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On a nilpotence conjecture of J. P. May

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Abstract

We prove a conjecture of J. P. May concerning the nilpotence of elements in ring spectra with power operations, that is, H_∞-ring spectra. Using an explicit nilpotence bound on the torsion elements in $K(n)$-local H_∞-algebras over E_n, we reduce the conjecture to the nilpotence theorem of Devinatz, Hopkins, and Smith. As corollaries, we obtain nilpotence results in various bordism rings including $MSpin_*$ and $MString_*$, results about the behavior of the Adams spectral sequence for E_∞-ring spectra, and the non-existence of E_∞-ring structures on certain complex-oriented ring spectra.

1. *Introduction*

Understanding the stable homotopy of the sphere has been a driving motivation of algebraic topology from its very beginning. Early landmark results include Serre's theorem that every element in the positive stems is of finite order and Nishida's theorem that every element in the positive stems is (smash-)nilpotent. This was vastly generalized by the nilpotence theorem of Devinatz, Hopkins, and Smith, which states that complex bordism is sufficiently fine a homology theory to detect nilpotence in general ring spectra. On the other hand, Nishida's proof used basic geometric constructions, namely extended powers, to transform the additive information of Serre's theorem into the multiplicative statement of nilpotence. This was made much more systematic and general in the work of May and his collaborators on H_∞-ring spectra, which in particular led to a specific nilpotence conjecture for this restricted class of ring spectra. In this note, we will establish his conjecture, as follows.

Theorem A. *Suppose that R is an H_∞-ring spectrum and $x \in \pi_* R$ is in the kernel of the Hurewicz homomorphism $\pi_* R \to H_*(R;\mathbb{Z})$. Then x is nilpotent.*

This result was conjectured by May and verified under the additional hypothesis that $px = 0$ for some prime p, in [**3**, Chapter II, Conjecture 2.7 and Theorem 6.2]. In the case where R is the sphere spectrum, Theorem A is equivalent to Nishida's nilpotence theorem. In contrast to the nilpotence theorem of [**5**], Theorem A does not require any knowledge of the complex cobordism of R, but we must add the hypothesis that R is H_∞.

We will prove the following result, which implies Theorem A. (An elementary argument shows that Theorem A implies Theorem B as well.)

Theorem B. *Suppose that R is an H_∞-ring spectrum and $x \in \pi_* R$ has nilpotent image, via the Hurewicz homomorphism, in $H_*(R;k)$ for $k = \mathbb{Q}$ and $k = \mathbb{Z}/p$ for each prime p. Then x is nilpotent.*

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Indeed, since each of the above Hurewicz homomorphisms factors through the integral Hurewicz homomorphism, we see that May's conjecture follows from Theorem B.

The outline of this note is as follows: In Section 2, we reduce Theorem B to the nilpotence theorem using designer-made power operations due to Rezk (Lemma 2.2). These operations give explicit nilpotence bounds in a $K(n)$-local context (Theorem 2.1) which complete the proof of the main theorem. In Section 3, we build on Strickland's foundational work on operations in Lubin–Tate theory to provide a proof of Lemma 2.2. We conclude with several applications which should be of independent interest as well as some speculative refinements of Theorem B in Section 4. These include results about the following topics.

(1) The nilpotence of elements in bordism rings (Proposition 4.4 and Theorems 4.6 and 4.7).
(2) The behavior of the Adams spectral sequence for E_∞-ring spectra (Proposition 4.10).
(3) The non-existence of E_∞-ring structures on certain complex-oriented ring spectra (Proposition 4.11).

For one of these applications, we will need the following fact: If R is an E_2-ring and $x \in \pi_k R$, then $R[x^{-1}]$ is canonically an E_2-ring. In Appendix A, we provide a quick proof of the more general statement for E_n-rings, provided n is at least 2.

In a preliminary version of this paper [**16**, Proposition 4.3], we observed that Theorem B implied half of an analog in Morava E-theory of Quillen's $\mathcal{F}$-isomorphism [**22**, Theorem 7.1]. We meanwhile found an independent proof of the full result, and this will be documented elsewhere.

2. *The proof of Theorem* B

Throughout this section, the notation and assumptions of Theorem B are in force.

Recall that, for each prime p and positive integer n, there are 2-periodic ring spectra $K(n)$ and E_n which are related by a map $E_n \to K(n)$ of ring spectra inducing the quotient map of the local ring $\pi_0 E_n$ to its residue field $\pi_0 K(n)$. The first family consists of the 2-periodic Morava K-theories, which play an important role in the Ravenel conjectures [**23**] and are especially amenable to computation. The second family consists of Lubin–Tate theories which satisfy certain universal properties that make them extremely rigid; in particular, each of them admits an essentially unique E_∞-algebra structure and a corresponding theory of power operations; see Section 3 for more details.

By the nilpotence theorem [**10**, Theorem 3.i], if we can show that x is nilpotent in $H_*(R;\mathbb{Q})$, $H_*(R;\mathbb{Z}/p)$, and $K(n)_*R$ for each prime p and positive integer n, then x is nilpotent. Since x has nilpotent image in $H_*(R;\mathbb{Q}) \cong \pi_*R \otimes \mathbb{Q}$, by replacing x with a suitable power, we may assume that it is torsion. Now, since x is torsion, it is zero in $H_*(R;\mathbb{Q})$ and by assumption, that x is nilpotent in $H_*(R;\mathbb{Z}/p)$ for each prime p. To show that x is nilpotent in $K(n)_*R$, we will show that it is nilpotent in the ring $\pi_* L_{K(n)}(E_n \wedge R)$ and then map to $K(n)_*R$. So Theorem B will follow from Theorem 2.1 below, applied to $T = L_{K(n)}(E_n \wedge R)$ and the image of x in T under the E_n-Hurewicz map.

To simplify notation in what follows, we have put $E = E_n$, $\check{E}(X) = L_{K(n)}(E \wedge X)$, and $\check{E}_*(X) = \pi_*\check{E}(X)$. The 'check' notation, for example, $\check{E}$, is meant to remind the reader that we are working in a $K(n)$-local category. In particular, the T that appears in Theorem 2.1 is $K(n)$-local; see the next section for our conventions.

Theorem 2.1. *Suppose that T is an H_∞-$\check{E}$-algebra and $x \in \pi_j T$.*

(1) *If j is even and $p^m x = 0$, then*

$$x^{(p+1)^m} = 0.$$

(2) *If j is odd, then $x^2 = 0$.*

Our proof of this will depend on the following unpublished result of Rezk [**26**, p. 12], which we will prove in Section 3.

LEMMA 2.2. *Suppose that T is an H_∞-$\check{E}$-algebra. Then there are operations Q and θ acting on $\pi_0 T$ and natural with respect to maps of H_∞-$\check{E}$-algebras satisfying the following*:

(1) $(-)^p = Q(-) + p\theta(-)$;
(2) Q *is additive*;
(3) $\theta(0) = 0$.

Proof of Theorem 2.1. The claim about odd degree elements is precisely [**25**, Proposition 3.14], so we may assume that j is even. Since $\pi_* T$ is a $\pi_* E$-algebra, either the periodicity generator in $\pi_2 E$ has non-trivial image in $\pi_2 T$ or $\pi_* T = 0$. In the latter case, the theorem holds vacuously, and so by dividing by a suitable power of the periodicity generator, we can furthermore assume that j is zero.

It follows from the first two items in Lemma 2.2 that if p were not a zero-divisor in $\pi_0 T$, then

$$\begin{aligned}\theta(p^m x) &= p^{pm-1}x^p - p^{m-1}Q(x)\\ &= p^{m-1}((p^{(p-1)m} - 1)x^p + x^p - Q(x)),\end{aligned}$$

which when combined with

$$p^m\theta(x) = p^{m-1}(x^p - Q(x))$$

yields

$$\theta(p^m x) = x^p(p^{pm-1} - p^{m-1}) + p^m\theta(x). \tag{2.3}$$

To see that (2.3) also holds in rings with p-torsion, consider $x \in \pi_0 T$ as a map $x\colon S^0 \to T$. Since the target is an H_∞-$\check{E}$-algebra, this map canonically extends, up to homotopy, through the free H_∞-$\check{E}$-algebra on S^0:

$$\begin{tikzcd} S^0 \arrow[r, "x"] \arrow[d, "\iota"'] & T \\ \check{E}(\mathbb{P}S^0). \arrow[ur, dashed, "P(x)"'] & \end{tikzcd} \tag{2.4}$$

Since $\check{E}_0(\mathbb{P}S^0)$ is torsion-free [**30**, Theorem 1.1], (2.3) holds in $\check{E}_0(\mathbb{P}S^0)$ with ι in place of x. After applying π_0 to equation (2.4), $P(x)$ induces a ring map sending $Q(\iota)$ and $\theta(\iota)$ to $Q(x)$ and $\theta(x)$, respectively, so (2.3) holds in $\pi_0 T$.

Now, since $p^m x = 0$, by multiplying (2.3) by x and using Lemma 2.2.(3), we see that $p^{m-1}x^{p+1} = 0$. The theorem now follows by induction on m. $\square$

REMARK. Since $2x^2 = 0$ for x in odd degrees, we could alternatively appeal to Theorem 2.1.(1) to conclude $x^6 = 0$. This weakening of Theorem 2.1.(2) suffices for proving Theorem B, but misses Rezk's sharp bound.

3. *Power operations in Morava E-theory*

Before proving Lemma 2.2, we first recollect enough results about the theory of E_∞- and H_∞-algebras from [**3**, **6**] to define their variants in E and $\check{E}$-modules.

Recall that the category of E_∞-ring spectra is equivalent to the category of algebras over the monad

$$\mathbb{P}(-) = \bigvee_{n \geqslant 0} \mathscr{O}(n)_+ \wedge_{\Sigma_n} (-)^{\wedge n},$$

acting on S-modules. Here $\mathscr{O}(n)$ is the nth space of an E_∞-operad, that is, any operad weakly equivalent to the commutative operad such that $\mathscr{O}(n)$ is a free Σ_n-space. The structure maps for the monad are derived from the structure maps for the operad in a straightforward way [**24**, Section 11]. The category of such algebras forms a model category and any two choices of E_∞-operad yield Quillen equivalent models [**7**, Theorem 1.6]. In fact, any such category is Quillen equivalent to a category of strictly commutative S-algebras.

The monad $\mathbb{P}$ descends to a monad on the homotopy category of S-modules and the category of H_∞-ring spectra is the category of algebras for this monad. Such spectra admit all of the structure maps of E_∞-ring spectra, but these maps only satisfy the required coherence conditions up to homotopy. There is a forgetful functor from the homotopy category of E_∞-ring spectra to H_∞-ring spectra which endows each E_∞-ring spectrum with power operations as defined below; see [**12**] for more details.

As shown in [**7**, **8**], each Lubin–Tate theory E admits an essentially unique E_∞-structure realizing the $\pi_* E$-algebra $E_*(E)$. Applying standard results from [**6**], we see that after taking a commutative model for E, the category of E-modules is a topological symmetric monoidal model category with unit E and smash product $\wedge_E$. The category of commutative E-algebras is Quillen equivalent to the category of E_∞-algebras in this category, which are in turn equivalent to the category of algebras over the following monad:

$$\mathbb{P}_E(-) = \bigvee_{n \geqslant 0} \mathscr{O}(n)_+ \wedge_{\Sigma_n} (-)^{\wedge_E n}.$$

By [**6**, Chapter VIII, Lemma 2.7], $\mathbb{P}_E$ respects $K(n)$-equivalences and descends to a monad $\mathbb{P}_{\check{E}}$ on the homotopy category of $K(n)$-local E-modules. The smash product in this category is the $K(n)$-localization of the smash product of E-modules, that is, $X \wedge_{\check{E}} Y := L_{K(n)}(X \wedge_E Y)$. We will call the category of algebras over $\mathbb{P}_{\check{E}}$ the category of H_∞-$\check{E}$-algebras. Since the equivariant natural equivalences

$$(\check{E}(-))^{\wedge_{\check{E}} n} \cong \check{E}((-)^{\wedge n})$$

induce a natural equivalence

$$\mathbb{P}_{\check{E}}(\check{E}(-)) \cong \check{E}(\mathbb{P}(-)), \tag{3.1}$$

we see that if R is an H_∞-ring spectrum, then $\check{E}(R)$ is an H_∞-$\check{E}$-algebra and acceptable input for Theorem 2.1.

Given an H_∞-$\check{E}$-algebra T, a map $x\colon S^0 \to T$ of spectra (that is, a map of E-modules $E \to T$), and an $\alpha \in \check{E}_0(B\Sigma_{p+})$, we obtain an operation

$$Q_\alpha \colon \pi_0 T \longrightarrow \pi_0 T$$

defined by the following composite:

$$Q_\alpha(x) \;:\; S^0 \xrightarrow{\alpha} \check{E}(B\Sigma_{p+}) \cong \mathscr{O}(p)_+ \wedge_{\Sigma_p} E^{\wedge_{\check{E}} p} \xrightarrow{D_p(x)} \mathscr{O}(p)_+ \wedge_{\Sigma_p} T^{\wedge_{\check{E}} p} \xrightarrow{\mu_p} T.$$

Here D_p is the functor associated to the pth extended power construction in E-modules and μ_p is the H_∞-$\check{E}$ structure map on T. It is clear that, by construction, Q_α is natural in maps of H_∞-$\check{E}$-algebras.

Example 3.2. The inclusion of the base point into $B\Sigma_p$ and the E-Hurewicz homomorphism induce a map $i\colon S^0 \to \check{E}(B\Sigma_{p+})$. The associated operation is the pth power map.

Since $\check{E}_*(B\Sigma_{p+})$ is a finitely generated free E_*-module and concentrated in even degrees, we have a duality isomorphism [**30**, Theorem 3.2]:

$$\check{E}_0(B\Sigma_{p+}) \cong \mathrm{Mod}_{\pi_0 E}(E^0(B\Sigma_{p+}), \pi_0 E).$$

Therefore, we can construct operations by defining the corresponding linear maps on $E^0(B\Sigma_{p+})$.

By an elementary diagram chase, the additive operations correspond to the subgroup Γ of $\check{E}_0(B\Sigma_{p+})$ defined by the following exact sequence:

$$0 \longrightarrow \Gamma \longrightarrow \check{E}_0(B\Sigma_{p+}) \longrightarrow \prod_{0<i<p} \check{E}_0((B\Sigma_i \times B\Sigma_{p-i})_+),$$

where the right-hand map is the product of the transfer homomorphisms; cf. [**25**, Section 6]. To rephrase this in terms of cohomology, let J be the ideal of $E^0(B\Sigma_{p+})$ generated by the cohomological transfer maps. Then the additive operations correspond to those $\pi_0 E$-module maps $E^0(B\Sigma_{p+}) \to \pi_0 E$ which factor through the quotient $E^0(B\Sigma_{p+})/J$.

Proof of Theorem 2.2. By [**25**, Proposition 10.3], we have a commutative solid arrow diagram of $\pi_0 E$-algebras:

$$\begin{tikzcd} E^0(B\Sigma_{p+}) \arrow[r, "r"] \arrow[d, "\varepsilon"'] & E^0(B\Sigma_{p+})/J \arrow[d, "\phi_2"] \arrow[l, dotted, bend left, "s"] \\ \pi_0 E \arrow[r, "\phi_1"] & \pi_0 E/p. \end{tikzcd}$$

Here ε is the map induced by the inclusion of a base point into $B\Sigma_p$. It is dual to the map i from Example 3.2 and corresponds to the pth power operation $(-)^p$. The maps r and ϕ_1 are the obvious quotient maps, while ϕ_2 is the unique map making the diagram commute.

By applying [**30**, Theorem 1.1] once again, we know that $E^0(B\Sigma_{p+})/J$ is a finitely generated free $\pi_0 E$-module. So the map r admits a section s of $\pi_0 E$-modules. By the discussion above, the composite map $\varepsilon \circ s \circ r$ determines an additive operation Q. Moreover,

$$\begin{aligned} \phi_1 \circ \varepsilon \circ (\mathrm{Id} - s \circ r) &= \phi_2 \circ r \circ (\mathrm{Id} - s \circ r) \\ &= \phi_2 \circ (r - r) \\ &= 0. \end{aligned}$$

It follows that

$$\epsilon - \epsilon \circ s \circ r = p \cdot f \tag{3.3}$$

for some homomorphism $f\colon E^0(B\Sigma_p) \to E^0$. If we let θ be the operation corresponding to f, then parts (1) and (2) of Lemma 2.2 will follow from (3.3) and our definitions.

To prove Lemma 2.2.(3), suppose $x = 0 \in \check{E}_0 R$. Then the extended power $D_p(x)$ factors through

$$E\Sigma_{p+} \wedge_{\Sigma_p} *^{\wedge_E p} \simeq *$$

and $Q_\alpha(0) = 0$ for all α. $\square$

4. *Applications*

We collect some applications of our main result in this section. Although all results can be applied toward H_∞-ring spectra, we state them in terms of E_∞-ring spectra since that is the case of greatest interest.

Proposition 4.1. *Suppose that $x \in \pi_* R$ is an* odd degree *element in the homotopy groups of an E_∞-ring spectrum R which has nilpotent image in $H_*(R; \mathbb{Z}/2)$. Then x is nilpotent.*

Proof. By Theorem B, it suffices to show that x has square zero image in rational and mod-p homology for p odd. This is a consequence of the fact that the homotopy groups of a commutative ring spectrum are graded-commutative, so odd degree elements square to 2-torsion by the sign rule. □

Proposition 4.2. *Suppose that R is an E_∞-ring spectrum such that $0 = m \cdot 1 \in \pi_0 R$ for some $m \neq 0$. Then, for every prime p and non-negative integer n, $K(n)_* R = 0$.*

Proof. By Theorem 2.1 applied with $T = \check{E}_n(R)$, the image of 1 in $K(n)_* R$ is nilpotent. □

Remark 4.3. The previous proposition immediately implies the following observation of Lawson: Any finite E_∞-ring spectrum R either has type 0, that is, $H_*(R; \mathbb{Q}) \neq 0$, or is weakly contractible. Indeed, if $H_*(R; \mathbb{Q}) \cong \pi_* R \otimes \mathbb{Q} = 0$, then Proposition 4.2 implies $K(n)_* R = 0$ for every prime p and non-negative integer n. Since the two-periodic Morava K-theory spectrum splits as a wedge of suspensions of the $2p^n - 2$-periodic Morava K-theory spectrum $\overline{K}(n)$, we see that $\overline{K}(n)_*(R) = 0$.

Since R is finite, the Atiyah–Hirzebruch spectral sequence for $\overline{K}(n)_* R$ collapses to $H_*(R; \mathbb{Z}/p)[v_n^{\pm 1}]$ for $n \gg 0$. This forces $H_*(R; \mathbb{Z}/p) = 0$ and hence R_p, the p-completion of R, is weakly contractible for all primes p. This combined with the fact that $\pi_* R$ is torsion implies that R is weakly contractible.

4.1. *Applications to bordism*

Recall that $\mathrm{BGL}_1 S$ is the classifying space for stable spherical fibrations [**18**]. This is an infinite loop space and associated to any infinite loop map $G \to \mathrm{GL}_1 S$ is an E_∞ Thom spectrum MG (see [**13**, Chapter IX]). Standard examples include the J-homomorphisms from SO, $Spin$, $String$, and their complex analogs. The homotopy groups of these geometric Thom spectra correspond to the bordism rings of the corresponding categories of (compact) manifolds [**27**, Chapter IV]. In this language, if a G-manifold M represents a bordism class $[M] \in \pi_* MG$, then the nilpotence of $[M]$ is equivalent to the statement that the cartesian powers M^n bound G-manifolds for all sufficiently large n.

Now all G-manifolds are oriented if and only if MG is $\mathbb{Z}$-orientable. A choice of orientation determines a Thom isomorphism $H_*(BG; \mathbb{Z}) \cong H_*(MG; \mathbb{Z})$. Even if G does not arise from one of our geometric examples, MG is $\mathbb{Z}$-orientable if and only if $\pi_0 MG \cong \mathbb{Z}$, and otherwise one has $\pi_0 MG \cong \mathbb{Z}/2$ (see [**18**, Chapter IX, Proposition 4.5]). The latter case occurs if and only if the map $f\colon G \to \mathrm{GL}_1 S$ does not lift to $\mathrm{SL}_1 S$, the connected component of the identity in $\mathrm{GL}_1 S$.

In the oriented case, we have Thom isomorphisms $H_*(BG; k) \cong H_*(MG; k)$ for any field k. In the geometric examples, this happens when the classifying map

$$f\colon G \longrightarrow O$$

lifts to SO. In these cases, the Thom isomorphism can be used to characterize the Hurewicz image of a G-bordism class $[M]$ in $H_*(MG; k)$ in terms of the k-characteristic numbers of M (at least if G is of finite type). These characteristic numbers can be calculated by pairing the fundamental class of M with the characteristic classes of the stable normal bundle of M (see [**31**, p. 401–402]). In particular, $[M]$ has trivial image in $H_*(MG; k)$ if and only if the k-characteristic numbers of M vanish.

In many important cases, these characteristic numbers have names. For example, when M is a manifold, not necessarily oriented, it still admits a fundamental class in $k = \mathbb{Z}/2$-homology and the $\mathbb{Z}/2$-characteristic numbers of M are the *Stiefel–Whitney numbers*. These numbers determine the image of $[M]$ in $H_*(MO; \mathbb{Z}/2)$. Stably complex manifolds, that is, U-manifolds, have fundamental classes in integral homology and their images in $H_*(MU; \mathbb{Z})$ are described in terms of their *Chern numbers*. Oriented manifolds also have *Pontryagin numbers* which determine their image in $H_*(MSO; \mathbb{Z}[1/2]) \hookrightarrow H_*(MSO; \mathbb{Q})$.

The previous discussion, Theorem B, and Proposition 4.1 now imply the following proposition.

PROPOSITION 4.4. *Suppose that $G \to \mathrm{SO}$ is a map of infinite loop spaces and G is of finite type* (*for example*, *the* 2*-connected cover $G = Spin$ or the* 6*-connected cover String*). *If M is a G-manifold whose rational and $\mathbb{Z}/p$-characteristic numbers vanish for all primes p, then M^n bounds a G-manifold for all sufficiently large n.*

Moreover, if M is an odd-dimensional manifold, then it suffices that the $\mathbb{Z}/2$-characteristic numbers vanish.

For the sake of completeness, we note that the unoriented case is simpler.

PROPOSITION 4.5. *Suppose that $f\colon G \to \mathrm{GL}_1 S$ is a map of based homotopy commutative H-spaces and non-trivial on π_0. Then MG is a homotopy commutative ring spectrum such that the mod-*2 *Hurewicz homomorphism*

$$\pi_* MG \longrightarrow H_*(MG; \mathbb{Z}/2)$$

*is a split injection. In particular, an element in $\pi_* MG$ is nilpotent if and only if its mod-*2 *Hurewicz image is nilpotent.*

Proof. It follows from the results of [**13**, Chapter IX] that MG is a homotopy commutative ring spectrum and, by the discussion above, $\pi_0 MG = \mathbb{Z}/2$. So by [**32**, Theorem 1.1], MG is an $H\mathbb{Z}/2$-module and the Hurewicz homomorphism splits. □

To understand how these results fit in with classical bordism ring calculations, we offer the following two results.

THEOREM 4.6. (1) *For a String*– (*respectively*, *Spin*–)*manifold M, the following are equivalent.*

(i) *For all sufficiently large n, M^n bounds a String*– (*respectively*, *Spin*–)*manifold*;
(ii) *M bounds an oriented manifold*;
(iii) *all Stiefel–Whitney and Pontryagin numbers of M vanish.*

(2) *For a $U\langle 6\rangle$*– (*respectively*, *SU*–)*manifold M, the following are equivalent.*
  (i) *For all sufficiently large n, M^n bounds a $U\langle 6\rangle$*– (*respectively*, *SU*–)*manifold*;
  (ii) *M bounds a stably complex manifold*;
  (iii) *all Chern numbers of M vanish.*

THEOREM 4.7. *Suppose that R is one of the following Thom spectra*: *MO, MSO, $MSpin$, $MString$, MU, MSU, or $MU\langle 6\rangle$. Then the kernel of the integral Hurewicz homomorphism $\pi_* R \to H_*(R; \mathbb{Z})$ is precisely the ideal of nilpotent elements, that is, the converse of Theorem* A *also holds for these R.*

$$\begin{tikzcd}
\pi_* MString \arrow[r] \arrow[d] & H_*(MString;\mathbb{Z}) \arrow[r, hook] \arrow[d, hook] & H_*(MString;\mathbb{Z}/2) \times H_*(MString;\mathbb{Q}) \arrow[d, hook] \\
\pi_* MSpin \arrow[r] \arrow[d] & H_*(MSpin;\mathbb{Z}) \arrow[r, hook] \arrow[d, hook] & H_*(MSpin;\mathbb{Z}/2) \times H_*(MSpin;\mathbb{Q}) \arrow[d, hook] \\
\pi_* MSO \arrow[r, hook] & H_*(MSO;\mathbb{Z}) \arrow[r, hook] & H_*(MSO;\mathbb{Z}/2) \times H_*(MSO;\mathbb{Q}).
\end{tikzcd}$$

FIGURE 4.1. *The Hurewicz homomorphisms for $\pi_* MString$, $\pi_* MSpin$, and $\pi_* MSO$.*

In general, the converse of Theorem A holds when the integral homology ring is reduced, so every nilpotent element in the homotopy ring is in the kernel of the Hurewicz map. For example, the case of MO in Theorem 4.7 follows from Proposition 4.5 and the identification of $H_*(MO;\mathbb{Z}/2) \cong H_*(BO;\mathbb{Z}/2)$ as a polynomial algebra. For the remainder of the claims in Theorem 4.7, we will show (see Figures 4.1 and 4.2) that the relevant integral homology ring is a subring of a reduced ring.

In the remainder of this subsection, we will simultaneously prove Theorems 4.6 and 4.7 by analyzing the Hurewicz homomorphisms. First we note that the equivalences (ii) $\iff$ (iii) in parts (1) and (2) of Theorem 4.6 are classical.

PROPOSITION 4.8. *Suppose that M is an oriented manifold and N is a stably complex manifold. Then*

1. *M bounds an oriented manifold if and only if the Stiefel–Whitney and Pontryagin numbers of M vanish* [**20**, Corollary 1];
2. *N bounds a stably complex manifold if and only if the Chern numbers of N vanish* [**31**, Corollary 20.26].

Now we consider the assertion about nilpotence in part (1) of Theorem 4.6. For this, we consider the diagram of graded-commutative rings in Figure 4.1. The vertical maps are induced by forgetting structure, the horizontal maps on the left are the integral Hurewicz homomorphisms, and the horizontal maps on the right are the product of the mod-2 reduction maps and the rationalization maps. The diagram commutes by the naturality of the Hurewicz homomorphisms.

LEMMA 4.9. *The maps in Figure 4.1 labeled with hooked arrows are injective. In particular, since the bottom right term is reduced, so are all of the displayed subrings. As a consequence, every nilpotent element in the displayed bordism groups maps to zero under the integral Hurewicz map.*

This lemma and Theorem A immediately imply that the kernels of the integral Hurewicz maps in Figure 4.1 are precisely the nilpotent elements. It also immediately follows that these classes are precisely the elements that map to zero in $\pi_* MSO$.

Proof (*of Lemma* 4.9). Since all of these Thom spectra are orientable, the homology of these Thom spectra are isomorphic as rings to the homology of their corresponding classifying spaces. All of these spaces are of finite type, so their homology is dual to their cohomology when working with field coefficients. We will use this fact repeatedly below.

We begin with the column on the right. The induced maps in mod-2 homology are injections by Stong's analysis of the associated Serre spectral sequences [**29**]. An easy argument with the Serre spectral sequence shows that the maps in rational homology are injections as well.

$$\begin{tikzcd}
\pi_* MU\langle 6\rangle \arrow[r] \arrow[d] & H_*(MU\langle 6\rangle;\mathbb{Z}) \arrow[r, hook] \arrow[d, hook] & H_*(MU\langle 6\rangle;\mathbb{Q}) \arrow[d, hook] \\
\pi_* MSU \arrow[r] \arrow[d] & H_*(MSU;\mathbb{Z}) \arrow[r, hook] \arrow[d, hook] & H_*(MSU;\mathbb{Q}) \arrow[d, hook] \\
\pi_* MU \arrow[r, hook] & H_*(MU;\mathbb{Z}) \arrow[r, hook] & H_*(MU;\mathbb{Q}).
\end{tikzcd}$$

FIGURE 4.2. *The Hurewicz homomorphisms for $\pi_* MU\langle 6\rangle$, $\pi_* MSU$, and $\pi_* MU$.*

Since all of our Thom spectra are of finite type, the injectivity of the horizontal homomorphisms on the right is equivalent to the claim that the only torsion in the integral homology groups is simple 2-torsion. We first show that all 2-torsion is simple, which is equivalent to the claim that the mod-2 Bockstein spectral sequence collapses at E_2. In the case of MSO, the calculation of the Bockstein action in [**31**, p. 513] shows that the E_2-page of the Bockstein spectral sequence is concentrated in even degrees and hence collapses. For $MSpin$ and $MString$, this follows from the given injections into $H_*(MSO;\mathbb{Z}/2)$ and the naturality and convergence of the Bockstein spectral sequences. Altogether this implies that the maps in homology with $\mathbb{Z}_{(2)}$-coefficients are injective.

To see that there is no odd primary torsion, we recall that the composite map of spectra

$$KO \longrightarrow KU \longrightarrow KO$$

corresponding to complexifying real vector bundles and then forgetting the complex structure is an equivalence after inverting 2. It follows that there is a similar retraction between the zeroth spaces of their higher connective covers. This implies that $H_*(BO\langle k\rangle;\mathbb{Z}[1/2])$ is a retract of $H_*(BU\langle k\rangle;\mathbb{Z}[1/2])$. It is well known that latter groups are torsion-free if $k \leqslant 4$, in which case $BU\langle k\rangle$ is either BU or BSU. By [**11**, Middle of p. 3485], $H^*(BString;\mathbb{Z}/p)$ is concentrated in even degrees for p odd and $H^*(BU\langle 6\rangle;\mathbb{Z}/p)$ is concentrated in even degrees for all primes p. By applying the Bockstein spectral sequence again, we see that $H_*(BString;\mathbb{Z}[1/2])$ and $H_*(BU\langle 6\rangle;\mathbb{Z})$ are torsion-free.

The vertical homomorphisms in the middle are now injective due to the commutativity of the diagram. The bottom composite horizontal homomorphism can be calculated by calculating the Stiefel–Whitney and Pontryagin numbers of the manifold. This map is an injection by the first half of Proposition 4.8. □

This completes the proof of the first half of Theorems 4.6 and 4.7. To conclude this subsection, we address the complex analogs. The relevant diagram of homology rings in this case appears in Figure 4.2.

The bottom left map in this diagram is injective by the second half of Proposition 4.8. The maps in rational homology are injections by an easy argument with the Serre spectral sequence. The injectivity of the horizontal arrows on the right is addressed during the proof of Lemma 4.9. Now one completes the proof of Theorem 4.6 using Theorem A as before.

4.2. *Differentials in the Adams spectral sequence and non-existence of E_∞-structures*

We can use our main result to establish differentials in the Adams spectral sequence, as follows.

PROPOSITION 4.10. *Suppose that R is a bounded below E_∞-ring spectrum such that $H_*(R;\mathbb{Z}/p)$ is of finite type. Let x be an element in positive filtration in the $H\mathbb{Z}/p$-based Adams spectral sequence converging to the homotopy of the p-completion R_p of R. Then either*

(1) *x does not survive the spectral sequence*;
(2) *x detects a non-trivial element in $\pi_* R_p \otimes \mathbb{Q}$; or*
(3) *x detects a nilpotent element in $\pi_* R_p$ and as a consequence all sufficiently large powers of x do not survive the spectral sequence.*

Proof. First we note that the hypotheses on R guarantee the convergence of the Adams spectral sequence and hence any element in $\pi_* R_p$ is detected somewhere in the spectral sequence. If our specified element x fails the first two properties, then it is a permanent cycle and detects a torsion element $z \in \pi_* R_p$. Since x is in positive filtration, z has trivial mod p Hurewicz image and is therefore nilpotent by Theorem B. Thus, for every sufficiently large n, the element x^n is a permanent cycle detecting $z^n = 0$ in homotopy. Such an element cannot survive the spectral sequence. □

Theorem B also implies the non-realizability of certain E_∞-ring spectra. To precisely state this result, recall that there are non-nilpotent elements called $v_n \in \pi_{2(p^n-1)} MU$ for each positive n and prime p. There are many choices for such elements, but for our purposes we can take any such element detected in positive Adams filtration.

PROPOSITION 4.11. *Let R be a bounded below ring spectrum (up to homotopy) under MU such that, for some prime p, the image of v_n^k in $\pi_* R_p$ is non-nilpotent p-torsion for some positive integers n and k. Moreover, suppose that $H_*(R; \mathbb{Z}/p)$ is of finite type. Then R does not admit the structure of an E_∞-ring spectrum.*

Proof. If R admitted an E_∞-ring structure, then so would its p-completion R_p. So assume that R_p is E_∞. Since maps of spectra never lower Adams filtration, the image of v_n in $\pi_* R_p$ must be detected in positive Adams filtration. Since this element is torsion and non-nilpotent, its existence is a contradiction to Proposition 4.10. □

EXAMPLE 4.12. The previous result implies that many ring spectra such as $R = MU/(p^i)$, $BP/(p^i v_n^k)$, or $ku/(p^i \beta^k)$, where β is the Bott element and i and k are positive integers, do not admit E_∞-ring structures.

4.3. *Conceivable refinements of Theorem B*

In deducing nilpotence in the homotopy of ring spectra from homological assumptions, there is an obvious tension between the class of ring spectra to allow and the homology theories used to test for nilpotence. On one extreme, the nilpotence theorem works for general ring spectra but needs the more sophisticated homology theory MU to test against. On the other extreme, Theorem B applies only to H_∞-ring spectra, but only needs the most elementary homology theories to test against.

An approximately intermediate result will be derived from Theorem 4.16, which is an unpublished result of Hopkins and Mahowald. Before proving this, we will now check that homotopy colimits of connective algebras are connective.

We will use the language of ∞-categories [**14**] and the formalism of ∞-operads [**15**, Chapter 2] in this subsection. We note that, at least when there are no nullary operations, the homotopy theory of ∞-operads has been shown to be Quillen equivalent to that of dendroidal sets [**9**], while the homotopy theory of dendroidal sets is Quillen equivalent to the homotopy theory of colored simplicial operads [**4**].

In this context, the symmetric monoidal smash product functor on spectra makes the associated ∞-category Sp into a symmetric monoidal ∞-category such that the smash product commutes with (homotopy) colimits in each variable [**15**, Proposition 4.1.3.10]. Using this symmetric monoidal structure, we obtain categories $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$ of $\mathcal{O}$-algebras in Sp for every ∞-operad $\mathcal{O}$ (see [**15**, Section 2.1.3]). For example, if $0 \leqslant k < \infty$, then the little k-cubes operad defines a topological category of operators whose operadic nerve is the ∞-operad E_k (see [**15**, Definition 5.1.0.2]). This ∞-operad corresponds to a simplicial operad with one color such that the ∞-category of E_k-ring spectra is the ∞-category $\mathrm{Alg}_{E_k}(\mathrm{Sp})$ (see [**15**, Definition 8.1.0.1]).

Since the smash product of two connective spectra is again connective, the smash product functor restricts to a symmetric monoidal structure on the subcategory $\mathrm{Sp}_{\geqslant 0}$ of connective spectra [**15**, Example 2.2.1.3]. This makes $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}_{\geqslant 0})$ into a subcategory of $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$.

LEMMA 4.13. *Let $\mathcal{O}$ be an ∞-operad with one color and let $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$ be the ∞-category of $\mathcal{O}$-algebras in spectra. Then the subcategory $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}_{\geqslant 0}) \subseteq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$ spanned by $\mathcal{O}$-algebras in connective spectra is closed under colimits.*

Proof. There is a free-forgetful adjunction [**15**, Proposition 3.1.3.11]

$$(\mathbb{P}_{\mathcal{O}}, U_{\mathcal{O}})\colon \mathrm{Sp} \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}). \tag{4.14}$$

Here $U_{\mathcal{O}}$ is the functor that sends an $\mathcal{O}$-algebra to its underlying spectrum. It is conservative [**15**, Lemma 3.2.2.6] and commutes with sifted colimits [**15**, Proposition 3.2.3.1], so it is monadic [**15**, Theorem 6.2.2.5]. The functor $\mathbb{P}_{\mathcal{O}}$ is described via

$$\mathbb{P}_{\mathcal{O}}(X) \simeq \coprod_{n \geqslant 0} (\mathcal{O}(n)_+ \wedge X^{\wedge n})_{h\Sigma_n}$$

by [**15**, Proposition 3.1.3.11] since $\mathcal{O}$ has one color. Since $\mathrm{Sp}_{\geqslant 0}$ is closed under smash powers, smashing with the spaces $\mathcal{O}(n)_+$, and colimits, we see that $\mathbb{P}_{\mathcal{O}}$ takes $\mathrm{Sp}_{\geqslant 0}$ to $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}_{\geqslant 0}) \subset \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$. Consider the composition

$$T_{\mathcal{O}} = \mathbb{P}_{\mathcal{O}} \circ U_{\mathcal{O}}\colon \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}).$$

Since the adjunction (4.14) is monadic, any $X \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$ can be obtained as the geometric realization of the simplicial bar construction $B(T_{\mathcal{O}}, X)_\bullet$, where $B(T_{\mathcal{O}}, X)_\bullet$ is a simplicial object in $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$ with $B(T_{\mathcal{O}}, X)_n = T_{\mathcal{O}}^{n+1}X$ (see [**15**, Proposition 6.2.2.12]).

Let $\mathcal{I}$ be an ∞-category and let $F\colon \mathcal{I} \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}_{\geqslant 0})$ be a functor. We can use the bar construction to compute $\varinjlim_{\mathcal{I}} F$. Namely, consider the functor $\widetilde{F}_\bullet\colon \mathcal{I} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}))$ given by $B(T_{\mathcal{O}}, F)_\bullet$. Then $F \simeq |\widetilde{F}_\bullet|$ as functors $\mathcal{I} \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$. Therefore, we have

$$\varinjlim_{\mathcal{I}} F = \varinjlim_{\mathcal{I}} |\widetilde{F}_\bullet| \simeq \left| \varinjlim_{\mathcal{I}} B(T_{\mathcal{O}}, F)_\bullet \right|. \tag{4.15}$$

By definition, $\varinjlim_{\mathcal{I}} B(T_{\mathcal{O}}, F)_\bullet$ refers to the simplicial $\mathcal{O}$-algebra $A_\bullet$ with $A_n = \varinjlim_{\mathcal{I}} T_{\mathcal{O}}^{n+1}F$, where the colimit is computed in $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$. Since $\mathbb{P}_{\mathcal{O}}$ is a left adjoint, it commutes with colimits and we can also write this as $A_n \simeq \mathbb{P}_{\mathcal{O}}(\varinjlim_{\mathcal{I}} U_{\mathcal{O}} T_{\mathcal{O}}^n F)$. Since $\mathrm{Sp}_{\geqslant 0} \subset \mathrm{Sp}$ is closed under colimits, and $\mathbb{P}_{\mathcal{O}}$ preserves connectivity, it follows that each $A_n \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$ is connective. Since the forgetful functor $U_{\mathcal{O}}$ commutes with geometric realizations, (4.15) now shows that $\varinjlim_{\mathcal{I}} F$ is connective as desired. □

THEOREM 4.16 (Hopkins–Mahowald). *For every prime p, the free E_2-ring R with $p = 0$ is the Eilenberg–MacLane spectrum $H\mathbb{Z}/p$.*

The following argument has also appeared in the preprint of Antolìn-Camarena and Barthel [**1**].

Proof. First we recall how R is constructed. Let $\mathbb{P}_2(-)$ (respectively, $\mathbb{P}_{2,p}(-)$) denote the free E_2-ring functor on spectra (respectively, $H\mathbb{Z}/p$-modules). Given an E_2-ring spectrum T and a map $f\colon X \to T$ of spectra, let $\tilde{f}$ denote the adjoint map $\mathbb{P}_2 X \to T$ of E_2-rings. Now R is given as the following homotopy pushout diagram of E_2-ring spectra:

$$\begin{tikzcd} \mathbb{P}_2(S^0) \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{0}"] & S^0 \arrow[d] \\ S^0 \arrow[r] & R. \end{tikzcd}$$

First we observe that R is connective, by Lemma 4.13. Next we claim that R is p-complete. By construction, $\pi_* R$ is a graded $\mathbb{Z}/p$-algebra and hence $\pi_n R$ is a $\mathbb{Z}/p$-module for each integer n. It follows that

$$\mathrm{Hom}(\mathbb{Z}[1/p], \pi_n R) = \mathrm{Ext}(\mathbb{Z}[1/p], \pi_n R) = 0$$

for each n, so R is p-complete by [**2**, Proposition 2.6].

By construction, R admits a canonical E_2-ring map $f\colon R \to H\mathbb{Z}/p$ extending the unit map $S \to H\mathbb{Z}/p$. Now f is a map between two p-complete spectra and hence an equivalence if and only if it induces an isomorphism in mod-p homology [**2**, Theorem 3.1]. The homology of $H\mathbb{Z}/p$ is the dual Steenrod algebra $\mathcal{A}_*$. To compute the homology of the source, we smash the defining homotopy pushout diagram for R with $H\mathbb{Z}/p$ and apply π_*.

After applying the natural equivalence $H\mathbb{Z}/p \wedge \mathbb{P}_2(-) \cong \mathbb{P}_{2,p}(H\mathbb{Z}/p \wedge -)$, we obtain the following homotopy pushout diagram of E_2-rings in $H\mathbb{Z}/p$-modules:

$$\begin{tikzcd} \mathbb{P}_{2,p}(H\mathbb{Z}/p) \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{0}"] & H\mathbb{Z}/p \arrow[d] \\ H\mathbb{Z}/p \arrow[r] & H\mathbb{Z}/p \wedge R. \end{tikzcd}$$

Since $p \simeq 0$ in $H\mathbb{Z}/p$-modules, we see that

$$H\mathbb{Z}/p \wedge R \simeq \mathbb{P}_{2,p}(H\mathbb{Z}/p \wedge S^1) \cong H\mathbb{Z}/p \wedge \mathbb{P}_2(S^1).$$

Now the generalized Snaith splitting theorem shows that $\mathbb{P}_2(S^1) \simeq \Sigma^\infty_+ \Omega^2 \Sigma^2 S^1$ (see [**19**, Section 6; **13**, VII Section 5; **17**, Theorem 6.1]). Using this splitting and two applications of the Serre spectral sequence, we see that $H_*(R; \mathbb{Z}/p)$ has the same Poincare series as $\mathcal{A}_*$, so it suffices to show that $H_* f$ is surjective.

The remainder of the argument is essentially that of [**3**, Chapter III, Proposition 4.8]. Since $H_* f$ is a map of graded-commutative rings, it is surjective if it surjects onto the indecomposables of $\mathcal{A}_*$. By [**3**, Chapter III, Theorems 2.2 and 2.3], the indecomposables of $\mathcal{A}_*$ are generated by the Bockstein class in degree 1 under the action of the Dyer–Lashof operations coming from the underlying E_2-ring structure. Since $H_* f$ commutes with these operations, it suffices to show that it hits the Bockstein class in degree 1.

Since R is connective and $\pi_0 R$ is a $\mathbb{Z}/p$-module, we see that

$$\pi_0 R \cong H_0(R; \mathbb{Z}) \cong H_0(R; \mathbb{Z}/p) \cong H_0(\Omega^2 S^3; \mathbb{Z}/p) = \mathbb{Z}/p.$$

It follows that $1 \in H_0(R; \mathbb{Z}/p)$ must be the target of a Bockstein operation and similarly for $1 \in \mathcal{A}_*$. Since $H_* f$ is a map of unital algebras commuting with the Bockstein operations, we see that $H_* f$ hits the Bockstein class generating $\mathcal{A}_1$. It now follows that f is a weak equivalence. $\square$

This leads to the following nilpotence result, whose assumptions lie roughly in between the nilpotence theorem and Theorem B. It is a generalization of Nishida's argument [**21**] on the nilpotence of order p elements in the stable stems.

Proposition 4.17. *Suppose that R is an E_2-ring, p is a prime, and $x \in \pi_* R$ is* simple p-*torsion and has nilpotent image under the Hurewicz homomorphism $\pi_* R \to H_*(R; \mathbb{Z}/p)$. Then x is nilpotent.*

Proof. It suffices to show that the localization $R[x^{-1}]$ is weakly contractible. Now $R[x^{-1}]$ is an E_2-ring by Theorem A.1 and $p = 0 \in \pi_0 R[x^{-1}]$. By Theorem 4.16, $R[x^{-1}]$ is an $H\mathbb{Z}/p$-algebra, and in particular a generalized Eilenberg–MacLane spectrum. Since $H_*(R[x^{-1}]; \mathbb{Z}/p) = 0$, $R[x^{-1}]$ must be weakly contractible. □

As mentioned earlier, it is also known that any *homotopy commutative* ring spectrum with $2 = 0$ is a generalized Eilenberg–MacLane spectrum; this is the main result of [**32**]. We do not know if analogs of these results hold with respect to higher-order torsion. For instance, we do not know if the free E_2-ring with $4 = 0$ is $K(n)$-acyclic for $0 < n < \infty$. Such a claim would strengthen our main result. We do note that the free E_n-ring with $p^k = 0$ is *not* a generalized Eilenberg–MacLane spectrum for $n, k \geqslant 2$ (unpublished).

Appendix. *Localizations of E_n-ring spectra*

In this appendix, we give a proof of the following result.

Theorem A.1. *Let $n \geqslant 2$. Let R be an E_n-ring and $S \subseteq \pi_* R$ be a multiplicative subset of homogeneous elements. Then there exists a unique E_n-ring $S^{-1}R$ under R such that the map $R \to S^{-1}R$ induces on π_* the localization map $\pi_*(R) \to S^{-1}\pi_*(R)$.*

The analog of this result for E_1-rings is [**15**, Section 8.2.4]. Specifically, in there, it is shown as follows.

Theorem A.2. *Let R be an E_1-ring. Let $S \subseteq \pi_* R$ be a multiplicative subset of homogeneous elements that satisfies the left Ore condition* [**15**, Definition 8.2.4.1]. *Consider the subcategory $\mathcal{C}_S \subseteq \mathrm{Mod}(R)$ consisting of those left R-modules M such that for $s \in S$, multiplication by s induces an isomorphism on $\pi_*(M)$. Then the following conditions are satisfied.*

(1) *The subcategory $\mathcal{C}_S$ is closed under arbitrary limits and colimits.*
(2) *The inclusion $\mathcal{C}_S \subseteq \mathrm{Mod}(R)$ admits a left adjoint denoted by $M \mapsto S^{-1}M$.*
(3) *The subcategory $\mathcal{C}_S$ is generated as a localizing subcategory by $S^{-1}R$, which is a compact object of $\mathcal{C}_S$.*
(4) *For any M, the adjunction map $M \to S^{-1}M$ induces the map $\pi_*(M) \to S^{-1}\pi_*(M)$ on homotopy groups.*
(5) *Let Nil_S be the collection of R-modules M such that $S^{-1}M$ is contractible. For each $s \in S$, let $R/s \in \mathrm{Mod}(R)$ denote the cofiber of the map $\Sigma^{|s|}R \to R$ given by right multiplication by s. Then Nil_S is the stable subcategory generated under colimits by the R/s for $s \in S$.*

Theorem A.2 is proved in [**15**] by constructing the ∞-category of $S^{-1}R$-modules, and appealing to the analog of the Schwede–Shipley theorem [**28**] for compactly generated, presentable stable ∞-categories. We will explain how this proof can be modified to prove

Theorem A.1 using the E_n-version of the Schwede–Shipley theorem [**15**, Proposition 8.1.2.6], which states that a presentable, stable E_{n-1}-monoidal ∞-category where the tensor structure preserves colimits in each variable is equivalent to $\mathrm{Mod}(R)$ for an E_n-ring R if and only if the unit $\mathbf{1}$ is a compact generator. In addition, given E_n-rings R, and R', to give a morphism of E_n-rings $R \to R'$ is equivalent to giving an E_{n-1}-monoidal functor $\mathrm{Mod}(R) \to \mathrm{Mod}(R')$. Here an E_{n-1}-monoidal functor $\mathrm{Mod}(R) \to \mathrm{Mod}(R')$ induces a map between the endomorphism algebras of the unit objects in these two categories and this is the desired E_n-ring map.

Proof of Theorem A.1. Since R is an E_n-ring, the ∞-category $\mathrm{Mod}(R)$ is naturally E_{n-1}-monoidal [**15**, Proposition 6.3.5.17]. Given $X, Y \in \mathrm{Mod}(R)$, we let $X \otimes_R Y$ denote the ordered E_{n-1}-monoidal product. It is well defined up to a *connected* space of choices since the connected components of the spaces in the E_{n-1}-operad are determined by linear orderings for $n = 2$ (while there is only one component for $n \geqslant 3$). Moreover, since $n \geqslant 2$, $\pi_*(R)$ is graded-commutative and the left Ore condition on S is automatically satisfied [**15**, Remark 8.2.4.2]. Therefore, it is possible to construct a theory of S-localization: that is, one can construct subcategories $\mathcal{C}_S, \mathrm{Nil}_S \subseteq \mathrm{Mod}(R)$ as in Theorem A.2.

We claim that the E_{n-1}-monoidal structure on $\mathrm{Mod}(R)$ is compatible with S-localization. In other words, there exists an E_{n-1}-monoidal structure on the subcategory $\mathcal{C}_S \subseteq \mathrm{Mod}(R)$, such that the S-localization $M \mapsto S^{-1}M$ is an E_{n-1}-monoidal functor.

This follows from [**15**, Proposition 2.2.1.9] if we can prove that, whenever we have maps of R-modules $M_1 \to M_2$ and $N_1 \to N_2$ that induce equivalences upon S-localizations, then $M_1 \otimes_R N_1 \to M_2 \otimes_R N_2$ induces an equivalence on S-localizations, because then, it will follow inductively that the operation of S-localization respects arbitrary k-fold operations from the operad E_{n-1}.

Now, to say that $M_1 \to M_2$ (respectively, $N_1 \to N_2$) induces an equivalence on S-localizations is to say that the cofibers belong to Nil_S.

It thus suffices to show that if $M, N \in \mathrm{Mod}(R)$ and *one* of M, N belongs to Nil_S, then $M \otimes_R N$ does. Suppose for definiteness that M belongs to Nil_S. To show that $M \otimes_R N \in \mathrm{Nil}_S$, consider the collection of all $N \in \mathrm{Mod}(R)$ such that $M \otimes_R N \in \mathrm{Nil}_S$. This collection contains R, as the unit, and it is a localizing subcategory. Therefore, it is equal to $\mathrm{Mod}(R)$ and we have proved the existence of an E_{n-1}-monoidal structure on $\mathcal{C}_S$, with $S^{-1}R$ as the unit; as this is a compact generator, we have $\mathcal{C}_S \simeq \mathrm{Mod}(S^{-1}R)$ as E_{n-1}-monoidal ∞-categories. It follows from [**15**, Proposition 8.1.2.6] that we acquire a natural E_n-ring structure on $S^{-1}R$. Moreover, we obtain a natural map $R \to S^{-1}R$ of E_n-rings from the E_{n-1}-monoidal functor $\mathrm{Mod}(R) \to \mathcal{C}_S$, by looking at endomorphisms of the unit.

We now prove uniqueness. From our construction of $\mathrm{Mod}(S^{-1}R)$ as a localization of $\mathrm{Mod}(R)$, it follows that, for any E_n-ring R', to give an E_{n-1}-monoidal functor $\mathrm{Mod}(S^{-1}R) \to \mathrm{Mod}(R')$ is equivalent to giving an E_{n-1}-monoidal functor $\mathrm{Mod}(R) \to \mathrm{Mod}(R')$ that takes R/s to 0 for each $s \in S$. In particular, it follows that if R' is an E_n-ring, then the mapping space $\mathrm{Hom}_{E_n}(S^{-1}R, R')$ of E_n-ring maps can be identified with the union of components of $\mathrm{Hom}_{E_n}(R, R')$ consisting of those E_n-ring maps $\phi : R \to R'$ that take each $s \in S$ to an invertible element of $\pi_*(R')$. Therefore, if R' is an E_n-ring under R such that the map $\pi_*(R) \to \pi_*(R')$ exhibits $\pi_*(R')$ as $S^{-1}\pi_*(R)$, then it follows that we obtain a map $S^{-1}R \to R'$ of E_n-rings that is necessarily an isomorphism. $\square$

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