ – preserves coequalizers in $R\text{-}\mathcal{M}od_\star$ the spectrum $T\star T$ is the colimit in $R\text{-}\mathcal{M}od_\star$ of

$$(3.6)\quad \begin{array}{ccc} R\star Q\star R\star Q & \underset{g_1\star R\star Q}{\overset{f_1\star R\star Q}{\rightrightarrows}} & R\star R\star Q \\ {\scriptstyle R\star Q\star f_1}\downarrow\downarrow{\scriptstyle R\star Q\star g_1} & & {\scriptstyle R\star f_1}\downarrow\downarrow{\scriptstyle R\star g_1} \\ R\star Q\star R & \underset{g_1\star R}{\overset{f_1\star R}{\rightrightarrows}} & R\star R \end{array}$$

and the colimits structure maps are induced from

$$t{\star}t : R{\star}R \to T{\star}T.$$

Since R is a commutative monoid in $\mathscr{E}_\infty$ the following diagram commutes (τ is the interchange):

$$\begin{array}{ccccc}
R{\star}Q{\star}R{\star}Q & \xrightarrow{R{\star}f{\star}R{\star}f} & R{\star}R{\star}R{\star}R & \xrightarrow{\mu_R{\star}\mu_R} & R{\star}R \\
\cong \downarrow R{\star}\tau{\star}R & & \cong \downarrow R{\star}\tau{\star}R & & \\
R{\star}R{\star}Q{\star}Q & \xrightarrow{R{\star}R{\star}f{\star}f} & R{\star}R{\star}R{\star}R & & \downarrow \mu_R \\
\downarrow \mu_R{\star}\mu_Q & & \downarrow \mu_R{\star}\mu_R & & \\
R{\star}Q & \xrightarrow{R{\star}f} & R{\star}R & \xrightarrow{\mu_R} & R
\end{array}$$

Hence, f_1 and correspondingly g_1 are E_∞ homomorphisms, and (3.6) is a diagram in $\mathscr{E}_\infty$. The folding maps, which define the various multiplications, induce a map of diagram (3.6) to T, giving rise to an R-module homomorphism $\mu_T : T{\star}T \to T$ satisfying

$$\mu_T \circ (t{\star}t) = t \circ \mu_R.$$

In particular, T is an E_∞ ring spectrum and t an E_∞ homomorphism. Now given an E_∞ homomorphism $h : R \to U$ such that $h \circ f = h \circ g$ then

$$h \circ \mu_R \circ (R{\star}f) = \mu_U \circ (h{\star}h) \circ (R{\star}f) = \mu_U \circ (h{\star}(h \circ f)) = h \circ \mu_R \circ (R{\star}g).$$

Hence, there is a unique R-module homomorphism

$$q : T \to U$$

such that $q \circ t = h$. Since $h{\star}h$ induces a map of (3.6) to $U{\star}U$, it follows that q is an E_∞ homomorphism. Conversely, given an E_∞ homomorphism $q : T \to U$ such that $q \circ t \circ f_1 = q \circ t \circ f_2$ as R-module homomorphisms then they agree as E_∞ homomorphisms, and hence $q \circ t \circ f = q \circ t \circ g$ because $R{\star}Q$ is the sum in $\mathscr{E}_\infty$ and μ_R is the folding map. Hence, the natural homeomorphism

$$\text{Equalizer } (R\text{-}\mathscr{M}od_\star(R, U) \underset{g_1^\star}{\overset{f_1^\star}{\rightrightarrows}} R\text{-}\mathscr{M}od_\star(R{\star}Q, U)) \cong R\text{-}\mathscr{M}od_\star(T, U)$$

restricts to the subspaces of E_∞ homomorphisms

$$\text{Equalizer } (\mathscr{E}_\infty(R, U) \underset{g^\star}{\overset{f^\star}{\rightrightarrows}} \mathscr{E}_\infty(Q, U)) \cong \mathscr{E}_\infty(T, U). \qquad \square$$

4. Topological Hochschild homology

4.1. Definition. Let R be an A_∞ ring spectrum and M be an R-bimodule (defined analogously to (3.1)). The *topological Hochschild homology* $THH(R;M)$ of R with coefficients in M is the topological realization in $\mathscr{S}\mathscr{U}$ of the simplicial spectrum $THH(R;M)_\bullet$

$$[n] \mapsto R\star\cdots\star R\star M = R^{\star n}\star M$$

(n copies of R) with the usual Hochschild structure maps

$$d^0 : R^{\star n}\star M \xrightarrow{R^{\star n-1}\star\xi_l} R^{\star n-1}\star M,$$

$$d^i : R^{\star n}\star M \xrightarrow{R^{\star n-i-1}\star\mu\star R^{\star i-1}} R^{\star n-1}\star M, \quad 0<i<n,$$

$$d^n : R^{\star n}\star M \xrightarrow{(R^{\star n-1}\star\xi_r)\circ\tau} R^{\star n-1}\star M,$$

$$s^i : R^{\star n}\star M \xrightarrow{R^{\star n-i}\star\eta\star R^i\star M} R^{\star n+1}\star M, \quad 0\le i\le n,$$

where $\mu : R\star R\to R$ is the multiplication, $\eta : S\to R$ the unit, ξ_l and ξ_r the left and right actions of R on M and τ the cyclic permutation

$$\tau : R\star\cdots\star R\star M \to R\star\cdots\star R\star M\star R.$$

If $M=R$ we write $THH(R)$ for $THH(R;R)$.

The topological realization of a simplicial spectrum $E_\bullet$ in $\mathscr{S}\mathscr{U}$ is the coend in $\mathscr{S}\mathscr{U}$ of the functor

$$\triangle^{op}\times\triangle \to \mathscr{S}\mathscr{U}, \qquad ([m],[n])\mapsto E_m\wedge\triangle(n)_+,$$

where $\triangle(n)$ is the standard n-simplex [5]. Since geometric realization commutes with any monad in $\mathscr{S}\mathscr{U}$ (cf. [13, Theorem 12.2] for the space-level version; the spectrum-level argument is similar, cf. [7, Ch. VI]), the geometric realization of a simplicial E_∞ ring is again an E_∞ ring. In particular, if R is an E_∞ ring and hence a commutative monoid in S-$\mathscr{M}od_\star$ the simplicial spectrum $THH(R)_\bullet$ lives in $\mathscr{E}_\infty$ so that $THH(R)\in\mathscr{E}_\infty$. Hence, we have an endofunctor

$$THH : \mathscr{E}_\infty \to \mathscr{E}_\infty.$$

Let $S^1_\bullet = \triangle(1)_\bullet/\partial\triangle(1)_\bullet$ denote the simplicial 1-sphere and $f_i^n : [n]\to[1]$ the map determined by $(f_i^n)^{-1}(1) = \{i,\dots,n\}$. Then

$$\triangle([n],[1]) = \{f_0^n, f_1^n, \dots, f_{n+1}^n\}.$$

The boundary and degeneracies are

$$d^j f_i^n = \begin{cases} f_{i-1}^{n+1} & \text{for } j<i, \\ f_i^{n+1} & \text{for } j \geq i, \end{cases}$$

$$s^j f_i^n = \begin{cases} f_{i-1}^{n+1} & \text{for } j<i, \\ f_i^{n+1} & \text{for } j \geq i. \end{cases}$$

Since $S_n^1 = \triangle([n],[1])/(f_0^n \sim f_{n+1}^n)$, and $\star$ is the coproduct in $\mathscr{E}_\infty$, we have the ring spectrum version of a well-known fact of classical Hochschild homology (e.g. see [11, Section 3]).

4.2. If $R \otimes S_\bullet^1$ denotes the simplicial object $[n] \to R \otimes S_n^1$ in $\mathscr{E}_\infty$ then $THH(R)_\bullet = R \otimes S_\bullet^1$.

Theorem B is a consequence of

4.3. Proposition. *For any $\mathscr{E}_\infty$ ring R the following diagram commutes up to natural isomorphism*

$$\begin{array}{ccc} \mathscr{T}op^{\triangle^{op}} & \xrightarrow{(R\otimes -)^{\triangle^{op}}} & \mathscr{E}_\infty^{\triangle^{op}} \\ \Big\downarrow T_1 & & \Big\downarrow T_2 \\ \mathscr{T}op & \xrightarrow{(R\otimes -)} & \mathscr{E}_\infty \end{array}$$

where we let T_i denote the respective topological realization functors.

Proof. Consider the diagram

$$\begin{array}{ccc} \mathscr{E}_\infty & \xrightarrow{\mathscr{E}_\infty(R,-)} & \mathscr{T}op \\ \Big\downarrow S_2 & & \Big\downarrow S_1 \\ \mathscr{E}_\infty^{\triangle^{op}} & \xrightarrow{\mathscr{E}_\infty(R,-)^{\triangle^{op}}} & \mathscr{T}op^{\triangle^{op}} \end{array}$$

where S_1 is the topologized singular functor right adjoint to T_1 and

$$S_2 : \mathscr{E}_\infty \to \mathscr{E}_\infty^{\triangle^{op}}, \qquad Q \mapsto ([n] \mapsto F(\triangle(n)_+, Q))$$

is the spectrum singular functor right adjoint to T_2. Since $\mathscr{E}_\infty(R,-)$ is right adjoint to $R \otimes$ —, the result now follows from

Claim. *There is a natural isomorphism*

$$\mathscr{E}_\infty(R,-)^{\triangle^{op}} \circ S_2 \cong S_1 \circ \mathscr{E}_\infty(R,-).$$

The left side sends an E_∞ ring Q to

$$[n] \mapsto \mathscr{E}_\infty(R, F(\triangle(n)_+, Q)).$$

The right side sends Q to

$$[n] \mapsto \mathscr{T}op(\triangle(n), \mathscr{E}_\infty(R,Q)).$$

The required natural isomorphism is the adjointness homeomorphism

$$\mathscr{E}_\infty(R, F(\triangle(n)_+, Q)) \cong \mathscr{T}op(\triangle(n), \mathscr{E}_\infty(R,Q)).$$

It remains to prove that the multiplication of $THH(R)$ is induced by the folding map $S^1 \sqcup S^1 \to S^1$. Since

$$R \otimes - : \mathscr{T}op \to \mathscr{E}_\infty \quad \text{and} \quad - \otimes S^1 : \mathscr{E}_\infty \to \mathscr{E}_\infty$$

are left adjoints they preserve sums. Hence there are canonical E_∞ isomorphisms

$$(R \star R) \otimes S^1 \cong THH(R) \star THH(R) \cong R \star (S^1 \sqcup S^1)$$

and the multiplication on $THH(R)$ is defined by either folding map

$$R \star R \to R \text{ or } S^1 \sqcup S^1 \to S^1 \qquad \square$$

4.4. There is also an internal realization

$$T_3 : \mathscr{E}_\infty^{\triangle^{op}} \to \mathscr{E}_\infty$$

sending a simplicial E_∞ ring spectrum $R_\bullet$ to the coend of the functor

$$\triangle^{op} \times \triangle \to \mathscr{E}_\infty, \qquad ([m],[n]) \mapsto R_m \otimes \triangle(n).$$

in $\mathscr{E}_\infty$. Applying the universal properties of the coend construction and of the tensor in $\mathscr{E}_\infty$ it is easy to see that the singular functor S_2 is right adjoint to T_3. Hence T_2 and T_3 are naturally isomorphic.

4.5. Proposition. *The topological realization in the sense of* [5] *of a simplicial E_∞ ring spectrum $R_\bullet$ is naturally isomorphic in $\mathscr{E}_\infty$ to the internal realization in the sense of* (4.4). *In particular, $THH(R)$ is the coend in $\mathscr{E}_\infty$ of*

$$\triangle^{op} \times \triangle \to \mathscr{E}_\infty, \qquad ([m],[n]) \mapsto THH(R)_m \otimes \triangle(n).$$

5. Maps of Bökstedt type

Let X be a topological space and R and T be E_∞ ring spectra. Since $R \otimes X$ and $R \wedge X_+$ are the X-parametrized coproducts of copies of R in $\mathscr{E}_\infty$ and $\mathscr{S}\mathscr{U}$, respectively, there are X-parametrized families of natural E_∞ inclusions $\{i_x : R \to R \otimes X\}$, respectively, spectrum level maps $\{j_x : R \to R \wedge X_+\}$ having the following universal property: given an X-parametrized family of E_∞ homomorphisms $f_x : R \to T$, respectively, an X-parametrized family of maps of spectra $g_x : R \to E$ there is a unique E_∞ homomorphisms $f : R \otimes X \to T$ such that $f \circ i_x = f_x$ for each $x \in X$, respectively, a unique map of spectra $g : R \wedge X_+ \to E$ such that $g \circ j_x = g_x$ for each $x \in X$.

If we take $f_x = id_R$ for all $x \in X$ we obtain an E_∞ homomorphisms $\rho_X : R \otimes X \to R$ such that $\rho_X \circ i_x = id_R$.

Considering the $i_x : R \to R \otimes X$ as maps of spectra we obtain an induced map

$$\lambda(R,X) : R \wedge X_+ \to R \otimes X$$

satisfying $\lambda(R,X) \circ j_x = i_x$, $x \in X$. Since i_x admits a retraction, we have

5.1. If R is not contractible and X is not empty $\lambda(R,X)$ is essential.

The universal property also implies

5.2. The following diagram commutes:

$$\begin{array}{ccccc}
(R \wedge X_+) \wedge Y_+ & \xrightarrow{\lambda(R,X)\wedge Y_+} & (R \otimes X) \wedge Y_+ & \xrightarrow{\lambda(R\otimes X,Y)} & (R \otimes X) \otimes Y \\
\downarrow{\scriptstyle\cong} & & & & \downarrow{\scriptstyle\cong} \\
R \wedge (X \times Y)^+ & & \xrightarrow{\lambda(R,X\times Y)} & & R \otimes (X \times Y)
\end{array}$$

Specializing to $X = Y = S^1$ we obtain a commutative diagram

$$\begin{array}{ccccc}
(R \wedge S^1_+) \wedge S^1_+ & \xrightarrow{\cong} & R \wedge (S^1 \times S^1)_+ & \xrightarrow{R\wedge\gamma_+} & R \wedge S^1_+ \\
\downarrow{\scriptstyle\lambda(R,S^1)\wedge S^1_+} & & \downarrow{\scriptstyle\lambda(R,S^1\times S^1)} & & \downarrow{\scriptstyle\lambda(R,S^1)} \\
(R \otimes S^1) \wedge S^1_+ & \xrightarrow{\lambda(R\otimes S^1,S^1)} (R \otimes S^1) \otimes S^1 \xrightarrow{\cong} & R \otimes (S^1 \times S^1) & \xrightarrow{\lambda(R\otimes\gamma)} & R \otimes S^1
\end{array}$$

where γ is the multiplication of S^1. Hence,

5.3. $\lambda(R,S^1) : R \wedge S^1_+ \to R \otimes S^1$ is S^1-equivariant.

Given a map $f:R\wedge X_+\to T$ of spectra such that each $f_x=f\circ j_x:R\to T$ is an E_∞ homomorphism, the f_x induce a unique E_∞ homomorphism $\hat{f}:R\otimes X\to T$ such that $\hat{f}\circ i_x=f_x$ for all $x\in X$. Since $f_x=\hat{f}\circ i_x=\hat{f}\circ\lambda(R,X)\circ j_x$ and $f_x=f\circ j_x$ we conclude

5.4. Let X be a topological space and R and T E_∞ ring spectra. Given a map of spectra $f:R\wedge X_+\to T$ such that each $f\circ j_x$ is an E_∞ homomorphism there exists a unique E_∞ homomorphism $\hat{f}:R\otimes X\to R$ such that $\hat{f}\circ\lambda(R,X)=f$.

5.5. For each $x\in X$ we have an R-module structure on $R\otimes X$ defined by

$$R*(R\otimes X)\xrightarrow{i_x*id}(R\otimes X)*(R\otimes X)\xrightarrow{multipl}R\otimes X.$$

Clearly, this defines an R-algebra structure on $R\otimes X$ turning i_x into an R-algebra homomorphism. This proves the first part of Theorem H. The second part is a well-known consequence of the first part [1, II.6.1]. It is straightforward to show that this structure is equivalent to an E_∞ algebra structure as defined in Definition 6.6 below.

6. Extensions of the result and final remarks

6.1. Definition. Let R be an A_∞ or E_∞ ring spectrum, K a unital right R-module and L a unital left R-module with structure maps ξ_K and ξ_L. The *reduced smash product over R of K and L* is the coequalizer in $S\text{-}\mathcal{M}od_\star$

$$K\star R\star L\underset{\xi_L}{\overset{\xi_K}{\rightrightarrows}}K\star L\xrightarrow{\eta}K\star_R L.$$

6.2. If K is a unital Q-R-bimodule, $K\star_R L$ is a unital Q-module since $Q\star$-preserves coequalizers by Lemma 2.14.

6.3. If R is an E_∞ ring spectrum any unital R-module is an R-bimodule in the obvious way and $\star_R$ defines a bifunctor

$$R\text{-}\mathcal{M}od_\star\times R\text{-}\mathcal{M}od_\star\to R\text{-}\mathcal{M}od_\star$$

6.4. Proposition. Let Q and R be A_∞ or E_∞ ring spectra, K a unital right Q-module, L a unital Q-R-bimodule and M a unital left R-module. Then there are natural isomorphisms

(1) $(K\star_Q L)\star_R M\cong K\star_Q(L\star_R M)$,
(2) $R\star_R M\cong M$,
(3) if R is an E_∞ ring spectrum, then $L\star_R M\cong M\star_R L$.

Proof. (1) follows from the fact that $K \star_Q -$ preserves coequalizers and that two coequalizers commute. (3) is trivial, and (2) holds because

$$R \star R \star M \underset{id \star \xi}{\overset{\mu \star id}{\rightrightarrows}} R \star M \xrightarrow{\xi} M$$

is a coequalizer by [12, VI. 7]. □

6.5. Corollary. *If R is an E_∞ ring spectrum R-$\mathcal{M}od_\star$ is a symmetric monoidal category with $\star_R$ as tensor product and R as unit.*

6.6. Definition. Let R be an E_∞ ring spectrum. An (E_∞, respectively) A_∞ *R-algebra* is a (commutative) monoid in the symmetric monoidal category R-$\mathcal{M}od_\star$.

We are interested in the commutative case. Let E_R-$\mathcal{A}lg$ denote the category of E_∞ R-algebras and R-algebra homomorphisms. We topologically enrich E_R-$\mathcal{A}lg$ in the obvious way so that the forgetful

$$U : E_R\text{-}\mathcal{A}lg \to R\text{-}\mathcal{M}od_\star$$

is continuous. U has a continuous left adjoint

$$F : R\text{-}\mathcal{M}od_\star \to E_R\text{-}\mathcal{A}lg$$

defined as follows: let $\mathcal{S}$ denote the category of finite sets $\underline{n} = \{1,\dots,n\}$ $n \geq 0$ with $\underline{0} = \phi$ and injections. For each $M \in R$-$\mathcal{M}od_\star$ define a functor

$$F_M : \mathcal{S} \to R\text{-}\mathcal{M}od_\star$$

by $F_M(\underline{n}) = M^{\star_R n}$ with $M^{\star_R 0} = R$ and for each ordered injection $f : \underline{m} \to \underline{n}$ by

$$F_M(f) : M^{\star_R m} \cong N_1 \star_R N_2 \star_R \cdots \star_R N_n \xrightarrow{\alpha} M^{\star_R n},$$

where $N_k = M$ if $k \in im(f)$ and $N_k = R$ otherwise. α is defined by the identity on M and by the unital structure on M

$$R \cong R \star S \longrightarrow R \star M \xrightarrow{\xi_M} M$$

F_M applied to a permutation is the obvious morphism. We define

$$F(M) = \operatorname{colim} F_M$$

in the category R-$\mathcal{M}od_\star$. Since F_M is a connected diagram this colimit is created in S-$\mathcal{M}od$, and since $M \star_R -$ preserves colimits of connected diagrams the concatenation of finite sets induces a multiplication on $F(M)$ making it a commutative monoid in R-$\mathcal{A}lg$. Extending the proofs of Section 3 to R-algebras and modules over R-algebras we obtain

6.7. Theorem. *E_R-$\mathscr{A}lg$ is topologically enriched and contains all indexed limits and colimits. All indexed limits are created in $\mathscr{S}\mathscr{U}$.*

Next let R be an E_∞ ring spectrum, A a commutative R-algebra and M an A-module.

6.8. Definition. The *topological Hochschild homology $THH_R(A;M)$* of A with coefficients in M is the topological realization in $\mathscr{S}\mathscr{U}$ of the simplicial R-module

$$[n] \mapsto A^{\star_R n} \star_R M$$

with the Hochschild structure maps.

As in Section 4 we obtain

6.9. Theorem. *Let R be an E_∞ ring spectrum and A an E_∞ R-algebra. Then there is a natural isomorphism in E_R-$\mathscr{A}lg$*

$$THH_R(A) \cong A \otimes S^1$$

and the multiplication of $THH_R(A)$ is induced by the folding map $S^1 \sqcup S^1 \to S^1$.

Elmendorf et al. [7] are contemplating another variant of a category of S-modules [8]. To distinguish it from Definition 2.3 we call its objects strong S-modules.

6.10. Definition (Elmendorf and May [8]). The category *sS-$\mathscr{M}od$* of *strong S-modules* is the subcategory of S-$\mathscr{M}od$ of all objects M for which the map $S \wedge_S M \to M$ is an isomorphism.

sS-$\mathscr{M}od$ is a coreflective subcategory of S-$\mathscr{M}od$ in the sense of [12, IV. 3] with the continuous coreflector

$$S\text{-}\mathscr{M}od \to sS\text{-}\mathscr{M}od, \quad R \mapsto S \wedge_S R.$$

Hence, [12, IV. Excercise 3.7].

6.11. Proposition. *As a topologically enriched category sS-$\mathscr{M}od$ contains all indexed colimits and they are created in S-$\mathscr{M}od$. It also contains all indexed limits which are obtained from the indexed limits in S-$\mathscr{M}od$ by applying the coreflector.*

Note that for strong unital S-modules $\wedge_S$ coincides with $\star$ so that the category S-$\mathscr{M}od_\star$ is redundant: we can work in sS-$\mathscr{M}od$. $\mathscr{E}_\infty$ has to be replaced by $s\mathscr{E}_\infty$, the subcategory of E_∞ ring spectra which are strong S-modules. If R is an E_∞ ring spectrum then $S \wedge_S R$ is an E_∞ ring spectrum in $s\mathscr{E}_\infty$ which is weakly equivalent to R. Hence, we may restrict our attention to $s\mathscr{E}_\infty$. Also observe that $s\mathscr{E}_\infty$ has the same indexed colimits as $\mathscr{E}_\infty$. If we now replace $\star$ by $\wedge_S$ throughout Sections 3 and 4 and the singular functor $S_2 : \mathscr{E}_\infty \to \mathscr{E}_\infty^{\Delta op}$ by the functor

$$s\mathscr{E}_\infty \to s\mathscr{E}_\infty^{\Delta op}, \quad Q \mapsto ([n] \mapsto S \wedge_S F(\Delta(n)_+, Q))$$

all proofs go through in this setting.

A final word to the notion of unital R-module. In [7, Ch. II] Elmendorf et al. showed that A_∞ and E_∞ ring spectra R are S-modules with structure maps $\eta : S \to R$ and $\mu : R \wedge_S R \to R$ such that μ is associative (and commutative in the E_∞ case) and that

$$\begin{array}{ccccc} S \wedge_S R & \xrightarrow{\eta \wedge id} & R \wedge_S R & \xleftarrow{id \wedge \eta} & R \wedge_S S \\ & \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\ & & R & & \end{array}$$

commutes. A module over an A_∞ or E_∞ ring spectrum R is an S-module M with a structure map $\xi : R \wedge_S M \to M$ satisfying the obvious associativity and unit conditions. Let R-$\mathscr{M}od$ denote the category of such R-modules and R-module homomorphisms. It is not difficult to show that our category R-$\mathscr{M}od_\star$ of unital R-modules is isomorphic to the category R-$\mathscr{M}od$ under R.

Acknowledgements

We thank A. Elmendorf, I. Kriz and P. May for keeping us informed about the progress of their work and the DFG for support during the preparation of this paper.

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