

UNIFORM TWISTED HOMOLOGICAL STABILITY

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Abstract. We prove a homological stability theorem for families of discrete groups (e.g. mapping class groups, automorphism groups of free groups, braid groups) with coefficients in a sequence of irreducible algebraic representations of arithmetic groups. The novelty is that the stable range is independent of the choice of representation. Combined with earlier work of Bergström–Diaconu–Petersen–Westerland this proves the Conrey–Farmer–Keating–Rubinstein–Snaith predictions for all moments of the family of quadratic L -functions over function fields, for sufficiently large odd prime powers.

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1. Introduction

1.1. **Four instances of the main theorem.** Recent work of [BDPW23] investigated connections between moments of families of quadratic L -functions over rational function fields, and stable homology of braid groups with twisted coefficients, proving in particular that an improved range of homological stability for the braid groups with these coefficients would imply a certain asymptotic formula for all moments. The goal of this paper is to prove such an improved stable range. In fact, the main theorem we prove here

will be applicable to many families of groups and coefficients. But rather than try to state the abstract form of the theorem in the introduction, we will give four important special cases. We emphasize that the hypotheses of the theorem are such that homological stability with twisted coefficients is already known in all cases. What is new is the *shape* of the stable range.

It will be convenient to formulate our homological stability theorems in terms of vanishing of relative homology. Recall that if $H \rightarrow G$ is a homomorphism, then vanishing of the relative homology group $H_d(G, H; \mathbb{Z})$ for all $d \leq n$ means exactly that $H_n(H; \mathbb{Z}) \rightarrow H_n(G; \mathbb{Z})$ is surjective, and $H_d(H; \mathbb{Z}) \rightarrow H_d(G; \mathbb{Z})$ is an isomorphism for $d < n$. The analogous statement is also true for homology with other coefficients.

1.1.1. Mapping class groups. Let Mod_g^1 be the mapping class group of an oriented genus g surface with a boundary component, and consider the natural map $\text{Mod}_g^1 \rightarrow \text{Sp}_{2g}(\mathbb{Z})$. Irreducible algebraic representations of the symplectic group Sp_{2g} are indexed by their highest weight, which is a partition λ of length $l(\lambda) \leq g$. For a partition λ , let $V_\lambda(g)$ be the irreducible rational representation of Sp_{2g} of highest weight λ , if $l(\lambda) \leq g$, and set $V_\lambda(g) = 0$ if $l(\lambda) > g$. We have a stabilisation map $\text{Mod}_{g-1}^1 \rightarrow \text{Mod}_g^1$ given by gluing on a torus.

Theorem 1.1. *The relative homology group $H_d(\text{Mod}_g^1, \text{Mod}_{g-1}^1; V_\lambda(g), V_\lambda(g-1))$ vanishes for $d < \frac{1}{3}g$.*

The best stable range known previously for these coefficients is that the above group vanishes for $d < \frac{2}{3}(g - |\lambda|)$ [GKRW19, Example 5.20]. This is of course better if $|\lambda| \ll g$, but in the regime $|\lambda| \gg g$ all previously known bounds gave no stable range at all.

1.1.2. Automorphism groups of free groups. Let $\text{Aut}(F_n)$ be the automorphism group of a free group on n generators, and consider the natural map $\text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z})$. We index irreducible algebraic representations of GL_n by pairs of partitions (λ, μ) such that $l(\lambda) + l(\mu) \leq n$. Let $V_{\lambda, \mu}(n)$ be the irreducible rational representation of GL_n associated to (λ, μ) if $l(\lambda) + l(\mu) \leq n$, and set $V_{\lambda, \mu}(n) = 0$ otherwise.

Theorem 1.2. *The relative homology group $H_d(\text{Aut}(F_n), \text{Aut}(F_{n-1}); V_{\lambda, \mu}(n), V_{\lambda, \mu}(n-1))$ vanishes for $d < \frac{1}{3}(n-1)$.*

The best range previously in the literature is that the above group vanishes for $d < \frac{1}{2}(n-1 - |\lambda| - |\mu|)$ [RWW17, Theorem 5.4].¹

1.1.3. Handlebody mapping class groups. Let HMod_g^1 be the mapping class group of a genus g handlebody with a marked disk on its boundary. There is a natural map $\text{HMod}_g^1 \rightarrow \text{Aut}(F_g)$, since $\text{Aut}(F_g)$ is the homotopy automorphism group of the handlebody. We may thus pull back the coefficients $V_{\lambda, \mu}(g)$ considered above to HMod_g^1 .

Theorem 1.3. *The relative homology group $H_d(\text{HMod}_g^1, \text{HMod}_{g-1}^1; V_{\lambda, \mu}(g), V_{\lambda, \mu}(g-1))$ vanishes for $d < \frac{1}{4}(g-1)$.*

The best previously known vanishing range is $d < \frac{1}{2}(g-1 - |\lambda| - |\mu|)$ [RWW17, Theorem 5.31].

¹This can in fact be improved to $d < \frac{4}{5}(n - |\lambda| - |\mu|)$. From [MPP19, Theorem 4.8] such a result would follow from a good enough vanishing range of relative homology with constant coefficients. The required stable range with rational coefficients follows from [HV98b, Proposition 1.2] combined with Galatius' theorem [Gal11] which tautologically gives surjectivity of the stabilisation map one degree above. Moreover, it is a folk theorem that the slope $\frac{4}{5}$ of [HV98b] can be improved to at least $\frac{7}{8}$.

1.1.4. *Braid groups and the Burau representation.* Let β_n be the braid group on n strands, and let $\beta_n \rightarrow \mathrm{Sp}_{n-1}(\mathbb{Z})$ be the *integral Burau representation*. The integral Burau representation of the braid group on an even number of strands takes values in the “odd symplectic groups” of Gelfand–Zelevinsky [GZ85], as we elaborate on in the body of the paper. The coefficient systems V_λ of the usual symplectic groups can be extended naturally to the odd symplectic groups, too.

Theorem 1.4. *The relative homology group $H_d(\beta_n, \beta_{n-1}; V_\lambda(n), V_\lambda(n-1))$ vanishes for $d < \frac{1}{12}(n-1)$.*

The best range in the existing literature is $d < \frac{1}{2}(n - |\lambda|)$ [RWW17, Theorem 5.22].² We expect the optimal slope in Theorem 1.4 to be exactly $\frac{1}{4}$.

The proofs of Theorems 1.1, 1.2 and 1.3 all use as input existing results in the literature establishing *high connectivity* of certain simplicial complexes associated to the respective families of groups. By contrast, the simplicial complexes relevant for Theorem 1.4 were not previously known to be highly connected, and we show this in Section 3 of this paper (with a probably-not-optimal range of connectivity). These complexes are associated to the sequence of congruence subgroups of the even/odd symplectic groups, given by the image of the integral Burau representation. High connectivity of the complexes associated to the even/odd symplectic groups themselves was recently established by Sierra–Wahl [SW24]. The idea of studying the system of all even/odd symplectic groups together for homological stability is an algebraic analogue of the set-up of Harr–Vistrup–Wahl [HVV22].

1.2. **Applications to moments of families of L -functions.** Theorem 1.4 has important implications in arithmetic statistics, in light of recent work of [BDPW23]. Let q be an odd prime power. For monic and square-free $d \in \mathbb{F}_q[t]$, let $L(s, \chi_d)$ denote the L -function associated to the Galois representation given by the first cohomology group of the affine hyperelliptic curve with equation $y^2 = d(x)$. Theorem 1.4, combined with the results of [BDPW23], implies the following:

Theorem 1.5. *For each $r \geq 1$, there is an explicit polynomial Q_r of degree $r(r+1)/2$ such that*

$$q^{-2g-1} \sum_{\substack{d \in \mathbb{F}_q[t] \\ \text{monic, squarefree} \\ \deg(d)=2g+1}} L(\tfrac{1}{2}, \chi_d)^r = Q_r(2g+1) + \mathcal{O}(4^{g(r+1)} q^{-(g+6)/12}).$$

Indeed, [BDPW23, Theorem 11.3.19] proves such a formula, but with the second factor in the error term given by $q^{-\theta(2g+1)/2}$, where θ is a *uniform stability bound* in the sense of [BDPW23, Definition 11.3.16]. It was conjectured in [BDPW23] that a nontrivial uniform stability bound exists, and our Theorem 1.4 implies that $\theta(n) = \frac{n+11}{12}$ is a uniform stability bound.

An important class of problems in analytic number theory is to understand the distribution of the central values $L(\frac{1}{2}, \pi)$, as $\{L(s, \pi)\}_{\pi \in P}$ varies over some naturally occurring family of L -functions. Our Theorem 1.5 is about the *moments* in the case of the family of quadratic extensions of $\mathbb{F}_q(t)$. Conrey–Farmer–Keating–Rubinstein–Snaith [CFK⁺05] have developed a “recipe” to predict the asymptotics of moments for large classes of families of L -functions. (See also [DGH03].) In the situation of Theorem 1.5, the CFKRS heuristics predict that the left-hand side of Theorem 1.5 is asymptotically $Q_r(2g+1)$. See [AK14, Conjecture 5], which also gives an explicit formula for Q_r . The conjecture was in this case known to hold for $r \leq 3$ [AK12, Flo17b], and for the highest three coefficients of Q_4 when $r = 4$ [Flo17a]. Theorem 1.5 shows that for every fixed r , the left-hand side is indeed asymptotically $Q_r(2g+1)$, with a power-saving error term, for all sufficiently large (but fixed) q . The fact that q may be fixed is important; allowing $q \rightarrow \infty$ leads to the case treated by Katz and Sarnak [KS99]. The CFKRS predictions have been proven correct in many cases before, but generally only for the first few moments

²This can in fact be improved to $d < n - 1 - |\lambda|$, as follows. Let \mathbf{R} be the free E_2 -algebra on a point. Using the knowledge of the rational homology of the braid groups, one sees that $H_{n,d}(\mathbf{R}/\sigma) = 0$ for $d < n - 1$, with notation as in [GKRW18]. An adaptation of [GKRW18, Theorem 19.2] in order to accommodate a vanishing line not through the origin, combined with [MPP19, Theorem 3.23] and [GKRW18, Lemma 19.4], gives the result.

(although see [Saw21a, Saw20, Saw21b]). Theorem 1.5 represents the first³ nontrivial family where the CFKRS asymptotics have been established for *all* moments (with the important caveat that there is no single value of q which works for all r).

In a nutshell, the connection between Theorem 1.5 and homological stability theorems is as follows. Using the Grothendieck–Lefschetz trace formula, the left-hand side of Theorem 1.5 may be written as the trace of Frobenius on the homology of β_{2g+1} , with coefficients in $(\wedge V_g(\frac{1}{2}))^{\otimes r}$. Here V_g denotes the reduced integral Burau representation, $\wedge(-)$ denotes the exterior algebra, and $\frac{1}{2}$ denotes a half-integer Tate twist. One can show that for $g > l(\lambda)$, the multiplicity of the symplectic irreducible representation V_λ in $(\wedge V_g(\frac{1}{2}))^{\otimes r}$ is given by a polynomial $p_{\lambda,r}$ in $(2g+1)$ of degree $r(r+1)/2$. The polynomial Q_r is given by

$$Q_r = \sum_{\lambda} c_{\lambda} \cdot p_{\lambda,r},$$

where c_{λ} is the trace of Frobenius on the stable homology of the braid group with coefficients in V_{λ} , which was calculated in [BDPW23]; one can show that this agrees with the conjectured formula for Q_r . Thus, the left-hand side is the trace of Frobenius on the homology of β_n , and the main term on the right-hand side is the trace of Frobenius on the homology of β_{∞} , suitably regularised. The stable homology is precisely what contributes equally to both left- and right-hand sides, and can be discarded when estimating the difference between the two. If the homologies of β_n and β_{∞} agree up to degree $\approx n/a$ then one gets an error term on the order of $q^{-n/2a}$, since by the Deligne bounds [Del80] the Frobenius eigenvalues on the k th homology have absolute value $\leq q^{-k/2}$. Importantly, we need to know this stability result for *all* λ contributing nontrivially, which is why it is crucial that the stable range can not depend on λ . It would also be interesting to improve our value $a = 12$.

For other examples of applications of homological stability to questions in arithmetic statistics, see [EVW16, EL23].

1.3. Polynomiality. The standard approach to proving homological stability with twisted coefficients is to prove stability for *polynomial coefficient systems*, a notion going back to work of Dwyer [Dwy80]. The coefficient systems occurring in Theorems 1.1, 1.2, 1.3, and 1.4 are all polynomial (of degree $|\lambda|$ or $|\lambda| + |\mu|$, respectively), and the best previously known stable ranges quoted from the literature in connection with Theorems 1.1–1.4 are all valid more generally for any polynomial coefficient system. Now for a *general* polynomial coefficient system, easy examples show that the stable range must be allowed to depend on the degree of polynomiality; in the generality of arbitrary polynomial coefficients the traditional bounds are going to be sharp. The *uniform* stable ranges obtained in Theorems 1.1–1.4, with no dependence on the degree of polynomiality, is a very particular feature of these specific coefficient systems defined by systems of irreducible algebraic representations. But these coefficient systems defined by irreducibles are also arguably the most important and well-studied examples of rational coefficient systems.

Let us make the discussion in the previous paragraph more explicit. We remind the reader that given a sequence of groups

$$\Gamma_0 \rightarrow \Gamma_1 \rightarrow \Gamma_2 \rightarrow \Gamma_3 \rightarrow \dots$$

for which we want homological stability, a *coefficient system* consists of a sequence

$$V(0) \rightarrow V(1) \rightarrow V(2) \rightarrow V(3) \rightarrow \dots$$

with each $V(n)$ a Γ_n -module, with suitably equivariant stabilisation maps. The *shift* of V is the coefficient system defined by $(\Sigma V)(n) = V(n+1)$, and V is inductively defined to be *polynomial of degree* $\leq d$ if $V \rightarrow \Sigma V$ is injective with cokernel polynomial of degree $\leq (d-1)$. The inductive definition of polynomiality makes it well suited for inductive arguments, and the standard proof of homological stability with polynomial coefficients goes by induction over the degree of polynomiality. The proof is a version of the *Quillen argument*, which is by far the most common technique for proving homological

³The aforementioned papers of Sawin do not treat the essential case where the individual terms are nonnegative.

stability results, and in fact the definition of polynomiality almost appears tailor-made to slot neatly into the Quillen argument.

The stable ranges obtained from the standard Quillen argument with polynomial coefficients are of the form⁴

$$H_i(\Gamma_n, \Gamma_{n-1}; V(n), V(n-1)) = 0 \quad \text{for } i < A \cdot n - B - d,$$

where A and B are constants, and d is the degree of polynomiality. In particular, the stable range depends on d , and in a situation where n and d go to infinity at the same rate the stable range may effectively become zero. This is not a bug, or an artifact of a poorly constructed argument — in general, it is simply not possible to do better:

Example 1.6. Consider the groups $\Gamma_n = \mathrm{Sp}_{2n}(\mathbb{Z})$, and let $V(n) = \mathbb{k}^{2n}$ be the defining representation of Sp_{2n} , for \mathbb{k} a field of characteristic zero. The family $\{V(n)^{\otimes r}\}$ is a polynomial coefficient system of degree r . The stabilisation maps

$$H_0(\Gamma_{n-1}; V(n-1)^{\otimes 2s}) \rightarrow H_0(\Gamma_n; V(n)^{\otimes 2s})$$

are isomorphisms for $n > s$, and this is *sharp*; stabilisation is not surjective for $n = s$. This follows by explicit computation. Working instead with cohomology, the first fundamental theorem of invariant theory for the symplectic group says that $H^0(\Gamma_n; V(n)^{\otimes 2s})$ is spanned by all ways of partitioning the set $\{1, \dots, 2s\}$ into 2-element blocks, each such partition corresponding to a way of inserting the symplectic form. The second fundamental theorem of invariant theory gives all relations between these generators: for $n > s$, there are none, but for $n \leq s$ there are always nontrivial relations. For a review the first and second fundamental theorems of invariant theory in this context, see e.g. [LZ15, Section 4]. ■

The conclusion, then, is that even in the most simple and natural examples, and even for $H_0(-)$, the stable range must be allowed to depend on the degree of polynomiality.

1.4. Work of Borel. Nevertheless, one starting point of the present paper is that there are important examples of twisted homological stability theorems in which one has a uniform stable range for particular coefficient systems of arbitrarily high degree. Of particular importance for the present paper are the results of Borel about stable cohomology of arithmetic groups [Bor74, Bor81].

One considers an arithmetic group Γ inside a semisimple algebraic group G over \mathbb{Q} . Borel shows firstly that in a stable range, the cohomology $H^*(\Gamma; \mathbb{R})$ can be computed as the L^2 -cohomology of the associated locally symmetric space for Γ . Moreover, the L^2 -cohomology can in a range be computed in terms of G -invariant differential forms on the symmetric space of G ; in particular, the latter does not depend on the particular choice of Γ . If $\{G_n\}$ is a family of classical groups and Γ_n is an arbitrary family of arithmetic subgroups, then for $n \rightarrow \infty$ the consequence is that $H^*(\Gamma_n; \mathbb{R})$ stabilises to an answer that can be computed purely representation-theoretically.

Borel's results also give twisted homological stability. If in the above setting V is a real irreducible algebraic representation of G , then one can similarly calculate $H^*(\Gamma; V)$ in terms of differential forms. However, the consequence is simply that if V is nontrivial then $H^*(\Gamma; V)$ vanishes in a stable range. What is important for us is that although Borel in his original paper gave a stable range depending on V , it is in fact possible to give a uniform stable range depending only on the group G , under mild hypotheses.

For example, let us return to the example of $\Gamma_n = \mathrm{Sp}_{2n}(\mathbb{Z})$. For any partition λ one obtains a sequence of representations $V_\lambda(n)$ of Γ_n , forming a polynomial coefficient system of degree $|\lambda|$, as we will explain in more detail in Section 4.2.1. Importantly, $H_*(\Gamma_n; V_\lambda(n))$ satisfies homological stability with a *uniform stable range*, independent of λ . This does not contradict Example 1.6: it is only for systems of irreducible representations that we see the uniform homological stability, and the decomposition of

⁴If the coefficient system is *split* then one can improve this to $A(n-d) - B$. This is better since, without additional bells and whistles, one always has $A \leq \frac{1}{2}$.

$V(n)^{\otimes d}$ into irreducibles depends nontrivially upon n , although it stabilises for n sufficiently large with respect to d .

Borel's work on stable cohomology is of a fundamentally different nature than any of the standard machinery for homological stability — at its core, it is a theorem about automorphic forms, proven by transcendental techniques. Borel's methods only prove homological stability rationally, and analogous integral assertions are just completely false.

1.5. The main theorem. Thus, in the very specific situation of families of arithmetic groups, we have natural examples of coefficient systems with uniform stable range. One may now instead ask about families of groups $\{\Gamma_n\}$ which come with a natural *map* to a sequence of arithmetic groups $\{Q_n\}$:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Gamma_n & \longrightarrow & \Gamma_{n+1} & \longrightarrow & \Gamma_{n+2} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & Q_n & \longrightarrow & Q_{n+1} & \longrightarrow & Q_{n+2} & \longrightarrow & \cdots \end{array}$$

The examples of Theorem 1.1, Theorem 1.2, Theorem 1.3, and Theorem 1.4 all fit into this pattern:

- (1) $\Gamma_n = \text{Mod}_n^1$ and $Q_n = \text{Sp}_{2n}(\mathbb{Z})$,
- (2) $\Gamma_n = \text{Aut}(F_n)$ and $Q_n = \text{GL}_n(\mathbb{Z})$,
- (3) $\Gamma_n = \text{HMod}_n^1$ and $Q_n = \text{GL}_n(\mathbb{Z})$,
- (4) $\Gamma_n = \beta_n$ and $Q_n \subset \text{Sp}_{n-1}(\mathbb{Z})$ the image of the integral Burau representation.

In each of these examples, homological stability of the family $\{\Gamma_n\}$ is known via the Quillen argument. Thus, they satisfy homological stability *integrally*, with *arbitrary* polynomial coefficient systems, but with a stable range depending on the degree of polynomiality. But one may instead take real (or rational) coefficients, and consider the *very specific* systems of polynomial coefficients obtained by pulling back a sequence of irreducible algebraic representations from the family of groups $\{Q_n\}$. Borel tells us that for this particular type of coefficients, the homology of Q_n stabilises with a uniform stable range, independent of the polynomial degree. It is natural to ask whether the same is true for the homology of Γ_n with the same coefficients. The goal of this paper is to prove that under mild hypotheses, the answer is in fact *yes*.

A heuristic picture of the main theorem is the following: suppose as above that we have a family of groups $G = \{\Gamma_n\}$ surjecting onto $Q = \{Q_n\}$, and suppose both families have nice homological stability properties (highly connected destabilisation or splitting complexes). If V is an *arbitrary* coefficient system on Q such that $H_*(Q_n; V(n))$ stabilises, must $H_*(\Gamma_n; V(n))$ stabilise? In general, the answer is *no* — but if we assume in addition that all shifts of V stabilise on Q , then the answer is *yes*, and the stable range on G can be bounded in terms of the stable range on Q of all shifts.

1.6. Sharpness. It is interesting that Theorem 1.1 and Theorem 1.2 are both close to optimal: there is very little room for a better uniform stable range, as the following examples show.

In the setting of mapping class groups, take λ to be the partition (1^{3k}) of length $3k$. Using that the stable homology of the mapping class group with symplectic coefficients is known [Loo96, Kaw08, RW18] one can see that $H_k(\text{Mod}_\infty^1; V_{(1^{3k})}) \cong \mathbb{Q}$. Then Theorem 1.1 shows that $H_k(\text{Mod}_{3k+1}^1; V_{(1^{3k})}(3k+1))$ is nontrivial, as it surjects onto the stable homology. On the other hand, $H_k(\text{Mod}_{3k-1}^1; V_{(1^{3k})}(3k-1))$ must vanish, since the coefficient module $V_{(1^{3k})}(3k-1)$ itself is zero. So we are at most one off from the optimal stable range.

For automorphism groups of free groups one may run an entirely similar argument. Take instead $\lambda = (1^{2k})$ and $\mu = (1^k)$. The stable homology of $\text{Aut}(F_n)$ with coefficients in $V_{\lambda, \mu}$ was recently computed by Lindell [Lin22], and in particular it turns out that $H_k(\text{Aut}(F_\infty), V_{(1^{2k}), (1^k)}) \cong \mathbb{Q}$. From Theorem 1.2 we get that $H_k(\text{Aut}(F_{3k+2}); V_{(1^{2k}), (1^k)}(3k+2))$ is nontrivial, again by surjectivity of the stabilisation map. But also $H_k(\text{Aut}(F_{3k-1}); V_{(1^{2k}), (1^k)}(3k-1)) = 0$ since the coefficient module vanishes, making Theorem 1.2 very close to optimal, too.

Similar considerations show that the best possible slope of stability in Theorem 1.4 is $\frac{1}{4}$, using that the stable homology is known from [BDPW23], and considering the homology of $V_{(2^{2k})}$ in degree k .

1.7. Relation to Torelli groups. Another motivation for proving uniform twisted homological stability theorems is the work of Hain [Hai97] giving quadratic presentations of the Malcev completion of the Torelli subgroup of the mapping class group for $g \geq 6$ (later improved to $g \geq 4$ [Hail5]). Focusing for simplicity on the case of surfaces with a boundary component, it turns out that to determine the generators, one needs to know $H^1(\text{Mod}_g^1; V_\lambda(g))$ for all irreducible representations λ , and to determine the relations, one needs to know $H^2(\text{Mod}_g^1; V_\lambda(g))$ for all λ . Our Theorem 1.1 shows that for $g \geq 10$, both these groups agree with their stable values, which are known by [Loo96] (see also [Kaw08, RW18]). With Theorem 1.1 as input, Hain’s arguments could be dramatically simplified, although with a worse range for g . By contrast, Hain’s strategy to control $H^2(\text{Mod}_g^1; V_\lambda(g))$ is instead to use a theorem of Kabanov [Kab98], which says that the restriction map from degree 2 intersection cohomology of V_λ on the Satake compactification of the moduli space of curves \mathcal{M}_g , to $H^2(\text{Mod}_g; V_\lambda(g))$, is an isomorphism for $g \geq 6$. This is proven by a delicate study of the boundary geometry. Kabanov’s theorem allows controlling the weights in the mixed Hodge structure and suffices to determine the relations in the Torelli Lie algebra. It is a well known open problem to prove an analogue of Hain’s results for the Torelli subgroups of $\text{Aut}(F_n)$, and one may hope that Theorem 1.2 will be useful in obtaining such an analogue. (Theorem 1.2 by itself can not be immediately applied to obtain a quadratic presentation of the Torelli Lie algebra of $\text{Aut}(F_n)$ — another key ingredient in Hain’s work is to argue that the Lie algebra in question is isomorphic to its associated graded for the lower central series, using Hodge theory, and the analogous statement is open in the $\text{Aut}(F_n)$ case.)

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2. A generic uniform twisted homological stability theorem

Our goal in this section is to formulate and prove a generic uniform twisted homological stability theorem, which takes as axioms certain desirable features of the groups $\{\Gamma_n\}$, their quotient arithmetic groups $\{Q_n\}$, and the class of coefficient systems to be considered. In Section 4 we will then verify these axioms in the cases described in Theorems 1.1, 1.2, 1.3, and 1.4. We will express this result in the language developed in [GKRW18], and rely on that work as well as [RW22] for some technical support.

2.1. Formulation. Our basic datum will be a morphism

$$p : (\mathbb{G}, \oplus, 0) \longrightarrow (\mathbb{Q}, \oplus, 0)$$

of braided strict monoidal groupoids, both of which have monoid of objects \mathbb{N} , with p the identity map on objects. We abbreviate

$$\Gamma_n := \text{Aut}_{\mathbb{G}}(n) \quad Q_n := \text{Aut}_{\mathbb{Q}}(n).$$

We assume that:

- (i) The induced maps $p_n : \Gamma_n \rightarrow Q_n$ are surjective, with kernels denoted K_n .
- (ii) Γ_0 is trivial, and hence Q_0 and K_0 are trivial too.
- (iii) The maps $\Gamma_a \times \Gamma_b \rightarrow \Gamma_{a+b}$ and $Q_a \times Q_b \rightarrow Q_{a+b}$ are injective.

2.1.1. *G- and Q-graded simplicial modules.* Fix a field \mathbb{k} of characteristic zero, and let $\text{sMod}_{\mathbb{k}}$ denote the category of simplicial \mathbb{k} -modules. We encourage the reader to mentally replace this by the category $\text{Ch}_{\mathbb{k}}$ of non-negatively graded chain complexes, if they are more comfortable in that context: the difference is purely technical, and allows us to directly quote results from [GKRW18], but at a conceptual level there is no difference. We will work in the categories $\text{sMod}_{\mathbb{k}}^{\mathbb{G}}$ and $\text{sMod}_{\mathbb{k}}^{\mathbb{Q}}$ of functors from \mathbb{G} or \mathbb{Q} to $\text{sMod}_{\mathbb{k}}$. These are endowed with the projective model structure, induced by the standard model structure on simplicial modules. For an object X of one of these categories, we define bigraded homology groups as the simplicial homotopy groups:

$$H_{n,d}(X) := \pi_d(X(n)).$$

This is the same as the homology of the associated chain complex of normalised chains. Equivalently, let $S_{\mathbb{k}}^d \in \text{sMod}_{\mathbb{k}}$ be the \mathbb{k} -linearisation of the simplicial d -sphere considered as a based simplicial set, and $S_{\mathbb{k}}^{n,d} := n_* S_{\mathbb{k}}^d$ denote the left Kan extension of this along the inclusion $\{n\} \rightarrow \mathbb{G}$. (Explicitly, $S_{\mathbb{k}}^{n,d}$ is supported on the object n , and evaluates here to $\mathbb{k}[\Gamma_n] \otimes_{\mathbb{k}} S_{\mathbb{k}}^d$ with its evident Γ_n -action.) This is cofibrant in $\text{sMod}_{\mathbb{k}}^{\mathbb{G}}$, and for an object X in this category we have $H_{n,d}(X) \cong [S_{\mathbb{k}}^{n,d}, X]_{\text{sMod}_{\mathbb{k}}^{\mathbb{G}}}$, the set of morphisms from $S_{\mathbb{k}}^{n,d}$ to X in the homotopy category of $\text{sMod}_{\mathbb{k}}^{\mathbb{G}}$. Similarly for $\text{sMod}_{\mathbb{k}}^{\mathbb{Q}}$.

Day convolution endows each of these categories with a braided monoidal structure, which we denote by \otimes (not be confused with the tensor product $\otimes_{\mathbb{k}}$ of simplicial \mathbb{k} -modules). The unit for this monoidal structure is the functor which is \mathbb{k} at $0 \in \mathbb{N}$ and 0 otherwise: in other words it is $S_{\mathbb{k}}^{0,0}$. Explicitly, in $\text{sMod}_{\mathbb{k}}^{\mathbb{G}}$ for example, it is given by the formula

$$(X \otimes Y)(n) := \text{colim}_{\oplus/n} X(-) \otimes_{\mathbb{k}} Y(-) \cong \bigoplus_{a+b=n} \text{Ind}_{\Gamma_a \times \Gamma_b}^{\Gamma_n} X(a) \otimes_{\mathbb{k}} Y(b).$$

A fundamental construction will be the left Kan extension $p_* : \text{sMod}_{\mathbb{k}}^{\mathbb{G}} \rightarrow \text{sMod}_{\mathbb{k}}^{\mathbb{Q}}$, given by

$$p_*(X)(n) := \text{colim}_{p/n} X(-) \cong \mathbb{k}[Q_n] \otimes_{\mathbb{k}[\Gamma_n]} X(n) \cong \mathbb{k} \otimes_{\mathbb{k}[K_n]} X(n).$$

It is strong monoidal [GKRW18, Lemma 2.13] with respect to the Day convolution monoidal structures described above. It is left adjoint to restriction p^* , and admits a left derived functor $\mathbb{L}p_*$. In grading n this is given by forming the homotopy K_n -orbits, giving a spectral sequence

$$(2.1) \quad E_{n,s,t}^2 = H_s(K_n; H_{n,t}(X)) \implies H_{n,s+t}(\mathbb{L}p_*(X)).$$

In chain complexes, this would be known as the hyperhomology spectral sequence.

2.1.2. *Coefficient systems.* In each case, the constant functor with value \mathbb{k} defines a commutative monoid object, denoted $\underline{\mathbb{k}}_{\mathbb{G}}$ and $\underline{\mathbb{k}}_{\mathbb{Q}}$ respectively. Following [GKRW18, Section 19.1], a *coefficient system* for the family of groups $\{Q_n\}$ is simply a left $\underline{\mathbb{k}}_{\mathbb{Q}}$ -module. Spelled out, this consists of a functor $V : \mathbb{Q} \rightarrow \text{sMod}_{\mathbb{k}}$, i.e. $\mathbb{k}[Q_n]$ -modules $V(n)$, equipped with $Q_a \times Q_b$ -equivariant maps

$$\phi_{a,b} : \mathbb{k} \otimes V(b) \longrightarrow V(a+b)$$

which are suitably associative. This recovers several well-known notions: if \mathbb{Q} is the groupoid of finite sets and bijections then a coefficient system is an *FI*-module of Church–Ellenberg–Farb [CEF15]; if \mathbb{Q} is the groupoid of finite-rank free modules over a ring and isomorphisms between them then a coefficient system is a *VIC*-module of Putman–Sam [PS17]; more generally, a coefficient system in this sense is the same as one in the sense of Randal-Williams–Wahl [RWW17], namely a functor from the enveloping homogeneous category $U\mathbb{Q}$ to \mathbb{k} -modules.

This structure in particular provides a map $\phi_{1,n-1} : V(n-1) \rightarrow V(n)$ equivariant with respect to $1 \oplus - : Q_{n-1} \rightarrow Q_n$, using which we can form the map

$$H_*(Q_{n-1}; V(n-1)) \longrightarrow H_*(Q_n; V(n))$$

as well as the associated relative homology groups $H_*(Q_n, Q_{n-1}; V(n), V(n-1))$. Similarly, a coefficient system for the groups $\{\Gamma_n\}$ is a left \mathbb{k}_G -module. As p^* is lax monoidal and $p^*\mathbb{k}_Q = \mathbb{k}_G$, any \mathbb{k}_Q -module V gives a \mathbb{k}_G -module p^*V . We often abuse notation by writing V for the latter too.

Given a coefficient system V , its *shift* is the functor ΣV given by $V(- \oplus 1)$. This again has the structure of a coefficient system, with the structure maps $(\Sigma\phi)_{a,b} = \phi_{a,b+1}$.

2.1.3. Derived indecomposables. The final ingredient for formulating our theorem is more technical. The commutative algebra object \mathbb{k}_G in the braided monoidal category $\text{sMod}_{\mathbb{k}}^G$ is not usually cofibrant (because the $\mathbb{k}[\Gamma_n]$ -module \mathbb{k} is not usually free). It cannot usually be replaced by a cofibrant object still having the structure of a *commutative algebra*, but it can be replaced by a cofibrant object having the weaker structure of a *unital E_2 -algebra* (cf. [GKRW18, Section 12]). We will use this concept in our proofs, but for stating the theorem we can work with a more elementary and explicit notion. (The notation is chosen to fit with [GKRW18].)

Define an object $\overline{T} \in \text{sMod}_{\mathbb{k}}^G$ by

$$\overline{T}(n) := \mathbb{k}[E_\bullet \Gamma_n],$$

where $E_\bullet \Gamma_n$ is the simplicial set given by the two-sided bar construction $B_\bullet(\Gamma_n, \Gamma_n, *)$: this is contractible, and has a free left Γ_n -action so \overline{T} is cofibrant. The monoidal structure on G gives homomorphisms $\Gamma_n \times \Gamma_m \rightarrow \Gamma_{n+m}$, and so by the functoriality and monoidality of $\mathbb{k}[E_\bullet(-)]$ gives maps

$$\overline{T}(n) \otimes_{\mathbb{k}} \overline{T}(m) \longrightarrow \overline{T}(n+m)$$

which are equivariant for $\Gamma_n \times \Gamma_m \rightarrow \Gamma_{n+m}$: these assemble to a morphism $\mu : \overline{T} \otimes \overline{T} \rightarrow \overline{T}$ in $\text{sMod}_{\mathbb{k}}^G$. The identity element $e \in \Gamma_0$ gives a morphism $\iota : S_{\mathbb{k}}^{0,0} \rightarrow \overline{T}$, and the strictness of the monoidal structure makes $\overline{\mathbf{T}} := (\overline{T}, \mu, \iota)$ into a unital and associative algebra object in $\text{sMod}_{\mathbb{k}}^G$. As each $E_\bullet \Gamma_n$ is contractible, there is an equivalence $\overline{\mathbf{T}} \xrightarrow{\sim} \mathbb{k}_G$ of associative algebras, exhibiting \overline{T} as a cofibrant replacement of \mathbb{k}_G in $\text{sMod}_{\mathbb{k}}^G$.

As the left Kan extension p_* is lax (in fact even strong) monoidal, the object

$$\overline{\mathbf{R}} := p_*(\overline{\mathbf{T}})$$

is then an associative algebra in $\text{sMod}_{\mathbb{k}}^Q$, and as $\overline{\mathbf{T}}$ is cofibrant in $\text{sMod}_{\mathbb{k}}^G$ it is a model for the derived Kan extension $\mathbb{L}p_*(\mathbb{k}_G)$. In particular it has

$$H_{n,d}(\overline{\mathbf{R}}) = \pi_d \left(\text{colim}_{p/n} \mathbb{k}[E_\bullet \Gamma_n] \right) \cong \pi_d \left(\text{hocolim}_{p/n} \mathbb{k} \right) \cong H_d(K_n; \mathbb{k}),$$

given by the homology of the kernels K_n of the surjective homomorphisms $p_n : \Gamma_n \rightarrow Q_n$. As these homology groups in degree 0 are all \mathbb{k} , Postnikov truncation provides a map of associative algebras $\overline{\mathbf{R}} \rightarrow \mathbb{k}_Q$, making \mathbb{k}_Q into a left module over $\overline{\mathbf{R}}$. There is also an augmentation $\epsilon : \overline{\mathbf{R}} \rightarrow S_{\mathbb{k}}^{0,0}$, so we may form the two-sided bar construction $B(S_{\mathbb{k}}^{0,0}, \overline{\mathbf{R}}, \mathbb{k}_Q) \in \text{sMod}_{\mathbb{k}}^Q$.

Remark 2.1. This bar construction is equivalent to the “derived indecomposables” $Q_{\mathbb{L}}^{\overline{\mathbf{R}}}(\mathbb{k}_Q)$ of \mathbb{k}_Q as a $\overline{\mathbf{R}}$ -module, as described in [GKRW18, Section 9.4.2]. It is not immediate that these agree, because \mathbb{k}_Q is not cofibrant in $\text{sMod}_{\mathbb{k}}^Q$ and so [GKRW18, Corollary 9.17] does not apply. To see that they nonetheless do agree, it suffices to show that the functor $B(S_{\mathbb{k}}^{0,0}, \overline{\mathbf{R}}, -)$ preserves weak equivalences between *arbitrary* $\overline{\mathbf{R}}$ -modules, and for this it suffices to show that $S_{\mathbb{k}}^{0,0} \otimes \overline{\mathbf{R}}^{\otimes p} \otimes -$ preserves weak equivalences between arbitrary objects of $\text{sMod}_{\mathbb{k}}^Q$. In fact tensoring with any object preserves weak equivalences in $\text{sMod}_{\mathbb{k}}^Q$. To see this, if $X \in \text{sMod}_{\mathbb{k}}^Q$ and $f : Y \xrightarrow{\sim} Z$ is a weak equivalence, then in grading n the map $X \otimes f$ is

$$\bigoplus_{a+b=n} \text{Ind}_{Q_a \times Q_b}^{Q_n} X(a) \otimes_{\mathbb{k}} Y(b) \longrightarrow \bigoplus_{a+b=n} \text{Ind}_{Q_a \times Q_b}^{Q_n} X(a) \otimes_{\mathbb{k}} Z(b).$$

As \mathbb{k} is a field the maps $X(a) \otimes_{\mathbb{k}} Y(b) \rightarrow X(a) \otimes_{\mathbb{k}} Z(b)$ are weak equivalences, and by assumption (iii) $\mathbb{k}[Q_n]$ is a free and hence flat $\mathbb{k}[Q_a \times Q_b]$ -module and so $\text{Ind}_{Q_a \times Q_b}^{Q_n}(-) = \mathbb{k}[Q_n] \otimes_{\mathbb{k}[Q_a \times Q_b]}(-)$ preserves weak equivalences.

2.1.4. *Statement of the theorem.* Our generic uniform twisted homological stability theorem then takes the following form.

Theorem 2.2. *In the setting just described, suppose that $\theta \in (0, \infty)$, $\tau \in (-\infty, 0]$, and \mathcal{V} is a class of coefficient systems on Q such that*

(I) *\mathcal{V} is closed under shifting,*

(II) *$H_d(Q_n, Q_{n-1}; V(n), V(n-1)) = 0$ for $d < \theta \cdot (n + \tau)$ and $n \geq 1$, and all $V \in \mathcal{V}$,*

and suppose also that $\nu \in (0, \infty)$ and $\xi \in (-\infty, 1]$ are such that

(III) *$H_{n,d}(B(S_{\mathbb{k}}^{0,0}, \overline{\mathbf{R}}, \underline{\mathbb{k}}_Q)) = 0$ for $d < \nu \cdot n + \xi$ and $n \geq 1$.*

Then $H_d(\Gamma_n, \Gamma_{n-1}; V(n), V(n-1)) = 0$ for $d < \min(\theta, \frac{\nu}{2-\xi}) \cdot (n + \tau)$ and all $V \in \mathcal{V}$.

2.2. **Methods for verifying axiom (III).** In practice, it can be difficult to directly verify axiom (III) (unless G happens to be the free braided monoidal groupoid on one generator — see Remark 4.15). We therefore provide two companion results, Proposition 2.3 and Proposition 2.5, which say that it suffices to estimate the connectivities of certain absolute invariants of the braided monoidal groupoids G and Q .

Proposition 2.3 and Proposition 2.5 should be thought of as being philosophically very similar. The former takes as input an estimate of the connectivities of the *destabilisation complexes* of G and Q , in the sense of [RW17, Definition 2.1], and the latter works instead with their *E_1 -splitting complexes* in the sense of [GKRW18, Definition 17.9]. An estimate of the connectivity of either of these families of complexes will imply an estimate of the connectivity of the other, under mild hypotheses [RW22, Proposition 7.1], but having both Proposition 2.3 and Proposition 2.5 will allow for better stable ranges in practice. Proposition 2.3 allows for a connectivity estimate of any slope, and a small offset, whereas Proposition 2.5 only treats the case that the E_1 -splitting complexes are *spherical*, i.e. what [GKRW18] calls the “standard connectivity estimate”; it seems that the splitting complexes are in any case most useful when they are spherical.

In order to define the space of destabilisations, we use assumption (iii) which implies that the stabilisation maps $\Gamma_{n-p-1} \rightarrow \Gamma_n$ are injective. Then the semi-simplicial set $W_{\bullet}(G, n)$ has p -simplices

$$W_p(G, n) := \text{colim}_{(-\oplus(p+1))/n} G(-, n) \cong \frac{\Gamma_n}{\Gamma_{n-p-1}}$$

and face maps induced by certain maps $-\oplus(p+1) \xrightarrow{\sim} -\oplus 1 \oplus p$ formed using the braided monoidal structure of G , see [RW17, Definition 2.1] for details.

Proposition 2.3. *Suppose that $\nu \in (0, \infty)$ and $\xi \in [-1, \infty)$ satisfy*

(i) *$\tilde{H}_d(W_{\bullet}(G; n); \mathbb{k}) = 0$ for $d < \nu \cdot n + \xi - 1$ and $n \geq 1$,*

(ii) *$\tilde{H}_d(W_{\bullet}(Q; n); \mathbb{k}) = 0$ for $d < \nu \cdot n + \xi - 1$ and $n \geq 1$.*

Then $H_{n,d}(B(S_{\mathbb{k}}^{0,0}, \overline{\mathbf{R}}, \underline{\mathbb{k}}_Q)) = 0$ for $d < \nu \cdot n + \xi$ and $n \geq 1$.

The proof is identical to that of [RW22, Theorem 6.2 (ii)], but we spell out the details.

Proof. We first determine the homology of $B(S_{\mathbb{k}}^{0,0}, \overline{\mathbf{R}}, \underline{\mathbb{k}}_Q)$ in degree zero and in grading zero. Recall from Remark 2.1 that this object agrees with the derived indecomposables $Q_{\mathbb{L}}^{\overline{\mathbf{R}}}(\underline{\mathbb{k}}_Q)$, and so following [GKRW18] we write $H_{n,d}^{\overline{\mathbf{R}}}(\underline{\mathbb{k}}_Q)$ for its homology groups. The map $H_{n,d}(\overline{\mathbf{R}}) = H_d(K_n; \mathbb{k}) \rightarrow H_{n,d}(\underline{\mathbb{k}}_Q)$ is an isomorphism when $d = 0$ and an epimorphism when $d = 1$ (as $H_{*,1}(\underline{\mathbb{k}}_Q) = 0$), so that the relative homology satisfies $H_{n,d}(\underline{\mathbb{k}}_Q, \overline{\mathbf{R}}) = 0$ for $d \leq 1$ or $n = 0$. As $H_{*,0}(\overline{\mathbf{R}}) = \underline{\mathbb{k}}_Q$, it follows from [GKRW18, Corollary 11.14 (ii)] that the Hurewicz map $\mathbb{k} \otimes_{\underline{\mathbb{k}}_Q} H_{*,*}(\underline{\mathbb{k}}_Q, \overline{\mathbf{R}}) \rightarrow H_{*,*}^{\overline{\mathbf{R}}}(\underline{\mathbb{k}}_Q, \overline{\mathbf{R}})$ is an isomorphism in

bidegrees (n, d) with $d \leq 2$, so that $H_{n,d}^{\overline{\mathbf{R}}}(\underline{\mathbb{k}}_{\mathbb{Q}}, \overline{\mathbf{R}}) = 0$ for $d \leq 1$. Using assumption (ii) that K_0 is trivial, one sees that $H_{n,d}^{\overline{\mathbf{R}}}(\underline{\mathbb{k}}_{\mathbb{Q}}, \overline{\mathbf{R}})$ vanishes for $n = 0$ too. The long exact sequence for a pair then shows that

$$(2.2) \quad H_{*,0}^{\overline{\mathbf{R}}}(\underline{\mathbb{k}}_{\mathbb{Q}}) = H_{*,0}^{\overline{\mathbf{R}}}(\overline{\mathbf{R}}) = \mathbb{k} \text{ supported in grading } 0,$$

$$(2.3) \quad H_{*,1}^{\overline{\mathbf{R}}}(\underline{\mathbb{k}}_{\mathbb{Q}}) = H_{*,1}^{\overline{\mathbf{R}}}(\overline{\mathbf{R}}) = 0,$$

$$(2.4) \quad H_{0,*}^{\overline{\mathbf{R}}}(\underline{\mathbb{k}}_{\mathbb{Q}}) = H_{0,*}^{\overline{\mathbf{R}}}(\overline{\mathbf{R}}) = \mathbb{k} \text{ supported in degree } 0.$$

At this point we change our perspective on $\overline{\mathbf{R}}$ slightly, by giving a more conceptual construction of it. The commutative algebra $\underline{\mathbb{k}}_{\mathbb{G}}$ in the braided monoidal category $\text{sMod}_{\mathbb{k}}^{\mathbb{G}}$ is in particular a unital E_2 -algebra object in this category. It has a unique augmentation $\epsilon : \underline{\mathbb{k}}_{\mathbb{G}} \rightarrow S_{\mathbb{k}}^{0,0}$, and we write $(\underline{\mathbb{k}}_{\mathbb{G}})_{>0}$ for its kernel, which is a nonunital E_2 -algebra. This admits a cofibrant replacement $\mathbf{T} \xrightarrow{\sim} (\underline{\mathbb{k}}_{\mathbb{G}})_{>0} \in \text{Alg}_{E_2}(\text{sMod}_{\mathbb{k}}^{\mathbb{G}})$, which is also cofibrant in $\text{sMod}_{\mathbb{k}}^{\mathbb{G}}$. This in turn can be unitalised-and-strictified to an associative monoid $\overline{\mathbf{T}}$ (see [GKRW18, Section 12.2.1]), which is cofibrant in $\text{sMod}_{\mathbb{k}}^{\mathbb{G}}$ by [GKRW18, Lemma 12.7 (i)] and is equivalent to $\underline{\mathbb{k}}_{\mathbb{G}}$ as an associative algebra. Using [GKRW18, Lemma 12.8], this $\overline{\mathbf{T}}$ is equivalent as an associative algebra to the explicit associative algebra of the same name we defined earlier. In particular, the associative algebra $\overline{\mathbf{R}}$ we defined earlier is equivalent as an $(E_1$ -algebra or) associative algebra to the unitalisation-and-strictification of the non-unital E_2 -algebra

$$\mathbf{R} := p_*(\mathbf{T}) \in \text{Alg}_{E_2}(\text{sMod}_{\mathbb{k}}^{\mathbb{Q}}).$$

Following [RW22, Section 5], let us write $\widetilde{\mathbf{S}}_{\mathbb{Q}} := \overline{E_2(S_{\mathbb{k}}^{1,0})}$ for the unitalisation-and-strictification of the free E_2 -algebra on the object $S_{\mathbb{k}}^{1,0} \in \text{sMod}_{\mathbb{k}}^{\mathbb{Q}}$. A choice of map $S_{\mathbb{k}}^{1,0} \rightarrow \mathbf{R}$ representing the canonical generator of $H_{1,0}(\mathbf{R}) = H_0(K_1; \mathbb{k})$ freely extends to an E_2 -map $E_2(S_{\mathbb{k}}^{1,0}) \rightarrow \mathbf{R}$, and applying $(\overline{\quad})$ to this gives a map of associative algebras $\widetilde{\mathbf{S}}_{\mathbb{Q}} \rightarrow \overline{\mathbf{R}}$. Postnikov truncation also gives a map of associative algebras $\widetilde{\mathbf{S}}_{\mathbb{Q}} \rightarrow \underline{\mathbb{k}}_{\mathbb{Q}}$.

We now translate the assumptions in the proposition into statements about derived indecomposables. Working momentarily in the category $\text{sMod}_{\mathbb{k}}^{\mathbb{G}}$, we can form the analogous associative algebra $\widetilde{\mathbf{S}}_{\mathbb{G}} := \overline{E_2(S_{\mathbb{k}}^{1,0})}$ in this category, and $\underline{\mathbb{k}}_{\mathbb{G}}$ is a module for it. Then [RW22, Theorem 5.1] shows that $H_{n,d}^{\widetilde{\mathbf{S}}_{\mathbb{G}}}(\underline{\mathbb{k}}_{\mathbb{G}}) \cong \widetilde{H}_{d-1}(W_{\bullet}(\mathbb{G}; n); \mathbb{k})$, which by assumption vanishes for $d-1 < \nu \cdot n + \xi - 1$ and $n \geq 1$. As $Q_{\mathbb{L}}^{\widetilde{\mathbf{S}}_{\mathbb{Q}}}(\overline{\mathbf{R}}) \simeq \mathbb{L}p_*(Q_{\mathbb{L}}^{\widetilde{\mathbf{S}}_{\mathbb{G}}}(\underline{\mathbb{k}}_{\mathbb{G}}))$, the spectral sequence (2.1) then implies that $H_{n,d}^{\widetilde{\mathbf{S}}_{\mathbb{Q}}}(\overline{\mathbf{R}}) = 0$ for $d-1 < \nu \cdot n + \xi - 1$ and $n \geq 1$ as well. Similarly, $H_{n,d}^{\widetilde{\mathbf{S}}_{\mathbb{Q}}}(\underline{\mathbb{k}}_{\mathbb{Q}}) \cong \widetilde{H}_{d-1}(W_{\bullet}(\mathbb{Q}; n); \mathbb{k})$, which vanishes for $d-1 < \nu \cdot n + \xi - 1$ and $n \geq 1$ by assumption.

Interchanging geometric realisations and using $B(\overline{\mathbf{R}}, \overline{\mathbf{R}}, \underline{\mathbb{k}}_{\mathbb{Q}}) \xrightarrow{\sim} \underline{\mathbb{k}}_{\mathbb{Q}}$ gives an equivalence

$$B(S_{\mathbb{k}}^{0,0}, \widetilde{\mathbf{S}}_{\mathbb{Q}}, \overline{\mathbf{R}}, \overline{\mathbf{R}}, \underline{\mathbb{k}}_{\mathbb{Q}}) \simeq B(S_{\mathbb{k}}^{0,0}, \widetilde{\mathbf{S}}_{\mathbb{Q}}, B(\overline{\mathbf{R}}, \overline{\mathbf{R}}, \underline{\mathbb{k}}_{\mathbb{Q}})) \simeq B(S_{\mathbb{k}}^{0,0}, \widetilde{\mathbf{S}}_{\mathbb{Q}}, \underline{\mathbb{k}}_{\mathbb{Q}}).$$

We filter the right $\overline{\mathbf{R}}$ -module $B(S_{\mathbb{k}}^{0,0}, \widetilde{\mathbf{S}}_{\mathbb{Q}}, \overline{\mathbf{R}})$ by its \mathbb{N} -grading, by setting

$$F^i B(S_{\mathbb{k}}^{0,0}, \widetilde{\mathbf{S}}_{\mathbb{Q}}, \overline{\mathbf{R}})(n) := \begin{cases} B(S_{\mathbb{k}}^{0,0}, \widetilde{\mathbf{S}}_{\mathbb{Q}}, \overline{\mathbf{R}})(n) & n \geq i \\ 0 & n < i. \end{cases}$$

The associated graded of this filtration may be canonically identified with $B(S_{\mathbb{k}}^{0,0}, \widetilde{\mathbf{S}}_{\mathbb{Q}}, \overline{\mathbf{R}})$ as an element in $\text{sMod}_{\mathbb{k}}^{\mathbb{Q}}$, but its $\overline{\mathbf{R}}$ -module structure is now trivial, i.e. factors through $\epsilon : \overline{\mathbf{R}} \rightarrow S_{\mathbb{k}}^{0,0}$. It induces a filtration of the object $B(B(S_{\mathbb{k}}^{0,0}, \widetilde{\mathbf{S}}_{\mathbb{Q}}, \overline{\mathbf{R}}), \overline{\mathbf{R}}, \underline{\mathbb{k}}_{\mathbb{Q}})$, having associated graded

$$\text{gr} B(B(S_{\mathbb{k}}^{0,0}, \widetilde{\mathbf{S}}_{\mathbb{Q}}, \overline{\mathbf{R}}), \overline{\mathbf{R}}, \underline{\mathbb{k}}_{\mathbb{Q}}) \simeq B(S_{\mathbb{k}}^{0,0}, \widetilde{\mathbf{S}}_{\mathbb{Q}}, \overline{\mathbf{R}}) \otimes B(S_{\mathbb{k}}^{0,0}, \overline{\mathbf{R}}, \underline{\mathbb{k}}_{\mathbb{Q}}).$$

It therefore gives a spectral sequence of signature

$$E_{n,p,q}^1 = \bigoplus_{a+b=p+q} \text{Ind}_{Q_{-p} \times Q_{n+p}}^{Q_n} H_{-p,a}^{\tilde{S}_Q}(\overline{\mathbf{R}}) \otimes H_{n+p,b}^{\overline{\mathbf{R}}}(\mathbb{k}_Q) \implies H_{n,p+q}^{\tilde{S}_Q}(\mathbb{k}_Q),$$

with $p \leq 0$ and differentials $d^r : E_{n,p,q}^r \rightarrow E_{n,p-r,q+r-1}^r$.

We now prove the statement in the proposition by induction on d . By (2.2) it holds for $d = 0$. Suppose then that it holds for all $d < D$. Similar to the proof of (2.4) we have that $H_{0,*}^{\tilde{S}_Q}(\overline{\mathbf{R}})$ is \mathbb{k} supported in degree zero, so $E_{n,0,D}^1 \cong H_{n,D}^{\overline{\mathbf{R}}}(\mathbb{k}_Q)$. By our discussion above and the inductive assumption, for $1 \leq r < n$ the target of the differential

$$d^r : E_{n,0,D}^r \longrightarrow E_{n,-r,D+r-1}^r = \bigoplus_{a+b=D-1} \text{Ind}_{Q_r \times Q_{n-r}}^{Q_n} H_{r,a}^{\tilde{S}_Q}(\overline{\mathbf{R}}) \otimes H_{n-r,b}^{\overline{\mathbf{R}}}(\mathbb{k}_Q)$$

vanishes as long as $D-1 < (\nu \cdot r + \xi) + (\nu \cdot (n-r) + \xi)$, so as long as $D < \nu \cdot n + \xi$ using that $\xi \geq -1$.

Similarly, the target of $d^n : E_{n,0,D}^n \rightarrow E_{n,-n,D+n-1}^n$ is identified with $H_{n,D-1}^{\tilde{S}_Q}(\overline{\mathbf{R}})$ by (2.4), so vanishes as long as $D-1 < \nu \cdot n + \xi$ and hence in particular as long as $D < \nu \cdot n + \xi$. Thus in this range $E_{n,0,D}^1 = E_{n,0,D}^\infty$. But $E_{n,0,D}^\infty$ is a subquotient of $H_{n,D}^{\tilde{S}_Q}(\mathbb{k}_Q)$, which by our earlier discussion vanishes as long as $D < \nu \cdot n + \xi$. Thus $E_{n,0,D}^1 \cong H_{n,D}^{\overline{\mathbf{R}}}(\mathbb{k}_Q)$ vanishes in this range. \square

Remark 2.4. A reason the proof of Proposition 2.3 is similar to that of [RW22, Theorem 6.2 (ii)] (a result originally proven in [Pat20]) is that they have a common generalization. In both cases, there is a map of rings $S \rightarrow R$ and an R -module M . The hypotheses are a vanishing line for the derived indecomposables of R and M as S -modules and the conclusion is a vanishing line for the derived indecomposables of M as an R -module. The results of [RW22] allows us to relate constructions like $W_\bullet(\mathbb{G}; n)$ and ‘‘central stability homology’’ to derived indecomposables.

In order to define the E_1 -splitting complex, we use assumption (iii). Then the semi-simplicial set $S_\bullet^{E_1}(\mathbb{G}; n)$ has p -simplices

$$S_p^{E_1}(\mathbb{G}; n) := \text{colim}_{n_0, \dots, n_{p+1} \in \mathbb{N}_{>0}^{p+2}} \mathbb{G}(n_0 \oplus \dots \oplus n_{p+1}, n) \cong \coprod_{\substack{n_0 + \dots + n_{p+1} = n \\ n_i > 0}} \frac{\Gamma_n}{\Gamma_{n_0} \times \dots \times \Gamma_{n_{p+1}}},$$

and face maps induced by the monoidal structure.

Proposition 2.5. *Suppose that*

- (i) $\tilde{H}_d(S_\bullet^{E_1}(\mathbb{G}; n); \mathbb{k}) = 0$ for $d < n - 2$,
- (ii) $\tilde{H}_d(S_\bullet^{E_1}(\mathbb{Q}; n); \mathbb{k}) = 0$ for $d < n - 2$.

Then $H_{n,d}(B(S_{\mathbb{k}}^{0,0}, \overline{\mathbf{R}}, \mathbb{k}_Q)) = 0$ for $d < \frac{2}{3} \cdot n$.

Proof. We work in the category $\text{sMod}_{\mathbb{k}}^Q$, and continue to use the nonunital E_2 -algebra \mathbf{R} in this category which we described in the proof of Proposition 2.3. We will apply [RW22, Theorem A.1] to the map of nonunital E_2 -algebras $\mathbf{R} \rightarrow \mathbf{S} \xrightarrow{\sim} (\mathbb{k}_Q)_{>0}$ given by a relative CW-approximation of the Postnikov truncation map.

We first establish a vanishing line for $H_{*,*}^{E_2}(\mathbf{R})$. Applied to $\mathbb{L}p_* Q_{\mathbb{L}}^{E_1}((\mathbb{k}_G)_{>0}) \simeq Q_{\mathbb{L}}^{E_1}(\mathbf{R})$, the spectral sequence (2.1) takes the form

$$E_{n,s,t}^2 = H_s(K_n; H_{n,t}^{E_1}((\mathbb{k}_G)_{>0})) \implies H_{n,s+t}^{E_1}(\mathbf{R}).$$

Combining Proposition 17.4 and Lemma 17.10 of [GKRW18] gives $H_{n,t}^{E_1}((\mathbb{k}_G)_{>0}) \cong \tilde{H}_{t-1}(S_\bullet^{E_1}(\mathbb{G}; n); \mathbb{k})$ for $n > 0$, so under assumption (i) we have $E_{s,t}^2 = 0$ for $t < n - 1$, and so $H_{n,d}^{E_1}(\mathbf{R}) = 0$ for $d < n - 1$. Using [GKRW18, Theorem 14.4] it follows that $H_{n,d}^{E_2}(\mathbf{R}) = 0$ for $d < n - 1$ too.

Similarly, using assumption (ii) it follows that $H_{n,d}^{E_2}(\mathbf{S}) \cong H_{n,d}^{E_2}((\mathbb{k}_Q)_{>0}) = 0$ for $d < n - 1$. The long exact sequence on E_2 -homology for the map $\mathbf{R} \rightarrow \mathbf{S}$ has the form

$$H_{n,d}^{E_2}(\mathbf{R}) \longrightarrow H_{n,d}^{E_2}(\mathbf{S}) \longrightarrow H_{n,d}^{E_2}(\mathbf{S}, \mathbf{R}) \xrightarrow{\partial} H_{n,d-1}^{E_2}(\mathbf{R}) \longrightarrow H_{n,d-1}^{E_2}(\mathbf{S}),$$

so $H_{n,d}^{E_2}(\mathbf{S}, \mathbf{R}) = 0$ for $d < n - 1$. To apply [RW22, Theorem A.1] we need a vanishing line of the form $d < \alpha \cdot n$ for these relative E_2 -homology groups, which we will obtain by considering the homology of \mathbf{R} and \mathbf{S} in low degrees.

The map $\mathbf{R} \rightarrow \mathbf{S}$ is an isomorphism on 0-th homology and an epimorphism on 1-st homology (as $H_{*,1}(\mathbf{S}) = 0$), so it follows from the Hurewicz theorem [GKRW18, Corollary 11.12] that $H_{n,d}^{E_2}(\mathbf{S}, \mathbf{R}) = 0$ for $d \leq 1$ and all n . As these relative homology groups also vanish for $d < n - 1$ as explained above, it follows that they vanish for $d < \frac{2}{3} \cdot n$.

This puts us in a situation to apply [RW22, Theorem A.1], which provides a map

$$H_{n,d}^{\overline{\mathbf{R}}}(\overline{\mathbf{S}}, \overline{\mathbf{R}}) \longrightarrow H_{n,d}^{E_2}(\mathbf{S}, \mathbf{R})$$

which is an isomorphism in degrees $d < \frac{2}{3} \cdot n$: thus the domain also vanishes in this range. The same vanishing line then holds for $H_{n,d}^{\overline{\mathbf{R}}}(\overline{\mathbf{S}}) \cong H_{n,d}(B(S_{\mathbb{k}}^{0,0}, \overline{\mathbf{R}}, \mathbb{k}_Q))$, as required. \square

2.3. Proof of Theorem 2.2.

2.3.1. *The pointwise monoidal structure.* We will make use of an additional monoidal structure on the categories $\text{sMod}_{\mathbb{k}}^{\mathbb{G}}$ and $\text{sMod}_{\mathbb{k}}^{\mathbb{Q}}$ which has not been exploited in [GKRW18], namely the pointwise (or Hadamard) tensor product

$$(A \boxtimes B)(n) := A(n) \otimes B(n).$$

The units for these monoidal structures are the constant functors $\mathbb{k}_{\mathbb{G}}$ and $\mathbb{k}_{\mathbb{Q}}$. As \mathbb{k} is a field, the functor $- \boxtimes -$ preserves weak equivalences in each variable. If $A \in \text{sMod}_{\mathbb{k}}^{\mathbb{G}}$ and $B \in \text{sMod}_{\mathbb{k}}^{\mathbb{Q}}$ then it enjoys a projection formula $p_*(A) \boxtimes B \cong p_*(A \boxtimes p^*(B))$; evaluated at n this is simply Frobenius reciprocity for induction and restriction along $p_n : \Gamma_n \rightarrow Q_n$. The interaction between the pointwise and the Day convolution monoidal structures goes under the name of a “duoidal structure” (see e.g. [GLF16, Section 2.1]), and is a natural distributivity law

$$(A \boxtimes B) \otimes (C \boxtimes D) \longrightarrow (A \otimes C) \boxtimes (B \otimes D).$$

This is merely a morphism, *not* an isomorphism.

2.3.2. *Homology with twisted coefficients.* Considering \mathbb{N} as a braided monoidal category with only identity morphisms, there are morphisms

$$r : \mathbb{G} \xrightarrow{p} \mathbb{Q} \xrightarrow{q} \mathbb{N}$$

of braided monoidal groupoids, where q is defined to be the identity on objects. We will consider objects of $\text{sMod}_{\mathbb{k}}^{\mathbb{N}}$: these again have bigraded homology groups, which are again corepresented by the bigraded sphere objects $S_{\mathbb{k}}^{n,d}$. If V is a $\mathbb{k}_{\mathbb{Q}}$ -module, then the derived Kan extension $\mathbb{L}q_*(V)$ satisfies

$$H_{n,d}(\mathbb{L}q_*(V)) \cong \pi_d \left(\text{hocolim}_{Q_n} V(n) \right) \cong H_d(Q_n; V(n)).$$

Similarly,

$$H_{n,d}(\mathbb{L}r_*(p^*V)) \cong \pi_d \left(\text{hocolim}_{\Gamma_n} p^*V(n) \right) \cong H_d(\Gamma_n; V(n)).$$

We can manipulate the latter object as

$$\begin{aligned}
\mathbb{L}r_*(p^*V) &\simeq \mathbb{L}q_*\mathbb{L}p_*(\underline{\mathbb{k}}_G \boxtimes p^*V) && \text{as } \underline{\mathbb{k}}_G \text{ is the unit for } \boxtimes \\
&\simeq \mathbb{L}q_*\mathbb{L}p_*(\mathbf{T}^+ \boxtimes p^*V) && \text{by homotopy invariance of } \boxtimes \\
&\simeq \mathbb{L}q_*p_*(\mathbf{T}^+ \boxtimes p^*V) && \text{as } \mathbf{T}^+ \boxtimes p^*V \text{ is cofibrant} \\
&\simeq \mathbb{L}q_*(\mathbf{R}^+ \boxtimes V) && \text{by the projection formula} \\
&\simeq \mathbb{L}q_*(\overline{\mathbf{R}} \boxtimes V) && \text{by homotopy invariance of } \boxtimes.
\end{aligned}$$

2.3.3. *The stability map.* The object $\overline{\mathbf{R}} \boxtimes V \in \text{sMod}_{\mathbb{k}}^{\mathbb{Q}}$ obtains the structure of a left $\overline{\mathbf{R}}$ -module by writing $\overline{\mathbf{R}} = \overline{\mathbf{R}} \boxtimes \underline{\mathbb{k}}_{\mathbb{Q}}$ and using the distributivity law

$$(\overline{\mathbf{R}} \boxtimes \underline{\mathbb{k}}_{\mathbb{Q}}) \otimes (\overline{\mathbf{R}} \boxtimes V) \longrightarrow (\overline{\mathbf{R}} \otimes \overline{\mathbf{R}}) \boxtimes (\underline{\mathbb{k}}_{\mathbb{Q}} \otimes V)$$

followed by the multiplication $\overline{\mathbf{R}} \otimes \overline{\mathbf{R}} \rightarrow \overline{\mathbf{R}}$ and the structure map $\underline{\mathbb{k}}_{\mathbb{Q}} \otimes V \rightarrow V$.

Under the isomorphisms $\mathbb{k} = H_0(K_1; \mathbb{k}) \cong H_{1,0}(\overline{\mathbf{R}}) \cong [S_{\mathbb{k}}^{1,0}, \overline{\mathbf{R}}]_{\text{sMod}_{\mathbb{k}}^{\mathbb{Q}}}$ the canonical generator is represented by a map $\sigma : S_{\mathbb{k}}^{1,0} \rightarrow \overline{\mathbf{R}}$. Using the left $\overline{\mathbf{R}}$ -module structure described above, we can form the map

$$(2.5) \quad \sigma \cdot - : S_{\mathbb{k}}^{1,0} \otimes (\overline{\mathbf{R}} \boxtimes V) \xrightarrow{\sigma \otimes id} \overline{\mathbf{R}} \otimes (\overline{\mathbf{R}} \boxtimes V) \xrightarrow{\sim} (\overline{\mathbf{R}} \boxtimes V).$$

Unravelling definitions shows that on $H_{n,d}(-)$ this has the form

$$\sigma \cdot - : H_d(K_{n-1}; V(n-1)) \longrightarrow H_d(K_n; V(n)),$$

where the K_i act trivially on the $V(i)$, as these are pulled back from the Q_i . On applying $\mathbb{L}q_*(-)$ to (2.5) we obtain a map $\sigma \cdot - : S_{\mathbb{k}}^{1,0} \otimes \mathbb{L}q_*(\overline{\mathbf{R}} \boxtimes V) \rightarrow \mathbb{L}q_*(\overline{\mathbf{R}} \boxtimes V)$, and again unravelling definitions shows that on $H_{n,d}(-)$ it has the form

$$\sigma \cdot - : H_d(\Gamma_{n-1}; V(n-1)) \longrightarrow H_d(\Gamma_n; V(n)).$$

This is the map to which the conclusion of Theorem 2.2 refers. Writing $(\overline{\mathbf{R}} \boxtimes V)/\sigma$ for the homotopy cofibre of (2.5), in order to prove that theorem we must establish the claimed vanishing line for the bigraded homology groups of the object $\mathbb{L}q_*((\overline{\mathbf{R}} \boxtimes V)/\sigma) \in \text{sMod}_{\mathbb{k}}^{\mathbb{N}}$.

2.3.4. *The filtration.* In view of Remark 2.1, axiom (III) says that $H_{n,d}^{\overline{\mathbf{R}}}(\underline{\mathbb{k}}_{\mathbb{Q}}) = 0$ for $d < \nu \cdot n + \xi$ and $n \geq 1$. Combined with equations (2.2), (2.3), (2.4) from the proof of Proposition 2.5 it follows that $H_{n,d}^{\overline{\mathbf{R}}}(\underline{\mathbb{k}}_{\mathbb{Q}}, \overline{\mathbf{R}})$ vanishes for $d < \nu \cdot n + \xi$, or for $d \leq 1$, or for $n = 0$. Applying [GKRW18, Theorem 11.21], it follows that there is a relative CW $\overline{\mathbf{R}}$ -module $\overline{\mathbf{R}} \rightarrow \mathbf{M} \xrightarrow{\sim} \underline{\mathbb{k}}_{\mathbb{Q}}$ which only has relative (n, d) -cells with $d \geq 2$, $n \geq 1$, and $d \geq \nu \cdot n + \xi$. There is a homotopy cofibre sequence

$$(2.6) \quad (\overline{\mathbf{R}} \boxtimes V)/\sigma \longrightarrow (\mathbf{M} \boxtimes V)/\sigma \longrightarrow ((\mathbf{M}/\overline{\mathbf{R}}) \boxtimes V)/\sigma,$$

as $- \boxtimes V$ preserves homotopy cofibre sequences, and $(-)/\sigma$ preserves homotopy cofibre sequences of left $\overline{\mathbf{R}}$ -modules. The skeletal filtration of the relative CW $\overline{\mathbf{R}}$ -module $\overline{\mathbf{R}} \rightarrow \mathbf{M}$ induces a skeletal filtration of the CW $\overline{\mathbf{R}}$ -module $\mathbf{M}/\overline{\mathbf{R}}$, having associated graded

$$\text{gr}(\mathbf{M}/\overline{\mathbf{R}}) \simeq \bigoplus_{\alpha} S_{\mathbb{k}}^{n_{\alpha}, d_{\alpha}} \otimes \overline{\mathbf{R}}$$

with $d_{\alpha} \geq 2$, $n_{\alpha} \geq 1$, and $d_{\alpha} \geq \nu \cdot n_{\alpha} + \xi$.

2.3.5. The spectral sequence argument. With these preparations completed, we now embark on the proof of Theorem 2.2 proper. In what follows, it will be convenient to declare that Γ_n and Q_n is the “empty group” if $n < 0$; the classifying space of the empty group is the empty space, and its homology vanishes in all degrees. With this convention, axiom (II) holds for all $n \in \mathbb{Z}$, even though it was only assumed for $n \geq 1$. In particular, when $n = 0$ we have $H_d(Q_0, Q_{-1}; V(0), V(-1)) = H_d(Q_0, V(0))$, and axiom (II) holds since $\tau \leq 0$.

We proceed by induction on d . If $d = 0$ then

$$H_0(\Gamma_n, \Gamma_{n-1}; V(n), V(n-1)) = H_0(Q_n, Q_{n-1}; V(n), V(n-1)),$$

as Γ_i acts on $V(i)$ via its surjection to Q_i , and the latter vanishes for $0 < \theta \cdot (n + \tau)$ by axiom (II), so for $0 < \min(\theta, \frac{\nu}{2-\xi}) \cdot (n + \tau)$ too. Suppose then that the conclusion of Theorem 2.2 holds for all $d < D$.

The skeletal filtration of the CW $\overline{\mathbf{R}}$ -module $\mathbf{M}/\overline{\mathbf{R}}$ described above induces a filtration of the object $\mathbb{L}q_*(((\mathbf{M}/\overline{\mathbf{R}}) \boxtimes V)/\sigma)$, yielding a spectral sequence

$$E_{n,s,t}^1 = \bigoplus_{\alpha \text{ s.t. } d_\alpha = s} H_{n,s+t}(\mathbb{L}q_*(((S_{\mathbb{k}}^{n_\alpha, d_\alpha} \otimes \overline{\mathbf{R}}) \boxtimes V)/\sigma)) \implies H_{n,s+t}(\mathbb{L}q_*(((\mathbf{M}/\overline{\mathbf{R}}) \boxtimes V)/\sigma)).$$

Unravelling definitions, we see that

$$\begin{aligned} H_{n,d}(\mathbb{L}q_*(((S_{\mathbb{k}}^{n_\alpha, d_\alpha} \otimes \overline{\mathbf{R}}) \boxtimes V)/\sigma)) &\cong H_{d-d_\alpha}(\Gamma_{n-n_\alpha}, \Gamma_{n-1-n_\alpha}; V(n), V(n-1)) \\ &= H_{d-d_\alpha}(\Gamma_{n-n_\alpha}, \Gamma_{n-1-n_\alpha}; \Sigma^{n_\alpha} V(n-n_\alpha), \Sigma^{n_\alpha} V(n-1-n_\alpha)). \end{aligned}$$

As $d_\alpha \geq 2$, and $\Sigma^{n_\alpha} V$ is in the class \mathcal{V} by axiom (I), these groups vanish by inductive hypothesis as long as $d \leq D + 1$ and $d - d_\alpha < \min(\theta, \frac{\nu}{2-\xi}) \cdot (n - n_\alpha + \tau)$. It follows from the spectral sequence that $H_{n,d}(\mathbb{L}q_*(((\mathbf{M}/\overline{\mathbf{R}}) \boxtimes V)/\sigma)) = 0$ as long as $d \leq D + 1$ and $d < \min(\theta, \frac{\nu}{2-\xi}) \cdot (n + \tau) + \inf_\alpha \{d_\alpha - \min(\theta, \frac{\nu}{2-\xi}) n_\alpha\}$.

We now consider the portion of the long exact sequence

$$H_{n,D+1}(\mathbb{L}q_*(((\mathbf{M}/\overline{\mathbf{R}}) \boxtimes V)/\sigma)) \xrightarrow{\partial} H_{n,D}(\mathbb{L}q_*((\overline{\mathbf{R}} \boxtimes V)/\sigma)) \longrightarrow H_{n,D}(\mathbb{L}q_*((\mathbf{M} \boxtimes V)/\sigma))$$

obtained by applying $\mathbb{L}q_*(-)$ to the homotopy cofibre sequence (2.6). The equivalence $\mathbf{M} \xrightarrow{\sim} \underline{\mathbb{k}}_{\mathbb{Q}}$, and the fact that the latter is the unit for $- \boxtimes -$, identifies the right-hand term with

$$H_D(Q_n, Q_{n-1}; V(n), V(n-1)),$$

which vanishes as long as $D < \theta \cdot (n + \tau)$ by axiom (II). With the discussion above it follows that the middle term vanishes as long as

$$D < \min(\theta \cdot (n + \tau), \min(\theta, \frac{\nu}{2-\xi}) \cdot (n + \tau) + \inf_\alpha \{d_\alpha - \min(\theta, \frac{\nu}{2-\xi}) n_\alpha - 1\}).$$

Using that $d_\alpha \geq \nu \cdot n_\alpha + \xi$, we see that

$$\begin{aligned} d_\alpha - \min(\theta, \frac{\nu}{2-\xi}) n_\alpha - 1 &\geq d_\alpha - \min(\theta, \frac{\nu}{2-\xi}) \frac{1}{\nu} (d_\alpha - \xi) - 1 \\ &= d_\alpha (1 - \min(\frac{\theta}{\nu}, \frac{1}{2-\xi})) + (\min(\frac{\theta}{\nu}, \frac{1}{2-\xi}) \xi - 1). \end{aligned}$$

Noting now that $1 - \min(\frac{\theta}{\nu}, \frac{1}{2-\xi}) \geq 0$, since $\xi \leq 1$, and that $d_\alpha \geq 2$, we find

$$\begin{aligned} d_\alpha (1 - \min(\frac{\theta}{\nu}, \frac{1}{2-\xi})) + (\min(\frac{\theta}{\nu}, \frac{1}{2-\xi}) \xi - 1) &\geq 2(1 - \min(\frac{\theta}{\nu}, \frac{1}{2-\xi})) + (\min(\frac{\theta}{\nu}, \frac{1}{2-\xi}) \xi - 1) \\ &= 1 - \min(\frac{\theta}{\nu}, \frac{1}{2-\xi}) (2 - \xi) \geq 0, \end{aligned}$$

so $H_{n,D}(\mathbb{L}q_*((\overline{\mathbf{R}} \boxtimes V)/\sigma))$ vanishes as long as $D < \min(\theta, \frac{\nu}{2-\xi}) \cdot (n + \tau)$, as required.

3. The destabilisation complexes for the image of the Burau representation

The goal in this section is to prove high connectivity of the complexes $W_\bullet(\mathbb{Q}; n)$ used in the proof of Theorem 1.4, i.e. the destabilisation complexes associated to the family of groups given by the image of the integral Burau representation. The reader who is more interested in Theorems 1.1, 1.2, and 1.3 may skip ahead to Section 4. The proof is an adaptation of an argument of Mirzaii–van der Kallen [MvdK01] in the case of the usual symplectic groups, taking as input also a result of Charney [Cha84] showing high connectivity of partial basis complexes with a congruence condition.

3.1. Even and odd symplectic groups. The images of the integral Burau representation form a family of congruence subgroups of the *even and odd* symplectic groups, as introduced by Gelfand–Zelevinsky [GZ85]. They were introduced to “interpolate” between the usual symplectic groups — in many situations, it may seem that there ought to be some such family of “missing” groups.

Definition 3.1. The *odd symplectic group* Sp_{2n-1} is the parabolic subgroup of Sp_{2n} stabilising a unimodular vector.

Remark 3.2. The even and odd symplectic groups may be considered as algebraic groups over \mathbb{Z} , but we will really only be interested in their integer points.

Remark 3.3. A closely related definition of odd symplectic group was independently proposed by Proctor [Pro88]. Proctor defines the group Sp_{2n-1}^P as the subgroup of GL_{2n-1} stabilising a skew-symmetric form of maximal rank. There is a natural surjection $\mathrm{Sp}_{2n-1} \rightarrow \mathrm{Sp}_{2n-1}^P$, and all of the representations of the odd symplectic groups which we will consider will factor through the projections to Proctor’s groups. Nevertheless, it will be important that we work with the Gelfand–Zelevinsky groups throughout. One reason is that Proctor’s groups do not even admit suitable stabilisations: the embedding of Sp_{2n-2} into Sp_{2n} does not factor through the evident embedding $\mathrm{Sp}_{2n-2} \hookrightarrow \mathrm{Sp}_{2n-1}^P$, but it obviously factors through Sp_{2n-1} . Our perspective will be that the Gelfand–Zelevinsky definition of odd symplectic group is for all purposes the correct one, but that it has the hiccup that the “defining” n -dimensional representation of Sp_n is not faithful for odd n ; the group Sp_n^P is the image of the defining representation. When discussing the representations of odd symplectic groups defined by highest weights, we may if we prefer tacitly work with Proctor’s groups instead.

The first goal of this section is to make the system of all even and odd symplectic groups into a braided monoidal groupoid. The monoidal structure is somewhat subtle. In particular, it does *not* restrict to the usual block-sum of matrices when restricted to the classical even symplectic groups. In fact, the even symplectic groups do not even form a subgroupoid, as the indexing in the groupoid is “off by one” compared to the classical one — the groupoid is of the form $\coprod_{n \geq 0} \mathrm{Sp}_{n-1}(\mathbb{Z})$, so to speak. Moreover, unlike the block-sum, the braiding is *not* a symmetry; it is genuinely braided monoidal.

3.1.1. A monoidal groupoid of even and odd symplectic groups. Let \mathbb{M} be the following groupoid. Objects are triples $(M, \langle -, - \rangle, \phi)$ where M is an abelian group, $\langle -, - \rangle$ is a skew-symmetric bilinear form on M , and $\phi : M \rightarrow \mathbb{Z}$ is linear. Morphisms are isomorphisms of this data. We generally denote an object of \mathbb{M} by M , leaving the remaining data implicit in the notation.

The category \mathbb{M} admits a monoidal structure given by

$$(M, \langle -, - \rangle_M, \phi_M) \oplus (M', \langle -, - \rangle_{M'}, \phi_{M'}) = (M \oplus M', \langle -, - \rangle_{M \oplus M'}, \phi_{M \oplus M'})$$

where for $x, y \in M$ and $x', y' \in M'$ we have

$$\langle x + x', y + y' \rangle_{M \oplus M'} = \langle x, y \rangle_M + \langle x', y' \rangle_{M'} + \phi_M(x)\phi_{M'}(y') - \phi_M(y)\phi_{M'}(x'),$$

and $\phi_{M \oplus M'}(x + x') = \phi_M(x) + \phi_{M'}(x')$. The sum of two morphisms is the block-sum.

Let \mathbb{Z} denote the object of \mathcal{M} given by the triple $(\mathbb{Z}, 0, \text{id})$. Let \mathcal{T} denote the full subgroupoid spanned by the objects $\mathbb{Z}^{\oplus n}$ for $n \geq 0$. Write $T_n = \text{Aut}_{\mathcal{T}}(\mathbb{Z}^{\oplus n})$. We denote the standard basis for $\mathbb{Z}^{\oplus n}$ by e_1, \dots, e_n . Explicitly, $\mathbb{Z}^{\oplus n}$ has the bilinear form

$$\langle e_i, e_j \rangle = \begin{cases} 1 & i < j \\ 0 & i = j, \\ -1 & i > j \end{cases}$$

and $\phi(e_i) = 1$ for all i . The monoidal structure furnishes maps $T_n \times T_m \rightarrow T_{n+m}$, given by block-sum of matrices.

Theorem 3.4. *There are isomorphisms $T_n \cong \text{Sp}_{n-1}(\mathbb{Z})$.*

Proof. Let $M = \mathbb{Z}^{\oplus n}$.

If n is odd, we claim first that there is a canonical T_n -invariant isomorphism $M \cong \ker(\phi) \oplus \mathbb{Z}v_n$, where $v_n \in M$ is the vector $e_1 - e_2 + e_3 - \dots + e_n$. Indeed it is clear that T_n preserves $\ker(\phi)$. But T_n also preserves v_n , since v_n is the unique vector in M satisfying $\phi(v_n) = 1$ and $\langle v_n, - \rangle$ is identically zero. It follows that T_n is just the group of automorphisms of $\ker(\phi)$ preserving the restriction of the form $\langle -, - \rangle$. But $\langle -, - \rangle$ is a symplectic form on $\ker(\phi)$.

If n is even, then $\langle -, - \rangle$ is a symplectic form on M . Via the symplectic form, we can identify $\phi \in M^\vee$ with a vector in M , which one verifies to be given by $v_n = e_1 - e_2 + \dots - e_n$. (That is, v_n is the unique vector satisfying $\langle v_n, x \rangle = \phi(x)$ for all $x \in M$.) Thus T_n is identified with the stabiliser of the unimodular vector v_n inside $\text{Sp}(M)$. \square

Corollary 3.5. *The element $v_n = e_1 - e_2 + \dots + (-1)^{n-1}e_n \in \mathbb{Z}^{\oplus n}$ is fixed by every element of T_n .*

Proof. This was observed in the proof of the previous theorem. \square

Definition 3.6. By the *defining representation* of T_n or $\text{Sp}_{n-1}(\mathbb{Z})$ we mean the representation $\ker(\phi)$.

Remark 3.7. A reader may object to Definition 3.6, saying that the action of T_n on $\mathbb{Z}^{\oplus n}$ ought to be considered its defining representation. But Definition 3.6 is forced upon us if we want the defining representation of $\text{Sp}_{2g}(\mathbb{Z})$ to be the usual representation of rank $2g$.

3.1.2. The Burau representation. The Burau representation is a well-studied sequence of representations of the braid groups, whose definition we now recall. In general the representation has coefficients in the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$. We will only be interested in its specialization to $t = -1$, which is called the *integral Burau representation*.

Definition 3.8. The (unreduced) *integral Burau representation* of the braid group β_n is the representation which maps the standard generator σ_i of β_n (where $i = 1, \dots, n-1$) to the block-matrix

$$\begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \mathbf{1}_{n-i-1} \end{pmatrix}$$

where $B = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

We could replace B by its transpose in this definition, and we would get an isomorphic representation of the braid group. Indeed, transposing B amounts to conjugating by the matrix $\text{diag}(1, -1, 1, \dots)$. For this reason there are differing conventions in the literature for exactly how the Burau matrices are defined.

Definition 3.9. The *reduced integral Burau representation* is the rank $n-1$ subrepresentation of the integral Burau representation, given by β_n acting on $\{(x_1, \dots, x_n) \in \mathbb{Z}^n : x_1 + \dots + x_n = 0\}$.

The topological significance of the reduced integral Burau representation is the following. A classifying space for the braid group β_n is the space $\text{Conf}_n(\mathbb{C})$ of square-free monic polynomials of degree n over \mathbb{C} . For each such square-free polynomial f , we may consider the nonsingular hyperelliptic algebraic curve $y^2 = f(x)$ inside \mathbb{C}^2 . The “universal family”

$$\{(f, x, y) \in \text{Conf}_n(\mathbb{C}) \times \mathbb{C}^2 : y^2 = f(x)\}$$

defines a surface bundle over $\text{Conf}_n(\mathbb{C})$, consisting of surfaces of genus g with 1 (resp. 2) punctures when $n = 2g + 1$ (resp. $2g + 2$). The monodromy action of the fundamental group of the base of the surface bundle, on the first homology of the fibre, is precisely the reduced integral Burau representation.

3.1.3. A braiding on the monoidal groupoid of symplectic groups. Using the distinguished vector v_n from Corollary 3.5, we can define a braiding on the monoidal category $(\mathbb{T}, \oplus, 0)$. There are in fact two natural choices: one may consider either the isomorphisms $b_{n,m} : \mathbb{Z}^{\oplus n} \oplus \mathbb{Z}^{\oplus m} \rightarrow \mathbb{Z}^{\oplus m} \oplus \mathbb{Z}^{\oplus n}$ defined by the formula

$$(3.1) \quad (x, y) \mapsto (y + 2\phi(x)v_m, (-1)^m x)$$

for $x \in \mathbb{Z}^{\oplus n}$ and $y \in \mathbb{Z}^{\oplus m}$, or the inverse of this braiding, which is given by

$$(3.2) \quad (x, y) \mapsto ((-1)^n y, x + (-1)^{n+1} 2\phi(y)v_n).$$

It is routine to verify that these isomorphisms preserve ϕ and $\langle -, - \rangle$, and satisfy the hexagon identities; we will not write the verification out.

The braiding (3.1) is the one that arises naturally in the situation of Theorem 1.4, since the homomorphism $\beta_n \rightarrow \text{Aut}(\mathbb{Z}^{\oplus n})$ induced by (3.1) agrees with the integral Burau representation, as one sees by computing the images of the standard generators of the braid group. This homomorphism has image in $\text{Aut}_{\mathbb{T}}(\mathbb{Z}^{\oplus n}) = T_n \cong \text{Sp}_{n-1}(\mathbb{Z})$, and restricting the defining representation of $\text{Sp}_{n-1}(\mathbb{Z})$ to β_n gives the reduced integral Burau representation.

On the other hand, the inverse braiding (3.2) gives rise to more conveniently described destabilisation complexes (Lemma 3.13). We will work with the braiding (3.2) by default in this section. This is harmless, since the resulting destabilisation complexes are in any case homeomorphic (Remark 3.15).

We write $T_n[2]$ for the subgroup of T_n of matrices reducing to the identity mod 2. Under the isomorphism of Theorem 3.4, the group $T_n[2]$ is identified with the principal level 2 congruence subgroup of $\text{Sp}_{n-1}(\mathbb{Z})$, i.e. the kernel of $\text{Sp}_{n-1}(\mathbb{Z}) \rightarrow \text{Sp}_{n-1}(\mathbb{Z}/2)$.

The action of the symmetric group \mathfrak{S}_n on $\mathbb{Z}^{\oplus n}$ does *not* preserve the form $\langle -, - \rangle$, but it does preserve the reduction of this form mod 2. Thus it makes sense to define Q_n to be the subgroup of T_n consisting of matrices reducing to a permutation matrix mod 2. We have a chain of subgroups $T_n[2] \subset Q_n \subset T_n$, and two short exact sequences

$$\begin{aligned} 1 \rightarrow T_n[2] \rightarrow Q_n \rightarrow \mathfrak{S}_n \rightarrow 1, \\ 1 \rightarrow T_n[2] \rightarrow T_n \rightarrow \text{Sp}_{n-1}(\mathbb{Z}/2) \rightarrow 1. \end{aligned}$$

We write \mathbb{Q} for the monoidal subgroupoid $\{Q_n\} \subset \{T_n\}$, which inherits a braiding from \mathbb{T} by observing that the matrix representing the braiding (3.2) reduces to a permutation matrix mod 2.

Theorem 3.10. [A’C79, BPS22]. *The image of the integral Burau representation $\beta_n \rightarrow \text{Sp}_{n-1}(\mathbb{Z})$ is the congruence subgroup Q_n .*

3.2. The complexes. With the braiding on \mathbb{T} and \mathbb{Q} in place, the destabilisation complexes $W_{\bullet}(\mathbb{T}; n)$ and $W_{\bullet}(\mathbb{Q}; n)$ are well-defined. Let us first make these complexes explicit. We focus on $W_{\bullet}(\mathbb{Q}; n)$, but it will be clear that the same description works mutatis mutandis in both cases.

Lemma 3.11. *The Q_n -stabiliser of $e_n \in \mathbb{Z}^{\oplus n}$ is isomorphic to Q_{n-1} , identified with the automorphisms of $\text{Span}\{e_1, \dots, e_{n-1}\} = \mathbb{Z}^{\oplus(n-1)}$. By induction, the stabiliser of e_{n-p}, \dots, e_n is Q_{n-p-1} . Similarly for T_n and $T_n[2]$.*

Proof. Note that $\text{Span}\{e_1, \dots, e_{n-1}\} = \{u \in \mathbb{Z}^n : \langle u, e_n \rangle = \phi(u)\}$. So the stabiliser of e_n also preserves $\text{Span}\{e_1, \dots, e_{n-1}\}$, which is equipped with the same bilinear form $\langle -, - \rangle$ and linear form ϕ as $\mathbb{Z}^{\oplus(n-1)}$. \square

Definition 3.12. A tuple (u_1, \dots, u_n) of vectors in $\mathbb{Z}^{\oplus n}$ is a *Q-basis*, if there is a matrix $A \in Q_n$ such that $Ae_i = u_i$ for $i = 1, \dots, n$. A tuple $(u_{n-p}, \dots, u_n) \in \mathbb{Z}^{\oplus n}$, where $p = 0, \dots, n-1$, is called a *partial Q-basis* if it can be completed to a Q-basis (u_1, \dots, u_n) .

Lemma 3.13. *The destabilization complex $W_\bullet(Q, n)$ is isomorphic to the semisimplicial set whose p -simplices are partial Q-bases of size $p+1$, and whose face maps are given by deleting entries.*

Proof. We need to unwind definitions from [RWW17, Section 2]. The set $W_p(Q, n)$ consists of equivalence classes of maps $\mathbb{Z}^{\oplus n} \rightarrow \mathbb{Z}^{\oplus n}$, with two maps being equivalent if they differ by precomposition by $\text{Aut}_{\mathbb{Q}}(\mathbb{Z}^{\oplus(n-p-1)})$ acting on $\mathbb{Z}^{\oplus(n-p-1)} \oplus \mathbb{Z}^{\oplus(p+1)} = \mathbb{Z}^{\oplus n}$. Lemma 3.11 implies that $W_p(Q, n)$ is in bijection with the set of all partial Q-bases of size $(p+1)$, via the function

$$(A : \mathbb{Z}^{\oplus n} \rightarrow \mathbb{Z}^{\oplus n}) \mapsto (Ae_{n-p}, \dots, Ae_n).$$

Under this identification, the face maps $d_i : W_p(Q, n) \rightarrow W_{p-1}(Q, n)$ may be written as follows:

$$d_0(u_{n-p}, \dots, u_n) = (u_{n-p+1}, \dots, u_n), \quad \text{and} \quad d_i = d_0 \circ b_{i,1}^{-1},$$

where $b_{i,1}^{-1}$ denotes acting by the inverse of the braiding (3.2), switching the first two factors of $\mathbb{Z}^{\oplus i} \oplus \mathbb{Z} \oplus \mathbb{Z}^{\oplus(p-i)} \cong \mathbb{Z}^{\oplus(p+1)}$. This action is given by

$$b_{i,1}^{-1}(u_{n-p}, \dots, u_n) = (w, u_{n-p}, \dots, \widehat{u}_{n-p+i}, \dots, u_n),$$

where $w = 2u_{n-p} - 2u_{n-p+1} + \dots \pm 2u_{n-p+i-1} \mp u_{n-p+i}$. The conclusion follows. \square

Remark 3.14. Although the destabilisation complexes are naturally semisimplicial sets, we will prefer to think of $W_\bullet(Q; n)$ as a simplicial complex. To see that this is harmless, note that if $\{u_i\}_{i \in I}$ is a set of vectors, then there is at most one total order of the set I for which $\langle u_i, u_j \rangle = 1$ for $i < j$ in I ; the existence of such an ordering may be considered as a property of a set of vectors. Thus partial Q-bases are canonically ordered. We refer the reader to [RWW17, Section 2.1] for a general discussion of simplicial complexes vs. semisimplicial sets tailored to the setting of this paper; in terms of the discussion in [RWW17, top of page 558], the complex $W_\bullet(Q; n)$ is an instance of situation (B), and we are free to treat $W_\bullet(Q; n)$ as a simplicial complex or a semisimplicial set, whichever is most convenient.

Remark 3.15. The analogue of Lemma 3.13 fails for the braiding (3.1). If we temporarily let Q^{rev} denote the category \mathbb{Q} with the opposite braiding (3.1), then $W_\bullet(Q, n)$ and $W_\bullet(Q^{rev}, n)$ are not isomorphic as semisimplicial sets in any natural way, but their associated simplicial complexes are isomorphic. It is easiest to illustrate this by an example. Note first that they have the same vertex set. If $(x, y, z) \in W_2(Q, n)$ then $(z, 2z - y, 2z - 2y + x)$ is a 2-simplex in $W_\bullet(Q^{rev}, n)$ with the same set of vertices, but with a different order. The condition that a collection of vertices forms a simplex is the same in both cases, but the induced correspondences on sets of p -simplices do not respect the face maps.

Remark 3.16. The complexes $W_\bullet(T; n)$ were proven to be slope $1/3$ connected by Sierra and Wahl [SW24]. Their argument is rather different from the one we use for $W_\bullet(Q; n)$, and does not appear to generalize to our setting.

In our proof of high connectivity, we will quote results from Charney and Mirzaii-van der Kallen. Their papers are neither written in the setting of simplicial complexes nor semisimplicial sets, but in terms of “posets of sequences”, following Maazen [Maa79]. To aid in translation, a “subposet of $\mathcal{O}(V)$ satisfying the chain condition” is exactly the same as a directed simplicial complex, in the sense of the following definition, with vertex set V .

Definition 3.17. A *directed simplicial complex* is a semisimplicial set W_\bullet such that $W_n \rightarrow W_0^{\times(n+1)}$ is injective for all n , and no simplex has a repeated vertex.

That is, whereas faces of a simplicial complex are finite subsets of V , faces of a directed simplicial complex are finite sequences of elements in V (without repetition), and a subsequence of a face is again a face.

3.2.1. *Transitivity arguments.* Denote by $\mathrm{Sp}_{2g}(\mathbb{Z}, 2)$ the principal level 2 congruence subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$. The following is proven in [Vas70, Section 2] as an instance of more general results where $(\mathbb{Z}, 2\mathbb{Z})$ may be replaced by a pair (R, I) satisfying a stable rank condition, and the symplectic group replaced with an orthogonal or unitary group.

Theorem 3.18 (Vaserstein). *The group $\mathrm{Sp}_{2g}(\mathbb{Z}, 2)$ acts transitively on the set of unimodular vectors in \mathbb{Z}^{2g} whose reduction mod 2 is a fixed vector in $(\mathbb{Z}/2)^{2g}$.*

Definition 3.19. Let M be a symplectic module. A *partial isotropic basis* for M is a partial basis w_1, \dots, w_p such that $\langle w_i, w_j \rangle = 0$ for all $1 \leq i, j \leq p$. A *partial symplectic basis* is a partial basis $a_1, b_1, \dots, a_p, b_p$ such that a_1, \dots, a_p and b_1, \dots, b_p are both isotropic, and $\langle a_i, b_j \rangle = \delta_{ij}$.

Lemma 3.20. *The group $\mathrm{Sp}_{2g}(\mathbb{Z}, 2)$ acts transitively on the set of partial isotropic bases (or partial symplectic bases) in \mathbb{Z}^{2g} whose reduction mod 2 is a fixed partial isotropic basis (or partial symplectic basis) in $(\mathbb{Z}/2)^{2g}$.*

Proof. This follows from Theorem 3.18 by an argument exactly like in [MvdK01, Lemma 5.3]. \square

Definition 3.21. For a vector $a = \sum_{i=1}^n a_i e_i \in \mathbb{Z}^{\oplus n}$, let $\rho(a) = \{i \in \{1, \dots, n\} : a_i \equiv 1 \pmod{2}\}$.

Lemma 3.22. *Let $p < n$. The orbit of (e_1, \dots, e_p) under Q_n is the set of vectors (u_1, \dots, u_p) such that*

- (1) $\{u_1, \dots, u_p, v_n\}$ is a partial basis.
- (2) $\langle u_i, u_j \rangle = 1$ for $i < j$.
- (3) The sets $\rho(u_1), \dots, \rho(u_p)$ are disjoint singletons.
- (4) $\phi(u_i) = 1$ for all i .

Proof. It is clear that every element in the orbit of (e_1, \dots, e_p) satisfies (1)–(4). Now take a tuple (u_1, \dots, u_p) satisfying conditions (1)–(4). We construct $A \in Q_n$ taking each u_i to e_i . Since $Q_n \rightarrow \mathfrak{S}_n$ is surjective, we can assume $\rho(u_i) = \{i\}$ for all i . It then suffices to construct $A \in T_n[2]$ taking u_i to e_i for all i . It suffices to prove this for $i = 1$; once we know this, we may without loss of generality take $u_1 = e_1$, in which case Lemma 3.11 gives the result by induction.

Case 1: n is odd. Then there is an isomorphism $T_n[2] \cong \mathrm{Sp}_{n-1}(\mathbb{Z}, 2)$ induced by the action of $T_n[2]$ on $\ker(\phi)$, which is a symplectic module of rank $n - 1$. By Lemma 3.20 there is an element $A \in T_n[2]$ taking $u_1 - v_n$ to $e_1 - v_n$, as both are unimodular vectors in $\ker(\phi)$ with the same reduction mod 2. But A also fixes v_n .

Case 2: n is even. Then $T_n[2]$ is the v_n -stabiliser inside $\mathrm{Sp}_n(\mathbb{Z}, 2)$. In this case (v_n, e_1) and (v_n, u_1) are hyperbolic pairs with the same reduction mod 2, and again the result follows by Lemma 3.20. \square

Remark 3.23. When $p = n$, the orbit of (e_1, \dots, e_n) can be described by replacing condition (1) in the preceding lemma with the condition that $\{u_1, \dots, u_n\}$ is a basis and that $u_1 - u_2 + \dots + (-1)^{n-1} u_n = v_n$.

Lemma 3.24. *Let $p < n/2$. The orbit of $\{e_1 - e_2, e_3 - e_4, \dots, e_{2p-1} - e_{2p}\}$ under Q_n is the set of vectors $\{x_1, \dots, x_p\}$ such that*

- (1) $\{x_1, \dots, x_p, v_n\}$ is a partial basis.
- (2) $\langle x_i, x_j \rangle = 0$ for all i, j .
- (3) The sets $\rho(x_1), \dots, \rho(x_p)$ are disjoint two-element sets.
- (4) $\phi(x_i) = 0$ for all i .

Proof. It is clear that every element in the orbit of $\{e_1 - e_2, e_3 - e_4, \dots, e_{2p-1} - e_{2p}\}$ satisfies (1)–(4). Take $\{x_1, \dots, x_p\}$ satisfying (1)–(4). We will construct $A \in Q_n$ taking each x_i to $e_{2i-1} - e_{2i}$. Since $Q_n \rightarrow \mathfrak{S}_n$ is surjective, by first acting with some lift of an appropriate permutation we may assume that $\rho(x_i) = \{2i - 1, 2i\}$ for all i . It then suffices to construct an element of $T_n[2]$ taking each x_i to $e_{2i-1} - e_{2i}$.

Case 1: n is odd. Then there is an isomorphism $T_n[2] \cong \mathrm{Sp}_{n-1}(\mathbb{Z}, 2)$ induced by the action of $T_n[2]$ on $\ker(\phi)$, which is a symplectic module of rank $n - 1$, and the family $\{x_1, \dots, x_p\}$ is a partial isotropic basis in this symplectic module. So the result follows by Lemma 3.20.

Case 2: n is even. Then $T_n[2]$ is the v_n -stabiliser inside $\mathrm{Sp}_n(\mathbb{Z}, 2)$. In this case $\{x_1, \dots, x_k, v_n\}$ is a partial isotropic basis instead, and again it follows by Lemma 3.20. \square

3.2.2. *The complexes used in the proofs.* Let us now define the complexes that will play the main role in our arguments.

Definition 3.25. Let Z_n be the simplicial complex whose p -simplices are sets $\{u_0, \dots, u_p\}$ of vectors in $\mathbb{Z}^{\oplus n}$, such that:

- (1) $\{u_0, \dots, u_p, v_n\}$ is a partial basis.
- (2) The sets $\rho(u_1), \dots, \rho(u_p)$ are pairwise disjoint.

The complex Z_n itself will not actually play any role in the arguments of the paper; however, all the complexes we care about will be subcomplexes of it.

Definition 3.26. We let $Y_n \subset Z_n$ be the subcomplex consisting of simplices $\{u_0, \dots, u_p\}$ such that $\phi(u_i) = 0$, and $|\rho(u_i)| = 2$, for all i .

Definition 3.27. We let $IX_n \subset Y_n$ be the subcomplex consisting of simplices $\{u_0, \dots, u_p\}$ such that $\langle u_i, u_j \rangle = 0$ for all i, j .

Remark 3.28. Lemma 3.24 says precisely that Q_n acts transitively on p -simplices of IX_n for $p < n/2 - 1$.

Definition 3.29. We let $X_n \subset Z_n$ be the subcomplex consisting of simplices (u_0, \dots, u_p) such that $\phi(u_i) = 1$ and $|\rho(u_i)| = 1$, for all i , and $\langle u_i, u_j \rangle = 1$ for all $i < j$.

Lemma 3.30. *The complex X_n is the $(n - 2)$ -skeleton of the destabilisation complex $W_\bullet(\mathbb{Q}, n)$.*

Proof. This follows from Lemma 3.22 and Lemma 3.13. \square

Remark 3.31. While simplices in Z_n are unordered, we think of simplices of X_n as being ordered, just as in Remark 3.14. In particular, the following definition applies to X_n (but not IX_n or Y_n).

Definition 3.32. For W a directed simplicial complex, the *left-link* of a simplex $\sigma = (v_0, \dots, v_p) \in W_p$ is the directed simplicial complex $\mathrm{LLk}_{W_\bullet}(\sigma)$ whose q -simplices are $(u_0, \dots, u_q) \in W_q$ such that $(u_0, \dots, u_q, v_0, \dots, v_p) \in W_{p+q+1}$.

Lemma 3.33. *If σ is a p -simplex of X_n , then the left-link of σ in X_n is isomorphic to X_{n-p-1} .*

Proof. Using the action of Q_n on X_n , we may assume that $\sigma = (e_{n-p}, \dots, e_n)$. Now note that $\mathrm{Span}\{e_1, \dots, e_{n-p-1}\} = \{u \in \mathbb{Z}^{\oplus n} : \langle e_i, u \rangle = \phi(u) \text{ for } i = n-p, \dots, n\}$. (This is the same argument as Lemma 3.11.) Since every vertex v of the left-link satisfies $\phi(v) = 1$ and $\langle v, e_i \rangle = 1$ for $i = n-p, \dots, n$, one sees that $\mathrm{LLk}_{X_n}(\sigma)$ is the full subcomplex of X_n spanned by those vertices which lie in $\mathrm{Span}\{e_1, \dots, e_{n-p-1}\}$. \square

Definition 3.34. For $\sigma = \{u_0, \dots, u_p\}$ a simplex in Z_n , we let $Z_n(\sigma)$ be the subcomplex of $\mathrm{Lk}_{Z_n}(\sigma)$ consisting of simplices $\{x_0, \dots, x_q\}$ such that $\langle x_i, u_j \rangle = 0$ for all i, j . For $W \subset Z_n$ a subcomplex (which will always be one of X_n, Y_n , or IX_n), we write similarly $W(\sigma) = W \cap Z_n(\sigma)$.

Lemma 3.35. *Let σ be a p -simplex in X_n . Then $IX_n(\sigma)$ is isomorphic to IX_{n-p-1} .*

Proof. Using the action of Q_n on X_n , we may assume that $\sigma = \{e_{n-p}, \dots, e_n\}$. As in the previous lemma we have $\text{Span}\{e_1, \dots, e_{n-p-1}\} = \{u \in \mathbb{Z}^{\oplus n} : \langle e_i, u \rangle = \phi(u) \text{ for } i = n-p, \dots, n\}$. Since every vertex v of $IX_n(\tau)$ satisfies $\phi(v) = 0$ and $\langle v, e_i \rangle = 0$ for $i = n-p, \dots, n$, one sees that $IX_n(\tau)$ is the full subcomplex of IX_n spanned by those vertices which lie in $\text{Span}\{e_1, \dots, e_{n-p-1}\}$. \square

Definition 3.36. If W is a simplicial complex with vertex set W_0 , and S is a set, then we define $W\langle S \rangle$ to be the simplicial complex with vertex set $W_0 \times S$, where $\{(w_0, s_0), \dots, (w_p, s_p)\}$ is a p -simplex of $W\langle S \rangle$ if and only if $\{w_0, \dots, w_p\}$ is a p -simplex of W . Similarly, if W is a semisimplicial set, then $W\langle S \rangle$ denotes the semisimplicial set whose n -simplices are $W_n \times S^{\times n+1}$, with the obvious face maps. This causes no ambiguity when W may be considered as either a simplicial complex or a semisimplicial set.

Lemma 3.37. *Let σ be a p -simplex in IX_n . Then $X_n(\sigma)$ is isomorphic to $X_{n-2p-2}\langle (2\mathbb{Z})^{p+1} \rangle$.*

Proof. Using the action of Q_n on IX_n , we may assume that $\sigma = \{e_{n-2p-1} - e_{n-2p}, \dots, e_{n-1} - e_n\}$. We will now show that $\{u_0, \dots, u_q\}$ is a simplex in $X_n(\sigma)$ if and only if

$$u_i = 2a_{i,0}(e_{n-2p-1} - e_{n-2p}) + \dots + 2a_{i,p}(e_{n-1} - e_n) + u'_i,$$

where $u'_i \in \text{Span}\{e_1, \dots, e_{n-2p-2}\}$, and $\{u'_0, \dots, u'_q\}$ is a simplex in X_{n-2p-2} viewed as a subcomplex of X_n .

Suppose that u is a vertex of $X_n(\sigma)$, i.e. u spans a simplex with σ in Z_n , it is orthogonal to each vertex of σ , $\phi(u) = 1$, and $\rho(u)$ is a singleton. The singleton $\rho(u)$ is disjoint from each of $\rho(e_{n-2p-1} - e_{n-2p}) = \{n-2p-1, n-2p\}, \dots, \rho(e_{n-1} - e_n) = \{n-1, n\}$, so u reduces modulo 2 to some e_i with $i \leq n-2p-2$. As the vectors

$$e_{n-2p-1} - e_{n-2p}, \dots, e_{n-1} - e_n, e_1, \dots, e_{n-2p-2}$$

span the orthogonal complement of the span of $e_{n-2p-1} - e_{n-2p}, \dots, e_{n-1} - e_n$, it follows that

$$u = 2a_0(e_{n-2p-1} - e_{n-2p}) + \dots + 2a_p(e_{n-1} - e_n) + u'$$

with $a_j \in \mathbb{Z}$ and $u' \in \text{Span}\{e_1, \dots, e_{n-2p-2}\}$ reducing modulo 2 to some e_i . In addition $\phi(u') = \phi(u) = 1$, and so we have checked that $u' \in X_{n-2p-2}$.

It remains to check that $\{u_0, \dots, u_q\}$ is a simplex in $X_n(\sigma)$ if and only if the corresponding $\{u'_0, \dots, u'_q\}$ is a simplex in X_{n-2p-2} . For this, we observe that $u_0, \dots, u_q, e_{n-2p-1} - e_{n-2p}, \dots, e_{n-1} - e_n, v_n$ is a partial basis in $\mathbb{Z}^{\oplus n}$ if and only if $u'_0, \dots, u'_q, v_{n-2p-2}$ is a partial basis in $\mathbb{Z}^{\oplus (n-2p-2)}$. \square

3.3. The connectivity argument. We now proceed to the argument for high connectivity of X_n . The road map is as follows:

- Deduce from Charney's work and a bad simplex argument that Y_n is highly connected.
- Use a nerve lemma of Mirzaii-van der Kallen to deduce high connectivity of IX_n from high connectivity of Y_n .
- Use the nerve lemma again to deduce high connectivity of X_n from high connectivity of IX_n .

Definition 3.38. Let V be a finitely generated free abelian group. Let \mathcal{B} be a partial basis of $V/2 = V \otimes \mathbb{Z}/2$. Let $PB(V; \mathcal{B})$ be the simplicial complex with simplices $\{x_0, \dots, x_p\}$ where

- (1) x_0, \dots, x_p is a partial basis of V .
- (2) $\{x_0, \dots, x_p\}$ modulo 2 is a subset of \mathcal{B} .

Recall that a simplicial complex W is *wCM of dimension d* if for all simplices $\sigma \subset W$, $\text{Lk}_W(\sigma)$ is $(d - \dim(\sigma) - 2)$ -connected. In particular, taking $\sigma = \emptyset$, W itself must be $(d - 1)$ -connected. The following is a special case of [Cha84, Theorem on p. 2094].

Theorem 3.39 (Charney). *Let V be a free abelian group of rank n and \mathcal{B} a partial basis of $V/2$. Then $PB(V; \mathcal{B})$ is wCM of dimension $(|\mathcal{B}| - 2)$.*

We will apply a version of the “bad simplex argument”. There are many variants of the bad simplex argument in the literature; the following statement is exactly [GRW18, Proposition 2.5].

Proposition 3.40. *Let X be a simplicial complex and $Y \subset X$ a full subcomplex. Suppose that there exists an integer n such that whenever a simplex $\sigma \subset X$ has no vertex in Y , then $\text{Lk}_X(\sigma) \cap Y$ is $(n - \dim(\sigma) - 1)$ -connected. Then the pair (X, Y) is n -connected.*

Lemma 3.41. *If a topological space is $\frac{k}{2}$ -connected, then it is $\frac{k-2}{4}$ -connected.*

Proof. We have $\lfloor \frac{k-2}{4} \rfloor \leq \max(\lfloor \frac{k}{2} \rfloor, -2)$. \square

Lemma 3.42. *Let σ be a q -dimensional simplex of Y_n , possibly empty. Then $Y_n(\sigma)$ is $\frac{n-2q-11}{4}$ -connected.*

Proof. Consider a family of vectors $\mathcal{B} \subset (\mathbb{Z}/2)^{\oplus n}$ such that $\{\rho(x)\}_{x \in \mathcal{B}}$ are pairwise disjoint 2-element sets, each of which is also disjoint from $\rho(v)$ for each vertex v of σ . Notice that \mathcal{B} is necessarily a partial basis mod 2, and that $|\mathcal{B}|$ can take any value up to $\lfloor \frac{n-2q-2}{2} \rfloor$. We choose such a family with $|\mathcal{B}| = \lfloor \frac{n-2q-3}{2} \rfloor$.

Let $V = \ker \phi \cap \sigma^\perp$, where σ^\perp denotes the subspace of vectors x satisfying $\langle x, v \rangle = 0$ for each vertex v of σ . Observe that in fact $\mathcal{B} \subset V/2$.

We notice that a simplex $\{u_0, \dots, u_p\}$ in $PB(V; \mathcal{B})$ satisfies all conditions of being in $Y_n(\sigma)$, except potentially that $\{u_0, \dots, u_p, v_n\}$ is a partial basis. If n is odd, this condition in fact also holds because $\mathbb{Z}^{\oplus n} \cong \ker \phi \oplus \text{Span}(v_n)$, as observed in Theorem 3.4. If n is even, then we note that $v_n \in V$, since $\phi = \langle v_n, - \rangle$ for n even. Therefore we have $PB(V; \mathcal{B}) \cap Y_n(\sigma) = \text{Lk}_{PB(V; \mathcal{B} \cup \{v_n\})}(v_n)$ in this case. The fact that $|\mathcal{B}| = \lfloor \frac{n-2q-3}{2} \rfloor$ ensures that also $\mathcal{B} \cup \{v_n\}$ is a partial basis mod 2 for V , when n is even.

Define a subcomplex $P_n \subset Y_n(\sigma)$ by

$$P_n = \begin{cases} PB(V; \mathcal{B}) & n \text{ odd,} \\ \text{Lk}_{PB(V; \mathcal{B} \cup \{v_n\})}(v_n) & n \text{ even.} \end{cases}$$

In either case P_n is $\frac{n-2q-9}{2}$ -connected by Theorem 3.39, hence $\frac{n-2q-11}{4}$ -connected by Lemma 3.41.

The goal now will be to show that the inclusion $P_n \hookrightarrow Y_n(\sigma)$ is $\frac{n-2q-11}{4}$ -connected using Proposition 3.40. Let $\tau \subset Y_n(\sigma)$ be a p -simplex with no vertex in P_n . We need to show that $\text{Lk}_{Y_n(\sigma)}(\tau) \cap P_n$ is $(\frac{n-2q-11}{4} - p - 1)$ -connected. Let $\mathcal{C}(\tau) = \{x \in \mathcal{B} : \rho(v) \cap \rho(x) = \emptyset \text{ for all } v \in \tau\}$. Then

$$\text{Lk}_{Y_n(\sigma)}(\tau) \cap P_n = \begin{cases} PB(V; \mathcal{C}(\tau)) & n \text{ odd,} \\ \text{Lk}_{PB(V; \mathcal{C}(\tau) \cup \{v_n\})}(v_n) & n \text{ even.} \end{cases}$$

In either case, since $|\mathcal{C}(\tau)| \geq |\mathcal{B}| - 2(p+1) = \lfloor \frac{n-2q-4p-7}{2} \rfloor$ it follows from Theorem 3.39 that $\text{Lk}_{Y_n(\sigma)}(\tau) \cap P_n$ is $\frac{n-2q-4p-13}{2}$ -connected. By Lemma 3.41 it is also $\frac{n-2q-4p-15}{4} = (\frac{n-2q-11}{4} - p - 1)$ -connected. \square

Theorem 3.43 (Mirzaii–van der Kallen). *Let F and W be simplicial complexes. Let $W_\tau : P(F) \rightarrow S(W)$ be an order reversing map from the poset of simplices in F to the poset of subcomplexes in W , ie. $\sigma \leq \sigma'$ in F implies $W_{\sigma'} \subseteq W_\sigma \subseteq W$. Assume the following conditions:*

- (1) *The union $\bigcup_{\sigma \in F} W_\sigma$ contains the $(N+1)$ -skeleton of W .*
- (2) *F is N -connected.*
- (3) *For a simplex σ in F , W_σ is $\min(N-1, N - \dim \sigma)$ -connected.*
- (4) *For a simplex $\tau \subset W$, the subcomplex $F_\tau = \{\sigma \in F : \tau \in W_\sigma\}$ of F is $(N - \dim \tau)$ -connected.*
- (5) *For every vertex v in F , there exists an N -connected Y_v such that $W_v \subseteq Y_v \subseteq W$.*

Then W is N -connected.

Indeed, this is [MvdK01, Theorem 4.7]. They formulate the result in terms of posets of sequences (i.e. directed simplicial complexes) rather than simplicial complexes, but every simplicial complex can be considered a directed simplicial complex after arbitrarily choosing a total order on the set of vertices, without changing the geometric realization. Moreover, rather than (1) their hypothesis is that $W = \bigcup_{\sigma \in F} W_\sigma$. But this is no loss of generality, as we may in any case replace W by its $(N + 1)$ -skeleton.

Lemma 3.44. IX_n is $\frac{n-11}{4}$ -connected.

Proof. Let σ be a p -dimensional simplex in Y_n , possibly empty. We will prove that $IX_n(\sigma)$ is $\frac{n-2p-13}{4}$ -connected by descending induction on p . For $p = -1$ this proves the assertion. For the base case of the induction we may take $p \gg 0$, where the conclusion is vacuously true. For the induction step, we will apply Theorem 3.43.

Let $F = Y_n(\sigma)$, $W = IX_n(\sigma)$, and for a simplex τ in F , let $W_\tau = IX_n(\sigma \cup \tau)$. Then of course $W_{\tau'} \subseteq W_\tau$ for $\tau \subseteq \tau'$. We check the other conditions in Theorem 3.43 with $N = \frac{n-2p-13}{4}$:

- (1) Take $\pi \subset W$ a simplex of dimension $q \leq \frac{n-2p-9}{2}$. It suffices to prove that π lies in W_τ for some τ , since $\frac{n-2p-9}{2} \geq N + 1$. Note that $\sigma \cup \pi$ is a simplex in Y_n of dimension $p + q + 1 \leq \frac{n-7}{2}$. Therefore by Lemma 3.42, $Y_n(\sigma \cup \pi)$ is nonempty. If τ is a nonempty simplex of $Y_n(\sigma \cup \pi)$, then π is a simplex in $IX_n(\sigma \cup \tau) = W_\tau$.
- (2) F is N -connected, by Lemma 3.42.
- (3) For a q -simplex τ in F , $W_\tau = IX_n(\sigma \cup \tau)$ is $\frac{n-2p-2q-15}{4}$ -connected by induction, so is in particular $\min(N - 1, N - q)$ -connected.
- (4) For a q -simplex π in W we have that $F_\pi = Y_n(\sigma \cup \pi)$. This is $(N - q)$ -connected, because $Y_n(\sigma \cup \pi)$ is $\frac{n-2p-2q-13}{4}$ -connected by Lemma 3.42 and

$$\frac{n - 2p - 2q - 13}{4} \geq \frac{n - 2p - 4q - 13}{4} = N - q.$$

- (5) Let v be a vertex of F . If τ is a simplex of $W_v = IX_n(\sigma \cup \{v\})$, then $\tau \cup \{v\}$ is a simplex of $IX_n(\sigma)$. Hence $IX_n(\sigma \cup \{v\}) \subseteq IX_n(\sigma \cup \{v\}) * v \subseteq IX_n(\sigma)$, and the cone $IX_n(\sigma \cup \{v\}) * v$ is contractible. \square

Let W be a semisimplicial set, and S a set. Recall from Definition 3.36 the construction $W\langle S \rangle$. For any $s \in S$, there is a natural map $W \rightarrow W\langle S \rangle$ given by $W \cong W\langle \{s\} \rangle \hookrightarrow W\langle S \rangle$. Recall also from Definition 3.17 the notion of directed simplicial complex.

Proposition 3.45. *Let W be a directed simplicial complex, S a nonempty set. If the left-link of $\sigma \in W_p$ is $(N - p - 1)$ -connected for all σ , and W is $\min(1, N - 1)$ -connected, then $W \rightarrow W\langle S \rangle$ is N -connected for all $s \in S$.*

See [Cha84, Proposition on page 2088].

Theorem 3.46. X_n is $\frac{n-12}{4}$ -connected.

Proof. We proceed by induction on n . If $n \leq 7$ then the claim is vacuous, which provides the base cases: we now suppose that $n \geq 8$. We will apply Theorem 3.43. Let $F = IX_n$ and $W = X_n$. For a simplex τ in F , let $W_\tau = X_n(\tau)$. Then of course $W_{\tau'} \subseteq W_\tau$ for $\tau \subseteq \tau'$. We check the other conditions for $N = \frac{n-12}{4}$:

- (1) The union $\bigcup_{\tau \in F} W_\tau$ contains the $(n - 4)$ -skeleton of W , so in particular contains the $(N + 1)$ -skeleton. To see this, take a p -simplex of X_n , $p \leq n - 4$. Using the action of Q_n on X_n we can assume that the simplex is (e_{n-p}, \dots, e_n) , which lies in $X_n(\{e_1 - e_2\}) = W_{\{e_1 - e_2\}}$.
- (2) F is $\frac{n-11}{4}$ -connected by Lemma 3.44, so is in particular N -connected.
- (3) For a q -simplex τ in F , we have

$$W_\tau = X_n(\tau) \cong X_{n-2q-2} \langle (2\mathbb{Z})^{q+1} \rangle$$

by Lemma 3.37. The complex X_{n-2q-2} is $\frac{n-2q-14}{4}$ -connected by induction. Applying Proposition 3.45 we see that the pair (W_τ, X_{n-2q-2}) is $\frac{n-2q-11}{4}$ -connected: indeed, the left-link of a p -simplex in X_{n-2q-2} is $\frac{n-2q-p-15}{4}$ -connected by Lemma 3.33 and induction, so in particular $(\frac{n-2q-11}{4} - p - 1)$ -connected, and X_{n-2q-2} itself is $\frac{n-2q-14}{4}$ -connected. Thus W_τ is also $\frac{n-2q-14}{4}$ -connected. As

$$\frac{n-2q-14}{4} \geq \frac{n-4q-12}{4} = N-q \quad \text{if } q \geq 1$$

and

$$\frac{n-2q-14}{4} \geq \frac{n-16}{4} = N-1 \quad \text{if } q = 0,$$

the complex W_τ is in particular $\min(N-1, N-q)$ -connected.

- (4) For a p -simplex π in X , the subcomplex F_π of F is $IX_n(\pi) \cong IX_{n-p-1}$ by Lemma 3.35, which is $\frac{n-p-12}{4}$ -connected by Lemma 3.44. As

$$\frac{n-p-12}{4} \geq N-p,$$

the complex F_π is in particular $(N-p)$ -connected.

- (5) Let v be a vertex of F . Using the action of Q_n on X_n we may assume $v = e_{n-1} - e_n$. By Lemma 3.37 we have $W_v \cong X_{n-2}\langle 2\mathbb{Z} \rangle$. Consider the diagram of inclusions

$$\begin{array}{ccc} X_{n-2} & \hookrightarrow & X_{n-2}\langle 2\mathbb{Z} \rangle = W_v \\ \downarrow & & \downarrow \\ X_{n-2} * \{e_n\} & \hookrightarrow & X_n = W \end{array}$$

where the top horizontal arrow is induced by $\{0\} \hookrightarrow 2\mathbb{Z}$, and the bottom horizontal arrow comes from the fact that if σ is a simplex of X_{n-2} then $\sigma \cup \{e_n\}$ is a simplex of X_n . We then have a chain of subcomplexes

$$W_v \subseteq W_v \cup_{X_{n-2}} (X_{n-2} * \{e_n\}) \subseteq W.$$

In step (3) we have seen that the pair (W_v, X_{n-2}) is N -connected. Hence the complex $W_v \cup_{X_{n-2}} (X_{n-2} * \{e_n\}) \simeq W_v / X_{n-2}$ is N -connected. \square

Corollary 3.47. *The complex $W_\bullet(\mathbb{Q}, n)$ is $\frac{n-12}{4}$ -connected.*

Proof. Combine Lemma 3.30 and Theorem 3.46. \square

Remark 3.48. Our proof that the complexes $W_\bullet(\mathbb{Q}; n)$ are highly connected can be adapted to show that $W_\bullet(\mathbb{T}; n)$ are highly connected, too. In fact the argument would significantly simplify. Without the congruence condition, the analogue of IX_n would be the complex of isotropic partial bases $\{u_0, \dots, u_p\}$ such that $\phi(u_i) = 0$ for all i , forming a partial basis with v_n . When n is odd this is simply the complex of isotropic unimodular sequences in $\ker(\phi)$, and when n is even this is the link of v_n in the complex of isotropic unimodular sequences in $\mathbb{Z}^{\oplus n}$. Both are proven to be highly connected in [MvdK01, Theorem 5.7]. From this one can deduce high connectivity of the analogue of X_n as in Theorem 3.46. This would however only show slope $\frac{1}{4}$ connectivity of $W_\bullet(\mathbb{T}; n)$ (the slope drops when doing the analogue of step (3) of the proof of Theorem 3.46), which is worse than the slope $\frac{1}{3}$ connectivity proven by Sierra-Wahl [SW24].

4. Verifying the axioms

The goal in this section is to deduce Theorems 1.1, 1.2, 1.3 and 1.4 from the generic Theorem 2.2, i.e. verifying that axioms (I), (II) and (III) of Theorem 2.2 are satisfied in the respective situations.

4.1. Stable real cohomology of arithmetic groups. Let G be a semisimple algebraic group over \mathbb{Q} , $\Gamma \subset G$ an arithmetic subgroup, and V an irreducible real representation of G . Borel [Bor74, Bor81] has famously calculated the cohomology $H^*(\Gamma; V)$ in a stable range. For the proof, Borel studies the homomorphism

$$(4.1) \quad (\mathcal{H}^k \otimes V)^G \rightarrow H^k(\Gamma; V),$$

where the left-hand side of (4.1) denotes the space of G -equivariant V -valued harmonic k -forms on the symmetric space of G . The left-hand side of (4.1) is very computable: for nontrivial V , it vanishes, and for constant coefficients it coincides with the cohomology of the compact dual of G . The goal, then, is to show that (4.1) is an isomorphism in a range of degrees.

Many authors have subsequently revisited and/or improved on Borel's work. We will make no attempt at a complete historical survey, but mention in particular [VZ84, Fra98, Gro13]. The recent papers [Tsh19, LS19, BS23] all show, under different hypotheses, that (4.1) is an isomorphism for all $k < \text{rank}_{\mathbb{R}}(G)$. This bound is in particular independent of V , unlike the one originally obtained by Borel. Intriguingly, [BS23] establish such a result by means of geometric group theory, rather than by automorphic methods, and V is allowed to be any unitary representation.

We will be interested in the groups $G = \text{Sp}_{2g}$ and $G = \text{SL}_n$, whose real ranks are g and $n - 1$, respectively. In these cases Borel's results (with subsequent improvements) read as follows:

Theorem 4.1. *Let $G = \text{Sp}_{2g}$, $\Gamma \subset G$ be an arithmetic subgroup, and V be an irreducible real representation of G . Then for $k < g$ one has $H^k(\Gamma; V) = 0$, if V is nontrivial; the cohomology with trivial coefficients agrees below degree g with a polynomial algebra with generators in degrees 2, 6, 10, ...*

Theorem 4.2. *Let $G = \text{SL}_n$, $\Gamma \subset G$ be an arithmetic subgroup, and V be an irreducible real representation of G . Then for $k < n - 1$ one has $H^k(\Gamma; V) = 0$, if V is nontrivial; the cohomology with trivial coefficients agrees below degree $n - 1$ with an exterior algebra with generators in degrees 5, 9, 13, ...*

Remark 4.3. The homomorphism (4.1) is known to be *injective* in a significantly larger range of degrees. In particular, for $G = \text{Sp}_{2g}$ and $G = \text{SL}_n$, one can show that (4.1) is injective for all $k < 2 \cdot \text{rank}_{\mathbb{R}}(G)$. See [GKT21, Section 4].

Remark 4.4. From Theorem 4.2 one also obtains a similar result for arithmetic subgroups $\Gamma \subset \text{GL}_n$. Indeed, letting $\Gamma' = \Gamma \cap \text{SL}_n$, one finds that $H^*(\Gamma; V) = H^*(\Gamma'; V)^{\Gamma/\Gamma'}$, and that Γ/Γ' acts trivially on the cohomology in the stable range.

4.2. The groups and the coefficients. We will be interested in sequences of representations of three families of algebraic groups:

$$\begin{aligned} \dots &\subset \text{Sp}_{2n} \subset \text{Sp}_{2n+2} \subset \text{Sp}_{2n+4} \subset \dots \\ \dots &\subset \text{GL}_n \subset \text{GL}_{n+1} \subset \text{GL}_{n+2} \subset \dots \\ \dots &\subset \text{Sp}_n \subset \text{Sp}_{n+1} \subset \text{Sp}_{n+2} \subset \dots \end{aligned}$$

In all three cases, there is a monoidal structure present: for any ring R , the families

$$\coprod_{n \geq 0} \text{Sp}_{2n}(R), \quad \coprod_{n \geq 0} \text{GL}_n(R), \quad \text{and} \quad \coprod_{n \geq 0} \text{Sp}_{n-1}(R)$$

form braided monoidal groupoids satisfying the conditions (ii) and (iii) of §2.1: their monoids of objects are the natural numbers, and writing Q_n for $\text{Aut}(n)$ one has that Q_0 is trivial, and $Q_a \times Q_b \rightarrow Q_{a+b}$ is injective. In the first two cases, the monoidal structure is the obvious one, given by block-sum of matrices, and the braiding is given by the evident symmetry. The monoidal structure in the third case was described in detail in Section 3.

4.2.1. *The coefficient systems.* Let us now define the coefficient systems of interest to us for the above three families of groups. All of our examples will come from “highest weight theory” — algebraic representations of the groups in question can be naturally indexed by partitions; fixing a partition defines a system of representations, and the family of coefficient systems thus defined will satisfy homological stability with a uniform stable range, say when $R = \mathbb{Z}$.

That said, we will not actually appeal to highest weight theory in defining the relevant coefficient systems — indeed, the irreducible representation of a classical group associated with a dominant weight is unique only up to noncanonical isomorphism, which is hardly enough data to define a meaningful coefficient system. We will therefore instead follow Weyl’s approach [Wey39] to constructing the irreducible representations of the classical groups, by realizing them inside tensor powers of the defining representation via the invariant theory of the symmetric groups. To avoid repetition, representations are always considered over a characteristic zero base field.

By a *partition* we mean a descending sequence of nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ which eventually reaches zero. We set $|\lambda| = \sum_i \lambda_i$, and $l(\lambda) = \max\{i : \lambda_i \neq 0\}$. Partitions with $|\lambda| = r$ are in bijection with irreducible representations of the symmetric group \mathfrak{S}_r , and we denote by σ_λ the representation associated with λ , a Specht module. We define the *Schur functor* S^λ associated to λ by

$$S^\lambda(V) := V^{\otimes r} \otimes_{\mathfrak{S}_r} \sigma_\lambda,$$

where $r = |\lambda|$. It is classical that $S^\lambda(V)$ is nonzero if and only if $\dim(V) \geq l(\lambda)$, and that $S^\lambda(V)$ is always an irreducible representation of $\mathrm{GL}(V)$.

A Schur functor applied to a direct sum decomposes as a product of Schur functors,

$$(4.2) \quad S^\lambda(V \oplus W) = \bigoplus_{\mu, \nu} c_{\mu, \nu}^\lambda \cdot S^\mu(V) \otimes S^\nu(W),$$

where $c_{\mu, \nu}^\lambda$ is a *Littlewood–Richardson coefficient*. Equation (4.2) tells us in particular that the shift (in the sense of coefficient systems) of a Schur functor is a direct sum of Schur functors: more precisely, one has

$$(4.3) \quad S^\lambda(V \oplus \mathbb{k}) \cong \bigoplus_{\mu} S^\mu(V),$$

where μ ranges over partitions obtained from λ by removing a number of boxes from its Young diagram, no two in the same column. Equation (4.3) is an instance of Pieri’s formula.

4.2.2. *Symplectic groups.* Let $V(n) = \mathbb{k}^{2n}$ be the defining representation of Sp_{2n} . Then $S^\lambda(V(n))$ is generally not irreducible as a representation of Sp_{2n} . Let $V(n)^{(r)} \subset V(n)^{\otimes r}$ be the subspace of *traceless tensors*, i.e. the joint kernel of the $\binom{r}{2}$ evident contraction maps $V(n)^{\otimes r} \rightarrow V(n)^{\otimes(r-2)}$. Define, for $r = |\lambda|$,

$$(4.4) \quad V_\lambda(n) := S^\lambda(V(n)) \cap V(n)^{(r)},$$

the intersection taking place inside $V(n)^{\otimes r}$. Then Weyl showed that $V_\lambda(n)$ is nonzero if and only if $n \geq l(\lambda)$, that V_λ is always an irreducible representation of Sp_{2n} , and that all irreducible representations of Sp_{2n} arise in this way.

The $\{V(n)\}$ form a coefficient system, with structure maps $\phi_{a,b} : \mathbb{k} \otimes V(b) \rightarrow V(a+b)$ given by the inclusion of the last $2b$ basis vectors. The tensor powers $\{V(n)^{\otimes r}\}$ inherit the structure of a coefficient system. The Schur functors $\{S^\lambda(V(n))\}$ form a sub-coefficient system of $\{V(n)^{\otimes r}\}$; as do the traceless tensors $\{V(n)^{(r)}\}$, since the $\phi_{a,b}$ are isometric embeddings. Thus their intersection $\{V_\lambda(n)\}$ is also a coefficient system. In fact it is a polynomial coefficient system for the symplectic groups, of degree $|\lambda|$, consisting by construction of a sequence of irreducible representations of the symplectic groups.

Lemma 4.5. *Suppose, in the situation of Theorem 2.2, that \mathbb{Q} is the groupoid $\coprod_n \mathrm{Sp}_{2n}(\mathbb{Z})$, or more generally a monoidal subgroupoid of $\coprod_n \mathrm{Sp}_{2n}(\mathbb{Q})$, consisting of arithmetic subgroups. Let \mathcal{V} be the family of coefficient*

systems obtained by taking direct sums of the coefficient systems $\{V_\lambda(n)\}$, where λ ranges over all partitions. Then axioms (I) and (II) of Theorem 2.2 are satisfied, with $\theta = 1$ and $\tau = 0$.

Proof. The shift of a Schur functor is a direct sum of Schur functors, according to (4.3). From this fact, and (4.4), it follows that the shift of the coefficient system $\{V_\lambda(n)\}$ likewise decomposes as a direct sum of coefficient systems $\{V_\mu(n)\}$. (Equation (4.3) must be applied twice, since $V(n+1) = V(n) \oplus \mathbb{k}^2$, but no matter.) Thus axiom (I) is satisfied. Axiom (II), i.e. the vanishing of $H_d(Q_n, Q_{n-1}; V(n), V(n-1))$, follows directly from Theorem 4.1 in the case of nontrivial coefficients, and similarly for trivial coefficients when $d < n-1$. For trivial coefficients and $d = n-1$ we need to know moreover the surjectivity of $H_{n-1}(Q_{n-1}; \mathbb{Q}) \rightarrow H_{n-1}(Q_n; \mathbb{Q})$. This follows from the fact that injectivity of (4.1) holds in a larger range, as noted in Remark 4.3. \square

4.2.3. *General linear groups.* The coefficient systems we will consider for the general linear groups, in connection with automorphisms of free groups and handlebody groups, are of a very similar nature. Let $V(n) = \mathbb{k}^n$ be the defining representation of GL_n . Define the subspace of *traceless tensors*

$$V(n)^{(r,s)} \subset V(n)^{\otimes r} \otimes (V(n)^*)^{\otimes s}$$

to be the joint kernel of the rs evident contraction maps

$$V(n)^{\otimes r} \otimes (V(n)^*)^{\otimes s} \rightarrow V(n)^{\otimes r-1} \otimes (V(n)^*)^{\otimes s-1}.$$

For a pair of partitions λ and μ , with $|\lambda| = r$ and $|\mu| = s$, define

$$V_{\lambda,\mu}(n) = (S^\lambda(V(n)) \otimes S^\mu(V(n)^*)) \cap (V(n)^{(r,s)}),$$

the intersection taking place inside $V(n)^{\otimes r} \otimes (V(n)^*)^{\otimes s}$. Then $V_{\lambda,\mu}(n)$ is nonzero if and only if $l(\lambda) + l(\mu) \leq n$, and $V_{\lambda,\mu}(n)$ is an irreducible representation of GL_n . All irreducible representations of the general linear groups arise in this way.

The $\{V(n)\}$ form a coefficient system, with structure maps $\phi_{a,b} : \mathbb{k} \otimes V(b) \rightarrow V(a+b)$ given by the inclusion of the last b basis vectors; similarly, the $\{V(n)^*\}$ form a coefficient system, with structure maps $\phi_{a,b} : \mathbb{k} \otimes V(b)^* \rightarrow V(a+b)^*$ given by the dual of the projection to the last b basis vectors. These endow the $\{V(n)^{\otimes r} \otimes (V(n)^*)^{\otimes s}\}$ with the structure of a coefficient system, and just as in the symplectic groups case the $\{V_{\lambda,\mu}(n)\}$ form a sub-coefficient system.

Lemma 4.6. *Suppose, in the situation of Theorem 2.2, that \mathbb{Q} is the groupoid $\coprod_n \mathrm{GL}_n(\mathbb{Z})$, or more generally a monoidal subgroupoid of $\coprod_n \mathrm{GL}_n(\mathbb{Q})$, consisting of arithmetic subgroups. Let \mathcal{V} be the family of coefficient systems obtained by taking direct sums of the coefficient systems $\{V_{\lambda,\mu}(n)\}$, where λ and μ range over all partitions. Then axioms (I) and (II) of Theorem 2.2 are satisfied, with $\theta = 1$ and $\tau = -1$.*

Proof. The proof is the same as for Lemma 4.5, using Theorem 4.2 instead of Theorem 4.1. \square

4.2.4. *Even and odd symplectic groups.* Let $\mathbb{T} = \coprod_n T_n = \coprod_n \mathrm{Sp}_{n-1}(\mathbb{Z})$ be the braided monoidal groupoid considered in Section 3. Let $V(n)$ denote the defining representation of $T_n \cong \mathrm{Sp}_{n-1}(\mathbb{Z})$; recall from Definition 3.6 that this is the $(n-1)$ -dimensional representation given by $\ker(\phi : \mathbb{Z}^{\oplus n} \rightarrow \mathbb{Z})$, tensored with \mathbb{k} . There are $T_a \times T_b$ -equivariant maps $\mathbb{k} \otimes (\mathbb{k}^{\oplus b}) \rightarrow \mathbb{k}^{\oplus a+b}$ given by the inclusion of the last b basis vectors, and these restrict to structure maps $\phi_{a,b} : \mathbb{k} \otimes V(b) \rightarrow V(a+b)$ making the $\{V(n)\}$ into a coefficient system. As in Subsection 4.2.2 we can define the subspace of traceless tensors $V(n)^{(r)} \subset V(n)^{\otimes r}$, by taking the joint kernel of all ways of contracting along the bilinear form $\langle -, - \rangle$, the only difference being that the form $\langle -, - \rangle$ is not perfect when n is even. We then set, for $r = |\lambda|$,

$$V_\lambda(n) = S^\lambda(V(n)) \cap V(n)^{(r)}.$$

As for the symplectic and general linear groups, the $\{V(n)^{\otimes r}\}$ form a coefficient system and the $\{V_\lambda(n)\}$ form a sub-coefficient system. By the same argument as before, the shift of $V_\lambda(n)$ is the direct sum of $V_\mu(n)$, with μ ranging over all partitions obtained from λ by removing a number of boxes from the Young diagram of λ , no two in the same column.

We caution the reader that the notation is at this point inconsistent: whereas $V_\lambda(n)$ denoted a representation of Sp_{2n} in Subsection 4.2.2, it now denotes a representation of Sp_{n-1} . It should be clear from context what is meant.

The representation $V_\lambda(n)$ of Sp_{n-1} is not irreducible (but indecomposable) when $n-1$ is odd. The odd symplectic groups are *not* semisimple, and one can not deduce directly from Borel's work that the family of coefficient systems \mathcal{V} satisfies uniform homological stability. We will verify the existence of a uniform stable range for arithmetic subgroups of the even and odd symplectic groups by means of a direct calculation in the following section, expressing the cohomology of the odd symplectic groups in terms of the cohomology of the even symplectic groups in a stable range.

4.3. A computation with Kostant's theorem. The goal of this section is to prove the following result.

Lemma 4.7. *Suppose, in the situation of Theorem 2.2, that \mathcal{Q} is the groupoid $\coprod_n \mathrm{Sp}_{n-1}(\mathbb{Z})$, or more generally a monoidal subgroupoid of $\coprod_n \mathrm{Sp}_{n-1}(\mathbb{Q})$ consisting of arithmetic subgroups. Let \mathcal{V} be the family of coefficient systems obtained by taking direct sums of the coefficient systems $\{V_\lambda(n)\}$, where λ ranges over all partitions. Then axioms (I) and (II) of Theorem 2.2 are satisfied, with $\theta = \frac{1}{2}$ and $\tau = -1$.*

Thus in this section Q_n denotes an arithmetic subgroup of Sp_{n-1} ; we caution the reader to be mindful of the shift-by-one.

4.3.1. *A first reduction.* We claim first that Lemma 4.7 is implied by the following lemma.

Lemma 4.8. *If $d < n$, then*

$$\dim H_d(Q_{2n}; V_\lambda(2n+1)) = \begin{cases} \dim H_d(Q_{2n-1}; \mathbb{Q}) & \text{if } l(\lambda) \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof that Lemma 4.8 implies Lemma 4.7. To prove Lemma 4.7 we need to show that both stabilisation maps

$$H_d(Q_{2n-1}; V(2n-1)) \rightarrow H_d(Q_{2n}; V(2n)) \rightarrow H_d(Q_{2n+1}; V(2n+1))$$

are isomorphisms for $d < n-1$, and both maps are surjective for $d = n-1$, for all $V \in \mathcal{V}$.

We begin with the first assertion. We already know that their *composite* is an isomorphism for $d < n-1$ and for all $V \in \mathcal{V}$, by Borel's results for the usual symplectic groups (Lemma 4.5). Thus, it will suffice to prove that $\dim H_d(Q_{2n-1}; V(2n-1)) = \dim H_d(Q_{2n}; V(2n))$ in this range to see that both arrows individually must be isomorphisms, too. Moreover, it suffices to prove this in the case that $V = \Sigma V_\lambda$ is a shift of a coefficient system associated with a partition, reasoning by induction on polynomial degree and using that ΣV_λ decomposes as a direct sum of V_λ and coefficient systems of strictly lower degree.

Now the left hand side in Lemma 4.8 is by definition $\dim H_d(Q_{2n}; (\Sigma V_\lambda)(2n))$. But moreover, when $d < n-1$, then the right hand side agrees with $\dim H_d(Q_{2n-1}; \Sigma V_\lambda(2n-1))$ — indeed, $\Sigma V_\lambda \cong \bigoplus_\mu V_\mu$, with μ ranging over partitions obtained from λ by removing boxes from the Young diagram, no two in the same column. In this direct sum all nontrivial summands have vanishing homology when $d < n-1$, and the direct sum contains a (single) copy of the trivial representation if and only if $l(\lambda) \leq 1$.

For the second assertion, note first of all that the composite of the two arrows is known to be surjective for $d = n-1$ by Lemma 4.5. When V is the constant coefficient system, Lemma 4.8 implies that $\dim H_d(Q_{2n-1}; \mathbb{Q}) = \dim H_d(Q_{2n}; \mathbb{Q})$ (take $\lambda = 0$). Moreover, since the inclusion $\mathrm{Sp}_{2n} \rightarrow \mathrm{Sp}_{2n+1}$ admits a left inverse (by projecting onto the Levi factor), the map $H_d(Q_{2n-1}; \mathbb{Q}) \rightarrow H_d(Q_{2n}; \mathbb{Q})$ is injective, hence an isomorphism. For $\lambda \neq 0$, a dimension count combined with Lemma 4.8 implies that $H_{n-1}(Q_{2n}; V_\lambda(2n)) = 0$, so that surjectivity holds tautologically. Indeed, $V_\lambda(2n)$ is a direct summand of $(\Sigma V_\lambda)(2n)$, and if $l(\lambda) \leq 1$ then the trivial representation \mathbb{Q} is another, distinct, summand. That is, $H_{n-1}(Q_{2n}; V_\lambda(2n))$ is a summand of $H_{n-1}(Q_{2n}; V_\lambda(2n+1))$, and it is distinct from $H_{n-1}(Q_{2n}; \mathbb{Q}) \cong H_{n-1}(Q_{2n-1}; \mathbb{Q})$, so it must vanish. \square

4.3.2. *Kostant's theorem.* Let \mathfrak{g} be a finite dimensional semisimple Lie algebra in characteristic zero. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra, with nilpotent radical \mathfrak{u} and Levi subalgebra \mathfrak{l} . Let $V(\lambda)$ be the finite-dimensional irreducible representation of \mathfrak{g} of highest weight λ . Kostant's theorem [Kos61] computes the Lie algebra cohomology $H^q(\mathfrak{u}; V(\lambda))$ as a representation of \mathfrak{l} .

We let W be the Weyl group of \mathfrak{g} , W_P the Weyl group of \mathfrak{l} , and W^P the set of distinguished coset representatives of W/W_P . We denote by ρ the half-sum of all positive roots of \mathfrak{g} . For $w \in W$ and λ a weight of \mathfrak{g} , we define

$$w \bullet \lambda = w(\lambda + \rho) - \rho.$$

If $w \in W^P$, then $w^{-1} \bullet (-)$ takes dominant weights of \mathfrak{g} to dominant weights of \mathfrak{l} . We denote by $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ the length function of W .

Theorem 4.9 (Kostant). *With notation as above we have*

$$H^q(\mathfrak{u}; V(\lambda)) = \bigoplus_{\substack{w \in W^P \\ \ell(w)=q}} V_{\mathfrak{l}}(w^{-1} \bullet \lambda),$$

where $V_{\mathfrak{l}}(\mu)$ denotes the irreducible representation of \mathfrak{l} of highest weight μ .

4.3.3. *The calculation.* Let $G = \mathrm{Sp}_{2n}$ and $P = \mathrm{Sp}_{2n-1}$, the parabolic subgroup fixing a vector. The Levi factor is $L = \mathrm{Sp}_{2n-2}$, and we denote by U the unipotent radical of P . From the short exact sequence

$$1 \rightarrow U \rightarrow P \rightarrow L \rightarrow 1$$

we obtain a short exact sequence of discrete groups

$$1 \rightarrow \Gamma \rightarrow Q_{2n} \rightarrow \Gamma' \rightarrow 1,$$

with Γ and Γ' arithmetic subgroups of U and L , respectively. We study the homology of Q_{2n} via the Lyndon–Hochschild–Serre spectral sequence

$$(4.5) \quad E_{pq}^2 = H_p(\Gamma'; H_q(\Gamma; V_{\lambda}(2n+1))) \implies H_{p+q}(Q_{2n}; V_{\lambda}(2n+1)).$$

By Nomizu's theorem [Nom54] we have $H_*(\Gamma; V_{\lambda}(2n+1)) = H_*(\mathfrak{u}; V_{\lambda}(2n+1))$, where \mathfrak{u} is the Lie algebra of U .

Let us now compute $H_*(\mathfrak{u}; V_{\lambda}(2n+1))$, i.e. for V_{λ} an irreducible representation of G , using Kostant's theorem. All algebraic representations of the usual symplectic group are self-dual (this fails for the odd symplectic groups). Applied to G and L this means in particular that Kostant's theorem in this case dualizes to a homology isomorphism

$$H_q(\mathfrak{u}; V(\lambda)) = \bigoplus_{\substack{w \in W^P \\ \ell(w)=q}} V_{\mathfrak{l}}(w^{-1} \bullet \lambda).$$

We have that W is the hyperoctahedral group, i.e. the group of signed permutations, on n letters. The vector ρ is $(n, n-1, \dots, 1)$, the smallest regular weight of Sp_{2n} . The subgroup W_P is the group of signed permutations on $n-1$ letters, which we think of as the subgroup stabilising the first entry. We have $w \in W^P$ precisely when

$$w^{-1}(\rho)_2 > w^{-1}(\rho)_3 > \dots > w^{-1}(\rho)_n > 0.$$

Thus for $w \in W^P$, $w^{-1}(\rho)_1$ can be any element of $\{\pm 1, \pm 2, \dots, \pm n\}$ and the remaining entries are uniquely determined by the condition of being positive and in descending order. For λ a dominant weight of G and $w \in W^P$, the dominant weight of L given by $w^{-1} \bullet \lambda$ is simply the vector obtained by deleting the first entry of the vector $w^{-1} \bullet \lambda$.

Example 4.10. Take $n = 5$. Let w be the unique element of W^P with $w^{-1}(\rho)_1 = -3$. Then we will have

$$w^{-1}(\rho) = (-3, 5, 4, 2, 1)$$

and for $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ we have

$$w^{-1} \bullet \lambda = (-8 - \lambda_3, 1 + \lambda_1, 1 + \lambda_2, \lambda_4, \lambda_5).$$

The corresponding dominant weight of the Levi factor Sp_8 is obtained by deleting the first entry in the vector, giving $(1 + \lambda_1, 1 + \lambda_2, \lambda_4, \lambda_5)$. \blacksquare

For $p < n - 1$, the group $H_p(\Gamma', H_q(\Gamma; V_\lambda(2n + 1)))$ appearing in the Lyndon–Hochschild–Serre spectral sequence (4.5) is nontrivial only when $H_q(\Gamma; V_\lambda(2n + 1)) = H_q(\mathfrak{u}; V_\lambda(2n + 1))$ contains a copy of the trivial representation of L . Indeed, $L \cong \mathrm{Sp}_{2n-2}$ and $\Gamma' \subset L$ is arithmetic, so this follows from Theorem 4.1. We are therefore led to ask for which values of λ and w the weight $w^{-1} \bullet \lambda$ gives the trivial representation of L , i.e. the all zero vector of length $(n - 1)$. By pondering Example 4.10, it is clear that this happens precisely when $\lambda = (\lambda_1, 0, \dots, 0)$ and moreover

$$w^{-1}(\rho) = (n, n - 1, n - 2, \dots, 1) \quad \text{or} \quad w^{-1}(\rho) = (-n, n - 1, n - 2, \dots, 1).$$

That is, we must have $l(\lambda) \leq 1$ and $w \in \{\mathrm{id}, \theta\}$, where θ denotes the element that swaps the sign of the first entry. We have $\ell(\mathrm{id}) = 0$ and $\ell(\theta) = 2n - 1$.

It follows that if $l(\lambda) > 1$, then $H_*(\Gamma; V_\lambda(2n + 1))$ does not contain a copy of the trivial representation in any degree. Using Borel’s vanishing result (Theorem 4.1) we see that $E_{p,q}^2 = 0$ in the Lyndon–Hochschild–Serre spectral sequence (4.5) for all $p < n - 1$. Moreover, we also have $E_{n-1,0}^2 = 0$, since $H_0(\Gamma; V_\lambda(2n + 1)) = 0$ — if it were nonzero, it would be a copy of the trivial representation. We therefore see that $H_d(Q_{2n}; V_\lambda(2n + 1))$ vanishes for $d < n$. On the other hand if $l(\lambda) \leq 1$, then $H_q(\Gamma; V_\lambda(2n + 1)) \cong \mathbb{Q}$ is a single copy of the trivial representation for $q = 0$ and $q = 2n - 1$. For $p < n - 1$ we therefore see the cohomology of Γ' with trivial coefficients along the rows $q = 0$ and $q = 2n - 1$ of the spectral sequence, and all other cohomology vanishes. Noting in addition that we know the entry $E_{n-1,0}^2$, and that no differential can kill it, this proves Lemma 4.8, and hence Lemma 4.7.

4.4. The groupoids and their connectivities. Let us now return to the four instances of our main theorem described in Section 1. In all our examples, we have $\mathsf{G} = \coprod_n \Gamma_n$ a family of groups arising as symmetries of a sequence of topological objects, and $\mathsf{Q} = \coprod_n Q_n$ a family of arithmetic groups arising by taking the induced symmetries on homology.

4.4.1. Mapping class groups. We take Γ_n to be the mapping class group of a genus n surface with a boundary component, and $Q_n = \mathrm{Sp}_{2n}(\mathbb{Z})$. The groupoid G has a monoidal structure given by “pair of pants”-product. In fact the pair of pants-product makes the classifying space $\coprod_n B\Gamma_n$ into an E_2 -algebra, as first observed by Miller [Mil86], which in particular endows G with a braiding. See [RWW17, Section 5.6] for a careful construction. The monoidal structure on Q is simply given by block-sum of matrices, which is in an evident way symmetric monoidal.

Surjectivity of $\Gamma_n \rightarrow \mathrm{Sp}_{2n}(\mathbb{Z})$ is a classical fact in surface topology — the symplectic group is generated by transvections, and each transvection can be lifted to a Dehn twist in the mapping class group.

Homological stability of the mapping class groups is due to Harer [Har85], with subsequent improvements by [Iva93, Boll2]. The splitting complexes $S_\bullet^{E_1}(\mathsf{G}; n)$ are spherical by [GKRW19, Theorem 3.4]. Homological stability for the groups $\mathrm{Sp}_{2n}(\mathbb{Z})$ is originally due to Charney [Cha87], and van der Kallen and Looijenga [vdKL11] later proved that the complexes $S_\bullet^{E_1}(\mathsf{Q}; n)$ are spherical. Proposition 2.5 therefore implies the following result:

Lemma 4.11. *If G is the groupoid of mapping class groups, and Q the groupoid of integral symplectic groups, then axiom (III) of Theorem 2.2 is satisfied with $\nu = \frac{2}{3}$ and $\xi = 0$.*

This is the last ingredient needed to apply Theorem 2.2 to deduce Theorem 1.1, uniform homological stability for mapping class groups.

4.4.2. *Automorphisms of free groups.* We take $\Gamma_n = \text{Aut}(F_n)$ the automorphism group of a free group on n generators, and $Q_n = \text{GL}_n(\mathbb{Z})$. The groupoid G is symmetric monoidal in an evident way, being equivalent to the maximal subgroupoid of the category of finitely generated free groups, with monoidal structure given by the free product. The monoidal structure on Q is again simply given by block-sum of matrices, with the obvious symmetry.

Surjectivity of $\Gamma_n \rightarrow \text{GL}_n(\mathbb{Z})$ is an equally classical fact as in the mapping class group case, proven by lifting elementary matrices to the analogues of Dehn twists inside $\text{Aut}(F_n)$.

Homological stability of $\text{Aut}(F_n)$ is originally due to Hatcher and Hatcher–Vogtmann [Hat95, HV98a, HV04]. The splitting complexes $S_\bullet^{E_1}(G; n)$ were proven to be spherical by Hepworth [Hep20, Theorem 4.4]. Homological stability for $\text{GL}_n(\mathbb{Z})$ is due to Charney [Cha80], in which she proves that the complexes $S_\bullet^{E_1}(Q; n)$ are spherical. Proposition 2.5 therefore gives:

Lemma 4.12. *If G is the groupoid of automorphisms of free groups, and Q the groupoid of integral general linear groups, then axiom (III) of Theorem 2.2 is satisfied with $\nu = \frac{2}{3}$ and $\xi = 0$.*

This is the last ingredient needed to apply Theorem 2.2 to deduce Theorem 1.2, uniform homological stability for automorphism groups of free groups.

4.4.3. *Handlebody groups.* We take Γ_n to be the mapping class group of a genus n handlebody with a marked disk on its boundary, and $Q_n = \text{GL}_n(\mathbb{Z})$. The monoidal structure on G is described in [RWW17, Section 5.7]. Surjectivity of $\Gamma_n \rightarrow Q_n$ can again be proven via Dehn twists.

Homological stability of handlebody mapping class groups is due to Hatcher–Wahl [HW10], who establish that the destabilisation complexes $W_\bullet(G; n)$ are $\frac{n-3}{2}$ -connected. Since the complexes $S_\bullet^{E_1}(Q; n)$ are spherical, as noted above, [RW22, Theorem 7.1] implies that the homology of $W_\bullet(Q; n)$ vanishes below degree $n - 1$, for all $n > 1$. By Proposition 2.3 we obtain the following result:

Lemma 4.13. *If G is the groupoid of handlebody mapping class groups, and Q the groupoid of integral general linear groups, then axiom (III) of Theorem 2.2 is satisfied with $\nu = \frac{1}{2}$ and $\xi = 0$.*

This is the last ingredient needed to apply Theorem 2.2 to deduce Theorem 1.3, uniform homological stability for handlebody groups.

4.4.4. *The integral Burau representation.* In proving Theorem 1.4, we will take G to be the free braided monoidal groupoid on one generator, so that $G = \coprod_n \beta_n$, and we let Q be defined as in Section 3. The map $G \rightarrow Q$ is the integral Burau representation.

Homological stability of braid groups goes back to Arnold [Arn70], predating Quillen’s work. The destabilisation complexes $W_\bullet(G; n)$ associated to the braid groups are in fact *contractible*, by a theorem of Damiolini [Dam13, Theorem 2.48] (see [HV17, Proposition 3.2] for a published reference). The complex $W_\bullet(Q; n)$ is $\frac{n-12}{4}$ -connected by Corollary 3.47. By Proposition 2.3, the following result follows:

Lemma 4.14. *If G is the groupoid of braid groups, and Q the image of the integral Burau representation in the even and odd symplectic groups, then axiom (III) of Theorem 2.2 is satisfied with $\nu = \frac{1}{4}$ and $\xi = -1$.*

This is the last ingredient needed to apply Theorem 2.2 to deduce Theorem 1.4, uniform homological stability for braid groups. This then implies Theorem 1.5 on moments of families of quadratic L -functions.

Remark 4.15. Deducing Lemma 4.14 from Proposition 2.3 is in fact needlessly convoluted. When G is the braid groupoid, one can more directly identify $B(S_k^{0,0}, \overline{\mathbf{R}}, \mathbb{k}_Q)$ with the suspension of $W_\bullet(Q; n)$, making Proposition 2.3 rather degenerate in this case; moreover, high connectivity of the destabilisation complexes of the braid groupoid is in any case implicitly used in the proof of Proposition 2.3. We have opted to present the argument in this way in order to make the methods of proof of Lemmas 4.11, 4.12, 4.13, and 4.14 as uniform as possible.

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