Advances in Mathematics 305 (2017) 994–1084

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Nilpotence and descent in equivariant stable homotopy theory ☆

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ARTICLE INFO

Article history:
Received 24 July 2015
Received in revised form 9 September 2016
Accepted 20 September 2016
Available online 13 October 2016
Communicated by A. Blumberg

Keywords:
Stable equivariant homotopy theory
Stable homotopy theory
Localization
Completion
Nilpotence
Koszul duality
Eilenberg–Moore spectral sequence
Infinity categories
Tensor triangulated categories
Spectral sequences
Descent
Unipotence
Equivariant topological K-theory

ABSTRACT

Let G be a finite group and let $\mathscr{F}$ be a family of subgroups of G. We introduce a class of G-equivariant spectra that we call *$\mathscr{F}$-nilpotent*. This definition fits into the general theory of torsion, complete, and nilpotent objects in a symmetric monoidal stable ∞-category, with which we begin. We then develop some of the basic properties of $\mathscr{F}$-nilpotent G-spectra, which are explored further in the sequel to this paper.
In the rest of the paper, we prove several general structure theorems for ∞-categories of module spectra over objects such as equivariant real and complex K-theory and Borel-equivariant MU. Using these structure theorems and a technique with the flag variety dating back to Quillen, we then show that large classes of equivariant cohomology theories for which a type of complex-orientability holds are nilpotent for the family of abelian subgroups. In particular, we prove that equivariant real and complex K-theory, as well as the Borel-

☆ The first author was supported by the NSF Graduate Research Fellowship under grant DGE-110640. The second author was partially supported by the DFG grant SFB 1085 – Higher Invariants, Regensburg. The third author was partially supported by the DFG grants: NO 1175/1-1 and SFB 1085 – Higher Invariants, Regensburg.

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http://dx.doi.org/10.1016/j.aim.2016.09.027

Real K-theory

equivariant versions of complex-oriented theories, have this property.

Contents

1. Introduction

1.1. Quillen's theorem

This is the first in a series of papers whose goal is to investigate certain phenomena in equivariant stable homotopy theory revolving around the categorical notion of *nilpotence*. Our starting point is the classical theorem of Quillen [63] on the cohomology of a finite group G, which describes $H^*(BG; k)$ for k a field of characteristic $p > 0$ up to a relation called $\mathcal{F}$-isomorphism.

This result is as follows. Given a cohomology class $x \in H^r(BG; k)$, it determines for each elementary abelian p-subgroup $A \leqslant G$, a cohomology class $x_A \in H^r(BA; k)$ via restriction. These classes $\{x_A\}_{A \leq G}$ are not arbitrary; they satisfy the following two basic relations:

1. If $A, A' \leqslant G$ are a pair of elementary abelian p-subgroups which are conjugate by an element $g \in G$, then x_A maps to $x_{A'}$ under the isomorphism $H^*(BA; k) \cong H^*(BA'; k)$ induced by conjugation by g.
2. If $A \leqslant A'$ is an inclusion of elementary abelian subgroups of G, then $x_{A'}$ maps to x_A under the restriction map $H^*(BA'; k) \to H^*(BA; k)$.

Let $E_p(G)$ denote the family of elementary abelian p-subgroups of G and consider the subring

$$R \subseteq \prod_{A \in E_p(G)} H^*(BA; k)$$

of all tuples $\{x_A \in H^*(BA; k)\}_{A \in E_p(G)}$ which satisfy the two conditions above. The product of the restriction maps lifts to a ring homomorphism

$$H^*(BG; k) \xrightarrow{\psi} R, \quad x \mapsto \{x_A\}_{A \in E_p(G)}. \tag{1.1}$$

Quillen's $\mathcal{F}$-isomorphism theorem states roughly that (1.1) is an isomorphism modulo nilpotence. More precisely:

Theorem 1.2 *([63, Theorem 7.1]). The map ψ is a uniform $\mathcal{F}_p$-isomorphism: in other words, there exist integers m and n such that*

1. *For every $x \in \ker \psi$, $x^m = 0$.*
2. *For every $x \in R$, x^{p^n} belongs to the image of ψ.*

Theorem 1.2 establishes the fundamental role of *elementary abelian* groups in the cohomology of finite groups, and is extremely useful in calculations, especially since there are large known classes of groups for which (1.1) is an injection (or at least an injection when one uses the larger class of all abelian subgroups); see for instance [62, Prop. 3.4, Cor. 3.5]. Since the cohomology of elementary abelian groups is known, Theorem 1.2 enables one to, for example, determine the *Krull dimension* of $H^*(BG; k)$ [63, Theorem 7.7].

1.2. Descent up to nilpotence

Theorem 1.2 by itself is a computational result about cohomology. However, as the authors learned from [19,7], it can be interpreted as a consequence of a more precise *homotopical* statement. In the homotopy theory $\mathrm{Fun}(BG, \mathrm{Mod}(k))$ of k-module spectra equipped with a G-action (equivalently, the derived category of $k[G]$-modules), the commutative algebra objects $\{k^{G/A}\}_{A \in E_p(G)}$ satisfy a type of *descent up to nilpotence:* more precisely, the thick $\otimes$-ideal they generate is all of $\mathrm{Fun}(BG, \mathrm{Mod}(k))$. From this, using a descent type spectral sequence, it is not too difficult to extract Theorem 1.2 (compare [51, §4.2]). However, the descent-up-to-nilpotence statement is much more precise and has additional applications.

The purpose of these two papers is, first, to formulate a general categorical definition that encompasses the Carlson–Balmer interpretation of Theorem 1.2. Our categorical definition lives in the world of *genuine* equivariant stable homotopy theory, and, for a finite group G, isolates a class of G-equivariant spectra for which results such as Theorem 1.2 hold with respect to a given family of subgroups. The use of genuine G-equivariant theories allows for additional applications. For instance, our application to equivariant complex K-theory gives a homotopical lifting of Artin's theorem and gives a categorical explanation of results of Bojanowska [13,14] and Bojanowska–Jackowski [15] on equivariant K-theory of finite groups. This application, which relies on an analysis of the descent spectral sequence, will appear in the second paper [54]. In addition, the methods of $\mathscr{F}$-nilpotence can be applied to equivariant versions of algebraic K-theory, which leads to Thomason-style descent theorems in the algebraic K-theory of ring spectra. We will return to this in a third paper [22].

We will give numerous examples of equivariant cohomology theories that fulfill this criterion. The specialization to Borel-equivariant mod p cohomology will recover results such as Theorem 1.2, as well as versions of Theorem 1.2 where k is replaced by any complex-oriented theory. Indeed, the second purpose of these papers is to prove $\mathcal{F}$-isomorphism theorems generalizing Theorem 1.2, using a careful analysis of the relevant descent spectral sequences.

1.3. $\mathscr{F}$-nilpotence

We now summarize the contents of this paper. The present paper is almost exclusively theoretical, and the computational results (i.e., analogs of Theorem 1.2) will be the focus of the sequel [54], so we refer to the introduction of the sequel for further discussion.

Let $\mathcal{C}$ be a presentable stable ∞-category with a compatible symmetric monoidal structure, i.e., such that $\otimes$ preserves colimits in each variable. Given an algebra object A of $\mathcal{C}$, one says, following Bousfield, that an object of $\mathcal{C}$ is *A-nilpotent* if it belongs to the thick $\otimes$-ideal generated by A.

The following is the main definition of this series of papers.

Definition *(See Definition 6.36 below).* Let G be a finite group, and let Sp_G denote the ∞-category of G-spectra (see Definition 5.10). Let $\mathscr{F}$ be a family of subgroups of G. We say that $M \in \mathrm{Sp}_G$ is *$\mathscr{F}$-nilpotent* if it is nilpotent with respect to the algebra object $\prod_{H\in\mathscr{F}} F(G/H_+, S^0_G) \in \mathrm{CAlg}(\mathrm{Sp}_G)$.

We will especially be interested in this definition for a ring G-spectrum R (up to homotopy, not necessarily structured). In this case, we will see that R is $\mathscr{F}$-nilpotent if and only if the geometric fixed points $\Phi^H R$ are contractible for any subgroup $H \leq G$ which does not belong to $\mathscr{F}$ (Theorem 6.41). In the sequel to this paper, we will show that if $R \in \mathrm{Sp}_G$ is a ring G-spectrum which is $\mathscr{F}$-nilpotent, then the R-cohomology of any G-space satisfies an analog of Theorem 1.2 (with the elementary abelian subgroups replaced by those subgroups in $\mathscr{F}$).

The first goal of this paper is to develop the theory of nilpotence in an appropriately general context. We have also taken the opportunity to discuss certain general features of symmetric monoidal stable ∞-categories, such as a general version of Dwyer–Greenlees theory [26], due to Hovey–Palmieri–Strickland [38], yielding an equivalence between complete and torsion objects. Similar ideas have also been explored in recent work of Barthel–Heard–Valenzuela [10]. This material is largely expository, but certain aspects (in particular, decompositions such as Theorem 2.30 and Theorem 3.20 below) rely on the theory of ∞-categories and have not always been documented in the classical literature on triangulated categories. Our presentation is intended to make it clear that the notion of $\mathscr{F}$-nilpotence is a natural generalization of a *bounded torsion* condition.

Let $R \in \mathrm{Alg}(\mathrm{Sp}_G)$ be an associative algebra, and suppose that R is $\mathscr{F}$-nilpotent. A major consequence of $\mathscr{F}$-nilpotence is an associated decomposition (Theorem 6.42) of the ∞-category of R-module G-spectra.

Theorem. *Suppose $R \in \mathrm{Alg}(\mathrm{Sp}_G)$ is $\mathscr{F}$-nilpotent. Let $\mathcal{O}_{\mathscr{F}}(G)$ be the category of G-sets of the form G/H, $H \in \mathscr{F}$. Then there is an equivalence of ∞-categories*

$$\mathrm{Mod}_{\mathrm{Sp}_G}(R) \simeq \varprojlim_{G/H\in\mathcal{O}_{\mathscr{F}}(G)^{op}} \mathrm{Mod}_{\mathrm{Sp}_H}(\mathrm{Res}^G_H R).$$

If R is an $\mathbb{E}_\infty$-algebra object, then the above equivalence is (canonically) symmetric monoidal too.

We expect the decomposition given above to have future applications, as in most practical situations where $\mathscr{F}$-nilpotence arises, it is easier to study modules in Sp_H over $\mathrm{Res}^G_H R$ (for $H \in \mathscr{F}$) than to study modules over R itself.

1.4. Equivariant module spectra

In the rest of this paper, we take a somewhat different direction, albeit with a view towards proving $\mathscr{F}$-nilpotence results. We analyze the structure of *modules* over certain equivariant ring spectra. These results generalize work of Greenlees–Shipley [31] in the rational setting.

Our first results concern the structure of the ∞-category $\mathrm{Fun}(BG, \mathrm{Mod}(R))$ where R is a complex-oriented $\mathbb{E}_\infty$-ring and G is a connected compact Lie group. The application of these results to $\mathscr{F}$-nilpotence statements will come from embedding a finite group in a unitary group. In case G is a product of copies of tori or unitary groups (see Theorem 7.37 for precise conditions), we describe $\mathrm{Fun}(BG, \mathrm{Mod}(R))$ as an ∞-category of *complete* modules over a (non-equivariant) ring spectrum. For instance, we prove the following result.

Theorem 1.3. *Let R be an even periodic $\mathbb{E}_\infty$-ring. Then we have an equivalence of symmetric monoidal ∞-categories,*

$$\mathrm{Fun}(BU(n), \mathrm{Mod}(R)) \simeq \mathrm{Mod}(F(BU(n)_+, R))_{\mathrm{cpl}},$$

where on the right we consider modules complete with respect to the augmentation ideal in $\pi_0(F(BU(n)_+, R)) \simeq \pi_0(R)[[c_1, \dots, c_n]]$.

One can think of Theorem 1.3 as a homotopy-theoretic (complex-orientable) version of the *Koszul duality* between DG modules over an exterior algebra (which is replaced by the group algebra $R \wedge U(n)_+$) and DG modules over a polynomial algebra (which is replaced by $F(BU(n)_+, R)$). Rationally, these results are due to Greenlees–Shipley [31]. Theorem 1.3 is useful because it is generally much easier to work with modules over the non-equivariant ring spectrum $F(BU(n)_+, R)$ than to analyze $U(n)$-actions directly.

The unitary group is especially well-behaved because its cohomology is torsion-free. A more general result (Theorem 7.35 below) runs as follows:

Theorem 1.4. *Let R be an $\mathbb{E}_\infty$-ring and let G be a compact, connected Lie group. Suppose $H^*(BG; \pi_0 R) \simeq H^*(BG; \mathbb{Z}) \otimes_{\mathbb{Z}} \pi_0 R$ and that this is a polynomial ring over $\pi_0 R$; suppose furthermore that the cohomological R-based Atiyah–Hirzebruch spectral sequence (AHSS) for BG degenerates (e.g., $\pi_*(R)$ is torsion-free). Then there is an equivalence of symmetric monoidal ∞-categories between $\mathrm{Fun}(BG, \mathrm{Mod}(R))$ and the symmetric monoidal ∞-category of R-complete $F(BG_+, R)$-modules.*

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a presentable, symmetric monoidal ∞-category where the tensor product commutes with colimits in each variable. Then there is an adjunction of symmetric monoidal ∞-categories between $\mathrm{Mod}(\mathrm{End}(\mathbf{1}))$ and $\mathcal{C}$, and we will discuss general criteria for this adjunction to be a localization. We call such $\mathcal{C}$ *unipotent.* When applied to ∞-categories of the form $\mathrm{Fun}(BG, \mathrm{Mod}(R))$, these general criteria will recover results such as Theorem 1.4.

We will then explain that results such as Theorem 1.3 lead to very quick and explicit proofs (via the flag variety) of results including the following:

Theorem 1.5. *Let R be a complex-orientable $\mathbb{E}_\infty$-ring. Then, if G is a finite group, the Borel-equivariant G-spectrum associated to R is $\mathscr{F}$-nilpotent for $\mathscr{F}$ the family of abelian subgroups.*

We will prove more precise results for particular complex-orientable theories in the sequel to this paper. The main observation is that the (very nontrivial) action of the unitary group $U(n)$ on the flag variety $F = U(n)/T$ becomes trivialized after smashing with a complex-oriented theory; the trivialization is a consequence of Theorem 1.3 (though can also be proved by hand). The use of the flag variety in this setting is of course classical, and the argument is essentially due to Quillen (albeit stated in a slightly different form).

Finally, we shall treat the cases of equivariant real and complex K-theory. Here again, we make a study of their module categories in the case of compact, connected Lie groups. Our main result is that, once again, under certain conditions the symmetric monoidal ∞-category of modules (in equivariant spectra) over equivariant real and complex K-theory can be identified with the symmetric monoidal ∞-category of modules over a *non-equivariant* $\mathbb{E}_\infty$-ring spectrum.

Theorem 1.6. *Let G be a compact, connected Lie group with $\pi_1(G)$ torsion-free. Then the respective symmetric monoidal ∞-categories $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$ and $\mathrm{Mod}_{\mathrm{Sp}_G}(KO_G)$ are equivalent to the symmetric monoidal ∞-categories of modules (in the ∞-category of spectra) over the categorical fixed points of KU_G and KO_G respectively.*

For equivariant complex K-theory, these results use (and give a modern perspective on) the theory of Künneth spectral sequences in equivariant K-theory developed by Hodgkin [34], Snaith [70], and McLeod [57]. By embedding a finite group in a unitary group, one obtains a quick proof that equivariant K-theory is nilpotent for the family of abelian subgroups; in the sequel we shall see that it is actually nilpotent for the family of cyclic subgroups. The condition that $\pi_1(G)$ should be torsion-free does not rule out torsion in $H^*(G; \mathbb{Z})$ (e.g., $G = \mathrm{Spin}(n)$) and that the conclusion holds in these cases is a special feature of K-theory.

To obtain the result for equivariant *real* K-theory, we prove a version of the theorem of Wood $KO \wedge \Sigma^{-2}\mathbb{CP}^2 \simeq KU$ in the equivariant setting (Theorem 9.8) below.

Theorem 1.7. *Let G be any compact Lie group. Then there is an equivalence of KO_G-modules $KO_G \wedge \Sigma^{-2}\mathbb{CP}^2 \simeq KU_G$.*

Here $\mathbb{CP}^2$ is considered as a pointed space with trivial G-action. We then develop an analogous $\mathbb{Z}/2$-Galois descent picture from equivariant complex to real K-theory (which is due to Rognes [65] for $G = 1$). In particular, we show that (for any compact Lie group G) the map $KO_G \to KU_G$ of $\mathbb{E}_\infty$-algebras in Sp_G is a faithful $\mathbb{Z}/2$-Galois extension. The Galois descent or homotopy fixed point spectral sequence is carefully analyzed; here the trichotomy of irreducible representations into real, complex, and quaternionic plays a fundamental role.

1.5. Notation

We will freely use the theory of ∞-categories (quasi-categories) as treated in [44] and the theory of symmetric monoidal ∞-categories, as well as that of rings and modules in them, developed in [48]. Note that we will identify the $\mathbb{E}_1$ and associative ∞-operads. In a symmetric monoidal ∞-category, we will let $\mathbb{D}A$ denote the dual of a dualizable object A. We refer to [48, §4.6.1] for a treatment of duality and dualizable objects. Homotopy limits and colimits in an ∞-category will be written as $\varprojlim$ and $\varinjlim$. We will abuse notation and often identify an ordinary category $\mathcal{C}$ with the associated quasi-category $N(\mathcal{C})$. In addition, we will frequently identify abelian groups (resp. commutative rings) with their associated Eilenberg–MacLane spectra when confusion is unlikely to arise.

Throughout, we will write $\mathcal{S}$ for the ∞-category of spaces and Sp for the ∞-category of spectra. For G a compact Lie group, the G-equivariant analogs will be denoted $\mathcal{S}_G$ and Sp_G. We will also write BG for both the classifying space of G and its associated ∞-category (∞-groupoid), so that, for an ∞-category $\mathcal{C}$, $\mathrm{Fun}(BG, \mathcal{C})$ denotes the ∞-category of objects in $\mathcal{C}$ equipped with a G-action.

Part 1. Generalities on symmetric monoidal ∞-categories

2. Complete objects

Consider the ∞-category $\mathrm{Mod}(\mathbb{Z})$ of modules over the Eilenberg–MacLane spectrum $H\mathbb{Z}$, or equivalently the (unbounded) derived ∞-category [48, §1.3.5] of the category of abelian groups. Fix a prime number p. Then there are four stable subcategories of $\mathrm{Mod}(\mathbb{Z})$ that one can define.

1. The subcategory $(\mathrm{Mod}(\mathbb{Z}))_{p-\mathrm{tors}}$ of p-torsion $\mathbb{Z}$-modules: that is, the smallest localizing[1] subcategory of $\mathrm{Mod}(\mathbb{Z})$ containing $\mathbb{Z}/p$. An object of $\mathrm{Mod}(\mathbb{Z})$ belongs to $(\mathrm{Mod}(\mathbb{Z}))_{p-\mathrm{tors}}$ if and only if all of its homotopy groups are p-power torsion.

[1] Recall that a subcategory of a presentable stable ∞-category is said to be *localizing* if it is a stable subcategory closed under colimits.

2. The subcategory $\mathrm{Mod}_{\mathbb{Z}[p^{-1}]}$ of $\mathbb{Z}[p^{-1}]$-modules: that is, those objects $M \in \mathrm{Mod}(\mathbb{Z})$ such that $M \otimes N$ is contractible for every $N \in (\mathrm{Mod}(\mathbb{Z}))_{p-\mathrm{tors}}$. This subcategory is closed under both arbitrary limits and colimits.
3. The subcategory $(\mathrm{Mod}(\mathbb{Z}))_{p-\mathrm{cpl}}$ of p-complete $\mathbb{Z}$-modules: that is, those M such that for any $N \in \mathrm{Mod}_{\mathbb{Z}[p^{-1}]}$, the space of maps $\mathrm{Hom}_{\mathrm{Mod}(\mathbb{Z})}(N, M)$ is contractible. This subcategory is closed under arbitrary limits and $\aleph_1$-filtered colimits (but not all colimits).
4. The subcategory $(\mathrm{Mod}(\mathbb{Z}))_{p-\mathrm{nil}}$ consisting of those $M \in \mathrm{Mod}(\mathbb{Z})$ such that some power of p annihilates M: that is, such that $1_M \in \pi_0\mathrm{Hom}_{\mathrm{Mod}(\mathbb{Z})}(M, M)$ is p-power torsion. This subcategory is only closed under *finite* limits and colimits, as well as retracts.

The first three subcategories satisfy a number of well-known relationships. For instance:

1. There is a *completion* (or $\mathbb{Z}/p$-localization) functor $\mathrm{Mod}(\mathbb{Z}) \to (\mathrm{Mod}(\mathbb{Z}))_{p-\mathrm{cpl}}$.
2. There is an *acyclization* or *colocalization* functor $\mathrm{Mod}(\mathbb{Z}) \to (\mathrm{Mod}(\mathbb{Z}))_{p-\mathrm{tors}}$.
3. There is a *localization* functor $L\colon \mathrm{Mod}(\mathbb{Z}) \to \mathrm{Mod}_{\mathbb{Z}[p^{-1}]}$.

Dwyer–Greenlees theory [26] implies that p-adic completion induces an equivalence

$$(\mathrm{Mod}(\mathbb{Z}))_{p-\mathrm{tors}} \simeq (\mathrm{Mod}(\mathbb{Z}))_{p-\mathrm{cpl}}.$$

Moreover, there is an *arithmetic square* for building any object X of $\mathrm{Mod}(\mathbb{Z})$ from $X[p^{-1}]$, the p-adic completion $\widehat{X}_p$, and a compatibility map. Namely, any $X \in \mathrm{Mod}(\mathbb{Z})$ fits into a pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widehat{X}_p \arrow[d] \\ X[p^{-1}] \arrow[r] & (\widehat{X}_p)[p^{-1}]. \end{tikzcd} \tag{2.1}$$

This picture and its generalizations (for instance, its version in chromatic homotopy theory) are often extremely useful in understanding how to build objects. The fourth subcategory $(\mathrm{Mod}(\mathbb{Z}))_{p-\mathrm{nil}}$ does not fit into such a functorial picture, but every object here is both p-torsion and p-complete, and the p-torsion is *bounded*.

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a presentable, symmetric monoidal, stable ∞-category where the tensor product commutes with colimits in each variable. Let A be an associative algebra object in $\mathcal{C}$. In the next two sections, we briefly review an axiomatic version of the above picture in $\mathcal{C}$ with respect to A. The main focus of this paper is the fourth subcategory of nilpotent objects in equivariant stable homotopy theory. We emphasize that these ideas are by no means new, and have been developed by several authors, including [18,33,38, 26,29,45,10].

2.1. The Adams tower and the cobar construction

As usual, let $(\mathcal{C}, \otimes, \mathbf{1})$ be a presentable, symmetric monoidal stable ∞-category where the tensor product commutes with colimits. Let $A \in \mathrm{Alg}(\mathcal{C})$ be an associative algebra object of $\mathcal{C}$. We begin with a basic construction.

Construction 2.2 *(The Adams tower).* Let $M \in \mathcal{C}$. Then we can form a tower in $\mathcal{C}$

$$\cdots \to T_2(A, M) \to T_1(A, M) \to T_0(A, M) \simeq M \tag{2.3}$$

as follows:

1. $T_1(A, M)$ is the fiber of the map $M \to A \otimes M$ induced by the unit $\mathbf{1} \to A$, so that $T_1(A, M)$ maps naturally to M.
2. More generally, $T_i(A, M) := T_1(A, T_{i-1}(A, M))$ with its natural map to $T_{i-1}(A, M)$.

Inductively, this defines the functors T_i and the desired tower. We will call this the *A-Adams tower* of M. Observe that the A-Adams tower of M is simply the tensor product of M with the A-Adams tower of $\mathbf{1}$.

We can write the construction of the Adams tower in another way. Let $I = \mathrm{fib}(\mathbf{1} \to A)$, so that I is a *nonunital* associative algebra in $\mathcal{C}$ equipped with a map $I \to \mathbf{1}$. We have a tower

$$\cdots \to I^{\otimes n} \to I^{\otimes (n-1)} \to \cdots \to I^{\otimes 2} \to I \to \mathbf{1},$$

and this is precisely the A-Adams tower $\{T_i(A, \mathbf{1})\}_{i \geq 0}$. The A-Adams tower for M is obtained by tensoring this with M.

Example 2.4. Take $\mathcal{C} = \mathrm{Mod}(\mathbb{Z})$ and $A = \mathbb{Z}/p$. Then the Adams tower $\{T_i(\mathbb{Z}/p, M)\}$ of an object $M \in \mathrm{Mod}(\mathbb{Z})$ is given by

$$\cdots \to M \xrightarrow{p} M \xrightarrow{p} M.$$

The A-Adams tower has two basic properties:

Proposition 2.5.

1. *For each i, the cofiber of $T_i(A, M) \to T_{i-1}(A, M)$ admits the structure of an A-module (internal to $\mathcal{C}$).*
2. *Each map $T_i(A, M) \to T_{i-1}(A, M)$ becomes nullhomotopic after tensoring with A.*

Proof. Suppose $i = 1$. In this case, the cofiber of $T_1(A, M) \to M$ is precisely $A \otimes M$ by construction. We have a cofiber sequence

$$T_1(A, M) \to M \to A \otimes M,$$

and the last map admits a section after tensoring with A. Therefore, the map $T_1(A, M) \to M$ must become nullhomotopic after tensoring with A. Since $T_i(A, M) = T_1(A, T_{i-1}(A, M))$, the general case follows. □

Corollary 2.6. *Suppose $M \in \mathcal{C}$ is an A-module up to homotopy. Then the successive maps $T_i(A, M) \to T_{i-1}(A, M)$ in the Adams tower are nullhomotopic.*

Proof. If $M = A \otimes N$, then we just saw that each of the maps in the Adams tower is nullhomotopic. If M is an A-module up to homotopy, then M is a *retract* (in $\mathcal{C}$) of $A \otimes M$, so each of the maps in the Adams tower $\{T_i(A, M)\}$ must be nullhomotopic as well. □

The construction of the Adams tower can be carried out even if A is only an algebra object in the *homotopy category* of $\mathcal{C}$: that is, one does not need the full strength of the associative algebra structure in $\mathcal{C}$. However, we will also need the following construction that does use this extra (homotopy coherent) structure.

Construction 2.7. Given $A \in \mathrm{Alg}(\mathcal{C})$, we can form a cosimplicial object in $\mathcal{C}$,

$$\mathrm{CB}^\bullet(A) = \left\{ A \rightrightarrows A \otimes A \substack{\to\\\to\\\to} \dots \right\} \in \mathrm{Fun}(\Delta, \mathcal{C}),$$

called the *cobar construction* on A. The cobar construction extends to an augmented cosimplicial object

$$\mathrm{CB}^\bullet_{\mathrm{aug}}(A) \colon N(\Delta_+) \to \mathcal{C},$$

(where Δ_+ is the augmented simplex category of finite ordered sets), where the augmentation is from the unit object $\mathbf{1}$. The augmented cosimplicial object $\mathrm{CB}^\bullet_{\mathrm{aug}}(A)$ admits a *splitting* [48, §4.7.3] after tensoring with A: that is, the augmented cosimplicial object $\mathrm{CB}^\bullet_{\mathrm{aug}}(A) \otimes A$ is split.

Although the cobar construction in the 1-categorical context is classical, for precision we spell out the details of how one may extract the cobar construction using the formalism of [48, §2.1–2.2]. By definition, since $\mathcal{C}$ is a symmetric monoidal ∞-category, one has an ∞-category $\mathcal{C}^\otimes$ together with a cocartesian fibration $\mathcal{C}^\otimes \to N(\mathrm{Fin}_*)$ where Fin_* is the category of pointed finite sets. The underlying ∞-category $\mathcal{C}$ is obtained as the fiber over the pointed finite set $\{0\} \cup \{*\}$. The associative operad has an operadic nerve $N^\otimes(\mathbf{E}_1)$ which maps to $N(\mathrm{Fin}_*)$, and the algebra object A defines a morphism $\phi_A \colon N^\otimes(\mathbf{E}_1) \to \mathcal{C}^\otimes$ over $N(\mathrm{Fin}_*)$. The crucial point is that we have a functor $N(\Delta_+) \to N^\otimes(\mathbf{E}_1)$ whose definition we will now recall.

To understand this functor, recall that the operadic nerve $N^{\otimes}(\mathbf{E}_1)$ (recall that we identify $\mathbf{E}_1$ and the associative operad) comes from the (ordinary) category described as follows:

1. The objects are finite pointed sets $S \in \mathrm{Fin}_*$.
2. Given S, T, to give a morphism $S \to T$ in $N^{\otimes}(\mathbf{E}_1)$ amounts to giving a morphism $\rho\colon S \to T$ in Fin_* and an *ordering* on each of the sets $\rho^{-1}(t)$ for $t \in T \setminus \{*\}$.

We now obtain a functor $N(\Delta_+) \to N^{\otimes}(\mathbf{E}_1)$ which sends a finite ordered set S to $S \sqcup \{*\}$; a morphism $S \to T$ in Δ_+ clearly induces a morphism $N^{\otimes}(\mathbf{E}_1)$ (using the induced ordering on the preimages). Composing, we obtain a functor

$$\psi_A\colon N(\Delta_+) \to N^{\otimes}(\mathbf{E}_1) \xrightarrow{\phi_A} \mathcal{C}^{\otimes}.$$

Now, let $\mathrm{Fin}_*^{\mathrm{ac}} \subset \mathrm{Fin}_*$ be the (non-full) subcategory with the same objects, but such that morphisms of pointed sets $\rho\colon S \to T$ are required to be *active*, i.e., such that $\rho^{-1}(*) = *$. Observe that ψ_A factors (canonically) over $\mathcal{C}^{\otimes} \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*^{\mathrm{ac}})$.

Finally, since $\mathcal{C}$ is a symmetric monoidal ∞-category, we have a functor

$$\bigotimes\colon \mathcal{C}^{\otimes} \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*^{\mathrm{ac}}) \to \mathcal{C}$$

that, informally, tensors together a tuple of objects. To obtain this, observe that for any object $S \in \mathrm{Fin}_*^{\mathrm{ac}}$, there is a *natural* map $f_S\colon S \to \{0\} \cup \{*\}$ such that $f_S^{-1}(0) = S \setminus \{*\}$. (The naturality holds on $\mathrm{Fin}_*^{\mathrm{ac}}$, not on the larger category Fin_*.) The functor $\bigotimes$ is the cocartesian lift of this natural transformation. Now, the (augmented) cobar construction is the composition

$$\mathrm{CB}^{\bullet}_{\mathrm{aug}}(A)\colon N(\Delta_+) \xrightarrow{\psi_A} \mathcal{C}^{\otimes} \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}_*^{\mathrm{ac}}) \xrightarrow{\bigotimes} \mathcal{C}.$$

Our first goal is to demonstrate the connection between the Adams tower and the cobar construction. Given a functor $X^\bullet\colon \Delta \to \mathcal{C}$, we recall that $\mathrm{Tot}_n(X^\bullet)$ is defined to be the homotopy limit of $X^\bullet|_{\Delta^{\leq n}}$ for $\Delta^{\leq n} \subset \Delta$ the full subcategory spanned by $\{[0], [1], \dots, [n]\}$. We will need the following important result. The notion of stability is self-dual, so we have dualized the statement in the cited reference.

Theorem 2.8 *(Lurie [48, Th. 1.2.4.1]; ∞-categorical Dold–Kan correspondence). Let $\mathcal{C}$ be a stable ∞-category. Then the functor*

$$\mathrm{Fun}(\Delta, \mathcal{C}) \to \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathcal{C}), \quad X^\bullet \mapsto \{\mathrm{Tot}_n(X^\bullet)\}_{n \geq 0}$$

establishes an equivalence between cosimplicial objects in $\mathcal{C}$ and towers in $\mathcal{C}$.

In particular, we will show (Proposition 2.14) that under the ∞-categorical Dold–Kan correspondence, the cobar construction and the Adams tower correspond to one another. This result is certainly not new, but we have included it for lack of a convenient reference.

Definition 2.9. Given a finite nonempty set S, we will let $\mathcal{P}(S)$ denote the partially ordered set of nonempty subsets of S ordered by inclusion. We will let $\mathcal{P}^+(S)$ denote the partially ordered set of *all* subsets of S ordered by inclusion.

Construction 2.10. Suppose given morphisms $f_s\colon X_s \to Y_s \in \mathcal{C}$ for each $s \in S$. Then we obtain a functor

$$F^+(\{f_s\})\colon \mathcal{P}^+(S) \to \mathcal{C}$$

whose value on a subset $S' \subset S$ is given by

$$F^+(\{f_s\})(S') = \bigotimes_{s_1 \notin S'} X_{s_1} \otimes \bigotimes_{s_2 \in S'} Y_{s_2}.$$

We will let $F(\{f_s\})\colon \mathcal{P}(S) \to \mathcal{C}$ denote the restriction of $F^+(\{f_s\})$.

Our first goal is to give a formula for the inverse limit of these functors $F(\{f_s\})$. This will be important in determining the partial totalizations of the Adams tower (Proposition 2.14 below).

Proposition 2.11. *Let S be a finite nonempty set and suppose given morphisms $f_s\colon X_s \to Y_s$ in $\mathcal{C}$ for each $s \in S$. Form a functor $F(\{f_s\})\colon \mathcal{P}(S) \to \mathcal{C}$ as in Construction 2.10. Then there is an identification, functorial in $\prod_{s\in S} \mathrm{Fun}(\Delta^1, \mathcal{C})$,*

$$\varprojlim_{\mathcal{P}(S)} F(\{f_s\}) \simeq \mathrm{cofib}\left(\bigotimes_{s\in S} \mathrm{fib}(X_s \to Y_s) \to \bigotimes_{s \in S} X_s \right).$$

Proof. We first explain the map. Let $\mathrm{Fun}(\Delta^1, \mathcal{C})$ denote the ∞-category of arrows in $\mathcal{C}$; it is itself a stable ∞-category. Observe that any object $X \to Y$ of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ fits into a cofiber sequence

$$(\mathrm{fib}(X \to Y) \to 0) \to (X \to Y) \to (Y \to Y). \tag{2.12}$$

Given an S-indexed family of objects $\{f_s\colon X_s \to Y_s\}$ of $\mathrm{Fun}(\Delta^1, \mathcal{C})$, we have associated an object $F^+(\{f_s\}) \in \mathrm{Fun}(\mathcal{P}^+(S), \mathcal{C})$. We obtain a functor

$$\prod_{s\in S} \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\mathcal{P}^+(S), \mathcal{C})$$

which is exact in each variable. Therefore, using (2.12), we obtain a natural morphism

$$F^+(\{\mathrm{fib}(X_s \to Y_s) \to 0\}) \to F^+(\{f_s\}),$$

in $\mathrm{Fun}(\mathcal{P}^+(S), \mathcal{C})$. Taking the cofiber of this morphism yields an object of $\mathrm{Fun}(\mathcal{P}^+(S), \mathcal{C})$ where the initial vertex is mapped precisely to $\mathrm{cofib}\left(\bigotimes_{s\in S} \mathrm{fib}(X_s \to Y_s) \to \bigotimes_{s\in S} X_s\right)$ and whose restriction to $\mathcal{P}(S)$ is identified with $F(\{f_s\})$.

By the universal property of the homotopy limit (since $\mathcal{P}^+(S)$ is the *cone* on $\mathcal{P}(S)$), this gives a natural morphism

$$\mathrm{cofib}\left(\bigotimes_{s\in S} \mathrm{fib}(X_s \to Y_s) \to \bigotimes_{s\in S} X_s\right) \to \varprojlim_{\mathcal{P}(S)} F(\{f_s\}) \in \mathcal{C}. \tag{2.13}$$

We need to argue that this morphism (2.13) is an equivalence. We first claim that if one of the morphisms $f_s \colon X_s \to Y_s$ is an equivalence, then (2.13) is an equivalence, i.e., that

$$F^+(\{f_s\}) \colon \mathcal{P}^+(S) \to \mathcal{C}$$

is a limit diagram. However, this follows from the dual of [48, Lem. 1.2.4.15] applied to $K = \mathcal{P}(S \setminus \{s\})$ as $K^{\triangleleft} = \mathcal{P}^+(S \setminus \{s\})$ and $K^{\triangleleft} \times \Delta^1 = \mathcal{P}^+(S)$; the fiber of the natural map of diagrams $K^{\triangleleft} \to \mathcal{C}$ thus obtained is contractible. Here $K^{\triangleleft}$ is the *left cone* over K [44, Notation 1.2.8.4].

Now, to show that (2.13) is an equivalence, we observe that both sides are exact functors in each $\mathrm{Fun}(\Delta^1, \mathcal{C})$ variable. We use induction on the number of $f_s \colon X_s \to Y_s$ with Y_s noncontractible. If all the $Y_s = 0$, both sides of (2.13) are contractible. Now suppose n of the Y_s's are not zero, and choose $s_1 \in S$ with $Y_{s_1} \neq 0$. In this case, we use the cofiber sequence (2.12). In order to show that (2.13) is an equivalence, it suffices to show that (2.13) becomes an equivalence after we replace f_{s_1} either by $Y_{s_1} \xrightarrow{\mathrm{id}} Y_{s_1}$ or $\mathrm{fib}(X_{s_1} \to Y_{s_1}) \to 0$. We have treated the first case in the previous paragraph, and the second case follows by the inductive hypothesis. □

Proposition 2.14. *The tower associated (via the Dold–Kan correspondence) to the cosimplicial object* $\mathrm{CB}^\bullet(A)$ *is precisely the tower*

$$\{\mathrm{cofib}(T_{n+1}(A, \mathbf{1}) \to \mathbf{1})\}.$$

In other words, we have equivalences $\mathrm{Tot}_n(\mathrm{CB}^\bullet(A)) \simeq \mathrm{cofib}(I^{\otimes(n+1)} \to \mathbf{1})$.

Proof. We compute $\mathrm{Tot}_n(\mathrm{CB}^\bullet(A))$. For this, we let $\mathcal{P}([n])$ denote the partially ordered set of nonempty subsets of $[n]$. There is a natural functor

$$\mathcal{P}([n]) \to \Delta^{\leq n}$$

which is right cofinal by [48, Lem. 1.2.4.17]. We can describe the composite functor

$$\mathcal{P}([n]) \to \Delta^{\leq n} \xrightarrow{\mathrm{CB}^\bullet(A)} \mathcal{C}$$

as follows: it is obtained by considering the unit maps $f_s\colon \mathbf{1} \to A$ for each $s \in [n]$ and forming $F(\{f_s\})$ as in Construction 2.10. Now, the homotopy limit is thus computed by Proposition 2.11 and it is as desired. The maps in the tower, too, are seen to be the natural ones. □

2.2. Complete objects

We review rudiments of the theory of Bousfield localization [18] in our setting. As before, $(\mathcal{C}, \otimes, \mathbf{1})$ is a presentable, symmetric monoidal stable ∞-category where $\otimes$ commutes with colimits in each variable, and $A \in \mathrm{Alg}(\mathcal{C})$.

Definition 2.15. We say that an object $X \in \mathcal{C}$ is A-*complete* or A-*local* if, for any $Y \in \mathcal{C}$ with $Y \otimes A \simeq 0$, the space of maps $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is contractible. The A-complete objects of $\mathcal{C}$ span a full subcategory $\mathcal{C}_{A-\mathrm{cpl}} \subset \mathcal{C}$.

Example 2.16. A motivating example to keep in mind throughout is $\mathcal{C} = \mathrm{Mod}(\mathbb{Z})$ and $A = \mathbb{Z}/p$. Here, the A-complete objects of $\mathrm{Mod}(\mathbb{Z})$ are referred to as *p-adically complete.*

Example 2.17. Suppose A has the property that tensoring with A is conservative. For instance, a duality argument shows that this holds if A is dualizable (cf. [48, §4.6.1]) and $\mathbb{D}A$ generates $\mathcal{C}$ as a localizing subcategory. Then every object of $\mathcal{C}$ is A-complete.

Example 2.18. Suppose $M \in \mathcal{C}$ admits the structure of an A-module. Then M is A-complete. In fact, suppose $X \in \mathcal{C}$ is such that $A \otimes X$ is contractible. Then

$$\mathrm{Hom}_{\mathcal{C}}(X, M) \simeq \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{C}}(A)}(A \otimes X, M) \simeq \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{C}}(A)}(0, M) \simeq 0.$$

It follows formally from the definitions that $\mathcal{C}_{A-\mathrm{cpl}}$ is closed under all limits in $\mathcal{C}$. The subcategory $\mathcal{C}_{A-\mathrm{cpl}}$ can equivalently be described as consisting of those objects $X \in \mathcal{C}$ such that if $Y \to Y'$ is a map that becomes an equivalence after tensoring with A, then $\mathrm{Hom}_{\mathcal{C}}(Y', X) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$ is an equivalence.

We invoke here the theory of Bousfield localization in the ∞-categorical context [44, §5.5.4]. In particular, we let S be the collection of morphisms $Y \to Y'$ in $\mathcal{C}$ which become an equivalence after tensoring with A. By [44, Prop. 5.5.4.16], this class S, as a strongly saturated class ([44, Def. 5.5.4.5]) is of small generation. We now invoke the basic existence result [44, Prop. 5.5.4.15], which implies that $\mathcal{C}_{A-\mathrm{cpl}}$ is a presentable ∞-category, and that the inclusion $\mathcal{C}_{A-\mathrm{cpl}} \subset \mathcal{C}$ has a left adjoint.

Definition 2.19. We will let $L_A\colon \mathcal{C} \to \mathcal{C}_{A-\mathrm{cpl}}$ denote the left adjoint to the inclusion $\mathcal{C}_{A-\mathrm{cpl}} \subset \mathcal{C}$ and refer to L_A as A-*completion.* We will also abuse notation and use L_A to denote the composition $\mathcal{C} \xrightarrow{L_A} \mathcal{C}_{A-\mathrm{cpl}} \subset \mathcal{C}$ when confusion is unlikely to arise.

When regarded as a functor $L_A: \mathcal{C} \to \mathcal{C}$ (as $\mathcal{C}_{A-\mathrm{cpl}} \subset \mathcal{C}$ is a full subcategory), we have a natural transformation $X \to L_A X$ for any X, with the properties:

1. The map $X \to L_A X$ becomes an equivalence after tensoring with A.
2. The object $L_A X$ is A-complete.

Remark 2.20. We recall that colimits in $\mathcal{C}_{A-\mathrm{cpl}}$ are computed by first computing the colimit in $\mathcal{C}$ and then applying localization L_A again. In particular, while the inclusion $\mathcal{C}_{A-\mathrm{cpl}} \subset \mathcal{C}$ need not preserve colimits, the composition $\mathcal{C}_{A-\mathrm{cpl}} \subset \mathcal{C} \stackrel{\otimes A}{\to} \mathcal{C}$ does.

Suppose $\phi: X \to Y$ is a map in $\mathcal{C}$ such that $\phi \otimes 1_A: X \otimes A \to Y \otimes A$ is an equivalence. Then for any Z, the map $\phi \otimes 1_Z$ has the same property. In view of [48, Prop. 2.2.1.9], $\mathcal{C}_{A-\mathrm{cpl}}$ inherits the structure of a symmetric monoidal ∞-category such that the functor $L_A: \mathcal{C} \to \mathcal{C}_{A-\mathrm{cpl}}$ is symmetric monoidal.

In this subsection, we will review several characterizations of A-complete objects, and describe the subcategory of complete objects as a homotopy limit of presentable ∞-categories. Throughout, the assumption that A is dualizable will be critical as it implies that tensoring with A commutes with homotopy limits. The first basic result is as follows.

Proposition 2.21. *Suppose A is dualizable. For any object $M \in \mathcal{C}$, the map $M \to \mathrm{Tot}(M \otimes \mathrm{CB}^\bullet(A))$ exhibits the target as the A-completion of M.*

Proof. In fact, the map $M \to \mathrm{Tot}(M \otimes \mathrm{CB}^\bullet(A))$ becomes an equivalence after tensoring with A. This follows because $M \otimes \mathrm{CB}^\bullet_{\mathrm{aug}}(A)$ becomes a split augmented cosimplicial object after tensoring with A. In addition, we use the fact that tensoring with A commutes with arbitrary homotopy limits (as A is dualizable). Moreover, $\mathrm{Tot}(M \otimes \mathrm{CB}^\bullet(A))$ is A-complete as it is the homotopy limit of a diagram of objects, each of which is an A-module and therefore A-complete (Example 2.18). $\square$

In view of Proposition 2.14, we find (with $I = \mathrm{fib}(\mathbf{1} \to A)$) an equivalence

$$L_A M \simeq \varprojlim_n \left[\mathrm{cofib}(I^{\otimes n+1} \to \mathbf{1}) \otimes M\right]. \tag{2.22}$$

This recovers the familiar formula for p-adic completion in $\mathrm{Mod}(\mathbb{Z})$, for example.

We now obtain the following criteria for A-completeness.

Proposition 2.23. *The following are equivalent for an object $M \in \mathcal{C}$ and for $A \in \mathrm{Alg}(\mathcal{C})$, assumed dualizable in $\mathcal{C}$.*

1. *The object M is A-complete.*
2. *The homotopy limit of the Adams tower $\{T_i(A, M)\}_{i \geq 0}$ is contractible.*
3. *The augmented cosimplicial object $\mathrm{CB}^\bullet_{\mathrm{aug}}(A) \otimes M$ is a limit diagram.*

Proof. (1) $\Leftrightarrow$ (3). This follows from Proposition 2.21.

To see that (1) $\Leftrightarrow$ (2), one can use the comparison between the Adams tower and the cobar construction (Proposition 2.14) and conclude. One can also argue directly; we leave this to the reader. $\square$

Corollary 2.24. *If $X \in \mathcal{C}$ is A-complete and $Y \in \mathcal{C}$ is dualizable, then $X \otimes Y$ is A-complete.*

Construction 2.25. The object $\mathbb{D}A \in \mathcal{C}$ admits the structure of an A-module. In fact, we have a map $A \otimes \mathbb{D}A \to \mathbb{D}A$ which is (doubly) adjoint to the multiplication map $A \otimes A \to A$ which makes A into an A-module. Alternatively, the module structure on $\mathbb{D}A$ comes from applying the *right* adjoint $\mathrm{Hom}_{\mathcal{C}}(A, \cdot)$ of the forgetful functor $\mathrm{Mod}_{\mathcal{C}}(A) \to \mathcal{C}$ to $\mathbf{1} \in \mathcal{C}$.

It will now be convenient to make the following further hypotheses on $\mathcal{C}$ and A, which will be in effect until the end of the subsection.

Hypotheses 2.26. $(\mathcal{C}, \otimes, \mathbf{1})$ is a presentable, symmetric monoidal stable ∞-category where $\otimes$ commutes with colimits in each variable. We assume furthermore that:

1. The unit $\mathbf{1}$ is compact.
2. The object A is dualizable (as already assumed).
3. The ∞-category $\mathcal{C}$ is generated as a localizing subcategory by dualizable objects.

Recall that in this setting, compactness of the unit implies compactness of all dualizable objects.

Proposition 2.27. *Let $\mathcal{D}$ be a family of dualizable generators for $\mathcal{C}$. Then the objects $\{\mathbb{D}A \otimes X\}_{X \in \mathcal{D}}$ form a system of compact generators for $\mathcal{C}_{A-\mathrm{cpl}}$.*

Proof. Fix $X \in \mathcal{D}$. By Construction 2.25, $\mathbb{D}A \otimes X$ belongs to $\mathcal{C}_{A-\mathrm{cpl}}$ as it is an A-module. We show that $\mathbb{D}A \otimes X$ is compact in $\mathcal{C}_{A-\mathrm{cpl}}$. Indeed,

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{D}A \otimes X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, A \otimes Y),$$

and we observe that the functor $\mathcal{C}_{A-\mathrm{cpl}} \to \mathcal{C}$, $Y \mapsto A \otimes Y$, commutes with colimits (Remark 2.20). Since X is compact in $\mathcal{C}$, we can now conclude that $\mathbb{D}A \otimes X$ is compact in $\mathcal{C}_{A-\mathrm{cpl}}$.

To show that the $\{\mathbb{D}A \otimes X\}_{X \in \mathcal{D}}$ generate $\mathcal{C}_{A-\mathrm{cpl}}$, it suffices (Lemma 7.6) to show that if $Y \in \mathcal{C}_{A-\mathrm{cpl}}$ is arbitrary and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{D}A \otimes X, Y)$ is contractible for all $X \in \mathcal{D}$, then Y is contractible. But this means that $\mathrm{Hom}_{\mathcal{C}}(X, A \otimes Y)$ is contractible for all $X \in \mathcal{D}$. Thus $A \otimes Y$ is contractible, so Y in turn is contractible by A-completeness. $\square$

Next, we include a result that describes complete objects for a tensor product of algebras. This result (and its variants for torsion and nilpotent objects) will be useful in the sequel.

Proposition 2.28. *Suppose $A, B \in \mathrm{Alg}(\mathcal{C})$ are dualizable in $\mathcal{C}$. Then an object $X \in \mathcal{C}$ is $(A \otimes B)$-complete if and only if X is both A-complete and B-complete.*

Proof. Suppose X is $(A \otimes B)$-complete. Then $X \simeq \mathrm{Tot}(X \otimes \mathrm{CB}^\bullet(A \otimes B))$. Each term in this totalization is an A-module, and therefore A-complete. Thus, the homotopy limit X is A-complete too. Similarly, X is B-complete.

Conversely, suppose X is both A-complete and B-complete. Then consider the bicosimplicial diagram

$$X \otimes \mathrm{CB}^\bullet(A) \otimes \mathrm{CB}^\bullet(B) \colon \Delta \times \Delta \to \mathcal{C}. \tag{2.29}$$

Since $A^{\otimes k} \otimes X$ is B-complete for any k (Corollary 2.24), and since X is A-complete, one sees that the homotopy limit of the bicosimplicial diagram (2.29) is X itself: indeed, one computes the bitotalization one factor at a time. However, every term in the bicosimplicial diagram (2.29) is an $(A \otimes B)$-module and thus $(A \otimes B)$-complete. Thus, X is $(A \otimes B)$-complete itself. $\square$

The final goal of this section is to describe the ∞-category $\mathcal{C}_{A-\mathrm{cpl}}$ as a homotopy limit, via descent theory, when A is actually a *commutative* algebra object, so that the cobar construction takes values in commutative algebra objects.[2] This is the one part of the present section where the language of ∞-categories is necessary, and the result will be useful to us in the sequel.

Consider the augmented cobar construction $\mathrm{CB}^\bullet_{\mathrm{aug}}(A) \colon \Delta^+ \to \mathrm{CAlg}(\mathcal{C})$. Taking module ∞-categories everywhere, we obtain a cosimplicial diagram of symmetric monoidal stable ∞-categories

$$\mathrm{Mod}_{\mathcal{C}}(A) \rightrightarrows \mathrm{Mod}_{\mathcal{C}}(A \otimes A) \substack{\to \\ \to \\ \to} \dots$$

receiving an augmentation from $\mathcal{C}$.

Theorem 2.30. *If $A \in \mathrm{CAlg}(\mathcal{C})$ is dualizable in $\mathcal{C}$, then $\mathcal{C}_{A-\mathrm{cpl}}$ can be recovered as the homotopy limit*

$$\mathcal{C}_{A-\mathrm{cpl}} \simeq \mathrm{Tot}\left(\mathrm{Mod}_{\mathcal{C}}(A) \rightrightarrows \mathrm{Mod}_{\mathcal{C}}(A \otimes A) \substack{\to \\ \to \\ \to} \dots\right),$$

in the ∞-category of symmetric monoidal ∞-categories.

Proof. We have an adjunction

$$(F, G) \colon \mathcal{C} \rightleftarrows \mathrm{Mod}_{\mathcal{C}}(A)$$

[2] The cobar construction on an associative algebra object does not live in the ∞-category of algebra objects.

where $F(X) = A \otimes X$ and G forgets the A-module structure. This adjunction descends to a similar adjunction

$$(F', G') \colon \mathcal{C}_{A-\mathrm{cpl}} \rightleftarrows \mathrm{Mod}_{\mathcal{C}}(A),$$

with the same formulas. As a result, the coaugmentation from $\mathcal{C}$ of the cosimplicial symmetric monoidal ∞-category $\mathrm{Mod}_{\mathcal{C}}(\mathrm{CB}^\bullet(A))$ descends to a coaugmentation from $\mathcal{C}_{A-\mathrm{cpl}}$, leading to the natural functor

$$\mathcal{C}_{A-\mathrm{cpl}} \to \mathrm{Tot}\left(\mathrm{Mod}_{\mathcal{C}}(A) \rightrightarrows \mathrm{Mod}_{\mathcal{C}}(A \otimes A) \overset{\to}{\underset{\to}{\to}} \dots\right). \tag{2.31}$$

We want to see that this functor is an equivalence of symmetric monoidal ∞-categories. This is a descent argument using the ∞-categorical monadicity theorem and an identification of the above homotopy limit, which appears in the proof of [47, Prop. 6.18].

For this, we consider the map (2.31) and replace all ∞-categories with their opposites to obtain a new map

$$(\mathcal{C}_{A-\mathrm{cpl}})^{op} \to \mathrm{Tot}\left(\mathrm{Mod}_{\mathcal{C}}(A)^{op} \rightrightarrows \mathrm{Mod}_{\mathcal{C}}(A \otimes A)^{op} \overset{\to}{\underset{\to}{\to}} \dots\right). \tag{2.32}$$

It suffices to show that (2.32) is an equivalence. For this, we invoke [48, Cor. 4.7.6.3]. The necessary condition on left adjointability is satisfied in view of [47, Lem. 6.15]. In order to apply [48, Cor. 4.7.6.3], it therefore suffices to show that tensoring with A, as a functor $\mathcal{C}_{A-\mathrm{cpl}} \to \mathrm{Mod}_{\mathcal{C}}(A)$, preserves A-split totalizations and is conservative. However, since A is dualizable, tensoring with A preserves *all* limits, and it is conservative on $\mathcal{C}_{A-\mathrm{cpl}}$ (since any object $X \in \mathcal{C}_{A-\mathrm{cpl}}$ with $X \otimes A \simeq 0$ must be contractible itself). Therefore, we can apply the comonadicity theorem and complete the proof. □

We emphasize that the above argument is standard [47, §6] in ∞-categorical descent theory. The main use of it here is to identify an ∞-category of *complete* objects with respect to a dualizable algebra object.

Example 2.33. Suppose A has the property that tensoring with A is *conservative* on $\mathcal{C}$. In this case, one sees easily that $\mathcal{C}_{A-\mathrm{cpl}} = \mathcal{C}$ and the above result, Theorem 2.30, is a descent theorem for $\mathcal{C}$ itself as a homotopy limit of modules over the tensor powers $\{A^{\otimes(n+1)}\}_{n\geq 0}$. In fact, by [51, Th. 3.36], the commutative algebra object A is *descendable* in $\mathcal{C}$, i.e., the thick $\otimes$-ideal it generates is all of $\mathcal{C}$. In particular, this descent theorem is [51, Prop. 3.21]. While the decomposition of $\mathcal{C}$ as a homotopy limit does not require compactness of the unit, the additional conclusion of descendability of A does.

3. A-torsion objects and A^{-1}-local objects

In this section, we describe the theory of *torsion* objects with respect to the algebra object $A \in \mathrm{Alg}(\mathcal{C})$, and the dual theory of A^{-1}-local objects. The main results are a general version (Theorem 3.9) of the Dwyer–Greenlees [26] equivalence between complete and torsion objects, which is due to Hovey–Palmieri–Strickland [38, Th. 3.3.5] and a version of the arithmetic square (Theorem 3.20). We continue to work under Hypotheses 2.26.

3.1. Torsion objects

Definition 3.1. The subcategory $\mathcal{C}_{A-\mathrm{tors}}$ of A-*torsion* objects in $\mathcal{C}$ is the smallest localizing subcategory of $\mathcal{C}$ containing $A \otimes X$, for $X \in \mathcal{C}$ dualizable.

As with $\mathcal{C}_{A-\mathrm{cpl}}$, our first goal is to make $\mathcal{C}_{A-\mathrm{tors}}$ explicit.

Construction 3.2. By [48, Cor. 1.4.4.2], $\mathcal{C}_{A-\mathrm{tors}}$ is a presentable ∞-category. In particular, by the adjoint functor theorem [44, Cor. 5.5.2.9], the fully faithful inclusion $\mathcal{C}_{A-\mathrm{tors}} \subset \mathcal{C}$ is a left adjoint and admits a right adjoint

$$\mathrm{AC}_A : \mathcal{C} \to \mathcal{C}_{A-\mathrm{tors}},$$

which is called the A-*acyclization functor*. For any object $X \in \mathcal{C}$, there is a natural (counit) map $\mathrm{AC}_A(X) \to X$ in $\mathcal{C}$.

Our first goal is to get a handle on AC_A. We begin by showing that $\mathcal{C}_{A-\mathrm{tors}}$ is a $\otimes$-ideal (Definition 4.1).

Proposition 3.3. *If $Y \in \mathcal{C}_{A-\mathrm{tors}}$ and $X \in \mathcal{C}$, then $X \otimes Y \in \mathcal{C}_{A-\mathrm{tors}}$.*

Proof. Consider the collection of $X \in \mathcal{C}$ such that $X \otimes Y \in \mathcal{C}_{A-\mathrm{tors}}$. By definition, this collection is localizing, so to show that it is all of $\mathcal{C}$, it suffices to show that it contains all X *dualizable.* So, we may assume that X is dualizable.

Fix a dualizable X. Consider the collection of all $Y' \in \mathcal{C}_{A-\mathrm{tors}}$ such that $X \otimes Y' \in \mathcal{C}_{A-\mathrm{tors}}$. The collection of such Y' is localizing, so to show that it is all of $\mathcal{C}_{A-\mathrm{tors}}$, it suffices to consider the case where $Y' = A \otimes Y''$ for $Y'' \in \mathcal{C}$ dualizable. But in this case $X \otimes Y' = X \otimes A \otimes Y''$ clearly belongs to $\mathcal{C}_{A-\mathrm{tors}}$: in fact, it is one of the generating objects. $\square$

As a result, we find that $\mathcal{C}_{A-\mathrm{tors}}$ is also the localizing $\otimes$-ideal generated by A. We will now write down an explicit formula for $\mathrm{AC}_A(X)$.

Construction 3.4. Recall the Adams tower

$$\cdots \to T_2(A, \mathbf{1}) \to T_1(A, \mathbf{1}) \to T_0(A, \mathbf{1}) \simeq \mathbf{1}$$

of the unit object $\mathbf{1}$. As A is dualizable, each of the objects in this tower is dualizable, so we can form the dual tower

$$\mathbf{1} \to U_1 \to U_2 \to \dots,$$

where $U_i := \mathbb{D}(T_i(A, \mathbf{1}))$. We define $U_A = \varinjlim U_i$ and let V_A be the fiber of $\mathbf{1} \to U_A$.

Equivalently, let $\mathrm{CB}^\bullet(A)\colon \Delta \to \mathcal{C}$ denote the cobar construction on A and form the pointwise dual

$$\mathbb{D}(\mathrm{CB}^\bullet(A))\colon \Delta^{op} \to \mathcal{C},$$

which maps via an augmentation to $\mathbb{D}(\mathbf{1}) \simeq \mathbf{1}$. Then $V_A = |\mathbb{D}(\mathrm{CB}^\bullet(A))|$.

Proposition 3.5. *For any $X \in \mathcal{C}$, we have a natural equivalence $\mathrm{AC}_A(X) \simeq V_A \otimes X$. Therefore, $X \in \mathcal{C}_{A-\mathrm{tors}}$ if and only if the natural map $V_A \otimes X \to X$ is an equivalence.*

Proof. We have a natural map $V_A \otimes X \to X$. In order to show that $V_A \otimes X$ is identified with $\mathrm{AC}_A(X)$, we need to show two things:

1. $V_A \otimes X$ belongs to $\mathcal{C}_{A-\mathrm{tors}}$.
2. For any $Y \in \mathcal{C}_{A-\mathrm{tors}}$, we have that $\mathrm{Hom}_{\mathcal{C}}(Y, U_A \otimes X)$ is contractible.

The latter condition comes from the natural cofiber sequence $V_A \otimes X \to X \to U_A \otimes X$. We now prove these claims.

1. It suffices to show that $V_A \in \mathcal{C}_{A-\mathrm{tors}}$, by Proposition 3.3. For this, it suffices to show that the cofiber of each map $\mathbf{1} \to U_i$ belongs to $\mathcal{C}_{A-\mathrm{tors}}$. By induction and the octahedral axiom, it suffices to show that for each $i \geq 0$, the cofiber of $U_i \to U_{i+1}$ belongs to $\mathcal{C}_{A-\mathrm{tors}}$. But this map is the dual to the map $T_{i+1}(A, \mathbf{1}) \to T_i(A, \mathbf{1})$, and the fiber of this map is of the form $A \otimes M$ for a dualizable object M. Now any object of the form $\mathbb{D}(A \otimes M) \simeq \mathbb{D}A \otimes \mathbb{D}M$ belongs to $\mathcal{C}_{A-\mathrm{tors}}$ as $\mathbb{D}A$ is an A-module (Construction 2.25) and thus a retract of $A \otimes \mathbb{D}A$. Thus, the cofiber of $U_i \to U_{i+1}$ belongs to $\mathcal{C}_{A-\mathrm{tors}}$.
2. Fix X arbitrary. The collection of Y for which $\mathrm{Hom}_{\mathcal{C}}(Y, U_A \otimes X)$ is contractible is localizing, so it suffices to prove the claim for $Y = A \otimes Y'$ with Y' dualizable. In this case, we have

$$\mathrm{Hom}_{\mathcal{C}}(Y, U_A \otimes X) \simeq \mathrm{Hom}_{\mathcal{C}}(Y', \mathbb{D}A \otimes U_A \otimes X).$$

Now the tower $\{A \otimes T_i(A, \mathbf{1})\}$ has the property that every map is null (Proposition 2.5), so by duality, every map $U_i \otimes \mathbb{D}A \to U_{i+1} \otimes \mathbb{D}A$ is nullhomotopic. In particular, $\mathbb{D}A \otimes U_A$ is contractible, which proves the claim. $\square$

Example 3.6. We consider $\mathcal{C} = \mathrm{Mod}(\mathbb{Z})$, $A = \mathbb{Z}/p$. In this case, the sequence $1 \to U_1 \to U_2 \to \dots$ becomes the sequence

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \dots,$$

so that $U_{\mathbb{Z}/p} = \mathbb{Z}[p^{-1}]$ and $V_{\mathbb{Z}/p} = \Sigma^{-1}(\mathbb{Z}[p^{-1}]/\mathbb{Z})$.

Proposition 3.7. *Let $\mathcal{D}$ be a collection of dualizable generators for $\mathcal{C}$. Then the objects $\{\mathbb{D}A \otimes X\}_{X\in\mathcal{D}}$ form a system of compact generators for $\mathcal{C}_{A-\mathrm{tors}}$.*

Proof. These objects are A-modules by Construction 2.25, so they belong to $\mathcal{C}_{A-\mathrm{tors}}$; since they are compact in $\mathcal{C}$, they are compact in $\mathcal{C}_{A-\mathrm{tors}}$. It suffices to show that they are generators. Here the argument proceeds as in Proposition 2.27: it reduces to showing that if $Y \in \mathcal{C}_{A-\mathrm{tors}}$ and $A \otimes Y$ is contractible, then Y is contractible. But in this case, $Q \otimes Y$ is contractible for any A-module Q. It follows that $Q' \otimes Y$ is contractible for any $Q' \in \mathcal{C}_{A-\mathrm{tors}}$, in particular, for $Q' = V_A$, so that $V_A \otimes Y$ is contractible. But $V_A \otimes Y \simeq Y$ as $Y \in \mathcal{C}_{A-\mathrm{tors}}$. Therefore, Y is contractible. □

Next, we include an analog of Proposition 2.28 for torsion objects.

Proposition 3.8. *Let $A, B \in \mathrm{Alg}(\mathcal{C})$ be dualizable in $\mathcal{C}$. Then an object $X \in \mathcal{C}$ is $(A \otimes B)$-torsion if and only if it is both A-torsion and B-torsion.*

Proof. If X is $(A \otimes B)$-torsion, then we know that X belongs to the localizing $\otimes$-ideal (see Definition 4.1) generated by $A \otimes B$. Therefore, X belongs to the localizing $\otimes$-ideal generated by A and is consequently A-torsion. Similarly, X must be B-torsion.

Suppose now that X is both A-torsion and B-torsion. Then $V_A \otimes X \simeq X$ and $V_B \otimes X \simeq X$, so $V_A \otimes V_B \otimes X \simeq X$. It suffices to show, as a result, that $V_A \otimes V_B$ is $(A \otimes B)$-torsion. For this, we construct sequences

$$\mathbf{1} \to U_1^{(A)} \to U_2^{(A)} \to \dots \quad \text{and} \quad \mathbf{1} \to U_1^{(B)} \to U_2^{(B)} \to \dots$$

as in Construction 3.4, such that $V_A \simeq \mathrm{fib}\left(\mathbf{1} \to \varinjlim U_i^{(A)}\right)$ and $V_B \simeq \mathrm{fib}\left(\mathbf{1} \to \varinjlim U_i^{(B)}\right)$. To show that $V_A \otimes V_B$ is $(A \otimes B)$-torsion, we first observe that, for each i, j, $\mathrm{cofib}\left(U_i^{(A)} \to U_{i+1}^{(A)}\right) \otimes \mathrm{cofib}\left(U_j^{(B)} \to U_{j+1}^{(B)}\right)$ is an $(A \otimes B)$-module and hence $(A \otimes B)$-torsion. The claim for $V_A \otimes V_B$ now follows by induction. □

We now state and briefly prove a version of [26, Th. 2.1], due to Hovey–Palmieri–Strickland in our context.

Theorem 3.9 *(Cf. [38, Th. 3.3.5]). The functor of A-completion establishes an equivalence of ∞-categories $L_A \colon \mathcal{C}_{A-\mathrm{tors}} \simeq \mathcal{C}_{A-\mathrm{cpl}}$ (whose inverse is given by AC_A).*

Proof. Let $X \in \mathcal{C}$ be a dualizable object. Then $\mathbb{D}A \otimes X \in \mathcal{C}$ belongs to both the subcategories $\mathcal{C}_{A-\mathrm{tors}}$ and $\mathcal{C}_{A-\mathrm{cpl}}$. Moreover, the $\mathbb{D}A \otimes X$ form a family of compact generators (as X ranges over the dualizable objects) for both subcategories $\mathcal{C}_{A-\mathrm{tors}}, \mathcal{C}_{A-\mathrm{cpl}}$, in view of

Proposition 2.27 and Proposition 3.7. Since L_A carries $\mathbb{D}A\otimes X$ to itself (as any A-module is A-complete), it follows formally that L_A induces an equivalence as stated.

In more detail, we let $\mathcal{C}'_A \subset \mathcal{C}$ denote the thick subcategory (Definition 4.1) generated by the $\{\mathbb{D}A \otimes X\}$ for the $X \in \mathcal{C}$ dualizable. Then $\mathcal{C}'_A \subset \mathcal{C}_{A-\mathrm{tors}} \cap \mathcal{C}_{A-\mathrm{cpl}}$ and identifies with a system of compact generators of each. Therefore, we have equivalences $\mathcal{C}_{A-\mathrm{tors}} \simeq \mathrm{Ind}(\mathcal{C}'_A), \mathcal{C}_{A-\mathrm{cpl}} \simeq \mathrm{Ind}(\mathcal{C}'_A)$ by [44, Prop. 5.3.5.11]. The functor $L_A \colon \mathcal{C}_{A-\mathrm{tors}} \to \mathcal{C}_{A-\mathrm{cpl}}$ preserves colimits, as the composition of the inclusion $\mathcal{C}_{A-\mathrm{tors}} \subset \mathcal{C}$ and $L_A \colon \mathcal{C} \to \mathcal{C}_{A-\mathrm{cpl}}$. It also takes the compact generators $\mathbb{D}A\otimes X$ to compact objects of $\mathcal{C}_{A-\mathrm{cpl}}$. It is therefore induced by left Kan extension of the identity $\mathcal{C}'_A \to \mathcal{C}'_A \subset \mathrm{Ind}(\mathcal{C}'_A)$ [44, Lem. 5.3.5.8] and is therefore an equivalence. $\square$

3.2. A^{-1}-local objects and fracture squares

We keep the notation of the previous subsection.

Definition 3.10. We say that an object $X \in \mathcal{C}$ is A^{-1}-local if, for any object $Y \in \mathcal{C}_{A-\mathrm{tors}}$, we have $\mathrm{Hom}_{\mathcal{C}}(Y, X) \simeq 0$. We let $\mathcal{C}[A^{-1}] \subset \mathcal{C}$ denote the full subcategory spanned by the A^{-1}-local objects.

This condition (for a fixed X) is preserved under colimits in Y. It follows that:

Proposition 3.11. *An object $X \in \mathcal{C}$ is A^{-1}-local if and only if $A \otimes X$ is contractible.*

Proof. This follows easily from duality. In fact, $A \otimes X$ is contractible if and only if $\mathrm{Hom}_{\mathcal{C}}(Y, A \otimes X)$ is contractible for all dualizable Y, and this holds if and only if $\mathrm{Hom}_{\mathcal{C}}(Y \otimes \mathbb{D}A, X)$ is contractible for such Y. This is equivalent to the condition that X be A^{-1}-local. $\square$

In particular, an object is A^{-1}-local if and only if it is S-local as S ranges over the collection of maps $\mathbb{D}A \otimes X \to 0$ for X a dualizable object of $\mathcal{C}$. It follows by general theory that one can construct an A^{-1}-localization of any object in $\mathcal{C}$. However, we can do so directly:

Construction 3.12. Recall (Construction 3.4) the sequence $\mathbf{1} \to U_1 \to U_2 \to \dots$ and the cofiber sequence

$$V_A \to \mathbf{1} \to U_A,$$

with $U_A = \varinjlim U_i$. Recall also that the cofiber of each $U_i \to U_{i+1}$ admits the structure of an A-module.

For any $X \in \mathcal{C}$, we consider the morphism $X \to X[A^{-1}] := X \otimes U_A$. As shown in the proof of Proposition 3.5, the object $X[A^{-1}]$ is indeed A^{-1}-local. Moreover, if Y is any A^{-1}-local object, the natural map $\mathrm{Hom}_{\mathcal{C}}(X[A^{-1}], Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an equivalence;

this follows because the fiber is $\mathrm{Hom}_{\mathcal{C}}(V_A \otimes X, Y)$ and $V_A \otimes X$ belongs to $\mathcal{C}_{A-\mathrm{tors}}$ (compare Proposition 3.5).

It follows from this that $X \to X[A^{-1}]$ is precisely A^{-1}-localization, i.e., the left adjoint to the inclusion $\mathcal{C}[A^{-1}] \subset \mathcal{C}$. Note that, unlike A-completion, A^{-1}-localization is *smashing*: it is given by tensoring with $\mathbf{1}[A^{-1}] \simeq U_A$.

Example 3.13. If $\mathcal{C} = \mathrm{Mod}(\mathbb{Z})$ and $A = \mathbb{Z}/p$, then A^{-1}-localization is precisely p^{-1}-localization, i.e., tensoring with $\mathbb{Z}[p^{-1}]$.

Remark 3.14. These types of localizations are called *finite localizations* in [33]; here we are localizing away from the compact objects $\mathbb{D}A \otimes X$ for X dualizable. We refer also to [38, §3.3] for a discussion of finite localizations.

Our final goal in this section is to develop the theory of fracture squares, and to show that $\mathcal{C}$ can be described using a combination of the A-complete and the A^{-1}-local categories. We begin by checking that equivalences can be detected after tensoring with A and after A^{-1}-localization.

Proposition 3.15. *Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. Then f is an equivalence if and only if both $1_A \otimes f\colon A \otimes X \to A \otimes Y$ and $f[A^{-1}]\colon X[A^{-1}] \to Y[A^{-1}]$ are equivalences.*

Proof. We prove the non-obvious implication. Let $f\colon X \to Y$ be a morphism such that $1_A \otimes f$ and $f[A^{-1}]$ are equivalences. Consider the localizing subcategory $\mathcal{A}$ of all $Z \in \mathcal{C}$ such that $1_Z \otimes f$ is an equivalence. By hypothesis, $\mathcal{A}$ contains A and U_A. To show that it contains $\mathbf{1}$ (which is what we want), it suffices to show that $V_A \in \mathcal{A}$. But V_A belongs to the smallest localizing subcategory containing the $A \otimes X$ for X dualizable. The hypotheses imply that $A \otimes X \in \mathcal{A}$ for any $X \in \mathcal{C}$, so that $V_A \in \mathcal{A}$ as desired. $\square$

We are now ready to set up the arithmetic square. Compare [26, Prop. 4.13].

Construction 3.16. For any $X \in \mathcal{C}$, we have a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & L_A X \arrow[d] \\ X[A^{-1}] \arrow[r] & (L_A X)[A^{-1}]. \end{tikzcd} \tag{3.17}$$

We will call this the *A-arithmetic fracture square* of X.

Proposition 3.18. *The fracture square (3.17) is cartesian.*

Proof. In fact, one checks that the square is cartesian after tensoring with A (which annihilates the domain and codomain of the bottom horizontal arrow, both of which are

A^{-1}-local), and one checks that the square is cartesian after applying A^{-1}-localization. Thus, by Proposition 3.15 we find that (3.17) is cartesian. $\square$

In particular, any object $X \in \mathcal{C}$ can be recovered from the A-localization $L_A X$, the A^{-1}-localization $X[A^{-1}]$, and the morphism $X[A^{-1}] \to (L_A X)[A^{-1}]$. Our next goal is to promote this to an equivalence of stable ∞-categories.

Construction 3.19. Let $\mathrm{FracSquare}_A$ be the stable ∞-category defined by the homotopy fiber product $\mathrm{FracSquare}_A = \mathrm{Fun}(\Delta^1, \mathcal{C}[A^{-1}]) \times_{\mathcal{C}[A^{-1}]} \mathcal{C}_{A-\mathrm{cpl}}$. Here:

1. The functor $\mathrm{Fun}(\Delta^1, \mathcal{C}[A^{-1}]) \to \mathcal{C}[A^{-1}]$ is given by evaluation at the vertex 1.
2. The functor $\mathcal{C}_{A-\mathrm{cpl}} \to \mathcal{C}[A^{-1}]$ is given by applying A^{-1}-localization.

In other words, to give an object in $\mathrm{FracSquare}_A$ amounts to giving a map of A^{-1}-local objects $X_1 \to X_2$, an A-complete object X_0, and an equivalence $X_2 \simeq X_0[A^{-1}]$.

We thus obtain a functor $\mathcal{C} \to \mathrm{FracSquare}_A$ sending X to the associated fracture square.

Theorem 3.20. *The functor $\mathcal{C} \to \mathrm{FracSquare}_A$ that sends $X \in \mathcal{C}$ to the associated arithmetic square is an equivalence of ∞-categories.*

Proof. We first check full faithfulness. For ease of notation, we will write $X_{tA} := (L_A X)[A^{-1}]$. Consider the triple $(X[A^{-1}] \to X_{tA}, L_A X, \mathrm{id}\colon (L_A X)[A^{-1}] \simeq X_{tA}) \in \mathrm{FracSquare}_A$ associated to $X \in \mathcal{C}$.

Fix a triple $(Y_1 \to Y_2, Y_0, \phi\colon Y_0[A^{-1}] \simeq Y_2)$ in $\mathrm{FracSquare}_A$ and an object $X \in \mathcal{C}$. Then the space of maps between the object of $\mathrm{FracSquare}_A$ associated to X and this triple is computed as the homotopy fiber product

$$\big(\mathrm{Hom}_{\mathcal{C}}(X[A^{-1}], Y_1) \times_{\mathrm{Hom}_{\mathcal{C}}(X[A^{-1}], Y_2)} \mathrm{Hom}_{\mathcal{C}}(X_{tA}, Y_2)\big) \times_{\mathrm{Hom}_{\mathcal{C}}(X_{tA}, Y_2)} \mathrm{Hom}_{\mathcal{C}}(L_A X, Y_0).$$

Using the identifications $\mathrm{Hom}_{\mathcal{C}}(X[A^{-1}], Y_i) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y_i)$ for $i = 1, 2$ and $\mathrm{Hom}_{\mathcal{C}}(L_A X, Y_0) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y_0)$, we identify this fiber product with

$$\mathrm{Hom}_{\mathcal{C}}(X, Y_1) \times_{\mathrm{Hom}_{\mathcal{C}}(X, Y_2)} \mathrm{Hom}_{\mathcal{C}}(X, Y_0) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y_1 \times_{Y_2} Y_0),$$

It follows that the functor $\mathcal{C} \to \mathrm{FracSquare}_A$ sending $X \in \mathcal{C}$ to the associated arithmetic square admits a right adjoint G that sends a triple $(Y_1 \to Y_2, Y_0, Y_0[A^{-1}] \simeq Y_2)$ to the pullback $Y_1 \times_{Y_2} Y_0$. As the composition $\mathcal{C} \to \mathrm{FracSquare}_A \to \mathcal{C}$ is homotopic to the identity by Proposition 3.18, we find that the left adjoint $\mathcal{C} \to \mathrm{FracSquare}_A$ is fully faithful. In order to show that we have an equivalence of ∞-categories, it therefore suffices to show that the right adjoint is conservative, since we have a *colocalization*. This is checked as follows: given $(Y_1 \to Y_2, Y_0, Y_0[A^{-1}] \simeq Y_2)$, we find that

$$(Y_1 \times_{Y_2} Y_0)[A^{-1}] \simeq Y_1, \quad L_A(Y_1 \times_{Y_2} Y_0) \simeq Y_0, \quad Y_2 \simeq (L_A(Y_1 \times_{Y_2} Y_0))\,[A^{-1}].$$

In particular, Y_0, Y_1, Y_2 can be recovered from the pullback, which implies that G is conservative as desired. $\square$

Remark 3.21. We have tacitly used the following two standard facts about ∞-categories. Given a (homotopy) fiber product $\mathcal{A} = \mathcal{A}_1 \times_{\mathcal{A}_3} \mathcal{A}_2$, then we can compute mapping spaces in $\mathcal{A}$ as the homotopy fiber product of mapping spaces in $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$. Second, in $\mathrm{Fun}(\Delta^1, \mathcal{B})$ for an ∞-category $\mathcal{B}$, we can compute maps between a pair of objects $(X_1 \to X_2), (Y_1 \to Y_2)$ via the homotopy fiber product $\mathrm{Hom}_{\mathcal{B}}(X_2, Y_2) \times_{\mathrm{Hom}_{\mathcal{B}}(X_1, Y_2)} \mathrm{Hom}_{\mathcal{B}}(X_1, Y_1)$.

4. Nilpotence

Let $\mathcal{C}$ be a symmetric monoidal, stable ∞-category whose tensor product functor is exact in each variable. Let $A \in \mathcal{C}$ be an algebra object. In this subsection, we develop the theory of nilpotence: that is, the generalization to our setting of those objects in $\mathrm{Mod}(\mathbb{Z})$ annihilated by a power of the prime number p. For the moment, we do *not* assume anything as strong as Hypotheses 2.26.

Recall that:

Definition 4.1. A full stable subcategory $\mathcal{C}' \subset \mathcal{C}$ is called *thick* if $\mathcal{C}'$ is also idempotent-complete, i.e., every idempotent endomorphism induces a splitting. A subcategory $\mathcal{I} \subset \mathcal{C}$ is called a *$\otimes$-ideal* if whenever $X \in \mathcal{I}$ and $Y \in \mathcal{C}$, we have $X \otimes Y \in \mathcal{I}$. One then obtains the notion of a *thick $\otimes$-ideal*, which is a full subcategory that is both a thick subcategory and a $\otimes$-ideal.

We list some common sources of thick $\otimes$-ideals.

Example 4.2. If $Z \in \mathcal{C}$, the collection of $X \in \mathcal{C}$ such that $X \otimes Z$ is contractible is a thick $\otimes$-ideal. Of course, in this case the collection is actually a localizing $\otimes$-ideal too.

Example 4.3. Let $f\colon B \to C$ be a morphism. Consider the collection of all $X \in \mathcal{C}$ such that $1_X \otimes f^{\otimes n}\colon X \otimes B^{\otimes n} \to X \otimes C^{\otimes n}$ is nullhomotopic for $n \gg 0$. This is a thick $\otimes$-ideal (which is generally not a localizing subcategory).

We can now make the main definition of this section.

Definition 4.4. Let $\mathcal{C}$ be as above and let $A \in \mathrm{Alg}(\mathcal{C})$.

1. We will say that an object of $\mathcal{C}$ is *A-nilpotent* ([18, Def. 3.7]) if it belongs to the thick $\otimes$-ideal generated by A (i.e., the smallest thick $\otimes$-ideal containing A). We will let $\mathrm{Nil}_A \subset \mathcal{C}$ be the full subcategory spanned by the A-nilpotent objects.

2. We will say that A is *descendable* (see [7] and [51, §3]) if the thick $\otimes$-ideal generated by A is all of $\mathcal{C}$.

Example 4.5. A is descendable if and only if the unit object $\mathbf{1}$ is A-nilpotent.

Example 4.6. Let M be an A-module in $\mathcal{C}$. Then M is A-nilpotent. In fact, M is a retract (in $\mathcal{C}$) of $A \otimes M$.

We now give an important characterization of A-nilpotence in terms of the Adams tower.

Proposition 4.7. *The following are equivalent for an object $M \in \mathcal{C}$ and $A \in \mathrm{Alg}(\mathcal{C})$:*

1. *M is A-nilpotent.*
2. *For all $N \gg 0$, the maps $T_N(A, M) \to M$ in the Adams tower are nullhomotopic.*
3. *There exists a finite tower in $\mathcal{C}$*

$$T'_N \to \cdots \to T'_2 \to T'_1 \to T'_0 \simeq M,$$

with the properties that:

- *For each i, the cofiber of $T'_i \to T'_{i-1}$ admits the structure of an A-module object in $\mathcal{C}$.*
- *The composite map $T'_N \to T'_0$ is nullhomotopic (in $\mathcal{C}$).*

If we write $I = \mathrm{fib}(\mathbf{1} \to A)$ as before, then the conditions state that M is A-nilpotent if and only if for some N, the map $M \otimes I^{\otimes N} \to M$ is nullhomotopic. Note that fits into Example 4.3.

Proof. We will prove the implications cyclically.

(1) $\implies$ (2). Suppose M is A-nilpotent; then we want to show that the maps $T_N(A, M) \to M$ are nullhomotopic for $N \gg 0$. Observe that the collection of $M \in \mathcal{C}$ for which this satisfied is a $\otimes$-ideal, since we have natural isomorphisms $T_i(A, M) \simeq T_i(A, \mathbf{1}) \otimes M$. It is easy to see that the collection of such M is in addition thick, since the passage from M to its Adams tower commutes with cofiber sequences and retracts. Therefore, the collection of M with the desired property is a thick $\otimes$-ideal. To show that it contains every A-nilpotent object, it suffices to show that it contains A itself. In other words, we need to show that the A-based Adams tower has this property for A. But in this case, the map $T_1(A, A) \to T_0(A, A)$ is already nullhomotopic as it is the fiber of the map $A \to A \otimes A$ (which has a section). This completes the proof that (1) $\implies$ (2).

(2) $\implies$ (3). We can take the Adams tower, in view of Proposition 2.5 and the hypotheses.

(3) $\implies$ (1). In this case, the cofiber of each map $T'_i \to T'_{i-1}$ is A-nilpotent since it admits the structure of an A-module (Example 4.6). Using the octahedral axiom

and induction, it follows that the cofiber of $T'_N \to T'_0$ is A-nilpotent, since the class of A-nilpotent objects is closed under cofiber sequences. Since this map is nullhomotopic, it follows that $M \simeq T'_0$ is a retract of the cofiber of $T'_N \to T'_0$ and is thus A-nilpotent itself. $\square$

The above result enables one to quantify the notion of A-nilpotence.

Definition 4.8. Suppose M is A-nilpotent. We will write $\exp_A(M)$ for the smallest integer $N \geq 0$ such that $T_N(A, M) = I^{\otimes N} \otimes M \to M$ is nullhomotopic and call it the A-*exponent of* M.

Using the axioms and Proposition 2.14, one sees easily the following result (whose proof we leave to the reader).

Proposition 4.9.

1. *If M is A-nilpotent and M' is a retract of M, then* $\exp_A(M') \leq \exp_A(M)$.
2. *If $M' \to M \to M''$ is a cofiber sequence of A-nilpotent objects, then* $\exp_A(M) \leq \exp_A(M') + \exp_A(M'')$.
3. *The exponent* $\exp_A(M)$ *is the smallest choice of N that one can take in the second (or third) condition of Proposition 4.7.*
4. *The exponent* $\exp_A(M)$ *is the smallest $N \geq 0$ such that the map* $M \to \mathrm{Tot}_{N-1}(M \otimes \mathrm{CB}^\bullet(A))$ *admits a retraction.*

Remark 4.10. The quantification of nilpotence in this way is a special case of older ideas. For example, it can be obtained as a special case of the discussion in [6, Sec. 2]. In fact, we see that if $\mathcal{A}$ is the full subcategory of $\mathcal{C}$ consisting of objects of the form $A \otimes X, X \in \mathcal{C}$, then $\exp_A(M)$ is precisely the $\mathcal{A}$-*level* [6, Def. 2.3] of M.

The idea also appears (earlier) in [21] as follows. When A is dualizable, one has a *stable projective class* (cf. [21, Sec. 2.3]) given by the pair $(\mathcal{A}, \mathcal{I})$ where $\mathcal{A}$ is in the previous paragraph and $\mathcal{I}$ consists of all maps $X \to Y$ such that $X \otimes A \to Y \otimes A$ is null. To see this, we observe that for any $X \in \mathcal{A}$, we have that $\mathbb{D}A \otimes X \in \mathcal{A}$ and the map $\mathbb{D}A \otimes X \to X$ splits after tensoring with A, as one sees by dualizing the fact that $1 \to A$ splits after tensoring with A. Using duality, one sees also that for any $X \in \mathcal{A}$ and $f \colon Y \to Z$ in $\mathcal{I}$, the map $\pi_*\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \pi_*\mathrm{Hom}_{\mathcal{A}}(X, Z)$ is zero. From this, it is easy to verify that one has a projective class, via [21, Lem. 3.2]. The collection of objects of A-nilpotence at most n is precisely $\mathcal{A}_n$ in the sense of [21, Sec. 3.2].

Example 4.11. Consider the usual test example of $\mathcal{C} = \mathrm{Mod}(\mathbb{Z})$ and $A = \mathbb{Z}/p$. In this case, an object X is A-nilpotent if and only if multiplication by p^n annihilates it for some n: that is, if $p^n \colon X \to X$ is nullhomotopic for some n. This follows from Proposition 4.7 in view of the explicit description of the Adams tower in this case (Example 2.4). Moreover, one sees that $\exp_A(M)$ is the smallest n such that p^n annihilates M. Note that in this case, the Adams spectral sequence is precisely the Bockstein spectral sequence.

The theory of exponents leads to an exhaustive *filtration* of the ∞-category $\mathrm{Nil}_A \subset \mathcal{C}$ of A-nilpotent objects,

$$\mathrm{Nil}_A^{(0)} \subset \mathrm{Nil}_A^{(1)} \subset \mathrm{Nil}_A^{(2)} \subset \cdots \subset \bigcup_i \mathrm{Nil}_A^{(i)} = \mathrm{Nil}_A,$$

where an object $X \in \mathrm{Nil}_A$ belongs to $\mathrm{Nil}_A^{(i)}$ if and only if $\exp_A(X) \leq i$. It is easy to see that $\mathrm{Nil}_A^{(i)} \subset \mathcal{C}$ is a stable full subcategory which is closed under retracts, arbitrary direct sums (although Nil_A is only closed under finite direct sums), and tensoring with arbitrary elements of $\mathcal{C}$. For example, $\mathrm{Nil}_A^{(0)} = \{0\}$ and $\mathrm{Nil}_A^{(1)}$ consists of the retracts of A-modules. Moreover, the cofiber of a map from an object in $\mathrm{Nil}_A^{(i)}$ to an object in $\mathrm{Nil}_A^{(j)}$ belongs to $\mathrm{Nil}_A^{(i+j)}$.

Corollary 4.12. *Given $\mathcal{C}$ and A as above, the collection of A-nilpotent objects is the thick subcategory generated by those objects in $\mathcal{C}$ which admit the structure of an A-module.*

Proof. This follows from the third property in Proposition 4.7. $\square$

Corollary 4.13. *Let $(\mathcal{C}, \otimes, \mathbf{1})$ and $(\mathcal{D}, \otimes, \mathbf{1})$ be symmetric monoidal stable ∞-categories where the tensor structure is compatible with colimits. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal, exact functor. Let $A \in \mathrm{Alg}(\mathcal{C})$ and let $M \in \mathcal{C}$. Then if M is A-nilpotent, $F(M)$ is $F(A)$-nilpotent, and in fact, one has $\exp_{F(A)}(F(M)) \leq \exp_A(M)$.*

Proof. The first assertion follows from the third condition of Proposition 4.7, since that condition is preserved under any exact lax symmetric monoidal functor. The second assertion follows using Proposition 4.9.3. $\square$

We now prove another characterization of A-nilpotence. In classical terms, this characterization states the A-Adams filtration on $\mathrm{Hom}_{\mathcal{C}}(\cdot, M)$ should have a uniform bound (namely, the A-exponent of M).

Proposition 4.14. *The following are equivalent for $M \in \mathcal{C}$:*

1. *M is A-nilpotent.*
2. *There exists an integer N such that given any sequence in $\mathcal{C}$*

$$M_N \overset{\phi_N}{\to} M_{N-1} \overset{\phi_{N-1}}{\to} \cdots \overset{\phi_2}{\to} M_1 \overset{\phi_1}{\to} M,$$

such that each ϕ_i becomes nullhomotopic after tensoring with A, the composition $\phi_1 \circ \cdots \circ \phi_N\colon M_N \to M$ is nullhomotopic.

If M is A-nilpotent, the smallest N satisfying condition 2. is $\exp_A(M)$.

Proof. (2) $\implies$ (1). Take the Adams tower of M, $\{T_i(A, M)\}$. Each successive map in this tower becomes null after tensoring with A. Therefore, there is a composition $T_N(A, M) \to M$ for $N \gg 0$ which is nullhomotopic, which by Proposition 4.7 implies that M is A-nilpotent.

(1) $\implies$ (2). Consider the map $M_1 \xrightarrow{\phi_1} M \to A \otimes M$. It fits into a commutative square

$$\begin{tikzcd} M_1 \arrow[r, "\phi_1"] \arrow[d] & M \arrow[d] \\ A \otimes M_1 \arrow[r, "1_A \otimes \phi_1"] & A \otimes M. \end{tikzcd}$$

By assumption, the bottom horizontal map is nullhomotopic. It follows that the composition $M_1 \to A \otimes M$ is nullhomotopic, and in particular we have a commutative diagram

$$\begin{tikzcd} & T_1(A, M) \arrow[d] \\ M_1 \arrow[ur, "\overline{\phi_1}"] \arrow[r, "\phi_1"] & M, \end{tikzcd}$$

i.e., ϕ_1 lifts to $T_1(A, M)$.

Now, it follows that $\phi_1 \circ \phi_2 \colon M_2 \to M$ lifts to $T_1(A, M)$ as well, via $\overline{\phi_1} \circ \phi_2$. The map $\overline{\phi_1} \circ \phi_2$ becomes nullhomotopic after tensoring with A since ϕ_2 does. Therefore, $\phi_1 \circ \phi_2$ lifts to $T_2(A, M)$. Similarly, $\phi_1 \circ \cdots \circ \phi_k$ lifts to $T_k(A, M)$. If $k \gg 0$, the map $T_k(A, M) \to M$ is nullhomotopic, so $\phi_1 \circ \cdots \circ \phi_k$ must be nullhomotopic.

We leave the identification of the smallest such choice of N and $\exp_A(M)$ to the reader. $\square$

Corollary 4.15. *Suppose $\mathcal{C}$ is a closed symmetric monoidal ∞-category. Suppose $M \in \mathcal{C}$ is A-nilpotent and $X \in \mathcal{C}$ is arbitrary. Then the internal mapping object $\underline{\mathrm{Hom}}(X, M)$ is A-nilpotent. That is,* Nil_A *is closed under cotensors.*

Proof. By Corollary 4.12, it suffices to consider the case where M is an A-module. But in this case, the internal mapping object $\underline{\mathrm{Hom}}(X, M)$ also inherits the structure of an A-module and is therefore A-nilpotent. (In fact, this argument shows that $\exp_A \underline{\mathrm{Hom}}(X, M) \leq \exp_A(M)$; when $X = M$ this implies that the A-exponent of M is equal to the A-exponent of its endomorphism algebra.) $\square$

Next, we show that whether or not an A-nilpotent object belongs to a thick $\otimes$-ideal can be tested after tensoring with A: a sort of descent property.

Proposition 4.16. *Let $M \in \mathcal{C}$ be A-nilpotent and let $\mathcal{I} \subset \mathcal{C}$ be any thick $\otimes$-ideal. Then we have $A \otimes M \in \mathcal{I}$ if and only if $M \in \mathcal{I}$.*

Proof. We will prove the non-trivial implication. The hypotheses imply that $Q \otimes M \in \mathcal{I}$ for any A-module object Q. Consider the Adams tower $\{T_i(A, M)\} \simeq \{T_i(A, \mathbf{1}) \otimes M\}$. The cofiber of each map $T_i(A, M) \to T_{i-1}(A, M)$ is of the form $Q \otimes M$ where Q admits the structure of an A-module. In particular, it belongs to $\mathcal{I}$. Since $\mathcal{I}$ is a thick subcategory, it follows that the cofiber of each $T_k(A, M) \to M$ belongs to $\mathcal{I}$ and, since these maps are null for $k \gg 0$, it follows that $M \in \mathcal{I}$ too. □

We now include the analog of Propositions 2.28, 3.8 in the nilpotent case.

Proposition 4.17. *Let $A, B \in \mathrm{Alg}(\mathcal{C})$ be algebra objects and let $X \in \mathcal{C}$. Then X is $(A \otimes B)$-nilpotent if and only if it is both A-nilpotent and B-nilpotent.*

Proof. Suppose X is both A-nilpotent and B-nilpotent. Then we want to show that X is $(A \otimes B)$-nilpotent. This is a straightforward application of Proposition 4.16 with $\mathcal{I} = \mathrm{Nil}_{A \otimes B}$. Indeed, since X is B-nilpotent, we find that $X \in \mathrm{Nil}_{A \otimes B}$ if and only if $B \otimes X \in \mathrm{Nil}_{A \otimes B}$. Since $B \otimes X$ is A-nilpotent, we find that $B \otimes X \in \mathrm{Nil}_{A \otimes B}$ if and only if $A \otimes B \otimes X \in \mathrm{Nil}_{A \otimes B}$. But $A \otimes B \otimes X$ is an $(A \otimes B)$-module and clearly belongs to $\mathrm{Nil}_{A \otimes B}$. This proves that X is $(A \otimes B)$-nilpotent. We leave the other (easier) direction to the reader. □

We now prove some (slightly) less formal results about A-nilpotence. In the rest of this section, the compactness and dualizability hypotheses will become important. In particular, for the rest of this section, we assume Hypotheses 2.26.

Proposition 4.18. *Let $\mathcal{C}, A$ satisfy Hypotheses 2.26 (so that in particular, $\mathcal{C}$ is presentable). Let $M \in \mathcal{C}$ be compact. Then the following are equivalent:*

1. *M is A-nilpotent.*
2. *M is A-torsion.*

Proof. If M is A-torsion, then M is a compact object of $\mathcal{C}_{A-\mathrm{tors}}$ (since M is compact in $\mathcal{C}$), which means that it belongs to the thick subcategory generated by the compact generators $\mathbb{D}A \otimes X$ for $X \in \mathcal{C}$ dualizable: in fact, $\mathcal{C}_{A-\mathrm{tors}}$ is equivalent to the Ind-completion of precisely this thick subcategory. This implies that M is A-nilpotent. The other direction is evident. □

Theorem 4.19. *Suppose $\mathcal{C}, A$ satisfy Hypotheses 2.26. Let $X \in \mathcal{C}$ be such that X admits a unital multiplication in $\mathrm{Ho}(\mathcal{C})$. Then the following are equivalent:*

1. *X is A-nilpotent.*
2. *X is A-torsion (equivalently, $X[A^{-1}]$ is contractible).*

Proof. We establish the non-obvious implication. Since X is A-torsion, the sequence $\mathbf{1} \simeq U_0 \to U_1 \to \dots$ with colimit U_A has the property that $U_A \otimes X \simeq 0$, so that in particular, the map $\mathbf{1} \to X \to U_A \otimes X$ is nullhomotopic. Since $\mathbf{1}$ is compact, this means that the composition $\phi_N \colon \mathbf{1} \to U_N \to U_N \otimes X$ is nullhomotopic for $N \gg 0$. As a result, the map $\psi_N \colon X \to U_N \otimes X$ must be nullhomotopic too, for such N, as one sees using the unitary multiplication on X. So X is a retract of $V_N \otimes X$, which is A-nilpotent. $\square$

Remark 4.20. Theorem 4.19 corresponds to the following simple observation: given a (discrete) unital ring R which is all p-power torsion, there exists a uniform n such that $p^n R = 0$ (namely, we can take n so large that $p^n.1 = 0 \in R$).

We now include a result that describes thick $\otimes$-ideals generated by a single dualizable object in terms of nilpotence.

Proposition 4.21. *Let $Y \in \mathcal{C}$ be dualizable and let $X \in \mathcal{C}$ be arbitrary. Let $R = Y \otimes \mathbb{D}Y$ be the internal ring of endomorphisms of Y, so $R \in \mathrm{Alg}(\mathcal{C})$. Then X belongs to the thick $\otimes$-ideal generated by Y if and only if X is R-nilpotent.*

Proof. Let $\mathcal{I}$ be the thick $\otimes$-ideal generated by Y. We want to show that an object belongs to $\mathcal{I}$ if and only if it is R-nilpotent. To show that $Z \in \mathcal{I}$ implies Z is R-nilpotent, it suffices to consider the case $Z = Y$. In this case, Y is an R-module and hence R-nilpotent.

Finally, to show that every R-nilpotent object belongs to $\mathcal{I}$, it suffices to show that R does. But $R \simeq \mathbb{D}Y \otimes Y$ and thus clearly belongs to $\mathcal{I}$. $\square$

Finally, we treat a special class of examples, where *every* torsion object is nilpotent. We will encounter this in the sequel to this paper when we discuss $\mathscr{F}$-nilpotence results. The main result is that such phenomena only arise when one has an idempotent splitting of the unit, so that the ∞-category itself decomposes as a product.

Proposition 4.22. *The following are equivalent for a dualizable algebra object $A \in \mathrm{Alg}(\mathcal{C})$:*

1. *We have an equality $\mathcal{C}_{A-\mathrm{tors}} = \mathcal{C}_{A-\mathrm{cpl}}$ of subcategories of $\mathcal{C}$ (i.e., an object is A-torsion if and only if it is A-complete).*
2. *The inclusion $\mathrm{Nil}_A \subset \mathcal{C}_{A-\mathrm{tors}}$ is an equality.*
3. *The inclusion $\mathrm{Nil}_A \subset \mathcal{C}_{A-\mathrm{cpl}}$ is an equality.*
4. *The map $\mathbf{1} \to L_A\mathbf{1} \times \mathbf{1}[A^{-1}]$ is an equivalence and the symmetric monoidal ∞-category $\mathcal{C}$ decomposes as a product $\mathcal{C} \simeq \mathcal{C}_{A-\mathrm{cpl}} \times \mathcal{C}[A^{-1}]$.*
5. *The localization $(L_A\mathbf{1})[A^{-1}]$ is contractible.*

Proof. We first show that (2) implies (1). Every torsion object is A-complete, since nilpotent objects are complete. To see that conversely, an A-complete object X is A-torsion, recall the equivalence $L_A : \mathcal{C}_{A-\mathrm{tors}} \cong \mathcal{C}_{A-\mathrm{cpl}}$ to write $X \cong L_A(L_A^{-1}(X)) \cong L_A^{-1}(X)$, where the second equivalence follows because the torsion object $L_A^{-1}(X)$ is nilpotent, and L_A acts as the identity on nilpotent objects.

Again via the equivalence L_A between $\mathcal{C}_{A-\mathrm{tors}}$ and $\mathcal{C}_{A-\mathrm{cpl}}$ (which acts as the identity of Nil_A), one sees easily that (2) and (3) are equivalent. Moreover, (3) implies that $L_A\mathbf{1}$ is A-torsion, so that (5) holds; if conversely (5) holds, the algebra $L_A\mathbf{1}$ is A-torsion and therefore A-nilpotent by Theorem 4.19, and any A-complete object, as a module over $L_A\mathbf{1}$, is also A-nilpotent, so that (3) holds.

We now show that (3) implies (4). Consider the natural map $\phi\colon \mathbf{1} \to L_A\mathbf{1} \times \mathbf{1}[A^{-1}]$. Observe that $L_A\mathbf{1}$ is A-complete by definition, so it is A-torsion by assumption. Thus, ϕ becomes an equivalence after A^{-1}-localizing since the first factor on the right-hand side has trivial A^{-1}-localization. Moreover, ϕ also becomes an equivalence after tensoring with A (this does not use any of our assumptions). As a result, ϕ is an equivalence by Proposition 3.15. It follows easily that one gets a decomposition of $\mathcal{C}$ as desired. (Note that the above decomposition of $\mathcal{C}$ also follows from Theorem 3.20.)

To see that (4) implies (1), observe that $X \in \mathcal{C}$ is A-torsion if and only if its image in $\mathcal{C}[A^{-1}]$ is contractible, if and only if, by the decomposition in (4), X is A-complete.

Finally, (1) implies (5) because $L_A\mathbf{1}$ is A-complete, hence A-torsion, so that $(L_A\mathbf{1})[A^{-1}] \cong *$. □

Part 2. G-equivariant spectra and $\mathscr{F}$-nilpotence

5. G-spectra

Let G be a compact Lie group. In this section, we quickly review the basic facts about the homotopy theory Sp_G of (genuine) G-spectra, which we will treat as an ∞-category. Since a full exposition of Sp_G using ∞-categories rather than model categories has not yet appeared in the literature, we have included a discussion, beginning with a review of the relationship between model and ∞-categories. This is by no means intended to be a treatment of the classical theory and we refer to sources such as [55,42,71,50,1] for introductions to equivariant stable homotopy theory.

For our purposes, we will take Sp_G to be the ∞-category associated to the symmetric monoidal model category of *orthogonal G-spectra*. Although it is possible to construct Sp_G purely ∞-categorically via the theory of *spectral Mackey functors* (cf. [11]), we will need to use the existence of models of certain $\mathbb{E}_\infty$-algebras in Sp_G (namely, equivariant real and complex K-theory), even though an ∞-categorical treatment (and new construction) is to appear in Lurie's forthcoming work on elliptic cohomology (see [43] for a survey).

5.1. Model categories and ∞-categories

In this subsection, we begin by recalling how one passes from (symmetric monoidal) model categories to (symmetric monoidal) ∞-categories. Suppose that $\mathcal{C}$ is a model category with weak equivalences $\mathcal{W}$.

Construction 5.1. Let $\mathcal{C}^c \subset \mathcal{C}$ be the full subcategory spanned by the cofibrant objects. The model category $\mathcal{C}$ *presents* an ∞-category $\underline{\mathcal{C}}$ which, by definition, is the ∞-categorical localization $\underline{\mathcal{C}} := \mathcal{C}^c[\mathcal{W}^{-1}]$ [48, Def. 1.3.4.15].

In case $\mathcal{C}$ is a *simplicial* model category, one knows that the localization $\underline{\mathcal{C}} = \mathcal{C}^c[\mathcal{W}^{-1}]$ can also be described as the homotopy coherent nerve of the fibrant *simplicial* category spanned by the cofibrant-fibrant objects of $\mathcal{C}$ [48, Th. 1.3.4.20]. Given a Quillen equivalence $(F, G)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ between model categories admitting *functorial* cofibrant and fibrant replacements, the induced functor $\underline{F}\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ (obtained via universal properties) is an equivalence of ∞-categories in view of [48, Lem. 1.3.4.21]. Note that the cited theorem assumes that $\mathcal{C}$ and $\mathcal{D}$ are in addition combinatorial, but only uses the functorial fibrant and cofibrant replacement functors. We recall that if $\mathcal{C}, \mathcal{D}$ are cofibrantly generated, then the existence of functorial factorizations is well-known. In fact, such factorizations are so fundamental to the theory they are sometimes part of the definition of a model category [37]. For a modern reference, see [64, §12.1].

Example 5.2. Let Top denote the model category of compactly generated topological spaces (i.e., weak Hausdorff k-spaces) with the Quillen model structure (where weak equivalences are weak homotopy equivalences and fibrations are Serre fibrations). Then $\underline{\mathrm{Top}}$ is the ∞-category $\mathcal{S}$ of spaces.

We next recall the construction of *symmetric monoidal* ∞-categories from symmetric monoidal model categories. We begin with some preliminaries.

Definition 5.3 *([67]).* Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a model category with a closed symmetric monoidal structure. Suppose the unit is cofibrant and that for cofibrations $c \to c', d \to d'$, the natural pushout-product map

$$c \otimes d' \sqcup_{c \otimes d} c' \otimes d \to c' \otimes d'$$

is a cofibration, and a weak equivalence if either of the maps $c \to c'$ or $d \to d'$ is a weak equivalence. In this case, $(\mathcal{C}, \otimes, \mathbf{1})$ is called a *symmetric monoidal model category.*

This definition appears in [67], which replaces cofibrancy of the unit with a slightly weaker condition. In the case where $\mathcal{C}$ has a symmetric monoidal model structure, this can be used to construct symmetric monoidal ∞-categories.

Construction 5.4 *([48, Prop. 4.1.3.4]).* Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ is a symmetric monoidal model category. As before, let $\mathcal{C}^c \subset \mathcal{C}$ be the full subcategory spanned by the cofibrant objects (so that $\mathcal{C}^c$ is a monoidal subcategory) and $\mathcal{W}$ the class of weak equivalences in $\mathcal{C}^c$. Then, since the class $\mathcal{W}$ is compatible with the symmetric monoidal structure on $\mathcal{C}^c$, the

∞-categorical localization $\underline{\mathcal{C}} = \mathcal{C}^c[\mathcal{W}^{-1}]$ inherits a symmetric monoidal structure such that $\mathcal{C}^c \to \underline{\mathcal{C}}$ is symmetric monoidal.[3]

In case $\mathcal{C}$ is a *simplicial* symmetric monoidal category, there is an equivalent version of this construction that is often easier to work with.

Construction 5.5. Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ is a simplicial symmetric monoidal model category. Let $\mathcal{C}^{cf} \subset \mathcal{C}$ denote the full subcategory spanned by the cofibrant-fibrant objects. Consider the colored operad in simplicial sets whose objects are ordered tuples of the objects of $\mathcal{C}^{cf}$ and such that the morphisms between $\{X_1, \ldots, X_n\} \in \mathcal{C}^{cf}$ and $Y \in \mathcal{C}^{cf}$ are given by the simplicial mapping object $\mathrm{Map}_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, Y)$. The associated ∞-operad defines a symmetric monoidal structure on the homotopy coherent nerve of $\mathcal{C}^{cf}$ which is canonically equivalent to the symmetric monoidal structure on $\underline{\mathcal{C}} = \mathcal{C}^c[\mathcal{W}^{-1}]$ from Construction 5.4 [48, Prop. 4.1.3.10, Cor. 4.1.3.16].

We note also that this construction is functorial in symmetric monoidal Quillen adjunctions. If $(\mathcal{C}, \otimes, \mathbf{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes, \mathbf{1}_{\mathcal{D}})$ are symmetric monoidal model categories and if $F \colon \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal left Quillen functor, then F induces a symmetric monoidal functor of ∞-categories

$$\underline{F} \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}},$$

by the universal property of localization.

Example 5.6. Let Top_* denote the model category of *pointed* compactly generated topological spaces with the usual Quillen model structure. Then Top_* is a symmetric monoidal model category with the smash product. We have a symmetric monoidal equivalence $\mathcal{S}_* \simeq \underline{\mathrm{Top}_*}$.

5.2. G-spaces and G-spectra

We will now review the ∞-categories of G-spaces and G-spectra, and some of the basic functoriality in G that they possess.

Construction 5.7. Let G be a compact Lie group. Let $\mathrm{Top}_{*,G}$ denote the category of pointed compactly generated topological spaces equipped with a G-action (fixing the basepoint). We regard $\mathrm{Top}_{*,G}$ as a model category where a morphism $X \to Y$ is a weak equivalence (resp. fibration) if and only if for each closed subgroup $H \leq G$, $X^H \to Y^H$ is a weak homotopy equivalence (resp. Serre fibration). The generating cofibrations in $\mathrm{Top}_{*,G}$ are the morphisms $(G/H \times S^{n-1})_+ \to (G/H \times D^n)_+$ for $n \geq 0$. Via the smash

[3] In practice, $\mathcal{C}$ is frequently a *simplicial* model category. However, in this construction, one considers $\mathcal{C}^c$ as a discrete category with weak equivalences, and ignoring the simplicial structure.

product of pointed G-spaces, this is a symmetric monoidal model category (with unit given by S^0). We refer to [50, III.1] for a treatment of this model category.

Similarly, there is a model category Top_G of (unpointed) compactly generated topological spaces equipped with a G-action, where the weak equivalences and fibrations are detected on fixed points for closed subgroups. Via the cartesian product of G-spaces, this is a symmetric monoidal model category.

Definition 5.8. The ∞-category $\mathcal{S}_G$ of *G-spaces* is the symmetric monoidal ∞-category associated to the symmetric monoidal model category Top_G of Construction 5.7. We define $\mathcal{S}_{G*}$ similarly from $\mathrm{Top}_{*,G}$ and call it the ∞-category of *pointed G-spaces*.

We now discuss the analog for G-spectra.

Example 5.9. The category $\mathrm{OrthSpec}_G$ of orthogonal G-spectra [50], equipped with the stable model structure and the smash product, is an example of a symmetric monoidal model category. The pushout-product axiom is [50, III.7.5], and the unit is cofibrant ("q-cofibrant") as well.

Definition 5.10. The symmetric monoidal ∞-category Sp_G of *G-spectra* is the symmetric monoidal ∞-category associated to the symmetric monoidal model category $\mathrm{OrthSpec}_G$ of Example 5.9. As is customary, we will denote the monoidal product by $\wedge$ and the unit by either S^0 or S^0_G (depending on whether the group is clear from the context). We will also write $F(X,Y)$ for the internal mapping object for $X, Y \in \mathrm{Sp}_G$ (i.e., the function spectrum).

One has a symmetric monoidal left Quillen functor

$$\Sigma^\infty \colon \mathrm{Top}_{*,G} \to \mathrm{OrthSpec}_G,$$

and as a result one obtains a symmetric monoidal left adjoint functor

$$\Sigma^\infty \colon \mathcal{S}_{G*} \to \mathrm{Sp}_G,$$

with right adjoint Ω^∞.

Example 5.11. For $H \leq G$ a closed subgroup, we consider the G-space G/H and the pointed G-space G/H_+. The suspension spectra $\{\Sigma^\infty_+ G/H \in \mathrm{Sp}_G\}_{H\leq G}$ form a system of compact generators of Sp_G as a localizing subcategory. This is the assertion (or definition) that a G-spectrum M is weakly contractible if and only if its H-homotopy groups $\pi_*^H(M) := \pi_* \mathrm{Hom}_{\mathrm{Sp}_G}(\Sigma^\infty_+ G/H, M)$ (for $H \leq G$ an arbitrary closed subgroup) all vanish, and that these homotopy groups commute with arbitrary wedges. For simplicity, we will often write G/H_+ for $\Sigma^\infty_+ G/H$.

Remark 5.12. For convenience we remark that the above ∞-categories are presentable and hence admit presentations by combinatorial model categories. In the case of $\mathcal{S}_G$ and $\mathcal{S}_{G*}$ this follows from the well known fact that the transitive orbit spaces form a set of compact projective generators for these categories and [44, Prop. 5.5.8.25]. The same argument applies to Sp_G, but one can more easily apply [48, Cor. 1.4.4.2].

Next, we review the interaction between Sp_G and the ∞-categories of genuine equivariant spectra for subgroups. Let G be a compact Lie group. Let $H \le G$ be a closed subgroup. There is a symmetric monoidal, colimit-preserving functor

$$\mathrm{Res}^G_H \colon \mathrm{Sp}_G \to \mathrm{Sp}_H \tag{5.13}$$

given by *restriction*. This arises from a symmetric monoidal, left adjoint functor of restriction on the category of equivariant orthogonal spectra.

We will use the following properties of restriction and its adjoints.

Proposition 5.14. *Let $H \le G$ be a closed subgroup. The restriction functors*

$$\mathrm{Res}^G_H \colon \mathrm{Sp}_G \to \mathrm{Sp}_H$$

admit left adjoints Ind^G_H and right adjoints Coind^G_H. For a sequence of subgroup inclusions $K \le H \le G$ there are natural equivalences $\mathrm{Ind}^G_H \circ \mathrm{Ind}^H_K \cong \mathrm{Ind}^G_K$ and $\mathrm{Coind}^G_H \circ \mathrm{Coind}^H_K \cong \mathrm{Coind}^G_K$.

Moreover, for $X \in \mathrm{Sp}_G$, $Y \in \mathrm{Sp}_H$, there are natural equivalences:

$$(\mathrm{Coind}^G_H Y) \wedge X \simeq \mathrm{Coind}^G_H(Y \wedge \mathrm{Res}^G_H X) \tag{5.15}$$

$$(\mathrm{Ind}^G_H Y) \wedge X \simeq \mathrm{Ind}^G_H(Y \wedge \mathrm{Res}^G_H X) \tag{5.16}$$

$$\mathrm{Ind}^G_H S^0_H \simeq \Sigma^\infty_+ G/H. \tag{5.17}$$

If G is finite, then we have a natural equivalence

$$\mathrm{Ind}^G_H Y \simeq \mathrm{Coind}^G_H Y. \tag{5.18}$$

Proof. At the level of the *homotopy* category, these properties are classical. We briefly describe how to upgrade them to ∞-categorical equivalences, although knowing this is not critical for the rest of the paper. Since we are working with presentable *stable* ∞-categories, the existence of a left or right adjoint to an exact functor can be checked at the level of the *homotopy category:* the condition is that said functor should preserve arbitrary coproducts (resp. products) [48, Prop. 1.4.4.1]. So, $\mathrm{Ind}^G_H, \mathrm{Coind}^G_H$ exist at the ∞-categorical level for purely abstract reasons, once one knows about their existence at the homotopy category level (although one can write down strict models for these as well; see [49, §9.2] for the finite group case). The property (5.15) comes from a natural map (from left to right) at the level of ∞-categories. Checking this map is an equivalence is done at the level of homotopy categories. The remaining claims are checked in the same way.

Finally, (5.18) is a special case of the *Wirthmüller isomorphism* (for compact Lie groups, see [42, II.6]). Once again, the map arises from universal properties, as explained in [27]. Namely, once we know that $\operatorname{Ind}_H^G(S_H^0) \simeq \operatorname{Coind}_H^G(S_H^0)$, then we get a natural map in Sp_G,

$$S_G^0 \to \operatorname{Coind}_H^G(S_H^0) \simeq \operatorname{Ind}_H^G(S_H^0).$$

Thus, for any $Z \in \mathrm{Sp}_G$, we have natural transformations

$$Z \simeq Z \wedge S_G^0 \to Z \wedge \operatorname{Ind}_H^G(S_H^0) \simeq \operatorname{Ind}_H^G(\operatorname{Res}_H^G(Z)), \tag{5.19}$$

where we used the projection formula (5.16) in the last step. Taking $Z = \operatorname{Coind}_H^G(Y)$ for $Y \in \mathrm{Sp}_H$, we get a natural map

$$\operatorname{Coind}_H^G(Y) \to \operatorname{Ind}_H^G(\operatorname{Res}_H^G(\operatorname{Coind}_H^G(Y))) \to \operatorname{Ind}_H^G(Y), \tag{5.20}$$

where the last map comes from the adjunction $(\operatorname{Res}_H^G, \operatorname{Coind}_H^G)$. The map (5.20) is the natural transformation that implements the Wirthmüller isomorphism. This is explained in [27] at the level of homotopy categories, but it makes sense at the ∞-categorical level. $\square$

5.3. Restriction as base-change

Let G, H be finite groups. We will now present another point of view on the restriction functor $\operatorname{Res}_H^G \colon \mathrm{Sp}_G \to \mathrm{Sp}_H$, as a sort of base change. This point of view is due (albeit in the setting of the homotopy category) to Balmer [8] and Balmer–Dell'Ambrogio–Sanders [9].

We begin with some generalities. Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}}), (\mathcal{D}, \otimes, 1_{\mathcal{D}})$ be presentable, symmetric monoidal stable ∞-categories where the tensor structures commute with colimits, and let $L \colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous, symmetric monoidal functor. Then its right adjoint R is naturally *lax* symmetric monoidal. Since L (resp. R) is symmetric monoidal (resp. lax symmetric monoidal), we obtain induced functors at the level of commutative algebra objects

$$L \colon \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{D}), \quad R \colon \mathrm{CAlg}(\mathcal{D}) \to \mathrm{CAlg}(\mathcal{C}). \tag{5.21}$$

Proposition 5.22. *The induced pair of functors (5.21) form an adjoint pair. (This would work for any ∞-operad replacing the commutative one.)*

Proof. This is the analog of [28, Prop. A.5.11] for symmetric ∞-operads. Their proof applies essentially without change here. $\square$

We will now derive a new adjunction from the above data.

Construction 5.23. From (5.21), we obtain a commutative algebra structure on $R(1_{\mathcal{D}}) \in \mathcal{C}$ which, thanks to the lax symmetric monoidal structure on R, naturally acts on $R(Y)$ for any $Y \in \mathcal{D}$. Thus, we get a functor $\overline{R}\colon \mathcal{D} \to \mathrm{Mod}_{\mathcal{C}}(R(1_{\mathcal{D}}))$, fitting into a commuting square

$$\begin{tikzcd} & \mathrm{Mod}_{\mathcal{C}}(R(1_{\mathcal{D}})) \arrow[d] \\ \mathcal{D} \arrow[ur, "\overline{R}"] \arrow[r, "R"] & \mathcal{C}. \end{tikzcd}$$

The functor $\overline{R}$ is limit-preserving, and we see that the composite functor

$$\overline{L}\colon \mathrm{Mod}_{\mathcal{C}}(R(1_{\mathcal{D}})) \xrightarrow{L} \mathrm{Mod}_{\mathcal{D}}(LR(1_{\mathcal{D}})) \xrightarrow{\otimes_{(LR(1_{\mathcal{D}}))} 1_{\mathcal{D}}} \mathcal{D}$$

is left adjoint to $\overline{R}$ by inspection. The first functor here (written L as well) takes a $R(1_{\mathcal{D}})$-module (in $\mathcal{C}$) and applies L to obtain a $LR(1_{\mathcal{D}})$-module in $\mathcal{D}$. The second functor is base-change along the morphism of commutative algebra objects $LR(1_{\mathcal{D}}) \to 1_{\mathcal{D}}$ in $\mathcal{D}$. Note that this composite functor $\overline{L}$ is symmetric monoidal (as the composite of symmetric monoidal functors), and $\overline{R}$ therefore acquires a lax symmetric monoidal structure.

We thus get a new adjunction

$$(\overline{L}, \overline{R})\colon \mathrm{Mod}_{\mathcal{C}}(R(1_{\mathcal{D}})) \rightleftarrows \mathcal{D}. \tag{5.24}$$

Example 5.25. For instance, we see that $\overline{L}$ carries the "induced" $R(1_{\mathcal{D}})$-module $R(1_{\mathcal{D}}) \otimes X$, for $X \in \mathcal{C}$, to

$$\overline{L}\,(R(1_{\mathcal{D}}) \otimes X) \simeq 1_{\mathcal{D}} \otimes_{LR(1_{\mathcal{D}})} L(R(1_{\mathcal{D}}) \otimes X) \simeq L(X) \in \mathcal{D}. \tag{5.26}$$

Our first goal in this subsection is to give a simple set of criteria for when the adjunction $(\overline{L}, \overline{R})$ is an equivalence.

Definition 5.27. We say that the adjunction (L, R) *satisfies the projection formula* if, for $X \in \mathcal{C}, Y \in \mathcal{D}$, the natural map

$$R(Y) \otimes X \to R(Y \otimes L(X)), \tag{5.28}$$

adjoint to the map

$$L(R(Y) \otimes X) \simeq LR(Y) \otimes L(X) \xrightarrow{\mathrm{counit} \otimes 1_{L(X)}} Y \otimes L(X),$$

is an equivalence in $\mathcal{C}$.

Proposition 5.29. *Suppose we have an adjunction $(L, R)\colon \mathcal{C} \rightleftarrows \mathcal{D}$; here as above $\mathcal{C}, \mathcal{D}$ are presentable, symmetric monoidal stable ∞-categories such that the tensor structure on each commutes with colimits in each variable, and L is a symmetric monoidal functor. Suppose the adjunction has the following three properties:*

1. *The adjunction (L, R) satisfies the projection formula.*
2. *The right adjoint R commutes with arbitrary colimits.*
3. *The right adjoint R is conservative.*

Then the new adjunction $(\overline{L}, \overline{R})\colon \mathrm{Mod}_{\mathcal{C}}(R(1_{\mathcal{D}})) \rightleftarrows \mathcal{D}$ of Construction 5.23 is an inverse equivalence of symmetric monoidal ∞-categories.

Proof. Consider the collection of objects $X \in \mathrm{Mod}_{\mathcal{C}}(R(1_{\mathcal{D}}))$ such that $X \to \overline{RL}(X)$ is an equivalence. We would like to show that this collection contains *every* object of $\mathrm{Mod}_{\mathcal{C}}(R(1_{\mathcal{D}}))$. By hypothesis 2, this class of objects forms a localizing subcategory, so it suffices to show that these maps are equivalences for the generators $X = R(1_{\mathcal{D}}) \otimes X'$, $X' \in \mathcal{C}$. In this case, we have by (5.26), a map

$$R(1_{\mathcal{D}}) \otimes X' \to \overline{RL}(X) \simeq \overline{R}(L(X')) \simeq R(1_{\mathcal{D}}) \otimes X',$$

using the projection formula. The composite map is an equivalence, and it follows that the unit map is always an equivalence.

It follows from this that the left adjoint $\overline{L}$ is fully faithful. In fact, $\overline{L}$ is necessarily a *colocalization*, and if $Y \in \mathcal{D}$, then the cofiber C of the map $\overline{LR}(Y) \to Y$ has the property that the space of maps $\mathrm{Hom}_{\mathcal{D}}(\overline{L}X, C)$ is contractible for any $X \in \mathrm{Mod}_{\mathcal{C}}(R(1_{\mathcal{D}}))$. Therefore, $\overline{R}C$ is contractible and hence RC is contractible. By assumption R is conservative, so C is contractible. In particular, the counit maps of the adjunction $(\overline{L}, \overline{R})$ are also equivalences. □

We now specialize to the case of interest, that of the adjunction $(\mathrm{Res}^G_H, \mathrm{Coind}^G_H)\colon \mathrm{Sp}_G \rightleftarrows \mathrm{Sp}_H$ for a *finite* group G and a subgroup $H \leq G$. Recall that Res^G_H is a symmetric monoidal functor, so that this does fit into the preceding discussion. We begin by identifying the relevant commutative algebra object $R(1_{\mathcal{D}})$.

Construction 5.30. Given any finite G-set T, the function spectrum $F(T_+, S^0_G)$ inherits a natural commutative algebra structure in Sp_G since T is tautologically a *commutative coalgebra* in G-spaces. This construction sends finite coproducts of G-sets to *products* in $\mathrm{CAlg}(\mathrm{Sp}_G)$, and it carries a point to the unit S^0_G. It is also compatible with restriction to subgroups. Note that as G-spectra, $F(T_+, S^0_G) \simeq T_+$.

In case $T = G/H_+$, we would like to identify $F(G/H_+, S^0_G) \in \mathrm{CAlg}(\mathrm{Sp}_G)$ with $\mathrm{Coind}^G_H(S^0_H)$ (which acquires a commutative algebra structure since Coind^G_H is lax symmetric monoidal). To do this, recall that giving a map of commutative algebra objects $F(G/H_+, S^0_G) \to \mathrm{Coind}^G_H(S^0_H)$ in $\mathrm{CAlg}(\mathrm{Sp}_G)$ amounts to giving a map

$\mathrm{Res}^G_H(F(G/H_+, S^0_G)) \to S^0_H$ in $\mathrm{CAlg}(\mathrm{Sp}_H)$. But this map can be obtained by using the fact that G/H has a natural H-fixed point, which gives (as in the previous paragraph) a decomposition of $\mathrm{Res}^G_H F(G/H_+, S^0_G)$ as a product of S^0_H and another commutative algebra object. The map $\mathrm{Res}^G_H(F(G/H_+, S^0_G)) \to S^0_H$ is the projection onto the S^0_H piece. One now checks (at the level of underlying equivariant spectra) that the adjoint map gives an equivalence

$$F(G/H_+, S^0_G) \simeq \mathrm{Coind}^G_H(S^0_H) \in \mathrm{CAlg}(\mathrm{Sp}_G). \tag{5.31}$$

The functor $\mathrm{Coind}^G_H \colon \mathrm{Sp}_H \to \mathrm{Sp}_G$ is lax symmetric monoidal, and therefore there is a natural lax symmetric monoidal lifting

$$\begin{tikzcd} & \mathrm{Mod}_{\mathrm{Sp}_G}(F(G/H_+, S^0_G)) \arrow[d] \\ \mathrm{Sp}_H \arrow[ur] \arrow[r, "\mathrm{Coind}^G_H"] & \mathrm{Sp}_G \end{tikzcd}$$

where $\mathrm{Mod}_{\mathrm{Sp}_G}(F(G/H_+, S^0_G))$ denotes the symmetric monoidal ∞-category of modules in Sp_G over $F(G/H_+, S^0_G) \simeq \mathrm{Coind}^G_H(S^0_H)$.

Theorem 5.32 *(Cf. Balmer–Dell'Ambrogio–Sanders [9]). The functor* $\mathrm{Sp}_H \to \mathrm{Mod}_{\mathrm{Sp}_G}(F(G/H_+, S^0_G))$ *is an equivalence of symmetric monoidal ∞-categories.*

Proof. This is a consequence of Proposition 5.29:

Firstly, we already observed that our adjunction satisfies the projection formula in (5.15).

Secondly, by the Wirthmüller isomorphism (5.18), $\mathrm{Coind}^G_H \simeq \mathrm{Ind}^G_H$ is both a left and a right adjoint, so it preserves all limits and colimits.

Finally, we need to see that Coind^G_H is conservative. Suppose $Y \in \mathrm{Sp}_H$ is such that $\mathrm{Coind}^G_H(Y)$ is contractible. It follows that $\mathrm{Res}^G_H \mathrm{Coind}^G_H(Y)$ is contractible. Since G is finite, this contains Y as a retract, so Y is contractible. $\square$

Warning. Just as in [9, Lem. 3.3] the above argument still goes through for an arbitrary compact Lie group G if we assume that H is a closed finite index subgroup of G. However, for more general H the functor $\mathrm{Coind}^G_H(\cdot)$ fails to be conservative.

For example, consider the C_2-spectrum X constructed as the image of the idempotent $\frac{1-\tau}{2}$ on $(C_2)_+ \otimes \mathbb{Q}$, where $\tau \in C_2$ is the nontrivial element. The C_2-fixed point spectrum of $(KU_{C_2} \wedge X \otimes \mathbb{Q})$ is contractible while $(KU_{C_2} \wedge X \otimes \mathbb{Q})$ is non-equivariantly noncontractible. If we embed C_2 into $U(1)$ we obtain a coinduced $KU_{U(1)}$-module spectrum $\mathrm{Coind}^{U(1)}_{C_2}(KU_{C_2} \wedge X \otimes \mathbb{Q})$ whose $U(1)$-fixed points are contractible. However, it follows from Theorem 8.2 that the coinduced spectrum is contractible. Hence $\mathrm{Coind}^{U(1)}_{C_2}(\cdot)$ is not conservative.

Theorem 5.32 is both philosophically and practically important to us: it lets us identify the main concern in this paper (descent with respect to the commutative algebra objects $F(G/H_+, S^0_G) \in \mathrm{CAlg}(\mathrm{Sp}_G)$) as a form of descent with respect to restriction of subgroups. It will also enable us to recast some of our results in terms of restriction to subgroups.

6. Completeness, torsion, and descent in Sp_G

Let G be a *finite* group, and consider the symmetric monoidal, stable ∞-category Sp_G of G-spectra. In this section, we will consider the phenomena of completeness, torsion, and descent (formulated abstractly in the earlier sections) in Sp_G with respect to commutative algebra objects of the form $\{F(G/H_+, S^0_G)\}$ as H ranges over a family (Definition 6.1) of subgroups of G. We will see that this theory is closely related to the Lewis–May geometric fixed point functors. Next, we treat the decomposition of the ∞-categories of $\mathscr{F}$-complete spectra as a homotopy limit over the $\mathscr{F}$-orbit category. We then make the primary definition (Definition 6.36) of this paper by introducing the notion of *nilpotence* with respect to a family of subgroups.

6.1. Families of subgroups and $\mathscr{F}$-spectra

We now review some further relevant terminology from equivariant homotopy theory. Let G be a finite group.

Definition 6.1. A *family of subgroups* of G is a nonempty collection $\mathscr{F}$ of subgroups of G such that if $H \in \mathscr{F}$ and if $H' \leq G$ is subconjugate to H, then $H' \in \mathscr{F}$. Given a family $\mathscr{F}$, we will let $A_{\mathscr{F}} = \prod_{H\in\mathscr{F}} F(G/H_+, S^0_G) \in \mathrm{CAlg}(\mathrm{Sp}_G)$.

Important examples of families (which will arise in practice) include the families of p-subgroups, abelian subgroups, elementary abelian subgroups, etc.

Definition 6.2. Fix a family of subgroups $\mathscr{F}$. Then:

1. A G-spectrum is *$\mathscr{F}$-torsion* (or an *$\mathscr{F}$-spectrum*) if it belongs to the smallest localizing subcategory[4] of Sp_G containing the $\{F(G/H_+, S^0_G) \simeq G/H_+\}_{H\in\mathscr{F}}$. In other words, the G-spectrum is $A_{\mathscr{F}}$-torsion.
2. A G-spectrum is *$\mathscr{F}$-complete* if it is $A_{\mathscr{F}}$-complete.
3. A G-spectrum is *$\mathscr{F}^{-1}$-local* if it is $A_{\mathscr{F}}^{-1}$-local.
4. The *$\mathscr{F}$-completion, $\mathscr{F}$-acyclization*, and *$\mathscr{F}^{-1}$-localization* functors (on Sp_G) respectively are the $A_{\mathscr{F}}$-completion, $A_{\mathscr{F}}$-acyclization, and $A_{\mathscr{F}}^{-1}$-localization functors.

[4] In this case, this is equivalent to the smallest localizing $\otimes$-ideal.

Definition 6.2 is certainly not new. Before we get to our main goal (Definition 6.36 below), we write down the localization, completion, and acyclization functors explicitly (compare [42, II.2, II.9] and [50, Def. IV.6.1]). In doing so, the following construction will be useful.

Construction 6.3. We associate to $\mathscr{F}$ a G-space $E\mathscr{F}$ and a pointed G-space $\widetilde{E}\mathscr{F}$ which are determined up to weak equivalence by the following properties [55, §V.4]:

$$E\mathscr{F}^K \simeq \begin{cases} * & \text{if } K \in \mathscr{F} \\ \emptyset & \text{otherwise} \end{cases}, \qquad \widetilde{E}\mathscr{F}^K \simeq \begin{cases} * & \text{if } K \in \mathscr{F} \\ S^0 & \text{otherwise.} \end{cases} \tag{6.4}$$

The relevance of this definition to us is given in the following proposition.

Proposition 6.5. *The $\mathscr{F}^{-1}$-localization $S^0_G[A^{-1}_{\mathscr{F}}] \in \mathrm{Sp}_G$ is given by the suspension spectrum $\Sigma^\infty \widetilde{E}\mathscr{F}$. The $\mathscr{F}$-acyclization $\mathrm{AC}_{A_{\mathscr{F}}}(S^0_G)$ is given by $\Sigma^\infty_+ E\mathscr{F}$.*

Proof. It suffices to prove the second claim because there is a fiber sequence of G-spectra $\Sigma^\infty_+ E\mathscr{F} \longrightarrow S^0_G \longrightarrow \Sigma^\infty \widetilde{E}\mathscr{F}$. The (unpointed) G-space $E\mathscr{F}$ admits a cell decomposition with cells of the form G/H for $H \in \mathscr{F}$, so $\Sigma^\infty_+ E\mathscr{F}$ is $A_{\mathscr{F}}$-torsion. Now, it suffices to show that the map $\Sigma^\infty_+ E\mathscr{F} \to S^0_G$ becomes an equivalence after tensoring with $A_{\mathscr{F}}$. Equivalently, in view of Theorem 5.32, it should become an equivalence after *restricting* to any subgroup $H \in \mathscr{F}$. However, $E\mathscr{F}$ is equivariantly contractible after such a restriction, so the claim follows. □

Note also that the classical simplicial model of $E\mathscr{F}$ as the geometric realization $|X^{\bullet+1}|$ for $X = \bigsqcup_{H \in \mathscr{F}} G/H$ reproduces the simplicial model of the $A_{\mathscr{F}}$-acyclization as given in Construction 3.4.

Similarly, one sees easily that:

Proposition 6.6. *The $\mathscr{F}$-completion of a G-spectrum X is given by the internal mapping spectrum $F(E\mathscr{F}_+, X)$.*

By Theorem 3.9, we have an equivalence of ∞-categories between $\mathscr{F}$-torsion spectra and $\mathscr{F}$-complete spectra (given by $\mathscr{F}$-completion).

6.2. Geometric fixed points

The purpose of this subsection is to review the relationship between the Lewis–May *geometric fixed points* functor and the theory of A^{-1}-localization. Recall first that Sp_G is a presentable, symmetric monoidal, stable ∞-category. As such, it receives a canonical symmetric monoidal, colimit-preserving functor

$$i_* \colon \mathrm{Sp} \to \mathrm{Sp}_G.$$

Definition 6.7. The lax symmetric monoidal functor $\mathrm{Sp}_G \to \mathrm{Sp}$ right adjoint to i_* is called the functor of *categorical fixed points,* and will be denoted by $X \mapsto X^G$ or $i_G^* X$. More generally, given a subgroup $H \le G$, the composition $\mathrm{Sp}_G \xrightarrow{\mathrm{Res}^G_H} \mathrm{Sp}_H \xrightarrow{(\cdot)^H} \mathrm{Sp}$ is called the functor of *categorical H-fixed points,* and is denoted $X \mapsto X^H$ or $i_H^* X$.

Remark 6.8. The same notation is used when G is a compact Lie group (so that H is now required to be a closed subgroup).

The fixed point functors i_H^*, for $H \le G$, are corepresented by G/H_+; for $H = G$ this follows because S^0_G is the unit, and in general it follows by the adjunction between induction and restriction. Since all ∞-categories in question are compactly generated, and i_* preserves compact objects (the sphere is compact in Sp_H too), it follows that the functor of categorical fixed points (for any subgroup $H \le G$) preserves colimits. As right adjoints, the functors $\{i_H^*\}_{H \le G}$ of course preserve limits. We note also the relation

$$(\mathrm{Coind}^G_H X)^G \simeq X^H, \quad X \in \mathrm{Sp}_H, \tag{6.9}$$

which follows easily by universal properties.

The functor $i_G^* = (\cdot)^G$ is only lax symmetric monoidal, and has a nontrivial value on G_+ (in fact, by (6.9), one sees easily $(G_+)^G \simeq S^0$). To obtain a fixed point functor with the expected geometric properties, we first force the non-trivial orbits to be contractible via localization before taking categorical fixed points.

Construction 6.10. Let $\mathcal{P}_G$ denote the family of proper subgroups of G. Consider as before $A_{\mathcal{P}_G} = \prod_{H \lneq G} F(G/H_+, S^0_G) \in \mathrm{CAlg}(\mathrm{Sp}_G)$ and form the $A_{\mathcal{P}_G}^{-1}$-localization $\mathrm{Sp}_G[{\mathcal{P}_G}^{-1}]$, which receives a symmetric monoidal functor $\mathrm{Sp}_G \to \mathrm{Sp}_G[{\mathcal{P}_G}^{-1}]$ that annihilates the G/H_+ for $H \lneq G$.

An important result in the theory [42, Cor. II.9.6] is that the localization $\mathrm{Sp}_G[{\mathcal{P}_G}^{-1}]$ recovers the ordinary ∞-category of non-equivariant spectra.

Theorem 6.11. *The composite functor* $\mathrm{Sp} \xrightarrow{i_*} \mathrm{Sp}_G \xrightarrow{(\cdot)[A_{\mathcal{P}_G}^{-1}]} \mathrm{Sp}_G[{\mathcal{P}_G}^{-1}]$ *is an equivalence of symmetric monoidal ∞-categories, with inverse given by the fully faithful embedding* $\mathrm{Sp}_G[{\mathcal{P}_G}^{-1}] \subset \mathrm{Sp}_G$ *followed by categorical fixed points.*

Proof. Observe that, as a localization of Sp_G, the ∞-category $\mathrm{Sp}_G[{\mathcal{P}_G}^{-1}]$ is generated as a localizing subcategory by the images of $\{G/H_+\}_{H \le G}$. By definition, however, we have forced the non-trivial orbits to be contractible. In other words, $\mathrm{Sp}_G[{\mathcal{P}_G}^{-1}]$ is generated as a localizing subcategory by the unit $\Sigma^\infty \widetilde{E}\mathcal{P}_G$, which is the $A_{\mathcal{P}_G}^{-1}$-localization of the equivariant sphere by Proposition 6.5. In particular, $\mathrm{Sp}_G[{\mathcal{P}_G}^{-1}]$ is a symmetric monoidal, stable ∞-category where the unit object is a compact generator.

By the symmetric monoidal version of the Schwede–Shipley theorem [48, Prop. 7.1.2.7], it suffices to show that the categorical fixed points of $\Sigma^\infty \widetilde{E}\mathcal{P}_G$ are S^0. Here we use that $\Sigma^\infty \widetilde{E}\mathcal{P}_G$ is the suspension spectrum of a space, so $i_G^* \widetilde{E}\mathcal{P}_G$ is connective and we have equivalences of spaces

$$\Omega^\infty i_G^* \widetilde{E}\mathcal{P}_G = \mathrm{Hom}_{\mathrm{Sp}_G}(S_G^0, \Sigma^\infty \widetilde{E}\mathcal{P}_G) \simeq \varinjlim_V \mathrm{Hom}_{\mathcal{S}_{G*}}(S^V, \Sigma^V \widetilde{E}\mathcal{P}_G),$$

where V ranges over finite-dimensional orthogonal representations of G and S^V denotes the one-point compactification of V. However, for any V, with fixed vectors $V_0 = V^G$, we have homotopy equivalences of mapping spaces

$$\begin{aligned}\mathrm{Hom}_{\mathcal{S}_{G*}}(S^V, \Sigma^V \widetilde{E}\mathcal{P}_G) &\simeq \mathrm{Hom}_{\mathcal{S}_{G*}}(S^{V_0}, \Sigma^V \widetilde{E}\mathcal{P}_G) \\ &\simeq \mathrm{Hom}_{\mathcal{S}_{G*}}(S^{V_0}, \Sigma^{V_0} \widetilde{E}\mathcal{P}_G) = \mathrm{Hom}_{\mathcal{S}_*}(S^{V_0}, S^{V_0}),\end{aligned}$$

as the pointed G-space $\widetilde{E}\mathcal{P}_G$ has contractible H-fixed points for $H \neq G$. In particular, we find that the natural map of spectra $S^0 \to i_G^* \widetilde{E}\mathcal{P}_G$ is an equivalence. $\square$

Definition 6.12. The composition

$$\mathrm{Sp}_G \xrightarrow{(\cdot)[A_{\mathcal{P}_G}^{-1}]} \mathrm{Sp}_G[\mathcal{P}_G^{-1}] \subseteq \mathrm{Sp}_G \xrightarrow{(\cdot)^G} \mathrm{Sp},$$

is called the *geometric fixed points functor* and is denoted Φ^G.

By construction, Φ^G is a symmetric monoidal, colimit-preserving functor $\Phi^G : \mathrm{Sp}_G \to \mathrm{Sp}$. More generally, for $H \leq G$, we define a symmetric monoidal functor $\mathrm{Sp}_G \xrightarrow{\mathrm{Res}_H^G} \mathrm{Sp}_H \xrightarrow{\Phi^H} \mathrm{Sp}$, which we will write as Φ^H and call the *geometric H-fixed points functor.*

We thus recover the following classical result.

Proposition 6.13. *A G-spectrum M is contractible if and only if $\Phi^H M \in \mathrm{Sp}$ is contractible for every subgroup $H \leq G$.*

Proof. Suppose $\Phi^H M$ is contractible for each $H \leq G$. By induction on G, we may assume that $\mathrm{Res}_H^G M$ is contractible for every $H \lneq G$. If we let $A_{\mathcal{P}_G} = \prod_{H \lneq G} F(G/H_+, S_G^0)$ as before, then we have that $M \wedge A_{\mathcal{P}_G} \in \mathrm{Sp}_G$ is contractible, using Theorem 5.32. Our assumption $\Phi^G M \cong *$ implies via Theorem 6.11 that $M[A_{\mathcal{P}_G}^{-1}]$ is contractible, so that M must be contractible too (Proposition 3.15). $\square$

6.3. Example: Borel-equivariant spectra

Our next goal is to identify the ∞-categories of $\mathscr{F}$-*complete* objects as a certain limit. We begin with the most basic case, when the family consists only of the trivial subgroup.

Definition 6.14. A G-spectrum $M \in \mathrm{Sp}_G$ is said to be *Borel-equivariant* (or *Borel-complete* or *cofree*) if it is complete with respect to the trivial family $\mathscr{F} = \{\{1\}\}$, i.e., complete for the algebra object $F(G_+, S^0_G) \in \mathrm{Sp}_G$. Equivalently, by Proposition 6.6, M is Borel-equivariant if and only if the natural map

$$M \to F(EG_+, M)$$

is an equivalence in Sp_G. The Borel-equivariant G-spectra span a full subcategory $(\mathrm{Sp}_G)_{\mathrm{Borel}} \subset \mathrm{Sp}_G$.

Proposition 6.15. *A G-spectrum $M \in \mathrm{Sp}_G$ is Borel-equivariant if and only if for every $X \in \mathrm{Sp}_G$ which is nonequivariantly contractible,*[5] *we have* $\mathrm{Hom}_{\mathrm{Sp}_G}(X, M) \simeq *$.

Proof. Indeed, for $X \in \mathrm{Sp}_G$ being nonequivariantly contractible is equivalent to the condition that $G_+ \wedge X \in \mathrm{Sp}_G$ is contractible in view of Theorem 5.32. As a result, the condition that $\mathrm{Hom}_{\mathrm{Sp}_G}(X, M) \simeq *$ for every nonequivariantly contractible X is *precisely* the condition of $A_{\{1\}} = F(G_+, S^0_G)$-completeness. □

Proposition 6.16. *Suppose $M \in \mathrm{Sp}_G$ is Borel-equivariant. Then for all $H \leq G$, $\mathrm{Res}^G_H M \in \mathrm{Sp}_H$ is Borel-equivariant.*

Proof. Using Proposition 6.15 we need to show that for every $X \in \mathrm{Sp}_H$ with $\mathrm{Res}^H_{\{1\}} X \cong *$, the mapping space $\mathrm{Hom}_{\mathrm{Sp}_H}(X, \mathrm{Res}^G_H M)$ is contractible.

This mapping space is always equivalent to $\mathrm{Hom}_{\mathrm{Sp}_G}(\mathrm{Ind}_H^G X, M)$ and since $\mathrm{Res}^G_{\{1\}} \mathrm{Ind}_H^G X$ is a wedge of copies of $\mathrm{Res}^H_{\{1\}} X$, and hence contractible, the lastly displayed mapping space is indeed contractible, using the assumption that M is Borel-equivariant and Proposition 6.15 again. □

The main result is that the ∞-category of Borel-equivariant spectra can be described as the ∞-category of spectra with a G-action.

Proposition 6.17. *We have a canonical equivalence of symmetric monoidal ∞-categories* $(\mathrm{Sp}_G)_{\mathrm{Borel}} \simeq \mathrm{Fun}(BG, \mathrm{Sp})$.

Proof. By Theorem 2.30 for $\mathcal{C} = \mathrm{Sp}_G$ and $A = F(G_+, S^0_G)$, we know that we have an equivalence

$$(\mathrm{Sp}_G)_{\mathrm{Borel}} \simeq \mathrm{Tot}\left(\mathrm{Mod}_{\mathrm{Sp}_G}(F(G_+, S^0_G)) \rightrightarrows \mathrm{Mod}_{\mathrm{Sp}_G}(F((G \times G)_+, S^0_G)) \overset{\to}{\underset{\to}{\to}} \dots\right).$$

[5] That is, such that $\mathrm{Res}^G_{\{1\}} X \in \mathrm{Sp}$ is contractible.

However, we know by Theorem 5.32 that $\mathrm{Mod}_{\mathrm{Sp}_G}(F(G_+, S^0_G)) \simeq \mathrm{Sp}$. Similarly, we obtain that $\mathrm{Mod}_{\mathrm{Sp}_G}(F((G^n)_+, S^0_G))$ can be identified with $\mathrm{Mod}_{\mathrm{Sp}}(F(G^{n-1}_+, S^0)) \simeq \prod_{G^{n-1}} \mathrm{Sp}$. Unwinding the definitions, we find that $(\mathrm{Sp}_G)_{\mathrm{Borel}}$ is identified with a totalization

$$(\mathrm{Sp}_G)_{\mathrm{Borel}} \simeq \mathrm{Tot}\left(\mathrm{Sp} \rightrightarrows \prod_G \mathrm{Sp} \overset{\to}{\underset{\to}{\to}} \dots\right),$$

which recovers precisely the functor category $\mathrm{Fun}(BG, \mathrm{Sp})$ for the standard simplicial decomposition of BG. $\square$

Stated more informally, a Borel-equivariant spectrum is determined by its restriction to Sp (i.e., its underlying spectrum) together with the induced G-action on it.

Remark 6.18. Another way to think of this result, in view of the equivalence (Theorem 3.9) between torsion and complete objects, is to observe that $(\mathrm{Sp}_G)_{\mathrm{Borel}}$ has a compact generator given by G_+ itself. The endomorphism algebra of G_+ (in view of the universal property of induction) is given by the group algebra of G, $\Sigma^\infty_+ G \in \mathrm{Alg}(\mathrm{Sp})$. Similarly, the induced object G_+ in $\mathrm{Fun}(BG, \mathrm{Sp})$ has endomorphisms given by $\Sigma^\infty_+ G$. As a result, both ∞-categories are identified with $\mathrm{Mod}_{\mathrm{Sp}}(\Sigma^\infty_+ G)$ by Lurie's version of the Schwede–Shipley theorem [48, Th. 7.1.2.1].

The symmetric monoidal equivalence of Proposition 6.17 shows also that, for a Borel-equivariant G-spectrum, the categorical fixed points can be identified with the *homotopy fixed points* of the underlying object of $\mathrm{Fun}(BG, \mathrm{Sp})$. In fact, if X is any G-spectrum, then X defines an underlying spectrum $X_u = \mathrm{Res}^G_{\{e\}} X$ with a G-action, and we have a natural map

$$X^G \to X_u^{hG} \simeq F(EG_+, X)^G.$$

Proposition 6.19. *Suppose X is a G-spectrum with underlying spectrum with G-action $X_u \in \mathrm{Fun}(BG, \mathrm{Sp})$. Then the following are equivalent:*

1. *X is Borel-equivariant.*
2. *For each subgroup $H \leq G$, the map $X^H \to X_u^{hH}$ is an equivalence of spectra.*

In particular, the notion of "Borel-equivariance" can be useful for formulating descent questions.

Proof. This follows from the fact that X is Borel-equivariant if and only if the Borel-completion map $X \to F(EG_+, X)$ is an equivalence of G-spectra (i.e., induces an equivalence on H-fixed points for each $H \leq G$), and the H-fixed points of $F(EG_+, X)$ are given by X_u^{hH} for $H \leq G$. $\square$

Example 6.20. Given a spectrum M and a finite group G, we define the *Borel-equivariant G-spectrum* $\underline{M} \in \mathrm{Sp}_G$ to be $F(EG_+, i_* M)$. By construction, $\underline{M}$ is the genuine G-spectrum that represents Borel-equivariant M-cohomology on G-spaces as one sees by calculating maps in the ∞-category $\mathrm{Fun}(BG, \mathrm{Sp})$. Under the correspondence of Proposition 6.17, $\underline{M}$ corresponds to the spectrum M with trivial G-action.

As another consequence, we note also that the theory of *modules* over the Borel-equivariant form of a non-equivariant ring spectrum R is closely related to ∞-categories of the form $\mathrm{Fun}(BG, \mathrm{Mod}(R))$ where $\mathrm{Mod}(R)$ is the ∞-category of *left* R-modules. This result connects the analysis of "representation" ∞-categories such as $\mathrm{Fun}(BG, \mathrm{Mod}(R))$ to the genuinely equivariant analysis we are carrying out here.

Corollary 6.21. *Let $R \in \mathrm{Alg}(\mathrm{Sp})$ be an $\mathbb{E}_1$-algebra. Then the functor*

$$\mathrm{Mod}_{\mathrm{Sp}_G}(\underline{R}) \to \mathrm{Fun}(BG, \mathrm{Mod}(R)),$$

is fully faithful when restricted to the compact objects.

Proof. This follows because the compact objects in $\mathrm{Mod}_{\mathrm{Sp}_G}(\underline{R})$ (which form the thick subcategory generated by $\underline{R} \wedge G/H_+$ for $H \leq G$) are automatically Borel-complete themselves. $\square$

Borel-equivariant spectra will yield most of the examples that we apply the $\mathscr{F}$-nilpotence theory to in this paper and the next. As a result, we now describe several important cases. Many deep theorems in algebraic topology state that specific equivariant spectra are, in fact, Borel-complete. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a presentable symmetric monoidal, stable ∞-category whose tensor product preserves colimits in each variable. Given a finitely generated ideal $I = (x_1, \ldots, x_n) \subset \pi_0 \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$, we can form the *$I$-adic completion* of an object X in $\mathcal{C}$ following techniques of [45]: it is the limit of the cofibers $X/(x_1^k, \ldots, x_n^k)$ or, alternatively, the Bousfield localization of X with respect to $\bigotimes_{i=1}^n \mathbf{1}/x_i$. For example, given an $\mathbb{E}_\infty$-algebra R in Sp_G and a finitely generated ideal $I \subset \pi_0 R^G$, we can form the I-adic completion of R.

Example 6.22. The Atiyah–Segal completion theorem [4] states that the Borel-completion of equivariant K-theory $KU_G \in \mathrm{Sp}_G$ is equivalent to the completion of KU_G at the augmentation ideal $I \subset \pi_0 i_G^* KU_G \simeq R(G)$ (the complex representation ring).

Example 6.23. The Segal conjecture, proved by Carlsson in [20], states that the Borel-completion of the sphere spectrum $S^0 \in \mathrm{Sp}_G$ is the completion at the augmentation ideal in $\pi_0 i_G^*(S_G^0)$, which is the Burnside ring.

This point of view on completion theorems has been articulated, for instance, by Greenlees–May in [30].

6.4. Example: genuine C_p-spectra

In this subsection (which may be skipped without loss of continuity), we digress to give a decomposition of the ∞-category of C_p-spectra using the material of the previous subsection together with the theory of fracture squares. This decomposition is a well-known folklore result, but we have included it for expository purposes.

First, let G be a finite group and let $\mathscr{F}$ be a family of subgroups. Given any $X \in \mathrm{Sp}_G$, we have an arithmetic square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widehat{X}_{\mathscr{F}} \arrow[d] \\ X[\mathscr{F}^{-1}] \arrow[r] & X_{t\mathscr{F}} := \left(\widehat{X}_{\mathscr{F}}\right)[\mathscr{F}^{-1}], \end{tikzcd}$$

which allows us to recover X from its $\mathscr{F}$-completion $\widehat{X}_{\mathscr{F}} := F(E\mathscr{F}_+, X)$, its $\mathscr{F}^{-1}$-localization $X[\mathscr{F}^{-1}] := X[A_{\mathscr{F}}^{-1}]$, and its $\mathscr{F}$*-Tate construction* $X_{t\mathscr{F}} := \left(\widehat{X}_{\mathscr{F}}\right)[\mathscr{F}^{-1}]$. Using Theorem 3.20, we can obtain a decomposition of the ∞-category Sp_G.

Suppose now $G = C_p$ for some prime p and $\mathscr{F} = \{\{1\}\}$. In this case, we have two simplifications. First, we know that the $\mathscr{F}^{-1}$-local objects are given by the ∞-category of spectra (Theorem 6.11) and that the $\mathscr{F}$-complete objects are given by $\mathrm{Fun}(BC_p, \mathrm{Sp})$ (by Proposition 6.17). As a result, we deduce:

Theorem 6.24. *We have an equivalence of ∞-categories:*

$$\mathrm{Sp}_{C_p} \simeq \mathrm{Fun}(\Delta^1, \mathrm{Sp}) \times_{\mathrm{Sp}} \mathrm{Fun}(BC_p, \mathrm{Sp}),$$

where:

1. *The functor* $\mathrm{Fun}(\Delta^1, \mathrm{Sp}) \to \mathrm{Sp}$ *is evaluation at the terminal vertex* 1.
2. *The functor* $\mathrm{Fun}(BC_p, \mathrm{Sp}) \to \mathrm{Sp}$ *is the Tate construction.*

Stated informally: to give a C_p-spectrum is equivalent to giving an object $X \in \mathrm{Fun}(BC_p, \mathrm{Sp})$, an object $Y \in \mathrm{Sp}$, and a map $Y \to X_{tC_p}$.

6.5. Decomposition of the ∞-category of $\mathscr{F}$-complete spectra

Let G be a finite group and let $\mathscr{F}$ be a family of subgroups. We denote by $\mathcal{O}(G)$ the orbit category of G, i.e., the category of finite transitive G-sets. The purpose of this subsection is to prove a generalization of Proposition 6.17: We identify the ∞-category of $\mathscr{F}$-complete objects in Sp_G with a (homotopy) limit over a subcategory of the orbit category. This gives a generalization of Proposition 6.17 which will, however, require additional effort to set up.

First, observe that we have a functor

$$\mathcal{O}(G)^{op} \to \mathrm{CAlg}(\mathrm{Sp}_G), \quad G/H \mapsto F(G/H_+, S^0_G).$$

Definition 6.25. We let $\mathcal{O}_{\mathscr{F}}(G) \subseteq \mathcal{O}(G)$ denote the full subcategory spanned by the G-sets with isotropy in $\mathscr{F}$, i.e., the G-sets $\{G/H\}_{H\in\mathscr{F}}$.

Let $\mathrm{Cat}_\infty^\otimes$ be the ∞-category of symmetric monoidal ∞-categories and symmetric monoidal functors. We now obtain a functor

$$\mathcal{O}(G)^{op} \to \mathrm{Cat}_\infty^\otimes, \quad G/H \mapsto \mathrm{Mod}_{\mathrm{Sp}_G}(F(G/H_+, S^0_G)) \simeq \mathrm{Sp}_H,$$

where the last equivalence comes from Theorem 5.32. Note that $\mathcal{O}(G)^{op}$ has an initial object $G/G = *$, which is mapped by the above functor to Sp_G. As a result, for any subcategory $\mathcal{I} \subset \mathcal{O}(G)^{op}$, we obtain a symmetric monoidal functor

$$\mathrm{Sp}_G \to \varprojlim_{G/H\in\mathcal{I}} \mathrm{Sp}_H. \tag{6.26}$$

We can now state our main result, which gives a decomposition of the ∞-category $(\mathrm{Sp}_G)_{\mathscr{F}-\mathrm{compl}}$ of $\mathscr{F}$-complete spectra (generalizing Proposition 6.17).

Theorem 6.27. *The above functor (6.26) with $\mathcal{I} = \mathcal{O}_{\mathscr{F}}(G)^{op}$ factors through the $\mathscr{F}$-completion of Sp_G and gives an equivalence of symmetric monoidal ∞-categories*

$$(\mathrm{Sp}_G)_{\mathscr{F}-\mathrm{compl}} \simeq \varprojlim_{G/H\in\mathcal{O}_{\mathscr{F}}(G)^{op}} \mathrm{Sp}_H.$$

Theorem 2.30 already gives a decomposition of the ∞-category of $\mathscr{F}$-complete (i.e., $A_{\mathscr{F}}$-complete) G-spectra; however, it is indexed over Δ. In order to deduce Theorem 6.27, we shall need some general preliminaries on cofinality.[6] These arguments are not new and appear, for instance, in the proof of [47, Prop. 5.7], which is closely related. Note also that this recovers Proposition 6.17 as the special case $\mathscr{F} = \{\{1\}\}$.

Let $\mathcal{C}$ be an ∞-category and $X \in \mathcal{C}$. Suppose that the products $X^n, n \geq 1$ exist in $\mathcal{C}$. In this case, we can form a simplicial object $X^{\bullet+1} \in \mathrm{Fun}(\Delta^{op}, \mathcal{C})$,

$$X^{\bullet+1} = \left(\dots X \times X \times X \substack{\to \\ \to \\ \to} X \times X \rightrightarrows X\right).$$

To construct this object, we adjoin a terminal object $*$ to $\mathcal{C}$; in this case the above simplicial object is the Čech nerve of $X \to *$.

[6] We will use the convention, following [44], that *cofinality* of a functor refers to the invariance of *co*limits.

Proposition 6.28. *Let $\mathcal{C}$ be an ∞-category and let $X \in \mathcal{C}$ be an object such that the products X^n exist for $n \geq 1$. Suppose that every object $Y \in \mathcal{C}$ admits a map $Y \to X$. Then the functor $X^{\bullet+1} : \Delta^{op} \to \mathcal{C}$ is cofinal.*

Proof. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor of ∞-categories. The ∞-categorical version of Quillen's Theorem A [44, Thm. 4.1.3.1] (due to Joyal) states that F is cofinal if and only if the fiber product $\mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{B/}$ is contractible for each $B \in \mathcal{B}$. Recall that the left fibration $\mathcal{B}_{B/} \to \mathcal{B}$ classifies the corepresentable functor $f_B = \mathrm{Hom}_{\mathcal{B}}(B, \cdot)\colon \mathcal{B} \to \mathcal{S}$, so the colimit of $f_B \circ F$ is given by the homotopy type of $\mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{B/}$ in view of the computability of colimits in $\mathcal{S}$ via the Grothendieck construction [44, Cor. 3.3.4.6]. It follows from this that F is cofinal if and only if for every corepresentable functor $f\colon \mathcal{B} \to \mathcal{S}$, the colimit of $f \circ F\colon \mathcal{A} \to \mathcal{S}$ is contractible.

Now, let $Y \in \mathcal{C}$ be arbitrary. In order to prove cofinality, we need to show that the geometric realization $\left|\mathrm{Hom}_{\mathcal{C}}(Y, X^{\bullet+1})\right|$ is weakly contractible. However, this geometric realization can be identified with the geometric realization $\left|\mathrm{Hom}_{\mathcal{C}}(Y, X)^{\bullet+1}\right|$, which is contractible as $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is nonempty by assumption. $\square$

We now review some further ∞-categorical preliminaries on colimits. Compare [44, Remark 5.3.5.9].

Construction 6.29. If $\mathcal{C}'$ is an ∞-category, then there exists an ∞-category $\mathcal{C}$ containing $\mathcal{C}'$ as a full subcategory such that $\mathcal{C}$ admits finite coproducts and is initial with respect to this property. For an ∞-category $\mathcal{D}$ with finite coproducts, one has an equivalence

$$\mathrm{Fun}(\mathcal{C}', \mathcal{D}) \simeq \mathrm{Fun}_{\sqcup}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}_{\sqcup}(\mathcal{C}, \mathcal{D})$ denotes the subcategory spanned by those functors preserving finite coproducts. This equivalence is given by left Kan extension.

As a result, the objects of $\mathcal{C}$ are given by formal finite coproducts of objects in $\mathcal{C}'$. The ∞-category $\mathcal{C}$ can be explicitly constructed as the smallest subcategory of the presheaf ∞-category $\mathcal{P}(\mathcal{C}')$ containing $\mathcal{C}'$ and closed under finite coproducts.

Lemma 6.30. *Suppose $\mathcal{C}$ is an ∞-category with finite coproducts and $f\colon \mathcal{C} \to \mathcal{D}$ is a functor, where $\mathcal{D}$ has all colimits. Suppose f preserves finite coproducts. Let $\mathcal{C}' \subset \mathcal{C}$ be a full subcategory such that $\mathcal{C}$ is obtained by freely adjoining finite coproducts to $\mathcal{C}'$ as in Construction 6.29. Then the natural map in $\mathcal{D}$*

$$\varinjlim_{\mathcal{C}'} f \to \varinjlim_{\mathcal{C}} f$$

is an equivalence.

Proof. It follows that f is the left Kan extension of $f|_{\mathcal{C}'}$, which forces the map on colimits to be an equivalence. $\square$

Proof of Theorem 6.27. We take $\mathcal{C}$ to be the category of all finite G-sets all of whose isotropy groups lie in $\mathscr{F}$, so that $\mathcal{C}$ is obtained by freely adjoining finite coproducts to $\mathcal{O}_{\mathscr{F}}(G)$. We now consider the functor

$$M\colon \mathcal{C}^{op} \to \mathrm{Cat}_\infty^\otimes, \quad T \mapsto \mathrm{Mod}_{\mathrm{Sp}_G}(F(T_+, S_G^0)).$$

This functor sends finite coproducts in $\mathcal{C}$ to products in $\mathrm{Cat}_\infty^\otimes$. Let $U \in \mathcal{C}$ be the G-set $\bigsqcup_{H \in \mathscr{F}} G/H$. Observe that any G-set in $\mathcal{C}$ admits a map to U. We have a functor $\Delta^{op} \to \mathcal{C}$ given by the simplicial object $\dots \substack{\to\\\to\\\to} U \times U \rightrightarrows U$, which is cofinal in view of Proposition 6.28. Therefore, dualizing the cofinality statement, we find that

$$(\mathrm{Sp}_G)_{\mathscr{F}-\mathrm{compl}} \simeq \varprojlim_{\Delta} M \circ f \simeq \varprojlim_{\mathcal{C}} M,$$

where the first equivalence is Theorem 2.30 (in fact, the cosimplicial diagram $M \circ f$ is precisely the cobar construction) and the second equivalence follows by cofinality. Finally, we use Lemma 6.30 to identify $\varprojlim_{\mathcal{C}^{op}} M$ with $\varprojlim_{\mathcal{O}_{\mathscr{F}}(G)^{op}} M|_{\mathcal{O}_{\mathscr{F}}(G)^{op}}$. □

It will also be convenient to have a slight refinement of Theorem 6.27 based on a further cofinality argument. For this, we consider a collection $\mathcal{A}$ of subgroups in $\mathscr{F}$ such that every subgroup in $\mathscr{F}$ is contained in an element of $\mathcal{A}$. We assume that $\mathcal{A}$ is closed under conjugation and intersections. As before, we let $\mathcal{O}_{\mathcal{A}}(G)$ be the subcategory of the orbit category of G spanned by the G-sets $\{G/H\}_{H \in \mathcal{A}}$. We have an inclusion

$$\mathcal{O}_{\mathcal{A}}(G) \subset \mathcal{O}_{\mathscr{F}}(G).$$

Proposition 6.31. *Let $\mathcal{A}$ be a collection of subgroups in $\mathscr{F}$ such that every subgroup in $\mathscr{F}$ is contained in an element of $\mathcal{A}$. We assume that $\mathcal{A}$ is closed under conjugation and intersections. Then the inclusion $\mathcal{O}_{\mathcal{A}}(G) \to \mathcal{O}_{\mathscr{F}}(G)$ is cofinal.*

Proof. This follows from [43, Cor. 4.1.3.3]. In fact, we need to show that for any $G/H \in \mathcal{O}_{\mathscr{F}}(G)$, the category $\mathcal{O}_{\mathcal{A}}(G)_{(G/H)/}$ has weakly contractible nerve. In fact, the category $\mathcal{O}_{\mathcal{A}}(G)_{(G/H)/}$ is equivalent to the opposite of the *poset* $\mathcal{P}$ of subgroups of $\mathcal{A}$ that contain H. To see this, observe that an object in $\mathcal{O}_{\mathcal{A}}(G)_{(G/H)/}$ is given by a map of G-sets $G/H \to G/K$ for some $K \in \mathcal{A}$, which is given by multiplication by some $g \in G$. By conjugating, we observe that this object is isomorphic to a map $G/H \to G/K'$ given by multiplication by 1, so that $K' \subset H$. Thus, the objects up to isomorphism can be put in correspondence with $\mathcal{P}$; one checks that the morphisms can as well. The hypotheses imply that $\mathcal{P}$ has a minimal element and is therefore weakly contractible. □

Combining with Theorem 6.27, we then obtain:

Corollary 6.32. *Suppose $\mathcal{A}$ is as above. We then obtain an equivalence of symmetric monoidal ∞-categories*

$$(\mathrm{Sp}_G)_{\mathscr{F}-\mathrm{compl}} \simeq \varprojlim_{G/H \in \mathcal{O}_{\mathcal{A}}(G)^{op}} \mathrm{Sp}_H.$$

Example 6.33. Suppose G is a p-group and $\mathscr{F}$ is the family of proper subgroups. Then, one can take $\mathcal{A}$ to be the collection of proper subgroups of G which contain the Frattini subgroup.

We note an important special case of Corollary 6.32, which is deduced by taking $\mathcal{A} = \{H\}$ for $H \trianglelefteq G$ normal:

Corollary 6.34. *Suppose $H \trianglelefteq G$ is a normal subgroup. Then there is a natural G/H-action on the symmetric monoidal ∞-category Sp_H together with a symmetric monoidal functor*

$$\mathrm{Sp}_G \to (\mathrm{Sp}_H)^{hG/H}$$

which exhibits $(\mathrm{Sp}_H)^{hG/H}$ as the $\mathscr{F}$-completion of Sp_G for $\mathscr{F}$ the family of subgroups of G that are contained in H.

Remark 6.35. Let G be a finite group. In this case, one can give an inductive decomposition of Sp_G as a homotopy limit of ∞-categories of the form $\mathrm{Fun}(BG', \mathrm{Sp})$ (for various finite groups G') using Theorem 6.27 and the arithmetic square (3.17).

6.6. *$\mathscr{F}$-nilpotence*

We keep the notation $A_{\mathscr{F}}$ from Definition 6.1.

Definition 6.36. Given a family $\mathscr{F}$ of subgroups of G, we will let $\mathscr{F}^{\mathrm{Nil}} \subset \mathrm{Sp}_G$ denote the subcategory of $A_{\mathscr{F}}$-nilpotent objects, or equivalently the thick $\otimes$-ideal generated by $\{(G/H)_+\}_{H \in \mathscr{F}}$. We will say that a G-spectrum X is *$\mathscr{F}$-nilpotent* if it belongs to $\mathscr{F}^{\mathrm{Nil}}$. In this case, we will refer to the integer $\exp_{\mathscr{F}}(X) := \exp_{A_{\mathscr{F}}}(X)$ as the *$\mathscr{F}$-exponent* of X.

Clearly, $\mathscr{F}$-nilpotent G-spectra are both $\mathscr{F}$-torsion and $\mathscr{F}$-complete, i.e., if $X \in \mathscr{F}^{\mathrm{Nil}}$, then

$$E\mathscr{F}_+ \wedge X \simeq X \simeq F(E\mathscr{F}_+, X).$$

As we discuss in the sequel [54], $\mathscr{F}$-nilpotence is equivalent to $\mathscr{F}$-completeness together with the very rapid convergence of the associated homotopy limit spectral sequence based on a cellular decomposition of $E\mathscr{F}_+$.

We now discuss some of the first properties of $\mathscr{F}$-nilpotent spectra. Combining Proposition 4.14 with Theorem 5.32, we find the following criterion for $\mathscr{F}^{\mathrm{Nil}}$:

Proposition 6.37. *An object $X \in \mathrm{Sp}_G$ belongs to $\mathscr{F}^{\mathrm{Nil}}$ if and only if there exists an integer $N \in \mathbb{Z}_{\geq 0}$ such that whenever*

$$Y_1 \to Y_2 \to \cdots \to Y_N \to X$$

are maps in Sp_G whose restriction to Sp_H is nullhomotopic for each $H \in \mathscr{F}$, then the composition $Y_1 \to \cdots \to X$ is nullhomotopic (in Sp_G). If this is the case, the minimal such N is $\exp_{\mathscr{F}}(X)$.

Next, we show that $\mathscr{F}$-nilpotence can be descended under restriction and ascended under induction.

Proposition 6.38. *Suppose $H \leq G$ and let $\mathscr{F}$ be a family of subgroups of G. Let $\mathscr{F}_H$ be the family of those subgroups of H which belong to $\mathscr{F}$.*

1. *If $X \in \mathrm{Sp}_G$ is $\mathscr{F}$-nilpotent, then $\mathrm{Res}^G_H X \in \mathrm{Sp}_H$ is $\mathscr{F}_H$-nilpotent.*
2. *If $Y \in \mathrm{Sp}_H$ is $\mathscr{F}_H$-nilpotent, then $\mathrm{Coind}^G_H X \simeq \mathrm{Ind}^G_H X \in \mathrm{Sp}_G$ is $\mathscr{F}'$-nilpotent for any family $\mathscr{F}'$ containing $\mathscr{F}_H$. In particular $\mathrm{Coind}^G_H X$ is $\mathscr{F}$-nilpotent.*

Proof. Both assertions follow by applying Corollary 4.13. $\square$

Proposition 6.39. *Let $\mathscr{F}, \mathscr{F}'$ be two families of subgroups of G. Then a G-spectrum is $(\mathscr{F} \cap \mathscr{F}')$-nilpotent if and only if it is both $\mathscr{F}$-nilpotent and $\mathscr{F}'$-nilpotent.*

Proof. This follows from Proposition 4.17. While it is not true that $A_{\mathscr{F}} \wedge A_{\mathscr{F}'} \simeq A_{\mathscr{F} \cap \mathscr{F}'}$, one sees easily that $A_{\mathscr{F}} \wedge A_{\mathscr{F}'}$ admits the structure of a module over $A_{\mathscr{F} \cap \mathscr{F}'}$, and that a G-spectrum is nilpotent for $A_{\mathscr{F} \cap \mathscr{F}'}$ if and only if it is $A_{\mathscr{F}} \wedge A_{\mathscr{F}'}$-nilpotent. $\square$

We next show that all $\mathscr{F}$-nilpotence questions can be reduced to the case where the family $\mathscr{F}$ is the family of *proper* subgroups.

Proposition 6.40. *A G-spectrum $X \in \mathrm{Sp}_G$ is $\mathscr{F}$-nilpotent if and only if for every subgroup $H \leq G$ with $H \notin \mathscr{F}$, the restriction $\mathrm{Res}^G_H X \in \mathrm{Sp}_H$ is nilpotent with respect to the family of proper subgroups of H.*

Proof. The "only if" direction follows from Proposition 6.38. Therefore, it suffices to show that if $\mathrm{Res}^G_H X \in \mathrm{Sp}_H$ is nilpotent for the family of proper subgroups for each $H \notin \mathscr{F}$, then X is $\mathscr{F}$-nilpotent.

Without loss of generality, $G \notin \mathscr{F}$. For each $H \leq G$, let $\mathscr{F}_H$ denote the family of subgroups of H which belong to $\mathscr{F}$. Then, by induction on $|G|$, we may assume that for every $H \lneq G$, we have that the H-spectrum $\mathrm{Res}^G_H X$ is $\mathscr{F}_H$-nilpotent. Inducing, it follows that $\mathrm{Ind}^G_H \mathrm{Res}^G_H X \simeq X \wedge G/H_+$ is $\mathscr{F}$-nilpotent for each $H \lneq G$ (Proposition 6.38). In particular, if $A = \prod_{H \lneq G} F(G/H_+, S^0_G)$, we find that $X \wedge A$ is $\mathscr{F}$-nilpotent. But since X

is A-nilpotent by hypothesis (as X is nilpotent for the family of proper subgroups), we conclude that X is $\mathscr{F}$-nilpotent by Proposition 4.16. □

Using this, we can give a criterion for when a *G-ring spectrum* is nilpotent for a family of subgroups.

Theorem 6.41. *Let $R \in \mathrm{Sp}_G$ be a G-ring spectrum (up to homotopy). Then $R \in \mathscr{F}^{\mathrm{Nil}}$ if and only if for all $H \notin \mathscr{F}$, the geometric fixed point spectrum $\Phi^H R$ is contractible.*

Proof. By Proposition 6.40, we can assume $\mathscr{F}$ is the family of all proper subgroups of G. In this case, we know by Theorem 4.19 that $R \in \mathscr{F}^{\mathrm{Nil}}$ if and only if R is $\mathscr{F}$-torsion, which happens if and only if the $\mathscr{F}^{-1}$-localization $\Phi^G R$ is contractible. □

Given a G-ring spectrum which is nilpotent for a family of subgroups, any module spectrum over it has the same nilpotence property. As a result, we can obtain a decomposition of the module ∞-category:

Theorem 6.42. *Suppose R is an $\mathbb{E}_n$-algebra in Sp_G which is $\mathscr{F}$-nilpotent. Let $\mathcal{A} \subset \mathscr{F}$ be a collection of subgroups of G closed under conjugation and intersection, and such that any subgroup of $\mathscr{F}$ is contained in a subgroup belonging to $\mathcal{A}$. Then there is an equivalence of $\mathbb{E}_{n-1}$-monoidal ∞-categories*

$$\mathrm{Mod}_{\mathrm{Sp}_G}(R) \simeq \varprojlim_{G/H \in \mathcal{O}_{\mathcal{A}}(G)^{op}} \mathrm{Mod}_{\mathrm{Sp}_H}(\mathrm{Res}^G_H R).$$

Proof. For any ∞-operad $\mathcal{O}$, the association $\mathcal{C} \mapsto \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ sends homotopy limits in symmetric monoidal ∞-categories $\mathcal{C}$ to homotopy limits of ∞-categories.

For the convenience of the reader, we give a brief explanation of this fact. Recall that [48, §2.1] to every symmetric monoidal ∞-category $\mathcal{C}$ one has an ∞-category $\mathcal{C}^{\otimes}$ equipped with a map $q\colon \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$, where Fin_* is the category of pointed finite sets. In addition, the ∞-operad $\mathcal{O}$ determines an ∞-category $\mathcal{O}^{\otimes}$ and a functor $p\colon \mathcal{O}^{\otimes} \to N(\mathrm{Fin}_*)$. Given a diagram indexed over an ∞-category $\mathcal{I}$ of symmetric monoidal ∞-categories $\mathcal{C}_i, i \in \mathcal{I}$, we have a symmetric monoidal ∞-category $\varprojlim_{i\in\mathcal{I}} \mathcal{C}_i$ such that $(\varprojlim_{i\in\mathcal{I}} \mathcal{C}_i)^{\otimes} = \varprojlim_{i\in\mathcal{I}} (\mathcal{C}_i^{\otimes})$. Finally, $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is a full subcategory of $\mathrm{Fun}_{N(\mathrm{Fin}_*)}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$. This construction sends homotopy limits in $\mathcal{C}$ to homotopy limits of ∞-categories (as $\mathcal{C} \mapsto \mathcal{C}^{\otimes}$ does), and one checks that the condition that describes $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ as a full subcategory of $\mathrm{Fun}_{N(\mathrm{Fin}_*)}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$ is compatible with homotopy limits too.

Given a system of symmetric monoidal ∞-categories $\mathcal{C}_i$ indexed by an ∞-category $\mathcal{I}$ and an algebra object $A = \{A_i\} \in \mathrm{Alg}(\varprojlim \mathcal{C}_i)$, we obtain a decomposition of ∞-categories

$$\mathrm{Mod}_{\left(\varprojlim_{i\in\mathcal{I}} \mathcal{C}_i\right)}(A) \simeq \varprojlim_{i\in\mathcal{I}} \mathrm{Mod}_{\mathcal{C}_i}(A_i).$$

This follows similarly using the ∞-operads controlling modules [48, §4.2].

Therefore, setting $\mathcal{C} = \mathrm{Sp}_{\mathscr{F}}$, $\mathcal{C}_{G/H} = \mathrm{Sp}_H$, and $A_{G/H} = \mathrm{Res}^G_H R$ as G/H ranges over $\mathcal{O}_{\mathcal{A}}(G)^{op}$, Corollary 6.32 gives the desired decomposition for the ∞-category of R-modules in Sp_G which are $\mathscr{F}$-complete, i.e.,

$$\mathrm{Mod}_{(\mathrm{Sp}_G)_{\mathscr{F}}}(R) \simeq \varprojlim_{G/H \in \mathcal{O}_{\mathcal{A}}(G)^{op}} \mathrm{Mod}_{\mathrm{Sp}_H}(\mathrm{Res}^G_H R).$$

Here $(\mathrm{Sp}_G)_{\mathscr{F}}$ is the ∞-category of $\mathscr{F}$-complete G-spectra. However, every R-module is automatically $\mathscr{F}$-complete since R is $\mathscr{F}$-nilpotent. This gives the desired claim. $\square$

Part 3. Unipotence for equivariant module spectra

7. $U(n)$-unipotence and the flag variety

In this section, we prove several results on actions of compact Lie groups on modules over a complex-oriented ring spectrum. Our main results state that actions of the unitary group are determined by their homotopy fixed points. For example, to give a KU-module equipped with a $U(n)$-action is equivalent to giving a $F(BU(n)_+, KU)$-module which is complete with respect to the augmentation ideal in $\pi_0 F(BU(n)_+, KU)$. We will use these techniques to prove that the Borel-equivariant forms of such theories are nilpotent for the family of abelian subgroups.

For our nilpotence statements, the strategy of our argument, which goes back to ideas of Quillen [63] and is used prominently by Hopkins–Kuhn–Ravenel [36], is that of *complex-oriented descent.* Let G be a finite group, and embed $G \leq U(n)$. The flag variety $U(n)/T$ defines a G-space with abelian stabilizers and which thus has a cell decomposition in terms of the orbits G/A_+, $A \leq G$ abelian. We will show that in equivariant stable homotopy theory, over a base such as Borel-equivariant MU, the flag variety actually splits up as a sum of copies of the unit. This is easily seen to imply the desired nilpotence statement.

7.1. Unipotence

We begin with some generalities on symmetric monoidal stable ∞-categories. Note that we do *not* assume additional hypotheses such as the compactness of the unit here.

Definition 7.1. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a presentable symmetric monoidal, stable ∞-category where the tensor product commutes with colimits in each variable. We say that $\mathcal{C}$ is *weakly unipotent* if the unit $\mathbf{1}$ generates $\mathcal{C}$ as a localizing subcategory.

Example 7.2. The module ∞-category $\mathrm{Mod}(R)$ for an $\mathbb{E}_\infty$-ring R is weakly unipotent. In fact, by the symmetric monoidal version of Schwede–Shipley theory [48, Prop. 7.1.2.7], it is the *basic* example of a weakly unipotent ∞-category: more precisely, any weakly unipotent (symmetric monoidal, stable, etc.) ∞-category where the unit is *compact* is equivalent to $\mathrm{Mod}(R)$ for R the $\mathbb{E}_\infty$-ring of endomorphisms of the unit.

Example 7.3. Consider $\mathrm{Fun}(BG, \mathrm{Mod}(R))$ for G a finite group and R an $\mathbb{E}_\infty$-ring. This is generally not weakly unipotent: unless (for some prime p) G is a p-group and p is nilpotent in $\pi_0 R$, the induced object $F(G_+, R) \simeq R \wedge G_+$ cannot belong to the localizing subcategory generated by the unit R for purely algebraic reasons.

In fact, if there exists a prime number $q \mid |G|$ such that q is not nilpotent in R, let $G_q \leq G$ be a q-Sylow subgroup. Given any $M \in \mathrm{Fun}(BG, \mathrm{Mod}(R))$ that belongs to the localizing subcategory generated by the unit, one sees by considering long exact sequences that $\pi_*(M)[q^{-1}]$ must have trivial G_q-action (equivalently, $\sum_{g \in G_q} g$ is an isomorphism on $\pi_*(M)[q^{-1}]$). However, this is not the case for the induced object $R \wedge G_+$.

The compactness of the unit is crucial in Example 7.2, and we do not know how to classify weakly unipotent symmetric monoidal ∞-categories $\mathcal{C}$ in general. As a result, the following definition (Definition 7.7) will play more of a role for us. Recall first that if $\mathcal{C}$ is as above, and $R = \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is the $\mathbb{E}_\infty$-ring of endomorphisms of the unit, then one has a basic adjunction

$$(\otimes_R \mathbf{1}, \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \cdot))\colon \mathrm{Mod}(R) \rightleftarrows \mathcal{C}, \tag{7.4}$$

where the left adjoint $\mathrm{Mod}(R) \to \mathcal{C}$ is determined by the condition that it sends the unit to the unit (in fact, it canonically becomes a symmetric monoidal functor). This adjunction is not an equivalence in general, but it restricts to an equivalence between perfect R-modules and the thick subcategory of $\mathcal{C}$ generated by the unit. Recall [48, Def. 7.2.4.1] that the ∞-category of perfect R-modules is the thick subcategory of $\mathrm{Mod}(R)$ generated by the unit.

Proposition 7.5. *$\mathcal{C}$ is weakly unipotent if and only if $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \cdot)\colon \mathcal{C} \to \mathrm{Mod}(R)$ is conservative.*

This follows from the following more general lemma.

Lemma 7.6. *Let $\mathcal{C}$ be a presentable stable ∞-category. Consider a set $\{X_\alpha\}_{\alpha \in A}$ of objects in $\mathcal{C}$. Then the following are equivalent:*

1. *Given $Y \in \mathcal{C}$, Y is contractible if and only if $\mathrm{Hom}_{\mathcal{C}}(X_\alpha, Y) \in \mathrm{Sp}$ is contractible for each $\alpha \in A$.*
2. *The smallest localizing subcategory containing the $\{X_\alpha\}_{\alpha \in A}$ is all of $\mathcal{C}$.*

Proof. Assume the second condition, and let $Y \in \mathcal{C}$ be an object such that $\mathrm{Hom}_{\mathcal{C}}(X_\alpha, Y)$ is contractible for each $\alpha \in A$. Consider now the collection of $X \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is contractible; it is seen to be a localizing subcategory, and since it contains the $\{X_\alpha\}$, it contains all of $\mathcal{C}$. In particular, $\mathrm{Hom}_{\mathcal{C}}(Y,Y)$ is contractible, which implies that Y is contractible.

Conversely, assume the first condition, i.e. that the $\mathrm{Hom}_{\mathcal{C}}(X_\alpha,\cdot)$ ($\alpha \in A$) are jointly conservative. The localizing subcategory $\mathcal{C}' \subset \mathcal{C}$ generated by the $\{X_\alpha\}_{\alpha\in A}$ is presentable [48, Cor. 1.4.4.2] (note that $\mathcal{C}$ is itself presentable, so the hypotheses of that corollary are met), so that the inclusion $\mathcal{C}' \to \mathcal{C}$ has a right adjoint B by the adjoint functor theorem [44, Cor. 5.5.2.9]. It follows that if $X \in \mathcal{C}$, then one has a natural fiber sequence

$$F(X) \to B(X) \to X,$$

where $B(X) \to X$ is the counit of the adjunction and $F(X)$ is defined to be the fiber. One sees that for any $Y \in \mathcal{C}'$, the spectrum $\mathrm{Hom}_{\mathcal{C}}(Y, F(X))$ is contractible. Taking in particular $Y = X_\alpha$ for $\alpha \in A$, we find that $F(X)$ is contractible by hypothesis and that $B(X) \to X$ is an equivalence, so X belongs to the localizing subcategory generated by the $\{X_\alpha\}$. $\square$

Definition 7.7. $\mathcal{C}$ is *unipotent* if the adjunction (7.4) is a localization, i.e., if $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\cdot)$ is fully faithful.

Remark 7.8. We do not know whether there exists a symmetric monoidal, presentable stable ∞-category $\mathcal{C}$ which is weakly unipotent but not unipotent.

More generally, one can ask the following question. Let $\mathcal{D}$ be a presentable stable ∞-category and let $X \in \mathcal{D}$ be a generator, i.e., an object such that $\mathrm{Hom}_{\mathcal{D}}(X,\cdot)\colon \mathcal{D} \to \mathrm{Sp}$ is conservative. By Lemma 7.6, this is equivalent to supposing that the localizing subcategory generated by X is all of $\mathcal{D}$. In this case, one obtains an adjunction

$$\mathrm{Mod}(\mathrm{End}_{\mathcal{D}}(X)) \rightleftarrows \mathcal{D}, \tag{7.9}$$

where the right adjoint is conservative.

Question 7.10. If X generates $\mathcal{D}$ as a localizing subcategory, is (7.9) a localization?

The answer to the abelian analog of Question 7.10 (in the Grothendieck case) is affirmative in view of the Gabriel–Popescu theorem [61]. However, in general the answer to Question 7.10 can be no.

Example 7.11. Let $\mathcal{C} = D(\mathbb{Z}_p)$ be the derived ∞-category of modules over the p-adic integers $\mathbb{Z}_p$. We claim that the object $X = \mathbb{Q}_p \oplus \mathbb{F}_p$ generates $\mathcal{C}$. In fact, the cofiber sequence

$$\mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$$

shows easily that the localizing subcategory generated by X contains $\mathbb{Z}_p$, since the localizing subcategory generated by $\mathbb{F}_p$ contains $\mathbb{Q}_p/\mathbb{Z}_p$. However, we claim that the map

$$\mathrm{Hom}_{\mathcal{C}}(X, \mathbb{Z}_p) \otimes_{\mathrm{Hom}_{\mathcal{C}}(X,X)} X \to \mathbb{Z}_p \tag{7.12}$$

is not an equivalence, so the associated adjunction (7.9) is not a localization.

Indeed, if (7.12) were an equivalence, then writing $\mathrm{Hom}_{\mathcal{C}}(X, \mathbb{Z}_p)$ as a filtered colimit (over some filtered ∞-category $\mathcal{J}$) of perfect $\mathrm{Hom}_{\mathcal{C}}(X, X)$-modules, one would conclude that $\mathbb{Z}_p = \varinjlim_{\mathcal{J}} Y_j$, where each Y_j belongs to the thick subcategory of $\mathcal{C}$ generated by X. Since $\mathbb{Z}_p \in \mathcal{C}$ is compact, it follows that $\mathbb{Z}_p$ is a retract of some Y_j. However, the functor $\mathcal{C} \mapsto \mathcal{C}$ given by $X \mapsto (\widehat{X}_p)[1/p]$ annihilates X (and thus anything in the thick subcategory that X generates) but does not annihilate $\mathbb{Z}_p$, a contradiction.

We now state and prove the basic criterion we will use throughout to prove that ∞-categories are unipotent.

Proposition 7.13 *(Unipotence criterion). Let $\mathcal{C}$ be a presentable, stable, symmetric monodical ∞-category where the tensor commutes with colimits in each variable, as above. Suppose $\mathcal{C}$ contains an algebra object $A \in \mathrm{Alg}(\mathcal{C})$ with the following properties:*

1. *A is compact and dualizable in $\mathcal{C}$.*
2. *$\mathbb{D}A$ generates $\mathcal{C}$ as a localizing subcategory.*
3. *A belongs to the thick subcategory generated by the unit.*

Then $\mathcal{C}$ is unipotent. More precisely, if $R = \mathrm{End}_{\mathcal{C}}(\mathbf{1})$, then the natural adjunction (7.4) exhibits $\mathcal{C}$ as the completion of $\mathrm{Mod}(R)$ at $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) \in \mathrm{Alg}(\mathrm{Mod}(R))$.

Proof. Let $A_R := \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A)$. Since the adjunction (7.4) establishes an equivalence between perfect R-modules and the thick subcategory of $\mathcal{C}$ generated by the unit, it follows using hypothesis 3, that $A_R \in \mathrm{Mod}(R)$ is a perfect (equivalently, dualizable) algebra object. Moreover, we have $A \simeq A_R \otimes_R \mathbf{1} \in \mathcal{C}$.

We claim that the adjunction (7.4) factors through the A_R-completion of $\mathrm{Mod}(R)$. To see this, it suffices to show that if $M \in \mathrm{Mod}(R)$ is A_R-acyclic, then $M \otimes_R \mathbf{1} \in \mathcal{C}$ is contractible. But we know that $(M \otimes_R A_R) \otimes_R \mathbf{1} \in \mathcal{C}$ is contractible, so the equivalent object $(M \otimes_R \mathbf{1}) \otimes A \in \mathcal{C}$ is too. Thus, $M \otimes_R \mathbf{1} \in \mathcal{C}$ is contractible since the second assumption implies that tensoring with A is conservative on $\mathcal{C}$.

Therefore, by the universal property of the A_R-completion (as a Bousfield localization), we get a new adjunction

$$\mathrm{Mod}(R)_{A_R-\mathrm{cpl}} \rightleftarrows \mathcal{C}, \tag{7.14}$$

which we claim is an inverse equivalence. To see this, we observe that $\mathbb{D}A_R$ is a compact generator for $\mathrm{Mod}(R)_{A_R-\mathrm{cpl}}$ by Proposition 2.27. Its image, $\mathbb{D}A$, is a compact generator for $\mathcal{C}$ by assumption. However, the left adjoint of the adjunction (7.14) is fully faithful on the thick subcategory generated by $\mathbb{D}A_R$ (as the left adjoint in (7.4) is fully faithful on the thick subcategory generated by the unit).

Therefore, the left adjoint carries the compact generator $\mathbb{D}A_R$ to a compact generator of $\mathcal{C}$, and is fully faithful on the thick subcategory generated by $\mathbb{D}A_R$. It follows that the adjunction (7.14) is an equivalence as desired: both ∞-categories are equivalent to $\mathrm{Mod}(\mathrm{End}_R(\mathbb{D}A_R))$. $\square$

We will also need the following criterion for unipotence. Although this criterion requires more hypotheses than Proposition 7.13, these additional hypotheses will easily be verified in the case of interest. The main benefit to the next criterion is that A is not assumed to belong to the thick subcategory generated by the unit: instead, this is deduced from the assumptions. In our main application, the last hypothesis will translate into the relevance of the Eilenberg–Moore spectral sequence.

Proposition 7.15 *(Second unipotence criterion). Let $\mathcal{C}$ be as above, and let $R = \mathrm{End}_{\mathcal{C}}(\mathbf{1}) \in \mathrm{CAlg}(\mathrm{Sp})$. Suppose $\mathcal{C}$ contains an algebra object $A \in \mathrm{Alg}(\mathcal{C})$ with the following properties:*

1. *A is compact and dualizable in $\mathcal{C}$.*
2. *$\mathbb{D}A$ is compact and generates $\mathcal{C}$ as a localizing subcategory.*
3. *The ∞-category $\mathrm{Mod}_{\mathcal{C}}(A)$ is generated as a localizing subcategory by the A-module A itself, and A is a compact object in $\mathrm{Mod}_{\mathcal{C}}(A)$.*
4. *The natural map of R-module spectra*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) \otimes_R \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes A) \tag{7.16}$$

is an equivalence.

Then the conclusion of Proposition 7.13 holds.

Proof. We claim that the natural map

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, M) \otimes_R \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, N) \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, M \otimes N) \tag{7.17}$$

is an equivalence for any $M, N \in \mathcal{C}$ which are A-nilpotent. It suffices to prove this for $M, N \in \mathrm{Mod}_{\mathcal{C}}(A)$ in view of Corollary 4.12. But for $M, N \in \mathrm{Mod}_{\mathcal{C}}(A)$, both sides of (7.17) commute with arbitrary colimits in M, N by the assumption that A is compact in $\mathrm{Mod}_{\mathcal{C}}(A)$. It thus suffices (since $\mathrm{Mod}_{\mathcal{C}}(A)$ is generated as a localizing subcategory by A) to see that (7.17) is an equivalence for $M, N = A$, which we have assumed as part of the hypotheses. The natural equivalence in (7.17) implies the R-module $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A)$ is dualizable (i.e., perfect) in $\mathrm{Mod}(R)$ since A is dualizable in $\mathcal{C}$. More generally, if X is any dualizable object in $\mathcal{C}$ which is A-nilpotent, then $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X) \in \mathrm{Mod}(R)$ is dualizable.

Let $\mathcal{D} \subset \mathcal{C}$ denote the thick subcategory generated by the unit and each $A \otimes X$ for $X \in \mathcal{C}$ a dualizable object. Observe that $\mathcal{D}$ is closed under duality as $\mathbb{D}A$ is a retract of $A \otimes \mathbb{D}A$ (see Construction 2.25). Moreover, the natural map (7.17) is an equivalence if $M, N \in \mathcal{D}$.

Let $\mathrm{Mod}^\omega(R)$ denote the ∞-category of perfect R-modules. As a result, we can restrict the adjunction $\mathrm{Mod}(R) \rightleftarrows \mathcal{C}$ to a new adjunction

$$\mathrm{Mod}^\omega(R) \rightleftarrows \mathcal{D}.$$

The right adjoint in this adjunction is strictly symmetric monoidal, so by Lemma 7.18 below, we can conclude that $\mathcal{D} \simeq \mathrm{Mod}^\omega(R)$ and that $\mathcal{D} \subseteq \mathcal{C}$ is in fact the thick subcategory generated by the unit. In particular, $A \in \mathcal{C}$ therefore belongs to the thick subcategory generated by the unit. We can now apply Proposition 7.13 to conclude. $\square$

Lemma 7.18. *Let $\mathcal{D}_1, \mathcal{D}_2$ be symmetric monoidal ∞-categories. Suppose every object of $\mathcal{D}_1, \mathcal{D}_2$ is dualizable. Suppose we have a symmetric monoidal functor $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ with a* strictly *symmetric monoidal right adjoint H. Then the adjunction (F, H) is an inverse equivalence of symmetric monoidal ∞-categories.*

Proof. We first show that H is a fully faithful functor. To see this, we fix $X, Y \in \mathcal{D}_2$ and use

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}_2}(X, Y) &\simeq \mathrm{Hom}_{\mathcal{D}_2}(\mathbf{1}, \mathbb{D}X \otimes Y) \\
&\simeq \mathrm{Hom}_{\mathcal{D}_1}(\mathbf{1}, H(\mathbb{D}X \otimes Y)) \\
&\simeq \mathrm{Hom}_{\mathcal{D}_1}(\mathbf{1}, H(\mathbb{D}X) \otimes H(Y)) \\
&\simeq \mathrm{Hom}_{\mathcal{D}_1}(\mathbf{1}, \mathbb{D}H(X) \otimes H(Y)) \\
&\simeq \mathrm{Hom}_{\mathcal{D}_1}(H(X), H(Y)),
\end{aligned}$$

as desired. Dualizing this argument, we can also conclude that F is fully faithful. Therefore, the adjunction is an inverse equivalence. $\square$

The preceding lemma is presumably well-known to category theorists. We will also need a converse of sorts to these results:

Corollary 7.19. *Let $\mathcal{C}$ be as above with $R = \mathrm{End}_{\mathcal{C}}(\mathbf{1})$. Suppose that $\mathcal{C}$ is unipotent. Then any compact object of $\mathcal{C}$ belongs to the thick subcategory generated by the unit. In particular, if $X, Y \in \mathcal{C}$ are two compact objects, then the natural map*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X) \otimes_R \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, Y) \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes Y)$$

is an equivalence.

Proof. The second assertion clearly follows from the first since it is true for $X = Y = \mathbf{1}$ and those pairs (X, Y) satisfying the assertion form a thick subcategory in each variable.

Now suppose $X \in \mathcal{C}$. Then since $\mathcal{C}$ is unipotent, we know that the natural map

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X) \otimes_R \mathbf{1} \to X$$

is an equivalence.

Now, by the theory of Ind-objects in ∞-categories [44, §5.3], we can write the R-module $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X)$ as a filtered colimit of perfect R-modules. That is, there exists a filtered ∞-category $\mathcal{I}$ and a functor $f : \mathcal{I} \to \mathrm{Mod}(R)$ such that:

1. For each $i \in \mathcal{I}$, $f(i) \in \mathrm{Mod}(R)$ is a perfect R-module.
2. $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X)$ is identified with $\varinjlim_{i \in \mathcal{I}} f(i)$.

Therefore, we find that

$$X \simeq \varinjlim_{i \in \mathcal{I}} \left(f(i) \otimes_R \mathbf{1} \right),$$

where each $f(i) \otimes_R \mathbf{1} \in \mathcal{C}$ belongs to the thick subcategory generated by the unit. When X is compact, it follows that X is a retract of $f(i) \otimes_R \mathbf{1}$ for some i, proving the claim. $\square$

7.2. The Eilenberg–Moore spectral sequence

We now connect the abstract discussion of unipotence above to a very classical question when $\mathcal{C} = \mathrm{Fun}(X, \mathrm{Mod}(R))$ for X a connected space and R an $\mathbb{E}_\infty$-ring, so that $\mathcal{C}$ parametrizes (by definition) local systems of R-modules on X.

Definition 7.20 *(Cf. [46, §1.1]).* Choose a basepoint $* \in X$, and consider the pullback square

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X, \end{tikzcd} \tag{7.21}$$

and the induced square of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} F(X_+, R) \arrow[r] \arrow[d] & R \arrow[d] \\ R \arrow[r] & F(\Omega X_+, R). \end{tikzcd} \tag{7.22}$$

We say that the *R-based Eilenberg–Moore spectral sequence (EMSS) is relevant for X* if (7.22) is a pushout of $\mathbb{E}_\infty$-rings, i.e., if the induced morphism

$$R \otimes_{F(X_+,R)} R \to F(\Omega X_+, R) \tag{7.23}$$

is an equivalence. If so, we obtain a strongly convergent Tor-spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^{\pi_*(F(X_+,R))}_{p,q}(\pi_*(R), \pi_*(R)) \implies \pi_*(F(\Omega X_+, R)), \tag{7.24}$$

which we call the *R-based Eilenberg–Moore spectral sequence (EMSS).*

If the R-based EMSS is relevant, the spectral sequence (7.24) reduces to the classical (cohomological) Eilenberg–Moore spectral sequence in case $R = Hk$ for k a field and if X has finitely generated homology in each degree.

Construction 7.25. We now give another interpretation of the R-based EMSS. Observe that the pullback square (7.21) can be interpreted as a square in $\mathcal{S}_{/X}$, the ∞-category of spaces over X. Recall that there is an equivalence of ∞-categories

$$\mathcal{S}_{/X} \simeq \mathrm{Fun}(X, \mathcal{S}) \simeq \mathrm{Fun}(X^{op}, \mathcal{S}),$$

by the Grothendieck construction [44, §2.1].

Here, since X is a connected space, $\mathrm{Fun}(X, \mathcal{S}) \simeq \mathrm{Fun}(B\Omega X, \mathcal{S})$ can be identified with the ∞-category of spaces equipped with an action of ΩX (where, as before, we implicitly choose a basepoint of X). In particular, when one works in $\mathrm{Fun}(X, \mathcal{S})$, one has the pullback square

$$\begin{tikzcd} \Omega X \times \Omega X \arrow[r] \arrow[d] & \Omega X \arrow[d] \\ \Omega X \arrow[r] & *, \end{tikzcd} \tag{7.26}$$

where ΩX is given the action of ΩX by left multiplication; this corresponds to $* \in \mathcal{S}_{/X}$ in view of the fiber sequence $\Omega X \to * \to X$. This pullback square in $\mathrm{Fun}(X, \mathcal{S})$ corresponds via the Grothendieck construction to the cartesian square (7.21) in $\mathcal{S}_{/X}$.

Consider now the functor $\mathcal{S}^{op} \to \mathrm{CAlg}_{R/}$, $Y \mapsto F(Y_+, R)$. We apply it to (7.26). We obtain a commutative algebra object $A \in \mathrm{Fun}(X, \mathrm{Mod}(R))$ given by $F(\Omega X_+, R)$ with the natural ΩX-action. In particular, we obtain a square in $\mathrm{CAlg}(\mathrm{Fun}(X, \mathrm{Mod}(R)))$,

$$\begin{tikzcd} F(*_+, R) \arrow[r] \arrow[d] & F(\Omega X_+, R) \arrow[d] \\ F(\Omega X_+, R) \arrow[r] & F((\Omega X \times \Omega X)_+, R). \end{tikzcd} \tag{7.27}$$

When we apply the lax symmetric monoidal functor $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \cdot)\colon \mathrm{Fun}(X, \mathrm{Mod}(R)) \to \mathrm{Mod}(F(X_+, R))$ to (7.27), we obtain (7.22), in view of the correspondence between (7.26) and (7.21).

Suppose now ΩX has the homotopy type of a finite cell complex. Then $F((\Omega X \times \Omega X)_+, R) \simeq A \otimes A \in \mathrm{Fun}(X, \mathrm{Mod}(R))$, i.e., (7.27) is a pushout of commutative algebra objects in $\mathcal{C}$. We thus obtain:

Proposition 7.28. *Suppose ΩX has the homotopy type of a finite cell complex. Let $A \in \mathcal{C} = \mathrm{Fun}(X, \mathrm{Mod}(R))$ be the commutative algebra object $A = F(\Omega X_+, R)$. Then the R-based EMSS is relevant for X if and only if the natural map of $F(X_+, R)$-modules*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) \otimes_{F(X_+,R)} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes A),$$

is an equivalence.

Proof. The square in (7.27) is a pushout in $\mathrm{CAlg}(\mathrm{Fun}(X, \mathrm{Mod}(R))$ since ΩX has the homotopy type of a finite cell complex. By applying the lax symmetric monoidal functor

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \cdot)\colon \mathrm{Fun}(X, \mathrm{Mod}(R)) \to \mathrm{Mod}(F(X_+, R))$$

to this pushout we obtain (7.22). Hence the R-based EMSS is relevant for X if and only if $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \cdot)$ takes this particular pushout to a pushout. □

We are now ready for the main result of this subsection.

Theorem 7.29. *Let G be a compact Lie group and R an $\mathbb{E}_\infty$-ring. Then the R-based EMSS is relevant for BG if and only if the symmetric monoidal ∞-category $\mathcal{C} = \mathrm{Fun}(BG, \mathrm{Mod}(R))$ is unipotent. In this case, the $F(BG_+, R)$-module R is perfect, and $\mathrm{Fun}(BG, \mathrm{Mod}(R))$ is identified with the symmetric monoidal ∞-category of R-complete $F(BG_+, R)$-modules.*

Proof. As above, we consider the algebra object $A = F(G_+, R) \in \mathcal{C}$. Using equivariant Atiyah duality (see [42, Th. III.5.1] for the genuinely equivariant result), one sees that A is some suspension of the induced object $R \wedge G_+ = \mathbb{D}A$. Since the induced object $R \wedge G_+$ is a compact generator for $\mathcal{C}$, it follows that A is a compact generator as well.

Suppose $\mathcal{C}$ is unipotent. Then we apply Corollary 7.19 to conclude that A belongs to the thick subcategory generated by the unit and that $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) \otimes_{F(X_+,R)} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes A)$ is an equivalence. It follows by Proposition 7.28 that the R-based EMSS is relevant for BG. The remaining assertions now follow from Proposition 7.13 applied to A.

Conversely, if $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) \otimes_{F(X_+,R)} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes A)$ is an equivalence, we want to apply Proposition 7.15 to conclude that $\mathcal{C}$ is unipotent. In order to do this, we need to analyze A-modules in $\mathcal{C}$. To do this, consider the inclusion $* \to BG$ and the induced adjunction

$$\mathcal{C} = \mathrm{Fun}(BG, \mathrm{Mod}(R)) \rightleftarrows \mathrm{Mod}(R),$$

where the left adjoint restricts to a basepoint and the right adjoint sends $M \in \mathrm{Mod}(R)$ to the coinduced object $F(G_+, M)$. In particular, the right adjoint carries the unit of $\mathrm{Mod}(R)$ to $A \in \mathcal{C}$. Using Proposition 5.29, it follows that we have an equivalence of symmetric monoidal ∞-categories $\mathrm{Mod}_{\mathcal{C}}(A) \simeq \mathrm{Mod}(R)$, so that $\mathrm{Mod}_{\mathcal{C}}(A)$ has A as compact generator. Therefore, we have all the ingredients to apply Proposition 7.15 and conclude the argument. $\square$

Corollary 7.30. *Suppose the R-based EMSS is relevant for BG for G a compact Lie group. Then it is relevant for R' if R' is any $\mathbb{E}_\infty$-ring such that there exists a map of $\mathbb{E}_1$-rings $R \to R'$.*

Proof. We use Theorem 7.29. As the EMSS is relevant for BG, it follows that $\mathcal{C} = \mathrm{Fun}(BG, \mathrm{Mod}(R))$ is unipotent, and the coinduced object $F(G_+, R)$ belongs to the thick subcategory generated by the unit by Corollary 7.19. By base-change to R', we find that $F(BG_+, R') \in \mathrm{Fun}(BG, \mathrm{Mod}(R'))$ belongs to the thick subcategory generated by R' with trivial G-action. Now we can apply Proposition 7.13 to $\mathcal{C}' = \mathrm{Fun}(BG, \mathrm{Mod}(R'))$ to obtain that $\mathcal{C}'$ is unipotent too, so that the R'-based EMSS is relevant for BG. $\square$

Proposition 7.31. *Let $R \to R'$ be a descendable morphism of $\mathbb{E}_\infty$-rings (i.e., R is descendable as a commutative algebra in $\mathrm{Mod}(R)$, cf. Definition 4.4). Suppose the map $F(BG_+, R) \otimes_R R' \to F(BG_+, R')$ is an equivalence. Then the R-based EMSS is relevant for BG if and only if the R'-based EMSS is relevant.*

Proof. The "only if" implication is given by Corollary 7.30. For the converse, we want to show that the natural map $R \otimes_{F(BG_+,R)} R \to F(G_+, R)$ is an equivalence. To do so, since $R \to R'$ is descendable, it suffices to show that the base-change to R' is an equivalence. But by hypothesis (and the fact that G is a compact Lie group), this is precisely the map $R' \otimes_{F(BG_+,R')} R' \to F(G_+, R')$. $\square$

Corollary 7.32. *Let $R \to R'$ be a descendable morphism of $\mathbb{E}_\infty$-rings such that R' is a perfect R-module, and let G be a compact Lie group. Then the R-based EMSS is relevant for BG if and only if the R'-based one is.*

Proof. In fact, since R' is a perfect R-module, the map $F(BG_+, R) \otimes_R R' \to F(BG_+, R')$ is an equivalence, so we can apply Proposition 7.31. $\square$

The relevance of the EMSS, especially over $H\mathbb{Z}$ and $H\mathbb{F}_p$, has been treated classically in numerous sources, e.g., [24,25], and is discussed for complex K-theory in [40]. A more recent development is the ambidexterity theory of Hopkins–Lurie [35]. For example, in [35, Th. 5.4.3], they show (as a special case) that for G a p-group the ∞-category of $K(n)$-local modules over Morava E-theory with G-action (at the prime p) is unipotent; the analogous assertion about the EMSS is earlier work of Bauer [12].

7.3. The categorical argument

In Theorem 7.29, we saw that the unipotence of ∞-categories of the form $\mathrm{Fun}(BG, \mathrm{Mod}(R))$, for G a compact Lie group and R an $\mathbb{E}_\infty$-ring, is equivalent to the relevance of the R-based EMSS for the space BG.

The purpose of this subsection is to obtain a basic and easily checked sufficient criterion for relevance of the EMSS. For a given compact connected Lie group G, this criterion will always be applicable to $\mathbb{E}_\infty$-rings such that $\pi_*(R)$ is torsion-free away from a finite number of primes (compare Theorem 7.40 below). Therefore, we will be able to prove that several such ∞-categories are unipotent.

In the next subsection, we shall give a slightly different (and more geometric) variant of the following argument. We have included both arguments in this paper. The present argument seems to be more widely applicable. However, the geometric one generalizes better to the genuinely equivariant setting.

Proposition 7.33. *Suppose G is a compact, connected Lie group. Then the $\mathbb{Z}$-based EMSS is relevant for BG (and thus, by Corollary 7.30, the R-based EMSS is relevant for BG if R is any discrete $\mathbb{E}_\infty$-ring).*

Proof. Using Proposition 7.13 with $\mathcal{C} = \mathrm{Fun}(BG, \mathrm{Mod}(\mathbb{Z}))$ and $A = F(G_+, \mathbb{Z})$, it suffices to show that the induced object $G_+ \wedge \mathbb{Z} \in \mathrm{Fun}(BG, \mathrm{Mod}(\mathbb{Z}))$ belongs to the thick subcategory generated by the unit. This follows easily by working up the (finite) Postnikov decomposition of $G_+ \wedge \mathbb{Z}$: each of the successive cofibers has trivial G-action, because G is connected, and finitely generated homotopy. □

Remark 7.34. Using a similar argument, combined with the fact that nontrivial representations of p-groups in characteristic p always have nontrivial fixed points, one can argue that if $\pi_0(G)$ is a p-group, then the $\mathbb{F}_p$-based EMSS is relevant for BG. As a result, we have an equivalence of symmetric monoidal ∞-categories between $\mathrm{Fun}(BG, \mathrm{Mod}(\mathbb{F}_p))$ and complete modules over $F(BG_+, \mathbb{F}_p)$ (cf. Theorem 7.29).

Here is our main result:

Theorem 7.35. *Let R be an $\mathbb{E}_\infty$-ring and let G be a compact, connected Lie group. Suppose $H^*(BG; \pi_0 R) \simeq H^*(BG; \mathbb{Z}) \otimes_{\mathbb{Z}} \pi_0 R$ and this is a polynomial ring over $\pi_0 R$. Suppose moreover that the cohomological R-based AHSS for BG degenerates (e.g., $\pi_*(R)$ is torsion-free). Then the R-based EMSS for BG is relevant, so that* $\mathrm{Fun}(BG, \mathrm{Mod}(R))$

is unipotent and equivalent to the symmetric monoidal ∞-category of R-complete $F(BG_+, R)$-modules.

Proof of Theorem 7.35. Without loss of generality, we may assume that R is connective by Corollary 7.30 and replacing R by $\tau_{\geq 0} R$ if necessary. Choose classes $x_1, \dots, x_r \in H^*(BG; \pi_0 R)$ which form a system of polynomial generators, so that $H^*(BG; \pi_0 R) \simeq \pi_0 R[x_1, \dots, x_r]$. Let $k_i = |x_i|$. Choose lifts $y_1, \dots, y_r \in \widetilde{R}^*(BG)$, which we can by the degeneration of the AHSS. For each i, y_i classifies a self-map $\Sigma^{-k_i} R \to R$ in the ∞-category $\mathrm{Fun}(BG, \mathrm{Mod}(R))$.

Consider the coinduced object $F(G_+, R) \in \mathrm{CAlg}(\mathrm{Fun}(BG, \mathrm{Mod}(R)))$. One has a unit map $R \to F(G_+, R)$. Observe also that the homotopy fixed points of $F(G_+, R)$ are given by R itself. As a result, for each i, the composite map

$$\Sigma^{-k_i} R \xrightarrow{y_i} R \to F(G_+, R)$$

is nullhomotopic; this follows because y_i restricts to zero in $\pi_*(R) = \pi_* F(*_+, R)$. In particular, for each i we obtain maps in $\mathrm{Fun}(BG, \mathrm{Mod}(R))$,

$$R/y_i \to F(G_+, R).$$

On homotopy fixed points, these classify maps of $F(BG_+, R)$-modules

$$F(BG_+, R)/y_i \to R$$

that extend the map $F(BG_+, R) \to R$ given by evaluating at a point. Using the $\mathbb{E}_\infty$-structure on $F(G_+, R)$, we obtain a map

$$\bigotimes_{i=1}^{r} R/y_i \to F(G_+, R). \tag{7.36}$$

We claim that (7.36) is an equivalence. In order to see this, it suffices (as the underlying R-modules of both objects are bounded below) to base change along the map $R \to \pi_0 R$, so that we may assume that R is discrete. In this case, it suffices to see that the map (7.36) induces an equivalence on homotopy fixed points by Proposition 7.33. But this is the claim that we have an equivalence of $F(BG_+, \pi_0 R)$-modules

$$F(BG_+, \pi_0 R)/(x_1, \dots, x_r) \simeq \pi_0 R,$$

which we have assumed. As a result, it follows that the coinduced object $F(G_+, R) \in \mathrm{Fun}(BG, \mathrm{Mod}(R))$ belongs to the thick subcategory generated by the unit, so we may apply Proposition 7.13 and conclude unipotence. $\square$

We now obtain a basic result for unipotence for actions of the classical compact Lie groups (with 2 inverted for the $SO(n)$ family). A classic textbook reference for the calculations of the cohomology of the relevant classifying spaces is [58].

Theorem 7.37. *Suppose R is an $\mathbb{E}_\infty$-ring with $\pi_*(R)$ torsion-free. Then $\mathrm{Fun}(BG, \mathrm{Mod}(R))$ is unipotent if G is a product of copies of $U(n), SU(n), Sp(n)$ for some n. If 2 is invertible in R, then one can also include factors of $SO(n), Spin(n)$. In particular, we have an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Fun}(BG, \mathrm{Mod}(R)) \simeq \mathrm{Mod}(F(BG_+, R))_{\mathrm{cpl}},$$

where on the right we consider $F(BG_+, R)$-modules which are complete with respect to the $F(BG_+, R)$-module R.

We also include a counterexample to show the necessity of inverting 2 in the presence of $SO(n)$-factors.

Example 7.38. The ∞-category $\mathrm{Fun}(BSO(3), \mathrm{Mod}(KU))$ is *not* unipotent. When one works 2-adically, $KU^*(BSO(3))$ is a power series ring on one variable, but $KU^*(SO(3))$ has 2-torsion, and the EMSS does not converge (compare the remark at the end of [40, §4]).

On the other hand, we shall see that $\mathrm{Fun}(BG, \mathrm{Mod}(KU))$ is unipotent if G has no torsion in π_1. So, there are other examples of unipotence not covered by Theorem 7.37 (such as $G = \mathrm{Spin}(n), n \geq 4$).

Finally, we note that we can recover results of Greenlees–Shipley [31].

Example 7.39. Let G be a connected compact Lie group. Then (by Proposition 7.33) for $R = \mathbb{Q}$, the ∞-category $\mathrm{Fun}(BG, \mathrm{Mod}(\mathbb{Q}))$ is unipotent. Therefore, we find that $\mathrm{Fun}(BG, \mathrm{Mod}(\mathbb{Q}))$ is equivalent, as a symmetric monoidal ∞-category, to modules over $F(BG_+, \mathbb{Q})$ which are complete with respect to the augmentation ideal. This is closely related to the main result of [31].

Strictly speaking, Greenlees–Shipley work in the genuine equivariant setting in [31]; however, they work with *free* G-spectra, so that it is equivalent to work in $\mathrm{Fun}(BG, \mathrm{Mod}(\mathbb{Q}))$. Moreover, they use torsion instead of complete $F(BG_+, \mathbb{Q})$-modules. Note also that $F(BG_+, \mathbb{Q})$ is equivalent to a free $\mathbb{E}_\infty$-ring over $\mathbb{Q}$ on a finite number of generators, since $H^*(BG; \mathbb{Q})$ is a polynomial ring.

Fix a compact connected Lie group G. In order to make the assumptions of Theorem 7.35 more explicit, we now determine the minimal integer n with respect to divisibility such that $H^*(BG; \mathbb{Z}[1/n])$ is a polynomial algebra. To formulate the result, recall that G contains a maximal semi-simple subgroup $G_{ss} \subset G$, and that G is homeomorphic to $G_{ss} \times T$ for a torus $T \subset G$. The group G_{ss} in turn is uniquely a finite product of simple groups, the simply-connected covers of which are simply connected, simple Lie groups; these are classified by their Lie algebras. We refer to the finite list of Lie algebras thus associated with G as the types occurring in G.

Theorem 7.40. *Let G be a compact connected Lie group and $n \geq 1$ an integer. Then the following are equivalent:*

1. *The $\mathbb{Z}[1/n]$-algebra $H^*(BG, \mathbb{Z}[1/n])$ is polynomial.*
2. *The $\mathbb{Z}[1/n]$-algebra $H^*(G, \mathbb{Z}[1/n])$ is exterior.*
3. *The integer n is divisible by each of the following primes p:*
   - *Each p which occurs as the order of an element of $\pi_1(G)$.*
   - *The prime $p = 2$ if G contains a factor of type $\mathrm{Spin}(N)$ for some $N \geq 7$, G_2, F_4, E_6, E_7 or E_8.*
   - *The prime $p = 3$ if G contains a factor of type F_4, E_6, E_7 or E_8.*
   - *The prime $p = 5$ if G contains a factor of type E_8.*

Example 7.41. The conditions in Theorem 7.40 depend on G only through G_{ss}. The algebra $H^*(BG; \mathbb{Z})$ itself is polynomial if and only if the semi-simple part of G is simply connected and contains none of the types listed; this holds for example for $G = U(N)$. The minimal n such that $H^*(BSO(N); \mathbb{Z}[1/n])$ is polynomial is $n = 2$: The cover $\mathrm{Spin}(N) \to SO(N)$ is simple and simply connected, and the map has degree 2.

Proof of Theorem 7.40. We first show the equivalence of 2. and 3. Using the obvious generalization from $\mathbb{Z}$- to $\mathbb{Z}[1/n]$-coefficients of [17, Proposition 1.2] for $X = G$, we see that $H^*(G; \mathbb{Z}[1/n])$ is exterior if and only if it is torsion-free, i.e., if and only if n is divisible by all primes p such that $H^*(G; \mathbb{Z})$ has non-trivial p-torsion. As $G \simeq G_{ss} \times T$, these are exactly the torsion primes of $H^*(G_{ss}; \mathbb{Z})$. By [17, Lemme 3.3], passage from G_{ss} to its simply connected cover exactly picks up those p which divide the (finite) order of $\pi_1(G_{ss})$. Finally, for the semi-simple, simply connected case, the torsion primes are listed in [17, Théorème 2.5]. To see that 1. is equivalent to 3., we observe that by [17, Théorème 4.5], condition 3. in Theorem 7.40 is also equivalent to $H^*(BG, \mathbb{Z}[1/n])$ being torsion free. By the natural adaptation of [16, Th. 19.1] to $\mathbb{Z}[1/n]$-coefficients, we find that $H^*(BG; \mathbb{Z}[1/n])$ is a polynomial algebra as desired in precisely these cases. $\square$

7.4. The geometric argument

Let E be an $\mathbb{E}_\infty$-ring which is complex-orientable as an $\mathbb{E}_1$-ring; as shown in Proposition 7.42 this condition is often satisfied in practice. In this subsection, we shall describe actions of compact Lie groups G on E-modules where G is a product of unitary groups. Rather than going through the EMSS as in the previous subsection, we shall use complex-orientability instead and Proposition 7.13. The use of complex-orientability also appears in [32], and our methods are closely related to theirs. However, our results will be strictly contained in Theorem 7.37.

Proposition 7.42. *Suppose that E is an $\mathbb{E}_2$-ring and π_*E is concentrated in even degrees. Then there is a morphism $MU \to E$ of $\mathbb{E}_1$-rings.*

Proof. By passing to connective covers we can assume E is connective. Using the obstruction theory of [2], it suffices to show that the composite $f\colon BU \xrightarrow{J} BGL_1S \to BGL_1E$ of based spaces is null-homotopic. For this purpose, we fix a cell structure on BU using only even dimensional cells and inductively extend a null-homotopy over the skeleta of BU. Since E is $\mathbb{E}_2$, BGL_1E is a loop space and hence simple so we can apply elementary obstruction theory [56, §18.5]. The relevant obstructions lie in $\widetilde{H}^{2n+2}(BU; \pi_{2n+2}BGL_1E) \cong \widetilde{H}^{2n+2}(BU; \pi_{2n+1}E) = 0$ for $n \geq 0$. This builds a compatible sequence of based null-homotopies $H_n\colon BU_{2n} \wedge I_+ \to BGL_1E$ and taking colimits gives the desired null-homotopy of f. $\square$

For simplicity, we begin with the case of $G = U(1)$. Choose a complex orientation $x\colon \Sigma^{-2}BU(1) \to E$. Observe that $\pi_*(F(BU(1)_+, E)) \simeq \pi_*(E)[[x]] := \varprojlim \pi_*(E)[x]/x^n$ where $x \in \pi_{-2}(F(BU(1)_+, E))$ is a class that maps to zero under the map $F(BU(1)_+, E) \to E$ given by evaluation at a point.

We will now give a geometric proof of unipotence in the case of $U(1)$-actions.

Theorem 7.43. *The ∞-category $\mathrm{Fun}(BU(1), \mathrm{Mod}(E))$ is unipotent. The functor of homotopy fixed points is fully faithful and embeds $\mathrm{Fun}(BU(1), \mathrm{Mod}(E))$ as the subcategory of x-complete objects in $\mathrm{Mod}(F(BU(1)_+, E))$.*

Proof. Let V be the standard one-dimensional complex representation of $U(1)$. Consider the Euler cofiber sequence of pointed spaces with an $U(1)$-action

$$S(V)_+ \to S^0 \to S^V. \tag{7.44}$$

Here $S(V)$ denotes the unit sphere in V and S^V denotes the one-point compactification of V. Note that $S(V)_+$ is *induced* from the trivial group: it is just $U(1)_+$ with the action by translation. After smashing with E, we get a cofiber sequence in $\mathrm{Fun}(BU(1), \mathrm{Mod}(E))$ given by

$$E \wedge U(1)_+ \to E \to E \wedge S^V. \tag{7.45}$$

Now, we use the $\mathbb{E}_1$-complex orientation of E to give the equivalence

$$E \wedge S^V \simeq \Sigma^2 E \in \mathrm{Fun}(BU(1), \mathrm{Mod}(E)), \tag{7.46}$$

in view of the theory of orientations of [2]. This argument is crucial. To see (7.46), it suffices to take $E = MU$, and in this case one knows that $MU \wedge \Sigma^{-2} S^V \in \mathrm{Fun}(BU(1), \mathrm{Mod}(MU))$ factors through $\mathrm{Fun}(BU(1), BGL_1(MU))$ via the composition

$$BU(1) \to BU \xrightarrow{J} BGL_1(S^0) \to BGL_1(MU),$$

for J the complex J-homomorphism. The composition is nullhomotopic, which implies that the local system of MU-modules $\Sigma^{-2} S^V \wedge MU$ over $BU(1)$ is trivial; this is the claim of (7.46).

Finally, in view of (7.45) and (7.46), we find that the induced object $E \wedge U(1)_+$ belongs to the thick subcategory generated by the unit. Dualizing, we find that the coinduced object $A = F(U(1)_+, E)$ belongs to the thick subcategory generated by the unit. Now, applying Proposition 7.13, we conclude that $\mathrm{Fun}(BU(1), \mathrm{Mod}(E))$ is unipotent and is equivalent to the ∞-category of modules over $F(BU(1)_+, E)$ which are complete with respect to the $F(BU(1)_+, E)$-module E. Since $E \simeq F(BU(1)_+, E)/x$, this completes the proof. $\square$

We now give the analog for any of the unitary or special unitary groups.

Theorem 7.47. *If $G = U(n), SU(n)$, then the ∞-category $\mathrm{Fun}(BG, \mathrm{Mod}(E))$ is unipotent and equivalent (via homotopy fixed points) to the ∞-category of E-complete E^{BG}-modules.*

Proof. We consider the case $G = U(n)$. We will use the criterion of Proposition 7.13. It suffices to show that the induced object $E \wedge U(n)_+$ belongs to the thick subcategory generated by the unit.

Here we can work by induction on n. Suppose the induced object $E \wedge U(n-1)_+ \in \mathrm{Fun}(BU(n-1), \mathrm{Mod}(E))$ belongs to the thick subcategory generated by the unit. Let V be the standard representation of $U(n)$ on $\mathbb{C}^n$. Now we have a cofiber sequence as in (7.44), in pointed $U(n)$-spaces. It reads

$$U(n)/U(n-1)_+ \to S^0 \to S^V.$$

Smashing with E, we get a cofiber sequence

$$E \wedge U(n)/U(n-1)_+ \to E \to \Sigma^{2n} E, \tag{7.48}$$

where we used the same “untwisting” argument as in Theorem 7.43 to identify $E \wedge S^V$ with $\Sigma^{2n} E$.

Now, by the inductive hypothesis, the induced object $E \wedge U(n-1)_+ \in \mathrm{Fun}(BU(n-1), \mathrm{Mod}(E))$ belongs to the thick subcategory generated by the unit. Inducing upwards to $U(n)$, it follows that the induced object $E \wedge U(n)_+ \in \mathrm{Fun}(BU(n), \mathrm{Mod}(E))$ belongs to the thick subcategory generated by $E \wedge U(n)/U(n-1)_+$. However, (7.48) shows that $E \wedge U(n)/U(n-1)_+$ belongs to the thick subcategory generated by the unit in $\mathrm{Fun}(BU(n), \mathrm{Mod}(E))$. Therefore, by transitivity, the induced object $E \wedge U(n)_+$ belongs to the thick subcategory generated by the unit, and we can apply Proposition 7.13 to conclude the proof. $\square$

7.5. *The flag variety*

In this subsection, we include the principal applications of our general categorical machinery to nilpotence results.

Let $T \subset U(n)$ be a maximal torus, and let $F \simeq U(n)/T$ be the *flag variety* of $\mathbb{C}^n$. Observe that F has an action of $U(n)$, as a topological space. Therefore, $F_+ \in \mathrm{Fun}(BU(n), \mathrm{Sp})$. Our goal is to show that this action can actually be trivialized over a complex-oriented base. These ideas go back to [63,36].

Proposition 7.49. *Let E be an $\mathbb{E}_\infty$-ring which admits an $\mathbb{E}_1$-complex orientation. Then we have an equivalence $E \wedge F_+ \simeq \bigoplus_{i=1}^{n!} \Sigma^{k(i)} E$ of objects in $\mathrm{Fun}(BU(n), \mathrm{Mod}(E))$, where the $k(i)$ are even integers.*

Proof. It suffices to prove this with F_+ replaced by its Spanier–Whitehead dual $\mathbb{D}F_+$. Since $F \simeq U(n)/T$, the Spanier–Whitehead dual $\mathbb{D}F_+$ is the coinduction of the unit from $\mathrm{Fun}(BT, \mathrm{Sp})$ to $\mathrm{Fun}(BU(n), \mathrm{Sp})$. Let $\mathcal{C} = \mathrm{Fun}(BU(n), \mathrm{Mod}(E))$. Now,

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbb{D}F_+ \wedge E) \simeq F(BT_+, E) \in \mathrm{Mod}(F(BU(n)_+, E))$$

and this is a free $F(BU(n)_+, E)$-module of rank $n!$ as $E^*(BT)$ is a free $E^*(BU(n))$-module of rank $n!$ with generators in even degrees by the general theory of complex-oriented ring spectra. By unipotence of $\mathrm{Fun}(BU(n), \mathrm{Mod}(E))$ (Theorem 7.37), this is enough to prove the claim. $\square$

Although we included the general unipotence results for their own interest, Proposition 7.49 can be seen directly. Consider the ∞-category $\mathrm{Mod}^\omega(\underline{E})$, where $\underline{E} \in \mathrm{Sp}_G$ is the Borel-equivariant form of E. It suffices to prove that $\underline{E} \wedge \mathbb{D}F_+$ is a free module over $\underline{E}$; this amounts to checking that the homotopy groups of $\underline{E} \wedge \mathbb{D}F_+$ are free over the homotopy groups of $\underline{E}$, for each subgroup $H \subset G$ (cf. Recollection 9.11 below). This, however, is the formula for the complex-oriented cohomology of a flag bundle (cf. [36, Prop. 2.4]). We leave the details to the reader.

Theorem 7.50. *Let E be an $\mathbb{E}_\infty$-ring which admits an $\mathbb{E}_1$-complex orientation. Let G be any compact Lie group. Then the thick subcategory of $\mathrm{Fun}(BG, \mathrm{Mod}(E))$ generated by the $E \wedge G/A_+$, as $A \leq G$ ranges over the abelian subgroups, contains the unit $E \in \mathrm{Fun}(BG, \mathrm{Mod}(E))$.*

Proof. Embed $G \leq U(n)$ for some n and consider the flag variety F. As an E-module with $U(n)$-action, we saw in Proposition 7.49 that the unit is a retract of $E \wedge F_+$ in $\mathrm{Fun}(BU(n), \mathrm{Mod}(E))$. Therefore, if we restrict to G and consider $E \wedge F_+$ as an object in $\mathrm{Fun}(BG, \mathrm{Mod}(E))$, it contains the unit as a retract. But F, as a space with G-action, has abelian stabilizers and thus admits a finite cell decomposition with cells of the form $G/A \times D^n$ by the equivariant triangulation theorem [39]. In particular, $E \wedge F_+ \in \mathrm{Fun}(BG, \mathrm{Mod}(E))$ belongs to the *stable* subcategory generated by the G/A_+ as $A \leq G$ ranges over the abelian subgroups. This proves the result. $\square$

Here again there is a variant for the orthogonal groups. Let $T \subset SO(n)$ be a maximal torus, and let $F' = SO(n)/T$ be the real flag variety. One has:

Theorem 7.51. *Let E be an $\mathbb{E}_\infty$-ring such that 2 is invertible in $\pi_0(E)$ and $\pi_*(E)$ is torsion-free. Then, as an object in $\mathrm{Fun}(BSO(n), \mathrm{Mod}(E))$, the flag variety $F'_+ \wedge E$ is equivalent to a direct sum of copies of shifts of the unit.*

Proof. We know by Theorem 7.37 that $\mathrm{Fun}(BSO(n), \mathrm{Mod}(E))$ is equivalent to the ∞-category of complete modules over $F(BSO(n)_+, E)$. The hypotheses imply that the AHSS for $E^*(BSO(n))$ degenerates (as the differentials are torsion valued) and we have that

$$E^*(BSO(n)) \simeq \pi_*(E)[u_1, \ldots, u_m], \quad E^*(BT) \simeq \pi_*(E)[t_1, \ldots, t_m],$$

where m is the rank of $SO(n)$. Moreover, $E^*(BT)$ is a free module over $E^*(BSO(n))$, so that the same reasoning as in Proposition 7.49 can be applied. $\square$

8. Equivariant complex K-theory

In this section, we will study the ∞-category of modules over $U(n)$-equivariant K-theory $KU_{U(n)}$ in the ∞-category of genuine $U(n)$-spectra. Our main result will show that the symmetric monoidal ∞-category $\mathrm{Mod}_{\mathrm{Sp}_{U(n)}}(KU_{U(n)})$ is equivalent to the ∞-category of modules in spectra over its categorical fixed points. More generally, we will be able to replace $U(n)$ with any compact Lie group G with $\pi_1(G)$ torsion-free for this. This is a *unipotence* result for modules over equivariant K-theory. Note that the unit is compact in the genuine equivariant setting, so the completeness and convergence issues of the previous section do not arise.

This result (which was known to Greenlees–Shipley for G a torus) gives a new point of view on the classical question, considered by Hodgkin, McLeod, and Snaith, of Künneth spectral sequences in equivariant complex K-theory. In the following section, we will also treat the case of equivariant real K-theory using Galois descent. For the purposes of nilpotence, it gives (when combined with the equivariant K-theory of the flag variety) an “explicit” proof of nilpotence with respect to the family of abelian groups (as in the previous section). In the sequel [54] to this paper, we shall in fact see that in this result, abelian subgroups can be replaced by the family of *cyclic* subgroups. However, the reduction to the abelian case is in some sense the most important (and the one that generalizes).

Remark 8.1. We will generally write $KO_G, KU_G \in \mathrm{Sp}_G$ for the G-spectra representing G-equivariant K-theory. When the group is clear, we will sometimes simply write KO, KU (especially when we want to describe the equivariant K-theory of a space).

8.1. The case of a torus

We begin with the (simpler) case of a torus. We will need to use the existence of $\mathbb{E}_\infty$-structures on equivariant real and complex K-theory. These $\mathbb{E}_\infty$-structures are established in work of Joachim [41], and appear in Schwede's theory of global spectra [66]. These results are also a consequence of forthcoming work of Lurie on elliptic cohomology announced in [43].

We begin with the following special case. Most of the ideas (if not the statement) appear in [32], and the result was known to Greenlees–Shipley.

Theorem 8.2. *Let T be a torus. Then the symmetric monoidal ∞-category $\mathrm{Mod}_{\mathrm{Sp}_T}(KU_T)$ of modules (in T-equivariant spectra Sp_T) over KU_T is equivalent to $\mathrm{Mod}(i_T^* KU_T)$.*

Proof. By the Thom isomorphism, KU_T is what Greenlees–Shipley [32] call *complex-stable*: that is, given a representation sphere S^V (for a *complex* representation V of the torus), we have an equivalence of KU_T-modules $S^V \wedge KU_T \simeq KU_T$. By [32, Lem. 4.4], we are done. □

For the reader's convenience, we recall the method of argument in the case that $T = U(1)$. Recall that the ∞-category of $U(1)$-spectra is generated as a localizing subcategory by the $U(1)/H_+$ as $H \leq U(1)$ ranges over the closed subgroups. The only possibilities are $H = U(1)$ (in which case $U(1)/H_+$ is the unit) or $H = \mu_n$ for some n, the group of nth roots of unity. In this case, one considers the one-dimensional complex representation V_n of $U(1)$ given by the character $z \mapsto z^n$. The unit sphere $S(V_n)_+$ gives precisely $(U(1)/\mu_n)_+$. The cofiber sequence $S(V_n)_+ \to S^0 \to S^{V_n}$ now shows that the ∞-category of $U(1)$-spectra is generated as a localizing subcategory by the representation spheres S^V for V a complex representation of $U(1)$. When one works over KU_T, though, the complex stability enables us to include only the unit.

8.2. The general case

The purpose of this section is to give the proof of our main unipotence result for equivariant complex K-theory.

Theorem 8.3. *Suppose G is a compact, connected Lie group such that $\pi_1(G)$ is torsion-free. Then $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$ is canonically equivalent, as a symmetric monoidal ∞-category, to the ∞-category of module spectra over the categorical fixed points $i_G^* KU_G$.*

To appreciate the possible simplification this result brings about for studying $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$, we briefly remark on the structure of the (non-equivariant) $\mathbb{E}_\infty$-ring $A := i_G^* KU_G$: it is even periodic with $\pi_1(A) = 0$ and $\pi_0(A) = R(G)$. Landweber exactness shows that as a multiplicative cohomology theory, one has $A^*(-) = KU^*(-) \otimes_{\mathbb{Z}} R(G)$. It seems an interesting question to ask if the $\mathbb{E}_\infty$-ring A can be built from KU in a similarly transparent fashion.

Theorem 8.3 is equivalent, by Morita theory, to the assertion that $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$ is generated, as a localizing subcategory, by the unit. When G is a product of copies of $U(1)$ (i.e., a torus), we have already seen the proof of this (Theorem 8.2). The general case proceeds by restriction to a maximal torus.

The key ingredient for the general case is given by:

Lemma 8.4. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free. Let $T \subset G$ be a maximal torus and let $F = G/T$ be the flag variety of G. Let X be a finite G-cell complex. Then $KU_G^*(F) \simeq KU_T^*(*)$ is a free $R(G)[\beta_2^{\pm 1}] = KU_G^*(*)$-module, and the canonical map*

$$KU_G^*(X) \otimes_{KU_G^*(*)} KU_G^*(F) \to KU_G^*(X \times F) \cong KU_T^*(X) \tag{8.5}$$

is an isomorphism.

Proof. Lemma 8.4 follows by combining work of Hodgkin, Snaith, and McLeod. By [70, Th. 3.6], the construction in [34] of a Künneth spectral sequence is relevant (i.e., converges to the desired limit) for $\pi_1(G)$ torsion-free if the natural map $R(G) \otimes_{R(T)} R(G)[\beta_2^{\pm 1}] \to KU_T^*(G/T)$ is an isomorphism. The main result of [57] shows that this is in fact the case if $\pi_1(G)$ is torsion-free, so there is a Künneth spectral sequence with an edge map of the form appearing in (8.5). This implies that (8.5) is an isomorphism once we know that the representation ring $R(T)$ is free over $R(G)$; this is a theorem of Pittie [60, Thm. 1]. $\square$

The map (8.5) can be rewritten as follows. As before, we let i_G^* denote categorical fixed points $i_G^* \colon \mathrm{Sp}_G \to \mathrm{Sp}$. The equivariant K-theory of a finite G-cell complex X is obtained as

$$KU_G^*(X) = \pi_{-*} i_G^* (\mathbb{D}X_+ \wedge KU_G),$$

where X_+ denotes the suspension spectrum of X in Sp_G and $\mathbb{D}$ denotes Spanier–Whitehead duality. We will need this in the following form.

Lemma 8.6. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free and let $T \subset G$ and $F = G/T$ be as above. Then for any $M \in \mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$, the map*

$$R(T) \otimes_{R(G)} \pi_* i_G^* M \to \pi_* i_T^* M \simeq \pi_* i_G^* (M \wedge \mathbb{D}F_+) \tag{8.7}$$

is an isomorphism.

The last map in (8.7) is an isomorphism for tautological reasons: $M \wedge \mathbb{D}F_+$ is the coinduction of the restriction of M to Sp_T in view of the projection formula.

Proof. We observe that there is a natural map, since the right-hand side is linear over $R(T)$. It is a natural transformation of homology theories in $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$, and Lemma 8.4 implies that it is an isomorphism if M is the Spanier–Whitehead dual of the suspension spectrum of a finite G-cell complex. This implies that it is true in general, since the duals of suspension spectra of finite G-cell complexes generate Sp_G under colimits. $\square$

Proof of Theorem 8.3. Let G be as hypothesized. If G is a torus, we are already done. Let $M \in \mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$. Suppose that $i_G^* M = 0$; we want to show that M is itself contractible. If we can prove this, then we will have proved Theorem 8.3 because as $i_G^*(-) \simeq \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)}(KU_G, -)$ we will then know that the compact unit $\mathbf{1}$ generates $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$ as a localizing subcategory.

Choose a maximal torus $T \subset G$. By Lemma 8.6, we find that $\pi_*(i_T^* M) = 0$. In view of Theorem 8.2, this implies that the *restriction* of M to T (i.e., as an object of $\mathrm{Mod}_{\mathrm{Sp}_T}(KU_T)$) is contractible. Since restriction is symmetric monoidal, it follows that for any G-space X, the restriction of $M \wedge X_+$ to Sp_T is contractible; in particular, for any such X,

$$i_T^*(M \wedge X_+) = 0.$$

But by Lemma 8.6 again, this implies that

$$i_G^*(M \wedge X_+) = 0,$$

and since we had this for any X, we find that $M = 0$ itself, as the Spanier–Whitehead duals of the finite G-cell complexes X generate Sp_G as a localizing subcategory. $\square$

Remark 8.8. Our analysis relied on deep work of Hodgkin, Snaith, and McLeod. In the case when G is a product of unitary groups (which is the essential case for nilpotence results), the results needed are much more elementary. Namely, instead of Lemma 8.6, one can use:

Lemma 8.9 *([68, Prop. 3.9]). Let X be a finite G-cell complex and let V be any complex G-representation. Let $\mathbb{P}(V)$ be the projectivization of V considered as a G-space. Then $KU_G^*(\mathbb{P}(V))$ is a free $R(G)[\beta_2^{\pm 1}] = KU_G^*(*)$-module, and the map*

$$KU_G^*(X) \otimes_{KU_G^*(*)} KU_G^*(\mathbb{P}(V)) \to KU_G^*(X \times \mathbb{P}(V)) \tag{8.10}$$

is an isomorphism.

One can carry out the above strategy of proof, using Lemma 8.9 to replace a copy of $U(n)$ by a copy of $U(n-1) \times U(1)$.

Although our analysis relied on classical work on the equivariant Künneth spectral sequence, the main result (Theorem 8.3) gives a new interpretation of this spectral sequence. Namely, let G be a compact, connected Lie group with $\pi_1(G)$ torsion-free. If X is a finite G-cell complex, then one has $KU_G^*(X) \simeq \pi_* i_G^*(KU_G \wedge \mathbb{D}X_+)$. In particular, to every such X, we can associate the KU_G-module $KU_G \wedge \mathbb{D}X_+$ and its categorical fixed points $M_X := i_G^*(KU_G \wedge \mathbb{D}X_+) \in \mathrm{Mod}(i_G^* KU_G)$. The homotopy groups of the spectrum M_X give precisely the equivariant K-theory of X.

Since we have seen a symmetric monoidal equivalence $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G) \simeq \mathrm{Mod}(i_G^* KU_G)$ (where the latter takes place in the world of *nonequivariant* spectra), it follows that the association $X \mapsto M_X \in \mathrm{Mod}(i_G^* KU_G)$ is symmetric monoidal. The Künneth spectral sequence can thus be recovered as the classical Tor-spectral sequence for modules over the (nonequivariant) $\mathbb{E}_\infty$-ring spectrum $i_G^* KU_G$.

Let G be a general compact connected Lie group. In this generality, we do not know how to describe the ∞-category $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$. However, we note that:

Proposition 8.11. *If G is connected, the flag variety $KU_G \wedge (G/T)_+$ is a compact generator of* $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$.

The argument presented here shows that the above result is a consequence of Atiyah's "holomorphic transfer" [3]. The identification of the holomorphic transfer and a spectrum-level transfer as a consequence of index theory is discussed in [23, §4.3] for the unitary group. See also [59] for a discussion of these transfer maps. We have spelled out the details for the convenience of the reader.

Proof. The key step is to show that the unit KU_G is a retract of $KU_G \wedge (G/T)_+$. Let τ denote the tangent bundle of the flag variety $F = G/T$ and let F^τ denote its Thom space, the latter considered as a pointed G-space. Since G/T is a complex manifold, we have a Thom isomorphism

$$G/T_+ \wedge KU_G \simeq F^\tau \wedge KU_G \in \mathrm{Mod}_{\mathrm{Sp}_G}(KU_G).$$

It suffices now to show that the unit is a retract of $F^\tau \wedge KU_G$.

To see this, embed the flag variety $F \subset W$ for W a real G-representation. As a result, we have an embedding $\tau \subset TW \simeq W \otimes_{\mathbb{R}} \mathbb{C}$ and a consequent Pontryagin–Thom collapse map

$$S^{W \otimes_{\mathbb{R}} \mathbb{C}} \to F^{\tau+\nu},$$

for ν the normal bundle of $\tau \subset TW$. After smashing with K-theory, we obtain a map $KU_G \to F^{\nu+\tau} \wedge KU_G \simeq F^{\tau} \wedge KU_G$ by the Thom isomorphism. We will show that there exists a map $F^{\tau} \to KU_G$ such that the induced composite $KU_G \to F^{\tau} \wedge KU_G \to KU_G$ is an equivalence. In other words, we will produce a class in $\widetilde{KU}^0_G(F^{\tau})$ whose pullback to $\widetilde{KU}^0_G(S^{W \otimes_{\mathbb{R}} \mathbb{C}}) \simeq R(G)$ is a unit.

Indeed, the pull-back map $\widetilde{KU}^0_G(F^{\tau}) \to \widetilde{KU}^0_G(S^{W \otimes_{\mathbb{R}} \mathbb{C}})$ is given by the *analytic index* by the Atiyah–Singer index theorem [5]. As a result, one has to produce a G-equivariant elliptic differential operator (or complex) on F whose index in $R(G)$ is one-dimensional. We can take the Dolbeaut complex of the complex manifold F. By a special case of the Borel–Weil–Bott theorem, the coherent cohomology $H^*(F, \mathcal{O})$ is one-dimensional and concentrated in degree zero (with trivial G-action). It follows that the associated class in $\widetilde{KU}^0_G(F^{\tau})$ (which is the Thom class of the complex tangent bundle) has the desired property, and we get the splitting.

As a result, for any KU_G-module M, the natural map $\pi_* i_G^* M \to \pi_* i_T^* M$ is canonically a (split) injection. We now leave it to the reader to show, imitating the proof of Theorem 8.3, that if M is such that $\pi_* i_T^* M = 0$, then M itself is contractible: in other words, the flag variety is a compact generator. $\square$

The above argument with the holomorphic transfer underscores the importance of the flag variety in proving the above statements: in fact, the argument in [70] regarding the Künneth spectral sequence goes through the holomorphic transfer too.

Remark 8.12. We can translate the proof of the main result of [70] into our language, too. Suppose $\pi_1(G)$ is torsion-free, so that $R(T) \otimes_{R(G)} R(T)[\beta_2^{\pm 1}] \simeq KU_G^*(G/T \times G/T)$ and $R(T)$ is free over $R(G)$ (by [57] and [60]). We apply Lemma 7.18 now to the thick subcategory of $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$ generated by the unit and the flag variety (which is self-dual by the Wirthmüller isomorphism). It follows from Lemma 7.18 that this thick subcategory is generated by the unit. However, Proposition 8.11 implies that this thick subcategory consists precisely of the compact objects, so $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$ is unipotent as desired.

8.3. The Borel-completion

We can Borel-complete Theorem 8.3 to obtain a strengthening of our “Koszul duality” result Theorem 7.37 in the case of (nonequivariant) K-theory. We find:

Theorem 8.13. *Let G be a compact, connected Lie group with $\pi_1(G)$ torsion-free. Then the ∞-category $\mathrm{Fun}(BG, \mathrm{Mod}(KU))$ is unipotent and equivalent as a symmetric monoidal ∞-category to KU-complete modules over $F(BG_+, KU)$.*

Proof. Let $\hat{R}(G), \hat{R}(T)$ be the completions of the representation rings $R(G), R(T)$ at the respective augmentation ideals $I_G \subset R(G), I_T \subset R(T)$. By the Atiyah–Segal completion theorem [4], these give precisely π_0 of $F(BG_+, KU)$ and $F(BT_+, KU)$. Note that the map $R(G) \to R(T)$ exhibits $R(T)$ as a finite module over $R(G)$ by [69, Prop. 3.2], and that the I_T-adic topology on $R(T)$ is equivalent to the I_G-adic one [69, Cor. 3.9]. In any event, we find that

$$\hat{R}(G) \otimes_{R(G)} R(T) \to \hat{R}(T) \tag{8.14}$$

is an isomorphism.

By Proposition 7.13, it suffices to show that the induced object $KU \wedge G_+ \in \mathrm{Fun}(BG, \mathrm{Mod}(KU))$ belongs to the thick subcategory generated by the unit. Inducing upwards from $\mathrm{Fun}(BT, \mathrm{Mod}(KU))$ (where we already know the result by Theorem 7.35), we see that it belongs to the thick subcategory generated by $KU \wedge (G/T)_+$, so it suffices to show that the flag variety G/T_+ belongs to the thick subcategory generated by the unit. Note first that the flag variety is self-dual in $\mathrm{Fun}(BG, \mathrm{Mod}(KU))$ in view of the Wirthmüller isomorphism and complex orientability. As a result, Lemma 7.18 shows that it suffices to prove that the natural map

$$(KU \wedge F_+)^{hG} \otimes_{F(BG_+, KU)} (KU \wedge F_+)^{hG} \to (KU \wedge F_+ \wedge F_+)^{hG} \tag{8.15}$$

is an equivalence. Indeed, we can then apply Lemma 7.18 for $\mathcal{C} = \mathrm{Mod}^{\omega}(F(BG_+, KU))$ the ∞-category of perfect $F(BG_+, KU)$-modules, and $\mathcal{D}$ the thick subcategory of $\mathrm{Fun}(BG, \mathrm{Mod}(KU))$ generated by the unit and $KU \wedge F_+$; the result implies that $\mathcal{C} = \mathcal{D}$.

However, in view of the discussion in (8.14), it is a consequence of the Atiyah–Segal completion theorem and Lemma 8.4 that (8.15) is an equivalence. $\square$

8.4. Applications to nilpotence

As before, we can obtain:

Corollary 8.16. *Let G be a compact, connected Lie group with torsion-free π_1 and let $T \subset G$ be a maximal torus. Let $F = G/T$. Then we have an equivalence in $\mathrm{Mod}_{\mathrm{Sp}_G}(KU_G)$,*

$$KU_G \wedge \mathbb{D}F_+ \simeq \bigoplus_m KU_G$$

for some integer m.

Proof. This follows from Theorem 8.3 and the fact (due to [60]) that the $KU_G^*(F) \simeq R(T)[t^{\pm 1}]$ is a free module over $R(G)[t^{\pm 1}]$. $\square$

Applying Theorem 8.13, one obtains:

Corollary 8.17. *Let G be a compact connected Lie group with torsion-free π_1 and let $F = G/T$ be the flag variety as before. Then, as an object of* $\mathrm{Fun}(BG, \mathrm{Mod}(KU))$, *$F_+ \wedge KU$ is a direct sum of copies of the unit.*

Finally, as before we can obtain the nilpotence statement. Again, the full strength of unipotence is not really necessary for this argument: the freeness of the flag variety is equivalent to the projective bundle formula in equivariant K-theory.

Corollary 8.18. *If G is a finite group, then $KU_G \in \mathrm{Sp}_G$ is nilpotent for the family of abelian subgroups.*

Proof. Embed $G \leq U(n)$ and consider the action of G on $F = U(n)/T$ for $T \subset U(n)$ a maximal torus. We have $\mathbb{D}F_+ \wedge KU_{U(n)} \simeq \bigoplus_1^{n!} KU_{U(n)}$ by Corollary 8.16. Restricting down to G, we have $\mathrm{Res}_G^{U(n)} KU_{U(n)} \simeq KU_G$ and we get

$$\mathbb{D}F_+ \wedge KU_G \simeq \bigoplus_1^{n!} KU_G.$$

Choosing a triangulation of F with abelian stabilizers in G, we can conclude the proof as before. $\square$

9. Equivariant real K-theory

Let G be a compact Lie group. In this section, we will analyze G-equivariant *real* K-theory. Our main goal is to extend the results in the previous section to KO_G-modules in Sp_G, as well as to develop a Galois descent picture for equivariant K-theory. In particular, we will obtain an $\mathscr{F}$-nilpotence result for KO_G for G finite (for $\mathscr{F}$ the family of abelian subgroups).

Our main tool, which we will start with, is an equivariant version of the equivalence $KO \wedge \Sigma^{-2}\mathbb{CP}^2 \simeq KU$. This will enable us to "descend" (via thick subcategory arguments) many results for KU_G to KO_G. As a result, we will prove a similar unipotence result for KO_G-modules. When combined with techniques from [53] for $G = U(n)$, we will be able to prove that the *equivariant* complexification map $KO_G \to KU_G$ is a faithful C_2-Galois extension of $\mathbb{E}_\infty$-rings in $\mathrm{Sp}_{U(n)}$ (which we will then deduce for any compact Lie group G). The Galois picture was first developed nonequivariantly by Rognes [65] and has numerous applications.

9.1. Complexification in equivariant K-theory

The key ingredient in the proof below of the equivariant version of Wood's theorem (Theorem 9.8) is an analysis of the complexification map

$$\widetilde{KO}_G^*(\mathbb{CP}^2) \to \widetilde{KU}_G^*(\mathbb{CP}^2).$$

This *mostly* reduces to a purely non-equivariant calculation, since $\mathbb{CP}^2$ is regarded here as a space with trivial G-action. First of all, by [68, Prop. 2.2], we have a natural isomorphism $\widetilde{KU}_G^*(\mathbb{CP}^2) \simeq R(G) \otimes \widetilde{KU}^*(\mathbb{CP}^2)$. However, the picture is somewhat more complicated for equivariant KO. In this subsection, we discuss the equivariant real and complex K-theory of spaces with trivial G-action and give a complete analysis of the (equivariant) complexification map.

Definition 9.1. Given an irreducible representation V of G over $\mathbb{C}$, recall that there are three possibilities:

1. The representation V is not self dual as a G-representation over $\mathbb{C}$. In this case, the real representation $V|_{\mathbb{R}}$ underlying V is irreducible, and $\mathrm{End}_{G,\mathbb{R}}(V|_{\mathbb{R}}) \simeq \mathbb{C}$.
2. The underlying real representation $V|_{\mathbb{R}}$ is not irreducible as a real representation. Thus, V contains an $\mathbb{R}$-subspace $V_{\mathbb{R}} \subset V$ which is G-stable. One has $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq V$. As a complex representation, we have $V \simeq V^*$. Moreover, $\mathrm{End}_G(V_{\mathbb{R}}) \simeq \mathbb{R}$.
3. The representation V is self-dual as a G-representation over $\mathbb{C}$, but $V|_{\mathbb{R}}$ is irreducible. In this case, $\mathrm{End}_{G,\mathbb{R}}(V|_{\mathbb{R}}) \simeq \mathbb{H}$.

Given an irreducible representation W of G over $\mathbb{R}$, there are three corresponding possibilities:

1. There is an isomorphism $\mathrm{End}_G(W) \simeq \mathbb{C}$, and W arises as the restriction of an irreducible complex representation V of type 1. If it arises as the restriction of V, it equivalently arises as the restriction of V^*. In this case, W is called *complex.*
2. There is an isomorphism $W \simeq V_{\mathbb{R}}$ for an irreducible V of type 2. In this case, $\mathrm{End}_G(W) = \mathbb{R}$. In this case, W is called *real.*
3. The representation W is the restriction of some V of type 3; in this case $\mathrm{End}_G(W) = \mathbb{H}$ and W is called *quaternionic.*

Definition 9.2. Given a compact Lie group, we let $RO(G)$ denote the Grothendieck group of real finite-dimensional G-representations, as usual. Thus, $RO(G)$ is a free abelian group on the isomorphism classes of irreducible G-representations over $\mathbb{R}$.

We let $RO^{\mathbb{R}}(G) \subset RO(G)$ denote the subgroup spanned by the classes of those representations which are real (in the sense of Definition 9.1), $RO^{\mathbb{C}}(G) \subset RO(G)$ the subgroup spanned by the classes of those representations which are complex, and $RO^{\mathbb{H}}(G)$

the subgroup spanned by those that are quaternionic. We thus obtain a decomposition $RO(G) = RO^{\mathbb{R}}(G) \oplus RO^{\mathbb{C}}(G) \oplus RO^{\mathbb{H}}(G)$.

We now need the following result, in which KSp denotes symplectic or quaternionic K-theory.

Proposition 9.3 *([68, pp. 133–134]). Let X be a finite CW complex given trivial G-action. Then we have natural isomorphisms*

$$KU^*_G(X) \simeq R(G) \otimes KU^*(X), \tag{9.4}$$

$$KO^*_G(X) \simeq RO^{\mathbb{R}}(G) \otimes KO^*(X) \oplus RO^{\mathbb{C}}(G) \otimes KU^*(X) \oplus RO^{\mathbb{H}}(X) \otimes KSp^*(X). \tag{9.5}$$

We note that the first isomorphism is C_2-equivariant for the complex conjugation on all sides: in particular, including the C_2-action on $R(G)$. The second decomposition arises as follows in degree zero (by suspending and using periodicity, one gets the general case). Given a complex G-representation V and a complex vector bundle $\mathcal{W}$ on X, we form the G-equivariant *real* vector bundle $V \otimes_{\mathbb{C}} \mathcal{W}$. The other two summands are interpreted similarly.

We will need to describe the complexification map $KO^*_G(X) \to KU^*_G(X)$ with respect to the above decompositions. Without loss of generality (up to replacing X by a suspension), we take $* = 0$.

1. On $RO^{\mathbb{R}}(G) \otimes KO^0(X)$, the complexification map behaves as follows: given a real G-representation V and a real vector bundle $\mathcal{W} \in KO^0(X)$, the class $[V] \otimes [\mathcal{W}]$ maps to $[V_{\mathbb{C}}] \otimes [\mathcal{W}_{\mathbb{C}}]$.
2. On $RO^{\mathbb{C}}(G) \otimes KU^0(X)$, the complexification map behaves as follows: given a complex G-representation V and a complex vector bundle $\mathcal{W} \in KU^0(X)$, the class $[V] \otimes [\mathcal{W}]$ maps to $[V] \otimes [\mathcal{W}] + [V^*] \otimes [\mathcal{W}^*]$. This follows from unwinding the definitions: one has to complexify the G-equivariant real vector bundle (which is the restriction of a G-equivariant complex vector bundle) $V \otimes_{\mathbb{C}} \mathcal{W}$.
3. On $RO^{\mathbb{H}}(G) \otimes KSp^0(X)$, the complexification map is the most complicated. Given a quaternionic representation V (where we interpret the $\mathbb{H}$ action on the *right*) and a quaternionic vector bundle $\mathcal{W}$ on X, the associated equivariant real vector bundle is $V \otimes_{\mathbb{H}} \mathcal{W}$.
   The complexification is therefore the equivariant complex vector bundle $V_{\mathbb{C}} \otimes_{\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}} \mathcal{W}_{\mathbb{C}}$. Here $V_{\mathbb{C}}$ has a right action of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$ and $\mathcal{W}_C$ has a left action of $M_2(\mathbb{C})$. In general, we recall that the category of left (resp. right) $M_2(\mathbb{C})$-modules is equivalent to the category of $\mathbb{C}$-vector spaces, and the $M_2(\mathbb{C})$-linear tensor product between a right and left $M_2(\mathbb{C})$-module corresponds to the $\mathbb{C}$-linear tensor product between vector spaces.
   In particular, we can describe the equivariant $\mathbb{C}$-vector bundle $V_{\mathbb{C}} \otimes_{\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}} \mathcal{W}_{\mathbb{C}}$ as follows. The $M_2(\mathbb{C})$-module with G-action $V_{\mathbb{C}}$ corresponds to a complex representation V' of G, and the $M_2(\mathbb{C})$-bundle $\mathcal{W}_{\mathbb{C}}$ over X corresponds to a $\mathbb{C}$-vector bundle $\mathcal{W}'$

over X. It is easy to see that V' satisfies $V' \simeq (V')^*$ and the underlying real representation V (which is still irreducible) is the underlying real representation of V'. In any event, the complexification carries $[V] \otimes [\mathcal{W}] \mapsto [V'] \otimes [\mathcal{W}']$.

In order to make this useful, we will need to describe more explicitly the map $\phi\colon KSp^0(X) \to KU^0(X)$ (which sends an $\mathbb{H}$-bundle $\mathcal{W}$ to the complex vector bundle associated to the $M_2(\mathbb{C})$-bundle $\mathcal{W}_{\mathbb{C}}$ via the Morita equivalence between $M_2(\mathbb{C})$ and $\mathbb{C}$). In particular, we will need to know that the following diagram is commutative

$$\begin{tikzcd} KSp^0(X) \arrow[r, "\phi"] \arrow[d, "\simeq"] & KU^0(X) \arrow[d, "\simeq"] \\ KO^4(X) \arrow[r] & KU^4(X). \end{tikzcd} \tag{9.6}$$

Here, the vertical arrows are Bott periodicity and the bottom horizontal map is the usual complexification from real to complex K-theory.

To see the commutativity of this diagram, we use the fact that the natural transformation $KSp^0 \to KU^0$ comes from a map of KO-module spectra $KSp \simeq \Sigma^4 KO \to KU$. Since this map, for $X = *$, carries the class of the $\mathbb{H}$-module $\mathbb{H}$ to the class of the $\mathbb{C}$-module $\mathbb{C}^2$, it induces multiplication by 2 in π_0. Therefore, one sees that the induced map of KO-module spectra $\Sigma^4 KO \to KU$ is the complexification map $\Sigma^4 KO \to \Sigma^4 KU$ followed by Bott periodicity $\Sigma^4 KU \simeq KU$.

9.2. The equivariant Wood theorem

We recall first:

Theorem 9.7 *(Wood). One has an equivalence of KO-module spectra $KU \simeq KO \wedge \Sigma^{-2}\mathbb{CP}^2$.*

A proof of this result (as well as an analog for TMF) can be found in [52]. In [52], the strategy is to take the C_2-action on KU given by complex conjugation and *define* $KO \simeq KU^{hC_2}$. Wood's theorem is proved by showing that $KU \wedge \Sigma^{-2}\mathbb{CP}^2$, as a spectrum with C_2-action, is the coinduced object $F(C_{2+}, KU)$.

The main goal of this subsection is to prove an equivariant analog of Theorem 9.7:

Theorem 9.8. *Let G be a compact Lie group. One has an equivalence of KO_G-modules in Sp_G, $KO_G \wedge \Sigma^{-2}\mathbb{CP}^2 \simeq KU_G$.*

We note that the $\mathbb{CP}^2$ that enters here is the ordinary one: that is, it is treated as a pointed space with trivial G-action. As in Theorem 9.7, the C_2-action on KU_G will play an important role in this analysis. However, unlike in the setting of Theorem 9.7, we do *not* want to assume an equivalence of the form $KO_G \simeq KU_G^{hC_2}$ in Sp_G; we will instead prove this as a corollary. Our strategy instead is to build on the known result (Theorem 9.7) and analyze directly the map

$$KO_G \wedge \mathbb{D}(\mathbb{CP}^2) \to KU_G \wedge \mathbb{D}(\mathbb{CP}^2) \tag{9.9}$$

in homotopy. It will be convenient to work with the (equivalent, up to a shift) Spanier–Whitehead duals, since this amounts to understanding the map $\widetilde{KO}_G^*(\mathbb{CP}^2) \to \widetilde{KU}_G^*(\mathbb{CP}^2)$. Our goal is to show that this map is injective with image the C_2-invariants in $\widetilde{KU}_G^*(\mathbb{CP}^2)$.

To begin with the proof of Theorem 9.8, we give a description of the equivariant real K-theory of $\mathbb{CP}^2$. Our technical tool is the following:

Proposition 9.10. *Let G be a compact Lie group. Then for any i, $\widetilde{KO}_G^i(\mathbb{CP}^2) \to \widetilde{KU}_G^i(\mathbb{CP}^2)$ is injective with image the C_2-invariants in $\widetilde{KU}_G^i(\mathbb{CP}^2)$.*

Proof. Consider the space $\mathbb{CP}^2$. We denote by ψ the complex conjugation action. Then we will need to use the following facts from *nonequivariant* K-theory:

1. $\widetilde{KO}^i(\mathbb{CP}^2) = \mathbb{Z}$ for i even and vanishes for i odd. For i even, we let $z_i \in \widetilde{KO}^i(\mathbb{CP}^2)$ be a generator.
2. For i even, $\widetilde{KU}^i(\mathbb{CP}^2) \simeq \mathbb{Z}^2$, generated by classes x_i, y_i with $\psi x_i = y_i$. For i odd, $\widetilde{KU}^i(\mathbb{CP}^2) = 0$.
3. Under complexification, the map $\widetilde{KO}^i(\mathbb{CP}^2) \to \widetilde{KU}^i(\mathbb{CP}^2)$ is injective with image precisely the C_2-invariants. That is, (up to multiplying by a sign) complexification maps $z_i \mapsto x_i + y_i$.

Observe first that by 2., we know that $\widetilde{KU}_G^*(\mathbb{CP}^2) \simeq R(G) \otimes \widetilde{KU}^*(\mathbb{CP}^2)$ is a coinduced C_2-representation. As a C_2-representation, $R(G)$ is a direct sum of copies of $\mathbb{Z}$ (one for each irreducible G-representation over $\mathbb{C}$ of type 1 or 3 in the sense of Definition 9.1) and $\mathbb{Z}[C_2]$ (one for each irreducible representation of type 2).

Now, we analyze the complexification map

$$c\colon \widetilde{KO}_G^i(\mathbb{CP}^2) \to \widetilde{KU}_G^i(\mathbb{CP}^2),$$

which is the effect of (9.9) in homotopy. For i odd, both sides vanish, so we may consider only the case i even. Using Proposition 9.3 applied to the suspensions of $\mathbb{CP}^2$, we obtain a decomposition of the left-hand-side into three pieces that we analyze separately, using the discussion in the previous subsection.

1. On $RO^{\mathbb{R}}(G) \otimes \widetilde{KO}^i(\mathbb{CP}^2)$, c is an injection with image given by $R^{\mathrm{real}}(G) \otimes \mathbb{Z}\{x_i + y_i\} \subset \widetilde{KU}_G^0(\mathbb{CP}^2)$. Here $R^{\mathrm{real}}(G)$ denotes the free abelian group on the com-

plex irreducible representations of type 2 in Definition 9.1. The map carries the class of $[V] \otimes z_i$ to $[V_{\mathbb{C}}] \otimes (x_i + y_i)$.

2. On $RO^{\mathbb{C}}(G) \otimes \widetilde{KU}^i(\mathbb{CP}^2)$, c behaves as follows. If $[V] \in RO^{\mathbb{C}}(G)$ is the class of an irreducible representation of G obtained as the restriction of a complex irreducible W, then c acts on the classes $[V] \otimes x_i, [V] \otimes y_i$ as:

$$c([V] \otimes x_i) = [W] \otimes x_i + [W^*] \otimes y_i, \quad c([V] \otimes y_i) = [W] \otimes y_i + [W^*] \otimes x_i.$$

3. Consider finally $RO^{\mathbb{H}}(G) \otimes \widetilde{KSp}^0(\mathbb{CP}^2)$. Let V be a $\mathbb{C}$-representation whose restriction to $\mathbb{R}$ is an irreducible quaternionic representation, denoted the same. Given a generator $w_i \in \widetilde{KSp}^i(\mathbb{CP}^2)$, the associated class in $\widetilde{KU}^i(\mathbb{CP}^2)$ (obtained by complexification together with Morita equivalence) is $x_i + y_i$, thanks to (9.6). Therefore, we have

$$c([V] \otimes w_i) = [V] \otimes (x_i + y_i).$$

From this, and the description of $R(G)$ as a C_2-representation, the proposition follows. □

The proof of Theorem 9.8 will require a little bookkeeping, and we begin with some recollections on equivariant homotopy groups.

Recollection 9.11. *Let G be a compact Lie group. Let $A \in \mathrm{Alg}(\mathrm{Sp}_G)$ be an associative algebra in Sp_G. For each $H \leq G$, we define $\pi_*^H(A) = \pi_* \mathrm{Hom}_{\mathrm{Sp}_G}(G/H_+, A) = \pi_* i_H^* A$. Each $\pi_*^H(A)$ is a ring, and as H varies these rings are equipped with restriction homomorphisms*

$$\mathrm{Res}_H^{H'} : \pi_*^{H'} A \to \pi_*^H A.$$

Given a module $M \in \mathrm{Mod}_{\mathrm{Sp}_G}(A)$, we can define the homotopy groups $\{\pi_*^H(M)\}_{H \leq G}$, which come with restriction homomorphisms of their own and form a module over $\{\pi_*^H(A)\}$.

Note that the maps of A-modules $A \to M$ are classified by the elements of $\pi_0^G(M)$. Suppose for instance that there exist elements $x_1, \ldots, x_n \in \pi_0^G(M)$ such that for each $H \leq G$, the $\left\{\mathrm{Res}_H^G x_i\right\} \subset \pi_0^H(M)$ form a basis for the $\pi_*^H(A)$-module $\pi_*^H(M)$. In this case, we get maps $x_i \colon A \to M$ which yield an equivalence of A-modules $A^n \simeq M$.

Proof of Theorem 9.8. Recall that $\widetilde{KU}_G^0(\mathbb{CP}^2) \simeq R(G) \otimes_{\mathbb{Z}} \mathbb{Z}\{x, y\}$ for classes x, y which are interchanged under complex conjugation. We have an equivalence of KU_G-modules

$$KU_G \vee KU_G \simeq KU_G \wedge \mathbb{D}(\mathbb{CP}^2) \tag{9.12}$$

classified by the elements x, y: the elements x, y produce a map, and it is an equivalence because the restrictions of x, y to $\widetilde{KU}^*_H(\mathbb{CP}^2)$ form an $R(H)$-basis for any $H \le G$. As a result, we can consider a map

$$f\colon KU_G \wedge \mathbb{D}(\mathbb{CP}^2) \to KU_G,$$

which, on homotopy, sends $x \mapsto 1$ and $y \mapsto 0$. For each subgroup $H \le G$, one sees that the induced map

$$\left(\pi_*^H(KU_G \wedge \mathbb{D}(\mathbb{CP}^2))\right)^{C_2} \to \pi_*^H(KU_G \wedge \mathbb{D}(\mathbb{CP}^2)) \simeq \widetilde{KU}^*_H(\mathbb{CP}^2) \to \pi_*^H(KU_G)$$

is an isomorphism. It follows easily that the composition

$$KO_G \wedge \mathbb{D}(\mathbb{CP}^2) \to KU_G \wedge \mathbb{D}(\mathbb{CP}^2) \simeq KU_G \vee KU_G \xrightarrow{f} KU_G$$

is an equivalence, by comparing with Proposition 9.10. $\square$

As a result, we can also obtain the homotopy fixed point relation between real and complex K-theory, equivariantly.

Corollary 9.13. *The natural map $KO_G \to KU_G^{hC_2}$ in Sp_G is an equivalence.*

Proof. It suffices to show that the natural map $KO_G \wedge \mathbb{D}(\mathbb{CP}^2) \to (KU_G \wedge \mathbb{D}(\mathbb{CP}^2))^{hC_2}$ is an equivalence, because the thick subcategory that $\Sigma^\infty \mathbb{CP}^2$ generates is all of finite spectra by the nilpotence of η. This in turn can be checked on π_*^H for each subgroup $H \le G$. Now the map $\pi_*^H(KO_G \wedge \mathbb{D}(\mathbb{CP}^2)) \to \pi_*^H(KU_G \wedge \mathbb{D}(\mathbb{CP}^2))$ is injective and has image the C_2-invariants in the target, by Proposition 9.10. However, we have

$$\pi_*^H((KU_G \wedge \mathbb{D}(\mathbb{CP}^2))^{hC_2}) \simeq \pi_*^H(KU_G \wedge \mathbb{D}(\mathbb{CP}^2))^{C_2}$$

because the homotopy fixed point spectral sequence degenerates: the C_2-representation is induced. $\square$

9.3. Unipotence and nilpotence results

Using Theorem 9.8, we will now prove an analog of Theorem 8.3 for KO.

Theorem 9.14. *Suppose G is a compact, connected Lie group such that $\pi_1(G)$ is torsion-free. Then $\mathrm{Mod}_{\mathrm{Sp}_G}(KO_G)$ is canonically equivalent, as a symmetric monoidal ∞-category, to the ∞-category of module spectra over the categorical fixed points $i_G^* KO_G$.*

Proof. Suppose M is a KO_G-module (in Sp_G) whose categorical fixed points are trivial, i.e., $i_G^* M$ is contractible. We need to show that M is contractible; by Lemma 7.6, this will suffice for the theorem. To see this, we observe that if $i_G^*(M)$ is contractible, then $i_G^*(M \wedge \mathbb{D}(\mathbb{CP}^2))$ is contractible as well. However, $M \wedge \mathbb{D}(\mathbb{CP}^2)$ is a KU_G-module by Theorem 9.8, so by Theorem 8.3, it follows that $M \wedge \mathbb{D}(\mathbb{CP}^2)$ is contractible. Now, the thick subcategory that $\mathbb{D}(\mathbb{CP}^2)$ generates in finite spectra contains the sphere S^0 by the nilpotence of η, so that M is contractible itself. □

Using similar logic, one easily obtains:

Proposition 9.15. *Let G be a finite group and $\mathscr{F}$ a family of subgroups of G. Then $KO_G \in \mathscr{F}^{\mathrm{Nil}}$ if and only if $KU_G \in \mathscr{F}^{\mathrm{Nil}}$. In particular, KO_G is nilpotent for the family of abelian subgroups.*

Remark 9.16. In the sequel [54] to this paper, we will give another approach to the $\mathscr{F}$-nilpotence of KO_G using the *spin orientation*. We will actually show that KO_G is nilpotent for the family of *cyclic* subgroups.

9.4. The Galois picture

Let G be a compact Lie group. In the theory of structured ring spectra, it is known by work of Rognes [65] that the complexification map $KO \to KU$ is a faithful C_2-Galois extension: that is, it behaves like a C_2-torsor in ordinary algebraic geometry. As a consequence, it is for instance possible to carry out a form of *Galois descent* along $KO \to KU$. In this final subsection, we prove that an analogous picture holds equivariantly. We refer to [51] for preliminaries on Galois theory in a symmetric monoidal, stable ∞-category.

Theorem 9.17. *The natural map $KO_G \to KU_G$, together with the C_2-action on KU_G, exhibits KU_G as a faithful C_2-Galois extension of KO_G in* Sp_G.

Proof. Choose an embedding $G \leq U(n)$. In this case, one obtains a symmetric monoidal, cocontinuous functor $\mathrm{Res}_G^{U(n)} : \mathrm{Sp}_{U(n)} \to \mathrm{Sp}_G$ that carries $KO_{U(n)}, KU_{U(n)}$ to KO_G, KU_G. As a result, it suffices to show that $KO_{U(n)} \to KU_{U(n)}$ is a faithful C_2-Galois extension in $\mathrm{Sp}_{U(n)}$.[7]

In this case, the equivalence of Theorem 9.14 shows that it suffices to prove that if $A = i_{U(n)}^* KU_{U(n)}$, then the natural map

$$A^{hC_2} \to A, \tag{9.18}$$

exhibits A as a faithful C_2-Galois extension of A (in the category of non-equivariant spectra).We will prove this using the affineness machinery of [53]; one can also argue directly using Theorem 9.7.

[7] Recall that faithful Galois extensions are preserved by symmetric monoidal left adjoints.

Observe that A is an even periodic $\mathbb{E}_\infty$-ring, with $\pi_0(A) \simeq R(U(n))$, with a C_2-action. It follows that we can associate to A a *formal group* over $R(U(n))$, given by $\operatorname{Spf} A^0(\mathbb{CP}^\infty) \simeq \operatorname{Spf} R(U(n))[[x]]$. One sees that the associated formal group law over $\operatorname{Spec} R(U(n))$ is isomorphic to $\widehat{\mathbb{G}}_m$ (i.e., $A^*(\mathbb{CP}^\infty) = KU^*_{U(n)}(\mathbb{CP}^\infty) = R(U(n))\hat{\otimes} KU^*(\mathbb{CP}^\infty)$, etc.). One concludes that the unique map of schemes

$$\operatorname{Spec} R(U(n)) \to \operatorname{Spec} \mathbb{Z}$$

is such that the formal group $\operatorname{Spf} A^0(\mathbb{CP}^\infty)$ over $\operatorname{Spec} R(U(n))$ is pulled back from $\widehat{\mathbb{G}}_m$ over $\operatorname{Spec}\mathbb{Z}$.

Now, A has a C_2-action. Thus, $\operatorname{Spec} R(U(n))$ has a C_2-action from complex conjugation, and the formal group over $\operatorname{Spf} A^0(\mathbb{CP}^\infty)$ has one too. In the language of [53], we obtain a diagram

$$\operatorname{Spec} R(U(n))/C_2 \to \operatorname{Spec} \mathbb{Z}/C_2 \to M_{FG},$$

for M_{FG} the moduli stack of formal groups. Observe now that the first map $\operatorname{Spec} R(U(n))/C_2 \to \operatorname{Spec}\mathbb{Z}/C_2$ is affine (as the C_2-quotient of the affine map $\operatorname{Spec} R(U(n)) \to \operatorname{Spec}\mathbb{Z}$) and the map $(\operatorname{Spec}\mathbb{Z})/C_2 \to M_{FG}$ is affine. Therefore, the composition $\operatorname{Spec} R(U(n))/C_2 \to \operatorname{Spec}\mathbb{Z}/C_2 \to M_{FG}$ is affine. By [53, Th. 5.8] (see also [53, §2.5]), we obtain that the map (9.18) exhibits A as a faithful C_2-Galois extension of A^{hC_2}. $\square$

Example 9.19. We briefly calculate the homotopy fixed point spectral sequence (HFPSS) for $\pi_*^G KO_G \simeq \pi_*^G(KU_G)^{hC_2}$ as a modification of the (classical) computation when $G = 1$. First of all, we know that $\pi_*^G KU_G \simeq R(G)[\beta_2^{\pm 1}]$ where $|\beta_2| = 2$. The C_2-action on $R(G)$ is such that

$$R(G) = \bigoplus_V \mathbb{Z} \oplus \bigoplus_{V'} \mathbb{Z} \oplus \bigoplus_W \mathbb{Z}[C_2].$$

Here V ranges over isomorphism classes of irreducible $\mathbb{C}$-representations of type 2 (in the sense of Definition 9.1), V' ranges over isomorphism classes of irreducible $\mathbb{C}$-representations of type 3. Finally, W ranges over isomorphism classes of complex representations of type 1, up to the action $W \mapsto W^*$. Moreover, the C_2-action on the Bott element is by the sign representation.

It follows easily that, at the E_2-page the HFPSS for $\pi_*^G(KO_G)$ is a direct sum of copies of the HFPSS for $\pi_*(KO)$, one for each V and V', together with a sum of copies, one for each W, of $\mathbb{Z}[\beta_2^{\pm 1}]$ concentrated on the 0-line. Observe that the last component is necessarily given by permanent cycles because these classes come from $KO_G^*(*)$.

We now analyze the remaining classes. Recall first that the E_2-page for $\pi_*(KO)$ is given by $\mathbb{Z}[\beta^{\pm 2}, \eta]/(2\eta)$ where β has bidegree $(s,t) = (0,2)$ and η has bidegree

$(s,t) = (1,2)$. It follows that the E_2-page for $\pi_*^G(KO_G)$, when we ignore the contributions from W's, is given by the free module over $\mathbb{Z}[\beta^{\pm 2}, \eta]/(2\eta) \cdot [\mathbf{1}]$

$$E_2^{*,*} = \mathbb{Z}[\beta^{\pm 2}, \eta]/(2\eta)\, \{[V], [V']\}\,.$$

We conclude that $d_2 = 0$ holds for degree reasons. In the spectral sequence for $\pi_*(KO)$, it is well-known that one has the differential $d_3(\beta^2) = \eta^3$. This differential must happen here, too, i.e. we have $d_3(\beta^2 \cdot [\mathbf{1}]) = \eta^3 \cdot [\mathbf{1}]$. The classes $[V]$ survive to $KO_G^0(*) = RO(G)$ and are therefore permanent cycles and by multiplicativity one has $d_3(\beta^2 \cdot [V]) = \eta^3 \cdot [V]$. However, the classes $[V']$ do not survive to $KO_G^0(*)$ and necessarily support differentials. Since $\beta^2 \cdot [V']$ survives to $KO_G^{-4}(*)$ (thanks to (9.6)), we find that $\beta^2 \cdot [V']$ is a permanent cycle. Using multiplicativity again, we get $d_3([V']) = \eta^3\beta^{-2} \cdot [V']$. This determines the entire spectral sequence as being the direct sum of shifts by 0 and 4 of the $\pi_*(KO)$-HFPSS as well as the degenerate components coming from complex representations.

Acknowledgments

We would like to thank Clark Barwick, John Greenlees, Mike Hopkins, Srikanth Iyengar, Jacob Lurie, Peter May, and David Treumann for helpful discussions. We would especially like to thank Paul Balmer, whose preprint [7] made us aware of the connections between Quillen stratification and thick subcategory methods. We thank the referee for a very careful reading and catching numerous typos.

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