Summary version for the Kervaire invariant workshop,
prepared in haste: caveat emptor.
There are corrections of typos, some changes of notations,
and some additions to establish context.

EQUIVARIANT ORTHOGONAL SPECTRA

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Abstract. This is a summary of the Memoir AMS [12]. It was prepared by the second author for the October 25-29, 2010, MSRI workshop on the Kervaire invariant, with nearly all of the material extraneous to that application excised. For a compact Lie group G, we construct a symmetric monoidal model category of orthogonal G-spectra whose homotopy category is equivalent to the classical stable homotopy category of G-spectra. We ignore all extra arguments needed for the compact Lie case but not needed for the finite case, and we abbreviate or omit most proofs. In particular, we describe change of universe, change of group, fixed point, orbit, and geometric fixed point functors

Contents

Introduction. Philosophically, orthogonal spectra are intermediate between S-modules and symmetric spectra, enjoying some of the best features of both. They are defined in the same diagrammatic fashion as symmetric spectra, but with orthogonal groups rather than symmetric groups building in the symmetries required to define an associative and commutative smash product. We let G be a compact Lie group, and we understand subgroups of G to be closed.

Orthogonal spectra are more suitable than symmetric spectra for our purposes although, for finite groups, symmetric G-spectra work similarly. They share two of the essential features of the original G-spectra of [11] that facilitate equivariant generalization. First, they are defined in a coordinate-free fashion. This makes it simple and natural to build in spheres associated to representations, which play a central role in the theory. Second, their weak equivalences are just the maps that induce isomorphisms of homotopy groups. This simplifies the equivariant generalization of the relevant homotopical analysis.

We define orthogonal G-spectra and show that the category of orthogonal G-spectra is a closed symmetric monoidal category in Section 1. We show that this category has a proper Quillen model structure whose homotopy category agrees with that established in [11] in Section 2, and we describe the induced model structures on the various categories of orthogonal ring and module G-spectra. The proofs are very much like those given in the nonequivariant case in [13] and will be omitted or abbreviated. Actually, the inclusion of symmetric spectra in [13] added complications associated with subtleties about weak equivalences that need not detain us here.

We also generalize the model theoretic framework to deal with families and cofamilies of subgroups of G, and we discuss change of universe functors, change of group functors, orbit functors, and categorical and geometric fixed point functors on orthogonal G-spectra in Section 3.

Implicitly, equivariant orthogonal spectra were first applied in [15], and a global form of the definition, with orthogonal G-spectra varying functorially in G, was implicitly exploited in the proof of the completion theorem for complex cobordism of Greenlees and May [8]. A version of the norm functor was first defined there, where it was the key technical tool, but we do not consider that here.

1. Equivariant orthogonal spectra

This chapter parallels [13, Part I], and we focus on points of equivariance. It turns out that we need to distinguish carefully between topological G-categories $\mathscr{C}_G$, which are enriched over G-spaces, and their G-fixed topological categories $G\mathscr{C}$, which are enriched over spaces. After explaining this in §1, we define orthogonal G-spectra in §2, discuss their smash product in §3, and reinterpret the definition in terms of diagram spaces in §4. Recall that G is a compact Lie group.

1.1. **Preliminaries on equivariant categories.** Let $\mathscr{T}$ denote the category of based spaces, where spaces are understood to be compactly generated (= weak Hausdorff k-spaces). Let $G\mathscr{T}$ denote the category of based G-spaces and based G-maps. Then $G\mathscr{T}$ is complete and cocomplete, and it is a closed symmetric monoidal category under its smash product and function G-space functors. For based G-spaces A and B, we write $F(A,B)$ for the function G-space of all continuous maps $A \longrightarrow B$, with G acting by conjugation. Thus

$$G\mathscr{T}(A,B) = F(A,B)^G.$$

That is, a G-map $A \longrightarrow B$ is a fixed point of $F(A,B)$.

It is useful to think of $G\mathscr{T}$ in a different fashion. Let $\mathscr{T}_G$ be the category of based G-spaces (with specified action of G) and non-equivariant maps, which we henceforward call "arrows" to avoid confusion between maps and G-maps. Thus

$$\mathscr{T}_G(A,B) = F(A,B).$$

Then $\mathscr{T}_G$ is enriched over $G\mathscr{T}$: its morphism spaces are G-spaces, and composition is given by G-maps. The objects of $G\mathscr{T}$ and $\mathscr{T}_G$ are the same. If we think of G as acting trivially on the collection of objects (after all, $gA = A$ for all $g \in G$), then we may think of $G\mathscr{T}$ as the G-fixed point category $(\mathscr{T}_G)^G$.

Observe that $\mathscr{T}_G$ is also closed symmetric monoidal under the smash product and function G-space functors, with S^0 as unit. Of course, limits and colimits of diagrams of G-spaces (taken in $\mathscr{T}$) only inherit sensible G-actions when the maps in the diagrams are G-maps, so that we are working in $G\mathscr{T}$.

Many of our equivariant categories will come in pairs like this: we will have a category $\mathscr{C}_G$ consisting of G-objects and nonequivariant "arrows", and a category $G\mathscr{C}$ with the same objects and the G-maps between them. We can think of $G\mathscr{C}$ as $(\mathscr{C}_G)^G$, although the notation would be inconvenient. Formally, $\mathscr{C}_G$ will be enriched over $G\mathscr{T}$, so that its hom sets $\mathscr{C}_G(C,D)$ are based G-spaces and composition is given by continuous G-maps. We call such a category a *topological G-category.* As in [13], when the morphism spaces of $\mathscr{C}_G$ are given without basepoints, we implicitly give them disjoint G-fixed basepoints. We emphasize that it is essential to think in terms of such topological G-categories $\mathscr{C}_G$ even when the categories of ultimate interest are the associated categories $G\mathscr{C}$ of G-objects and G-maps between them. Note that, when constructing model structures, we must work in $G\mathscr{C}$ in order to have limits and colimits.

A *continuous G-functor* $X : \mathscr{C}_G \longrightarrow \mathscr{D}_G$ between topological G-categories is a functor X such that

$$X : \mathscr{C}_G(C,D) \longrightarrow \mathscr{D}_G(X(C), X(D))$$

is a map of G-spaces for all pairs of objects of $\mathscr{C}_G$. In terms of elementwise actions, this means that $gX(f)g^{-1} = X(gfg^{-1})$. It follows that X takes G-maps to G-maps. From now on, all functors defined on topological categories are assumed to be continuous.

A *natural G-map* $\alpha : X \longrightarrow Y$ between G-functors $\mathscr{C}_G \longrightarrow \mathscr{D}_G$ consists of G-maps $\alpha : X(C) \longrightarrow Y(D)$ such that the evident naturality diagrams

$$\begin{array}{ccc} X(C) & \longrightarrow & X(D) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Y(C) & \longrightarrow & Y(D) \end{array}$$

commute in $\mathscr{D}_G$ for all arrows (and *not* just all G-maps) $C \longrightarrow D$.

For background, we give the definitions of the categories $\mathscr{P}_G$ and $G\mathscr{P}$ of G-prespectra and their full subcategories $\mathscr{S}_G$ and $G\mathscr{S}$ of G-spectra. See [11] or [16] for more details. In fact, we have such categories for any G-universe U, and we write $\mathscr{P}_G^U$, etc, when necessary for clarity.

Definition 1.1. A *G-universe* U is a sum of countably many copies of each real G-inner product space in some set of irreducible representations of G that includes the trivial representation; U is *complete* if it contains all irreducible representations;

U is *trivial* if it contains only trivial representations. An *indexing G-space* in U is a finite dimensional sub G-inner product space of U. When $V \subset W$, we write $W - V$ for the orthogonal complement of V in W. Define $\mathscr{V}(U)$ to be the collection of all real G-inner product spaces that are isomorphic to indexing G-spaces in U.

Write S^V for the one-point compactification of V, and write $\Sigma^V A = A \wedge S^V$ and $\Omega^V A = F(S^V, A)$ for the resulting generalized loop and suspension functors.

Definition 1.2. A G-prespectrum X consists of based G-spaces $X(V)$ for indexing G-spaces $V \subset U$ and based G-maps $\sigma : \Sigma^{W-V} X(V) \longrightarrow X(W)$ for $V \subset W$; here σ is the identity if $V = W$, and the evident transitivity diagram must commute when $V \subset W \subset Z$. An arrow $f : X \longrightarrow Y$ of prespectra consists of based maps $f(V) : X(V) \longrightarrow Y(V)$ that commute with the structure maps σ; f is a G-map if the $f(V)$ are G-maps. A G-spectrum is a G-prespectrum whose adjoint structure G-maps $\tilde{\sigma} : X(V) \longrightarrow \Omega^{W-V} X(W)$ are homeomorphisms of G-spaces.

When U is the trivial universe, $G\mathscr{S}$ is the category of *naive* G-spectra, or spectra with G-actions. When U is a complete universe, $G\mathscr{S}$ is the category of *genuine* G-spectra, and the adjective is omitted: these G-spectra are the objects of the equivariant stable homotopy category of [11].

Remark 1.3. From the point of view of enriched category theory, we have the category $G\mathscr{C}$ of G-objects and G-maps, which we view as enriched over $\mathscr{T}$: all of our categories are topological, meaning that the categorical hom sets are based topological spaces and composition is given by continuous based maps. We can also view the category $G\mathscr{C}$ as enriched over the category of G-spaces, with the enrichment given by the "enriched hom" G-spaces $\mathscr{C}_G(C, D)$. From that point of view the "category" $\mathscr{C}_G$ is a red herring, an artifact of our special situation rather than something intrinsically relevant to the mathematics. Its "arrows", the points of the $\mathscr{C}_G(C, D)$, are special to the concrete nature of our equivariant situation and should not be thought of as morphisms in a category of their own. Our G-functors and natural G-maps are just examples of the category theorists' $G\mathscr{T}$-enriched functors and $G\mathscr{T}$-enriched natural transformations. The naturality may be expressed conceptually by the commutative diagram of G-spaces

$$\begin{array}{ccc} \mathscr{C}_G(C,D) & \xrightarrow{X} & \mathscr{D}_G(X(C),X(D)) \\ {\scriptstyle Y}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ \mathscr{D}_G(Y(C),Y(D)) & \xrightarrow[\alpha^*]{} & \mathscr{D}_G(X(C),Y(D)), \end{array}$$

with no mention of arrows. For accessibility and to parallel more closely the nonequivariant theory, we have chosen to avoid the language of enriched category theory and to treat $\mathscr{C}_G$ concretely. Our orthogonal G-spectra are G-functors, thought of as objects in a category of diagrams. Their domain categories are of the form $\mathscr{C}_G$, and not $G\mathscr{C}$, with arrows as morphisms. We find it generally more convenient to talk about orthogonal G-spectra concretely as ordinary functors with additional structure rather than as enriched functors in the category theorists' preferred language. The reader familiar with this language may view the use of $\mathscr{C}_G$ as just a notational device to record the use of the $G\mathscr{T}$ enrichment of $G\mathscr{C}$.

1.2. **The definition of orthogonal G-spectra.** As with G-spectra, we have several kinds of orthogonal G-spectra, depending on a choice of a set of irreducible representations of G. The reader is warned that, as explained in [13, 7.1], non-trivial orthogonal G-spectra are never G-spectra in the sense of the Definition 1.2.

Definition 1.4. Let $\mathscr{V} = \mathscr{V}(U)$ for some universe U. Define $\mathscr{I}_G^{\mathscr{V}}$ to be the (unbased) topological G-category whose objects are those of $\mathscr{V}$ and whose arrows are the linear isometric isomorphisms, with G acting by conjugation on the space $\mathscr{I}_G^{\mathscr{V}}(V,W)$ of arrows $V \longrightarrow W$.

Define a canonical G-functor $S_G^{\mathscr{V}} : \mathscr{I}_G^{\mathscr{V}} \longrightarrow \mathscr{T}_G$ by sending V to S^V. Clearly $\mathscr{I}_G^{\mathscr{V}}$ is a symmetric monoidal category under direct sums of G-inner product spaces, and the functor $S_G^{\mathscr{V}}$ is strong symmetric monoidal.

Variant 1.5. We could relax the conditions on $\mathscr{V}$ by allowing any cofinal subcollection $\mathscr{W}$ of $\mathscr{V}$ that is closed under finite direct sums. Here "cofinal" means that, up to G-isomorphism, every V in $\mathscr{V}$ is contained in some W in $\mathscr{W}$. We shall need the extra generality when we consider change of groups.

We usually abbreviate $\mathscr{I}_G = \mathscr{I}_G^{\mathscr{V}}$ and $S_G = S_G^{\mathscr{V}}$. The case of central interest is $\mathscr{V} = \mathscr{A}\ell\ell$, the collection of all finite dimensional real G-inner product spaces, but we shall work with the general case until we specify otherwise. From here, the basic categorical definitions and constructions of [13] go through without essential change. The only new point to keep track of is which arrows are G-maps and which are not. We give a quick summary. We shall not spell out diagrams, referring to [13] instead. We choose and fix a skeleton $sk\mathscr{I}_G$ of $\mathscr{I}_G$.

Definition 1.6. An *$\mathscr{I}_G$-space* is a (continuous) G-functor $X : \mathscr{I}_G \longrightarrow \mathscr{T}_G$. Let $\mathscr{I}_G\mathscr{T}$ be the category whose objects are the $\mathscr{I}_G$-spaces X and whose arrows are the natural transformations $X \longrightarrow Y$. Let $G\mathscr{I}\mathscr{T}$ be the category of $\mathscr{I}_G$-spaces and natural G-maps, so that

$$G\mathscr{I}\mathscr{T}(X,Y) = \mathscr{I}_G\mathscr{T}(X,Y)^G.$$

It is essential to keep in mind the distinction between arrows and G-maps of $\mathscr{I}_G$-spaces. We are interested primarily in the G-maps.

Definition 1.7. For $\mathscr{I}_G$-spaces X and Y, define the *"external" smash product* $X \bar{\wedge} Y$ by

$$X \bar{\wedge} Y = \wedge \circ (X \times Y) : \mathscr{I}_G \times \mathscr{I}_G \longrightarrow \mathscr{T}_G;$$

thus $(X \bar{\wedge} Y)(V,W) = X(V) \wedge Y(W)$. For an $\mathscr{I}_G$-space Y and an $(\mathscr{I}_G \times \mathscr{I}_G)$-space Z, define the *external function $\mathscr{I}_G$-space* $\bar{F}(Y,Z)$ by

$$\bar{F}(Y,Z)(V) = \mathscr{I}_G\mathscr{T}(Y, Z\langle V\rangle),$$

where $Z\langle V\rangle(W) = Z(V,W)$.

Remark 1.8. The definition generalizes to give the external smash product functor

$$\mathscr{I}_G^{\mathscr{V}}\mathscr{T} \times \mathscr{I}_G^{\mathscr{V}'}\mathscr{T} \longrightarrow (\mathscr{I}_G^{\mathscr{V}} \times \mathscr{I}_G^{\mathscr{V}'})\mathscr{T}.$$

Definition 1.9. An *orthogonal G-spectrum* is an $\mathscr{I}_G$-space $X : \mathscr{I}_G \longrightarrow \mathscr{T}_G$ together with a natural structure G-map $\sigma : X \bar{\wedge} S_G \longrightarrow X \circ \oplus$ such that the evident unit and associativity diagrams commute [13, §§1,8]. Let $\mathscr{I}_G\mathscr{S}$ denote the topological G-category of orthogonal G-spectra and arrows $f : X \longrightarrow Y$ that commute with the structure G-maps. Explicitly, the following diagrams must commute, where the σ are G-maps but the f are non-equivariant in general:

$$\begin{array}{ccc} X(V)\wedge S^W & \xrightarrow{\sigma} & X(V\oplus W) \\ {\scriptstyle f\wedge \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ Y(V)\wedge S^W & \xrightarrow[\sigma]{} & Y(V\oplus W) \end{array}$$

If these diagrams commute, then so do the diagrams obtained by replacing f by gf for $g\in G$, so that $\mathscr{I}_G\mathscr{S}(X,Y)$ is indeed a sub G-space of $\mathscr{I}_G\mathscr{T}(X,Y)$. Let $G\mathscr{I}\mathscr{S}$ denote the category of orthogonal G-spectra and the G-maps between them, so that

$$G\mathscr{I}\mathscr{S}(X,Y) = \mathscr{I}_G\mathscr{S}(X,Y)^G.$$

Orthogonal G-spectra are G-prespectra by neglect of structure.

Definition 1.10. Let $\mathscr{V} = \mathscr{V}(U)$. Define a discrete subcategory (identity morphisms only) of $\mathscr{I}_G$ whose objects are the indexing G-spaces in U. By restricting functors $\mathscr{I}_G \longrightarrow \mathscr{T}_G$ to this subcategory and using structure maps for $V\oplus(W-V)$, where $V\subset W$, we obtain forgetful functors

$$\mathbb{U} : \mathscr{I}_G\mathscr{S} \longrightarrow \mathscr{P}_G \quad \text{and} \quad \mathbb{U} : G\mathscr{I}\mathscr{S} \longrightarrow G\mathscr{P}.$$

Working with orthogonal G-spectra, we have an equivariant notion of a functor with smash product (FSP). It was used when defining the norm functor in [8].

Definition 1.11. An $\mathscr{I}_G$-FSP is an $\mathscr{I}_G$-space X with a unit G-map $\eta : S \longrightarrow X$ and a natural product G-map $\mu : X \,\bar{\wedge}\, X \longrightarrow X\circ\oplus$ of functors $\mathscr{I}_G\times\mathscr{I}_G \longrightarrow \mathscr{T}_G$ such that the evident unit, associativity, and centrality of unit diagrams commute [13, 22.3]. An $\mathscr{I}_G$-FSP is commutative if the evident commutativity diagram also commutes.

We have the topological G-category of $\mathscr{I}_G$-FSP's and its G-fixed point category of G-maps of $\mathscr{I}_G$-FSP's. An $\mathscr{I}_G$-FSP is an orthogonal G-spectrum with additional structure.

Lemma 1.12. *An $\mathscr{I}_G$-FSP has an underlying orthogonal G-spectrum with structure G-map*

$$\sigma = \mu\circ(\mathrm{id}\,\bar{\wedge}\,\eta) : X \,\bar{\wedge}\, S \longrightarrow X\circ\oplus.$$

We emphasize that all structure maps (σ, η, μ) in the definitions above must be G-maps, while their naturality requires their commutation with arrows.

1.3. **The smash product of orthogonal G-spectra.** Just as nonequivariantly, we can reinterpret FSP's in terms of a point-set level internal smash product on the category of orthogonal G-spectra that is associative, commutative, and unital up to coherent natural isomorphism.

Theorem 1.13. *The category $\mathscr{I}_G\mathscr{S}$ of orthogonal G-spectra has a smash product $\wedge_{S_G}$ and function spectrum functor F_{S_G} under which it is a closed symmetric monoidal category with unit S_G.*

Passing to G-fixed points on morphism spaces, we obtain the following corollary.

Corollary 1.14. *The category $G\mathscr{I}\mathscr{S}$ is also closed symmetric monoidal under $\wedge_{S_G}$ and F_{S_G}.*

After this section, we will abbreviate $\wedge_{S_G}$ to $\wedge$ and F_{S_G} to F, but the more cumbersome notations clarify the definitions. Monoids and commutative monoids are defined in any symmetric monoidal category, such as $\mathscr{I}_G\mathscr{S}$ and $G\mathscr{I}\mathscr{S}$. The external notion of an $\mathscr{I}_G$-FSP translates to the internal notion of a monoid in $\mathscr{I}_G\mathscr{S}$.

Theorem 1.15. *The categories of $\mathscr{I}_G$-FSP's and of commutative $\mathscr{I}_G$-FSP's are isomorphic to the categories of monoids in $\mathscr{I}_G\mathscr{S}$ and of commutative monoids in $\mathscr{I}_G\mathscr{S}$.*

We are only interested in G-maps between these, and we adopt a more familiar language.

Definition 1.16. A (commutative) orthogonal ring G-spectrum is a (commutative) monoid in $G\mathscr{I}\mathscr{S}$.

Theorem 1.15 asserts that (commutative) orthogonal ring G-spectra are the same as (commutative) $\mathscr{I}_G$-FSP's. That is, they are the same structures, but specified in terms of the internal rather than the external smash product.

We outline the proof of Theorem 1.13, which is the same as in [13]. We first construct a smash product $\wedge$ on the category of $\mathscr{I}_G$-spaces [13, 21.4]. This internalization of the external smash product is given by left Kan extension and is characterized by the adjunction homeomorphism of based G-spaces

$$\mathscr{I}_G\mathscr{T}(X\wedge Y, Z)\cong(\mathscr{I}_G\times\mathscr{I}_G)\mathscr{T}(X\,\bar{\wedge}\,Y, Z\circ\oplus). \tag{1.17}$$

An explicit description of $\wedge$ is given in [13, 21.4]. There is one key subtle point. The Kan extension is a kind of colimit, and our G-categories of diagrams do not admit colimits in general. However, the assumption that the maps

$$X:\mathscr{I}_G(V,W)\longrightarrow\mathscr{T}_G(X(V),X(W))$$

given by an $\mathscr{I}_G$-space X must be G-maps ensures that the equivalence relation that defines the Kan extension is G-invariant, producing a well-defined $\mathscr{I}_G$-space $X\wedge Y:\mathscr{I}_G\longrightarrow\mathscr{T}_G$ from $\mathscr{I}_G$-spaces X and Y.

There is an internal function $\mathscr{I}_G$-space functor F constructed from $\bar{F}$ [13, 21.6].

Proposition 1.18. *The category of $\mathscr{I}_G$-spaces is closed symmetric monoidal under $\wedge$ and F. Its unit object is the functor $\mathscr{I}_G\longrightarrow\mathscr{T}_G$ that sends 0 to S^0 and sends $V\neq 0$ to a point.*

We can reinterpret orthogonal G-spectra in terms of the internal smash product.

Proposition 1.19. *The $\mathscr{I}_G$-space S_G is a commutative monoid in $\mathscr{I}_G\mathscr{T}$, and the category of orthogonal G-spectra is isomorphic to the category of S_G-modules.*

From here, we imitate algebra, thinking of $\wedge$ and F as analogues of $\otimes$ and Hom.

Definition 1.20. For orthogonal G-spectra X and Y, thought of as right and left S_G-modules, define $X\wedge_{S_G}Y$ to be the coequalizer in the category of $\mathscr{I}_G$-spaces (constructed spacewise) displayed in the diagram

$$X\wedge S_G\wedge Y \underset{\mathrm{id}\wedge\mu'}{\overset{\mu\wedge\mathrm{id}}{\rightrightarrows}} X\wedge Y\longrightarrow X\wedge_{S_G}Y,$$

where μ and μ' are the given actions of S_G on X and Y. Then $X\wedge_{S_G}Y$ inherits an orthogonal G-spectrum structure from the orthogonal G-spectrum structure on

X or, equivalently, Y. The function orthogonal G-spectrum $F_{S_G}(X,Y)$ is defined dually in terms of a suitable equalizer [13, §22]

$$F_{S_G}(Y,Z) \longrightarrow F(Y,Z) \rightrightarrows F(Y \wedge S_G, Z).$$

Theorem 1.13 follows from the definitions and the universal property (1.17).

1.4. **A description of orthogonal G-spectra as diagram G-spaces.** As in [13, 2.1], there is a category $\mathscr{J}_G = \mathscr{J}_G^{\mathscr{V}}$ constructed from $\mathscr{I}_G$ and S_G such that if we define $\mathscr{J}_G$-spaces exactly as in Definition 1.6, then a *$\mathscr{J}_G$-space* is the same structure as an *orthogonal G-spectrum*. This reduces the study of orthogonal G-spectra to a special case of the conceptually simpler study of diagram G-spaces. Rather than repeat the cited formal definition, we give a more concrete alternative description of $\mathscr{J}_G$ in terms of Thom complexes.

Definition 1.21. We define the topological G-category $\mathscr{J}_G^{\mathscr{V}}$. The objects of $\mathscr{J}_G^{\mathscr{V}}$ are the same as the objects of $\mathscr{I}_G^{\mathscr{V}}$. For objects V and V', let $\mathscr{I}(V,V')$ be the (possibly empty) G-space of linear isometries from V to V'; G acts by conjugation. Of course, a linear isometry is necessarily a monomorphism, but, in contrast to our definition of the category $\mathscr{I}_G$, we no longer restrict attention to linear isometric isomorphisms. Let $E(V,V')$ denote the subbundle of the product G-bundle $\mathscr{I}(V,V')\times V'$ consisting of the points (f,x) such that $x \in V' - f(V)$. The G-space $\mathscr{J}_G^{\mathscr{V}}(V,V')$ of arrows $V \longrightarrow V'$ in $\mathscr{J}_G^{\mathscr{V}}$ is the Thom G-space of $E(V,V')$; it is obtained from the fiberwise one-point compactification of $E(V,V')$ by identifying the points at infinity, and it is interpreted to be a point if $\mathscr{I}(V,V')$ is empty. Define composition

$$\circ : \mathscr{J}_G^{\mathscr{V}}(V',V'') \wedge \mathscr{J}_G^{\mathscr{V}}(V,V') \longrightarrow \mathscr{J}_G^{\mathscr{V}}(V,V'') \tag{1.22}$$

by $(g,y)\circ(f,x) = (g\circ f, g(x)+y)$. The points $(\mathrm{id}_V, 0)$ give identity arrows. Observe that $\mathscr{J}_G^{\mathscr{V}}$ is symmetric monoidal under the operation $\oplus$ specified by $V \oplus V'$ on objects and

$$(f,x)\oplus(f',x') = (f\oplus f', x+x')$$

on arrows. Let $G\mathscr{J}^{\mathscr{V}}$ be the G-fixed category with the same objects, so that

$$G\mathscr{J}^{\mathscr{V}}(V,W) = \mathscr{J}_G^{\mathscr{V}}(V,W)^G.$$

We usually abbreviate $\mathscr{J}_G = \mathscr{J}_G^{\mathscr{V}}$. If $\dim V = \dim V'$, then a linear isometry $V \longrightarrow V'$ is an isomorphism and $\mathscr{J}_G(V,V') = \mathscr{I}_G(V,V')_+$. This embeds $\mathscr{I}_G$ as a sub symmetric monoidal category of $\mathscr{J}_G$. If $V \subset V'$, then

$$\mathscr{J}_G(V,V') \cong O(V')_+ \wedge_{O(V'-V)} S^{V'-V}.$$

In particular, the functor $\mathscr{J}_G(0,-) : \mathscr{I}_G \longrightarrow \mathscr{T}_G$ coincides with S_G. The category of $\mathscr{J}_G$-spaces is symmetric monoidal, as in Proposition 1.18 but with unit S_G, and we have the following result.

Theorem 1.23. *The symmetric monoidal category of orthogonal G-spectra is isomorphic to the symmetric monoidal category of $\mathscr{J}_G$-spaces.*

Using this reinterpretation, we see immediately that the category $G\mathscr{I}\mathscr{S}$ is complete and cocomplete, with limits and colimits constructed levelwise. The category $\mathscr{I}_G\mathscr{S}$ is tensored and cotensored over the category $\mathscr{T}_G$ of based G-spaces. For an orthogonal G-spectrum X and a based G-space A, the tensor $X \wedge A$ is given by the levelwise smash product, $(X \wedge A)(V) = X(V) \wedge A$, and the cotensor $F(A,X)$ is given similarly by the levelwise function space. We have both

$$\mathscr{I}_G\mathscr{S}(X \wedge A, Y) \cong \mathscr{T}_G(A, \mathscr{I}_G\mathscr{S}(X,Y)) \cong \mathscr{I}_G\mathscr{S}(X, F(A,Y)) \tag{1.24}$$

and, by passage to fixed points,

$$G\mathscr{I}\mathscr{S}(X \wedge A, Y) \cong G\mathscr{T}(A, \mathscr{I}_G\mathscr{S}(X,Y)) \cong G\mathscr{I}\mathscr{S}(X, F(A,Y)). \tag{1.25}$$

From the enriched category point of view of Remark 1.3, these adjunctions give that $G\mathscr{I}\mathscr{S}$ is tensored and cotensored over $G\mathscr{T}$. Here the enriched category point of view is clearly the right one to take. When we specialize these adjunctions to spaces A with trivial G-action, we may replace $\mathscr{I}_G\mathscr{S}(X,Y)$ with the categorical hom space $G\mathscr{I}\mathscr{S}(X,Y)$ and the enrichment over $\mathscr{T}$ takes a more elementary form. We define homotopies between maps of orthogonal G-spectra by use of the cylinders $X \wedge I_+$, and similarly for G-homotopies between G-maps.

We also use $\mathscr{J}_G$ to define represented orthogonal G-spectra that give rise to left adjoints to evaluation functors, as in [13, §3].

Definition 1.26. For an object V of $\mathscr{I}_G$, define the orthogonal G-spectrum V^* represented by V by $V^*(W) = \mathscr{J}_G(V,W)$. In particular, $0^* = S_G$. Define the *shift desuspension spectrum functors* $F_V : \mathscr{T}_G \longrightarrow \mathscr{I}_G\mathscr{S}$ and the *evaluation functors* $Ev_V : \mathscr{I}_G\mathscr{S} \longrightarrow \mathscr{T}_G$ by $F_V A = V^* \wedge A$ and $Ev_V X = X(V)$. Then F_V and Ev_V are left and right adjoint:

$$\mathscr{I}_G\mathscr{S}(F_V A, X) \cong \mathscr{T}_G(A, Ev_V X).$$

To mesh with notations used elsewhere, especially [11], we give these functors alternative names.

Notations 1.27. Let $\Sigma^\infty = F_0$ and $\Omega^\infty = Ev_0$. These are the suspension orthogonal G-spectrum and zeroth space functors. Note that $\Sigma^\infty A = S_G \wedge A$. Similarly, let $\Sigma^\infty_V = F_V$ and $\Omega^\infty_V = Ev_V$; we let $S^{-V} = \Sigma^\infty_V S^0$ and call it the *canonical* $(-V)$*-sphere*.

As in [13, 1.8], we have the following commutation with smash products.

Lemma 1.28. *There is a natural isomorphism*

$$F_V A \wedge F_W B \cong F_{V\oplus W}(A \wedge B).$$

As in [13, 1.6], but with an evident tensor product of functors notation, we have the following description of general orthogonal G-spectra in terms of represented ones. Observe that V^* varies contravariantly in V, so that we have a contravariant functor $\mathbb{D} : \mathscr{J}_G \longrightarrow \mathscr{I}_G\mathscr{S}$ specified by $\mathbb{D}V = V^*$.

Lemma 1.29. [1] *The evaluation maps $V^* \wedge X(V) \longrightarrow X$ of $\mathscr{I}_G$-spectra X, thought of as $\mathscr{J}_G$-spaces, induce a natural isomorphism*

$$\mathbb{D} \otimes_{\mathscr{J}_G} X = \int^{V \in sk\mathscr{I}_G} V^* \wedge X(V) \longrightarrow X.$$

The definitions and results of this section have analogues for prespectra. Recall Definition 1.10.

[1] In [9], it is said that X is the homotopy colimit of the spectra $S^{-V} \wedge X(V)$.

Definition 1.30. Let $\mathscr{V} = \mathscr{V}(U)$. We have a subcategory $\mathscr{K}_G = \mathscr{K}_G^{\mathscr{V}}$ of $\mathscr{J}_G$ such that a $\mathscr{K}_G$-space is the same thing as a G-prespectrum. The objects of $\mathscr{K}_G$ are the indexing G-spaces in U; the G-space $\mathscr{K}_G(V, V')$ of arrows is $S^{V'-V}$ if $V \subset V'$ and a point otherwise. The forgetful G-functor $\mathbb{U} : \mathscr{I}_G\mathscr{S} \longrightarrow \mathscr{P}_G$ has a left adjoint prolongation functor $\mathbb{P}$. It is the tensor product $\mathbb{P}X = \mathbb{D} \circ \iota \otimes_{\mathscr{K}_G} X$, where $\iota : \mathscr{K}_G \longrightarrow \mathscr{J}_G$ is the inclusion. See also [13, §3].

2. Model categories of orthogonal G-spectra

We explain the model structures on the category of orthogonal G-spectra and on its various categories of rings and modules. The material here is parallel to the material of [13, §§5-12]. We focus on points of equivariance. One new equivariant feature is the notion of a G-topological model category, which is an equivariant analogue of the classical notion of a topological (or simplicial) model category. To make sense of this, we must take into account the dichotomy between $\mathscr{C}_G$ and $G\mathscr{C}$: only $G\mathscr{C}$ can have a model structure, but use of $\mathscr{C}_G$ is essential to encode the G-topological structure, which is used to prove the model axioms.

2.1. The model structure on G-spaces.

We take for granted the generalities on nonequivariant topological model categories explained in [13, §5]. In particular we have the notion of a compactly generated model category, for which the small object argument for verifying the factorization axioms requires only sequential colimits. All of our examples of model categories will be of this form. However there are a few places where equivariance plays a role. We discuss these and then recall the model structure we need on the category $G\mathscr{T}$ of based G-spaces.

We begin with a topological G-category $\mathscr{C}_G$ and its G-fixed category $G\mathscr{C}$ of G-maps. We assume that $G\mathscr{C}$ is complete and cocomplete and that $\mathscr{C}_G$ is tensored and cotensored over $G\mathscr{T}$, so that (1.24) and (1.25) hold with $\mathscr{I}_G\mathscr{S}$ and $G\mathscr{I}$ replaced by $\mathscr{C}_G$ and $G\mathscr{C}$. The discussion in [13, §5] applies to $G\mathscr{C}$. One place where equivariance is relevant is in the Cofibration Hypothesis, [13, 5.3]. That uses the concept of an h-cofibration in $G\mathscr{C}$, namely a map that satisfies the homotopy extension property (HEP) in $G\mathscr{C}$. Since the maps in $G\mathscr{C}$ are G-maps, the HEP is understood to be equivariant. That is, h-cofibrations in $G\mathscr{C}$ satisfy the G-HEP. As in [13], we write q-cofibration and q-fibration for model cofibrations and cofibrations, but we write cofibrant and fibrant rather than q-cofibrant and q-fibrant.

A more substantial point of equivariance concerns the notion of a topological model category. As defined in [13, 5.12], that notion remembers only that $G\mathscr{C}$ is tensored and cotensored over $\mathscr{T}$, which is insufficient for our applications. We shall return to this point and define the notion of a "G-topological model category" after giving the model structure on $G\mathscr{T}$.

Definition 2.1. Let I be the set of cell h-cofibrations

$$i : (G/H \times S^{n-1})_+ \longrightarrow (G/H \times D^n)_+$$

in $G\mathscr{T}$, where $n \geq 0$ (S^{-1} being empty) and H runs through the (closed) subgroups of G. Let J be the set of h-cofibrations

$$i_0 : (G/H \times D^n)_+ \longrightarrow (G/H \times D^n \times I)_+$$

and observe that each such map is the inclusion of a G-deformation retract.

Recall that, for unbased spaces A and B, $(A\times B)_+ \cong A_+ \wedge B_+$. Recall too that, for a based H-space A and a based G-space B,

$$G\mathscr{T}(G_+ \wedge_H A, B) \cong H\mathscr{T}(A, B). \tag{2.2}$$

If A is a G-space, then we have a natural homeomorphism of G-spaces

$$G_+ \wedge_H A \cong (G/H)_+ \wedge A, \tag{2.3}$$

where G acts diagonally on the right; it sends the class of $g \wedge a$ to $gH \wedge ga$. Also, for a based space A regarded as a G-trivial G-space,

$$G\mathscr{T}(A, B) \cong \mathscr{T}(A, B^G) \tag{2.4}$$

and therefore

$$G\mathscr{T}((G/H)_+ \wedge A, B) \cong \mathscr{T}(A, B^H). \tag{2.5}$$

As a right adjoint, the G-fixed point functor preserves limits. It also preserves some, but not all, colimits.

Lemma 2.6. *The functor $(-)^G$ on based G-spaces preserves pushouts of diagrams one leg of which is a closed inclusion and colimits of sequences of inclusions. For a based space A and a based G-space B, $F(A,B)^G \cong F(A, B^G)$. For based G-spaces A and B, $(A \wedge B)^G \cong A^G \wedge B^G$.*

Definition 2.7. A map $f : A \longrightarrow B$ of G-spaces is a weak equivalence or Serre fibration if each $f^H : A^H \longrightarrow B^H$ is a weak equivalence or Serre fibration; by (2.5), f is a Serre fibration if and only if it satisfies the RLP (right lifting property) with respect to the maps in J. Note that a relative G-cell complex is a relative I-cell complex as defined in [13, 5.4].

Theorem 2.8. *$G\mathscr{T}$ is a compactly generated proper G-topological model category with respect to the weak equivalences, Serre fibrations, and retracts of relative G-cell complexes. The sets I and J are the generating q-cofibrations and the generating acyclic q-cofibrations.*

We must explain what it means for $G\mathscr{T}$ to be a "G-topological" model category. We revert to our general categories $\mathscr{C}_G$ and $G\mathscr{C}$, and we suppose that $G\mathscr{C}$ has a given model structure. For maps $i : A \longrightarrow X$ and $p : E \longrightarrow B$ in $G\mathscr{C}$, let

$$\mathscr{C}_G(i^*, p_*) : \mathscr{C}_G(X, E) \longrightarrow \mathscr{C}_G(A, E) \times_{\mathscr{C}_G(A,B)} \mathscr{C}_G(X, B) \tag{2.9}$$

be the map of G-spaces induced by $\mathscr{C}_G(i, \mathrm{id})$ and $\mathscr{C}_G(\mathrm{id}, p)$ by passage to pullbacks.

Definition 2.10. A model category $G\mathscr{C}$ is *G-topological* if the map $\mathscr{C}_G(i^*, p_*)$ is a Serre fibration (of G-spaces) when i is a q-cofibration and p is a q-fibration and is a weak equivalence (as a map of G-spaces) when, in addition, either i or p is a weak equivalence.

The point is that we must go beyond the category $G\mathscr{C}$ to the category $\mathscr{C}_G$ to formulate this equivariant notion. From the point of view of enriched category theory of Remark 1.3, this is the obviously right enriched version of the standard definitions of a simplicial or topological model category. It follows on passage to G-fixed point spaces that $G\mathscr{C}$ is also nonequivariantly topological, in the sense of [13, 5.12], but we need the equivariant version. The nonequivariant version has the following significance.

Lemma 2.11. *The pair (i,p) has the lifting property if and only if $G\mathscr{C}(i^*,p_*)$ is surjective.*

As in [13, §5], we will need two pairs of analogues of the maps $\mathscr{C}_G(i^*,p_*)$. First, for a map $i : A \longrightarrow B$ of based G-spaces and a map $j : X \longrightarrow Y$ in $G\mathscr{C}$, passage to pushouts gives a map

$$i\Box j : (A \wedge Y) \cup_{A\wedge X} (B \wedge X) \longrightarrow B \wedge Y \tag{2.12}$$

and passage to pullbacks gives a map

$$F_\Box(i,j) : F(B,X) \longrightarrow F(A,X) \times_{F(A,Y)} F(B,Y), \tag{2.13}$$

where $\wedge$ and F denote the tensor and cotensor in $\mathscr{C}_G$.

Second, assume that $\mathscr{C}_G$ is a closed symmetric monoidal category with product $\wedge_\mathscr{C}$ and internal function object functor $F_\mathscr{C}$. Then, for maps $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ in $G\mathscr{C}$, passage to pushouts gives a map

$$i\Box j : (Y \wedge_\mathscr{C} W) \cup_{X\wedge_\mathscr{C} W} (X \wedge_\mathscr{C} Z) \longrightarrow Y \wedge_\mathscr{C} Z, \tag{2.14}$$

and passage to pullbacks gives a map

$$F_\Box(i,j) : F_\mathscr{C}(Y,W) \longrightarrow F_\mathscr{C}(X,W) \times_{F_\mathscr{C}(X,Z)} F_\mathscr{C}(Y,Z). \tag{2.15}$$

Inspection of definitions gives adjunctions relating these maps.

Lemma 2.16. *Let $i : A \longrightarrow B$ be a map of based G-spaces and let $j : X \longrightarrow Y$ and $p : E \longrightarrow F$ be maps in $G\mathscr{C}$. Then there are natural isomorphisms of G-maps*

$$\mathscr{C}_G((i\Box j)^*,p_*) \cong \mathscr{T}_G(i^*,\mathscr{C}_G(j^*,p_*)_*) \cong \mathscr{C}_G(j^*,F_\Box(i,p)_*).$$

Therefore, passing to G-fixed points, $(i\Box j,p)$ has the lifting property in $G\mathscr{C}$ if and only if $(i,\mathscr{C}_G(j^*,p_*))$ has the lifting property in $G\mathscr{T}$.

Lemma 2.17. *Let i, j, and p be maps in $G\mathscr{C}$, where $\mathscr{C}_G$ is closed symmetric monoidal. Then there is a natural isomorphism of G-maps*

$$\mathscr{C}_G((i\Box j)^*,p_*) \cong \mathscr{C}_G(i^*,F_\Box(j,p)_*).$$

Returning to $\mathscr{T}_G$ and using Lemma 2.16, we see by a formal argument that the following lemma is equivalent to the assertion that $G\mathscr{T}$ is G-topological.

Lemma 2.18. *Let $i : A \longrightarrow X$ and $j : B \longrightarrow Y$ be q-cofibrations of G-spaces. Then $i\Box j$ is a q-cofibration and is acyclic if i or j is acyclic.*

2.2. **The level model structure on orthogonal G-spectra.** We here give the category $G\mathscr{I}\mathscr{S}$ of orthogonal G-spectra and G-maps a level model structure, following [13, §3]; maps will mean G-maps throughout. We need three definitions, the first of which concerns nondegenerate basepoints. A G-space is said to be nondegenerately based if the inclusion of its basepoint is an unbased h-cofibration (satisfies the G-HEP in the category of unbased G-spaces). As in [20, Prop. 9], a based h-cofibration between nondegenerately based G-spaces is an unbased h-cofibration. Each morphism space $\mathscr{J}_G(V,W)$ is nondegenerately based.

Definition 2.19. An orthogonal G-spectrum X is nondegenerately based if each $X(V)$ is a nondegenerately based G-space.

Definition 2.20. Define FI to be the set of all maps $F_V i$ with $V \in sk\mathscr{I}_G$ and $i \in I$. Define FJ to be the set of all maps $F_V j$ with $V \in sk\mathscr{I}_G$ and $j \in J$, and observe that each map in FJ is the inclusion of a G-deformation retract.

Definition 2.21. We define five properties of maps $f : X \longrightarrow Y$ of orthogonal G-spectra.

(i) f is a *level equivalence* if each map $f(V) : X(V) \longrightarrow Y(V)$ of G-spaces is a weak equivalence.
(ii) f is a *level fibration* if each map $f(V) : X(V) \longrightarrow Y(V)$ of G-spaces is a Serre fibration.
(iii) f is a *level acyclic fibration* if it is both a level equivalence and a level fibration.
(iv) f is a *q-cofibration* if it satisfies the LLP with respect to the level acyclic fibrations.
(v) f is a *level acyclic q-cofibration* if it is both a level equivalence and a q-cofibration.

Theorem 2.22. *The category $G\mathscr{I}\mathscr{S}$ of orthogonal G-spectra is a compactly generated proper G-topological model category with respect to the level equivalences, level fibrations, and q-cofibrations. The sets FI and FJ are the generating q-cofibrations and the generating acyclic q-cofibrations, and the following identifications hold.*

(i) *The level fibrations are the maps that satisfy the RLP with respect to FJ or, equivalently, with respect to retracts of relative FJ-cell complexes, and all orthogonal G-spectra are level fibrant.*
(ii) *The level acyclic fibrations are the maps that satisfy the RLP with respect to FI or, equivalently, with respect to retracts of relative FI-cell complexes.*
(iii) *The q-cofibrations are the retracts of relative FI-cell complexes.*
(iv) *The level acyclic q-cofibrations are the retracts of relative FJ-cell complexes.*

Moreover, every cofibrant orthogonal G-spectrum X is nondegenerately based.

The proof is the same as that of [13, 6.5]. As there, the following analogue of [13, 5.5] plays a role.

Lemma 2.23. *Every q-cofibration is an h-cofibration.*

The following analogue of [13, 3.7] also holds. The proof depends on 1.28 and Lemma 2.18 and thus on the fact that products of orbit spaces are triangulable as G-CW complexes.

Lemma 2.24. *If $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ are q-cofibrations, then*

$$i\Box j : (Y \wedge W) \cup_{X\wedge W} (X \wedge Z) \longrightarrow Y \wedge Z$$

is a q-cofibration which is level acyclic if either i or j is level acyclic. In particular, if Z is cofibrant, then $i \wedge \mathrm{id} : X \wedge Z \longrightarrow Y \wedge Z$ is a q-cofibration, and the smash product of cofibrant orthogonal G-spectra is cofibrant.

Let $[X, Y]^{\ell}_G$ denote the set of maps $X \longrightarrow Y$ in the level homotopy category $\mathrm{Ho}_{\ell}G\mathscr{I}\mathscr{S}$ and let $\pi(X, Y)_G$ denote the set of homotopy classes of maps $X \longrightarrow Y$. Then $[X, Y]^{\ell}_G \cong \pi(\Gamma X, Y)_G$, where $\Gamma X \longrightarrow X$ is a cofibrant approximation of X.

Fiber and cofiber sequences of orthogonal G-spectra behave the same way as for based G-spaces, starting from the usual definitions of homotopy cofibers Cf and homotopy fibers Ff [13, 6.8]. We record the analogue of [13, 6.9]. Most of the proof is the same as there. Some statements, such as the last clause of (i), are most easily proven by using (2.5) and Lemma 2.6 to reduce them to their nonequivariant counterparts by levelwise passage to fixed points.

Theorem 2.25. (i) *If A is a based G-CW complex and X is a nondegenerately based orthogonal G-spectrum, then $X \wedge A$ is nondegenerately based and*

$$[X \wedge A, Y]^{\ell}_G \cong [X, F(A,Y)]^{\ell}_G$$

for any Y. If $f : X \longrightarrow Y$ is a level equivalence of nondegenerately based orthogonal G-spectra, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a level equivalence.

(ii) *For nondegenerately based X_i, $\bigvee_i X_i$ is nondegenerately based and*

$$[\bigvee_i X_i, Y]^{\ell}_G \cong \prod_i [X_i, Y]^{\ell}_G$$

for any Y. A wedge of level equivalences of nondegenerately based orthogonal G-spectra is a level equivalence.

(iii) *If $i : A \longrightarrow X$ is an h-cofibration and $f : A \longrightarrow Y$ is any map of orthogonal G-spectra, where A, X, and Y are nondegenerately based, then $X \cup_A Y$ is nondegenerately based and the cobase change $j : Y \longrightarrow X \cup_A Y$ is an h-cofibration. If i is a level equivalence, then j is a level equivalence.*

(iv) *If i and i' are h-cofibrations and the vertical arrows are level equivalences in the following commutative diagram of nondegenerately based orthogonal G-spectra, then the induced map of pushouts is a level equivalence.*

$$\begin{array}{ccccc} X & \xleftarrow{i} & A & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \xleftarrow[i']{} & A' & \longrightarrow & Y' \end{array}$$

(v) *If X is the colimit of a sequence of h-cofibrations $i_n : X_n \longrightarrow X_{n+1}$ of nondegenerately based orthogonal G-spectra, then X is nondegenerately based and there is a $\lim^1$ exact sequence of pointed sets*

$$* \longrightarrow \lim{}^1 [\Sigma X_n, Y]^{\ell}_G \longrightarrow [X, Y]^{\ell}_G \longrightarrow \lim [X_n, Y]^{\ell}_G \longrightarrow *$$

for any Y. If each i_n is a level equivalence, then the map from the initial term X_0 into X is a level equivalence.

(vi) *If $f : X \longrightarrow Y$ is a map of nondegenerately based orthogonal G-spectra, then Cf is nondegenerately based and there is a natural long exact sequence*

$$\cdots \to [\Sigma^{n+1} X, Z]^{\ell}_G \to [\Sigma^n Cf, Z]^{\ell}_G \to [\Sigma^n Y, Z]^{\ell}_G \to [\Sigma^n X, Z]^{\ell}_G \to \cdots \to [X, Z]^{\ell}_G.$$

We shall also need a variant of the level model structure, called the positive level model structure, as in [13, §14]. It is obtained by ignoring representations V that do not contain a positive dimensional trivial representation. We can obtain a similar model structure by ignoring only $V = 0$, but that would not give the right model structure for some of our applications.

Definition 2.26. Define the positive analogues of the classes of maps specified in Definition 2.21 by restricting attention to those levels V with $V^G \neq 0$.

Definition 2.27. Let F^+I and F^+J be the sets of maps in FI and FJ that are specified in terms of the functors F_V with $V^G \neq 0$.

Theorem 2.28. *The category $G\mathscr{I}\mathscr{S}$ is a compactly generated proper G-topological model category with respect to the positive level equivalences, positive level fibrations, and positive level q-cofibrations. The sets F^+I and F^+J are the generating sets* *of positive q-cofibrations and positive level acyclic q-cofibrations. The positive q-cofibrations are the q-cofibrations that are homeomorphisms at all levels V such that $V^G = 0$.*

As in [13, §14], this is one case of a general relative version of Theorem 2.22.

Variants 2.29. There are other variants. Rather than using varying categories $\mathscr{I}_G^{\mathscr{V}}$, we could work with orthogonal G-spectra defined with respect to $\mathscr{V} = \mathscr{A}\ell\ell$ and define a "$\mathscr{V}$-level model structure" by restricting to those levels V that are isomorphic to representations in $\mathscr{V}$ when defining level equivalences, level fibrations, and the generating sets of q-cofibrations and acyclic q-cofibrations. This allows us to change $\mathscr{V}$ by changing the model structure on a single fixed category of orthogonal G-spectra; see Remark 3.9.

Remark 2.30. Everything in this section applies verbatim to the category $G\mathscr{P}$ of G-prespectra. Recall 1.10 and 1.30. Because $\mathscr{K}_G$ contains all objects of $\mathscr{J}_G$, the forgetful functor $\mathbb{U} : G\mathscr{I} \longrightarrow G\mathscr{P}$ creates the level equivalences and level fibrations of orthogonal G-spectra. That is, a map f of orthogonal G-spectra is a level equivalence or level fibration if and only if $\mathbb{U}f$ is a level equivalence or level fibration of prespectra. In particular $(\mathbb{P}, \mathbb{U})$ is a Quillen adjoint pair [13, A.1].

2.3. **The homotopy groups of G-prespectra.** By 1.10, an orthogonal G-spectrum has an underlying G-prespectrum indexed on a universe U such that $\mathscr{V}(U) = \mathscr{V}$. The homotopy groups of orthogonal G-spectra are defined to be the homotopy groups of their underlying G-prespectra. We discuss the homotopy groups of G-prespectra here. We first define Ω-G-spectra (more logically, prespectra).

Definition 2.31. A G-prespectrum X is an *Ω-G-spectrum*, if each of its adjoint structure maps $\tilde{\sigma} : X(V) \longrightarrow \Omega^{W-V}X(W)$ is a weak equivalence of G-spaces. An orthogonal G-spectrum is an *orthogonal Ω-G-spectrum* if each of its adjoint structure maps is a weak equivalence or, equivalently, if its underlying G-prespectrum is an Ω-G-spectrum.

It is convenient to write
$$\pi_q^H(A) = \pi_q(A^H)$$
for based G-spaces A.

Definition 2.32. For subgroups H of G and integers q, define the homotopy groups $\pi_q^H(X)$ of a G-prespectrum X by
$$\pi_q^H(X) = \mathrm{colim}_V\, \pi_q^H(\Omega^V X(V)) \quad \text{if} \quad q \geq 0,$$
where V runs over the indexing G-spaces in U, and
$$\pi_{-q}^H(X) = \mathrm{colim}_{V \supset \mathbb{R}^q}\, \pi_0^H(\Omega^{V-\mathbb{R}^q} X(V)) \quad \text{if} \quad q > 0.$$
A map $f : X \longrightarrow Y$ of G-prespectra is a *π_*-isomorphism* if it induces isomorphisms on all homotopy groups. A map of orthogonal G-spectra is a π_*-isomorphism if its underlying map of G-prespectra is a π_*-isomorphism.

As H varies, the $\pi_q^H(X)$ define a contravariant functor from the homotopy category $hG\mathscr{O}$ of orbits to the category of Abelian groups, but the functoriality need not be considered in the development of the model structures. We shall later use the terms "π_*-isomorphism" and "weak equivalence" interchangeably, but we prefer to use the term π_*-isomorphism here to avoid confusion among the different model structures on G-prespectra and orthogonal G-spectra. We state the results of this section for G-prespectra but, since the forgetful functor $\mathbb{U}$ preserves all relevant constructions, they apply equally well to orthogonal G-spectra. The previous section gives $G\mathscr{P}$ a level model structure.

Lemma 2.33. *A level equivalence of G-prespectra is a π_*-isomorphism.*

The nonequivariant version [13, 7.3] of the following partial converse is trivial. The equivariant version is the key result, [11, I.7.12], in the classical development of the equivariant stable homotopy category, and it is also the key result here. While the result there is stated for G-spectra, the argument is entirely homotopical and applies verbatim to Ω-G-spectra and is given in detail in the Memoir [12].

Theorem 2.34. *A π_*-isomorphism between Ω-G-spectra is a level equivalence.*

Using that space-level constructions commute with passage to fixed points, as in Lemma 2.6, all parts of the following equivariant analogue of [13, 7.4] follow from or are proven in the same way as the corresponding part of that result. As there, the nondegenerate basepoint hypotheses in Theorem 2.25 are not needed here.

Theorem 2.35. (i) *A map of G-prespectra is a π_*-isomorphism if and only if its suspension is a π_*-isomorphism.*

(ii) *The homotopy groups of a wedge of G-prespectra are the direct sums of the homotopy groups of the wedge summands, hence a wedge of π_*-isomorphisms of G-prespectra is a π_*-isomorphism.*

(iii) *If $i : A \longrightarrow X$ is an h-cofibration and a π_*-isomorphism of G-prespectra and $f : A \longrightarrow Y$ is any map of G-prespectra, then the cobase change $j : Y \longrightarrow X \cup_A Y$ is a π_*-isomorphism.*

(iv) *If i and i' are h-cofibrations and the vertical arrows are π_*-isomorphisms in the comparison of pushouts diagram of Theorem 2.25(iv), then the induced map of pushouts is a π_*-isomorphism.*

(v) *If X is the colimit of a sequence of h-cofibrations $X_n \longrightarrow X_{n+1}$, each of which is a π_*-isomorphism, then the map from the initial term X_0 into X is a π_*-isomorphism.*

(vi) *For any map $f : X \longrightarrow Y$ of G-prespectra and any $H \subset G$, there are natural long exact sequences*

$$\cdots \longrightarrow \pi_q^H(Ff) \longrightarrow \pi_q^H(X) \longrightarrow \pi_q^H(Y) \longrightarrow \pi_{q-1}^H(Ff) \longrightarrow \cdots,$$

$$\cdots \longrightarrow \pi_q^H(X) \longrightarrow \pi_q^H(Y) \longrightarrow \pi_q^H(Cf) \longrightarrow \pi_{q-1}^H(X) \longrightarrow \cdots,$$

and the natural map $\eta : Ff \longrightarrow \Omega Cf$ is a π_*-isomorphism.

Equivariant stability requires consideration of general representations $V \in \mathscr{V}$, rather than just the trivial representation as in (i).

Theorem 2.36. *Let $V \in \mathscr{V}$. A map $f : X \longrightarrow Y$ of G-prespectra is a π_*-isomorphism if and only if $\Sigma^V f : \Sigma^V X \longrightarrow \Sigma^V Y$ is a π_*-isomorphism*

Half of the theorem is given by the following lemma, which will be used in our development of the stable model structure.

Lemma 2.37. *Let $V \in \mathscr{V}$. If $f : X \longrightarrow Y$ is a map of G-prespectra such that $\Sigma^V f : \Sigma^V X \longrightarrow \Sigma^V Y$ is a π_*-isomorphism, then f is a π_*-isomorphism.*

Proof. By Proposition 2.39 below, $\Omega^V\Sigma^V f$ is a π_*-isomorphism. The conclusion follows by naturality from the following lemma. $\square$

Lemma 2.38. *For G-prespectra X and $V \in \mathscr{V}$, the unit $\eta : X \longrightarrow \Omega^V\Sigma^V X$ of the (Σ^V, Ω^V) adjunction is a π_*-isomorphism.*

Proposition 2.39. *If $f : X \longrightarrow Y$ is a π_*-isomorphism of G-prespectra and A is a finite based G-CW complex, then $F(\mathrm{id}, f) : F(A, X) \longrightarrow F(A, Y)$ is a π_*-isomorphism.*

The analogue for smash products is a little more difficult and gives the converse of Lemma 2.37 that is needed to complete the proof of Theorem 2.36.

Theorem 2.40. *If $f : X \longrightarrow Y$ is a π_*-isomorphism of G-prespectra and A is a based G-CW complex, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a π_*-isomorphism.*

We can reduce this to the case $A = G/H_+$ by use of Theorem 2.35, but that case seems hard to handle directly. One first proves the following partial result directly. The rest drops out model theoretically in the next section.

Lemma 2.41. *If $f : X \longrightarrow Y$ is a level equivalence of G-prespectra and A is a based G-CW complex, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a π_*-isomorphism.*

2.4. **The stable model structure on orthogonal G-spectra.** We give the categories of orthogonal G-spectra and G-prespectra stable model structures and prove that they are Quillen equivalent. The arguments are like those in the nonequivariant context of [13], except that we work with π_*-isomorphisms rather than the equivariant analogue of the stable equivalences used there. All of the statements and most of the proofs are identical in $G\mathscr{I}\mathscr{S}$ and $G\mathscr{P}$. Definition 2.21 specifies the level equivalences, level fibrations, level acyclic fibrations, q-cofibrations, and level acyclic q-cofibrations in these categories.

Definition 2.42. Let $f : X \to Y$ be a map of orthogonal G-spectra or G-prespectra.

(i) f is an *acyclic q-cofibration* if it is a π_*-isomorphism and a q-cofibration.
(ii) f is a *q-fibration* if it satisfies the RLP with respect to the acyclic q-cofibrations.
(iii) f is an *acyclic q-fibration* if it is a π_*-isomorphism and a q-fibration.

Theorem 2.43. *The categories $G\mathscr{I}\mathscr{S}$ and $G\mathscr{P}$ are compactly generated proper G-topological model categories with respect to the π_*-isomorphisms, q-fibrations, and q-cofibrations. The fibrant objects are the Ω-G-spectra.*

The set of generating q-cofibrations is the set FI specified in Definition 2.1. The set K of generating acyclic q-cofibrations properly contains the set FJ specified there. As nonequivariantly [13, §§8, 9], it is defined in terms of the following maps $\lambda_{V,W}$, which turn out to be π_*-isomorphisms.

Definition 2.44. For $V, W \in \mathscr{V}$, define $\lambda_{V,W} : F_{V\oplus W}S^W \longrightarrow F_V S^0$ to be the adjoint of the map

$$S^W \longrightarrow (F_V S^0)(V \oplus W) \cong O(V \oplus W)_+ \wedge_{O(W)} S^W$$

that sends w to the class of $e \wedge w$, where $e \in O(V \oplus W)$ is the identity element.

The following observation is the reason these maps play an important role.

Lemma 2.45. *For any orthogonal G-spectrum or G-prespectrum X,*

$$\lambda^*_{V,W} : \mathscr{I}_G\mathscr{S}(F_V S^0, X) \longrightarrow \mathscr{I}_G\mathscr{S}(F_{V\oplus W}S^W, X)$$

coincides with $\tilde{\sigma} : X(V) \longrightarrow \Omega^W X(V \oplus W)$ under the canonical homeomorphisms

$$X(V) = \mathscr{T}_G(S^0, X(V)) \cong \mathscr{I}_G\mathscr{S}(F_V S^0, X)$$

and

$$\Omega^W X(V \oplus W) = \mathscr{T}_G(S^W, X(V \oplus W)) \cong \mathscr{I}_G\mathscr{S}(F_{V\oplus W}S^W, X).$$

The following result is the equivariant version of [13, 8.6]. It is proven separately in the cases of G-prespectra and orthogonal G-spectra, using [13, 4.1] in the first case and [13, 4.4] in the second.

Lemma 2.46. *For all based G-CW complexes A, the maps*

$$\lambda_{V,W} \wedge \mathrm{id} : F_{V\oplus W}\Sigma^W A \cong F_{V\oplus W}S^W \wedge A \longrightarrow F_V S^0 \wedge A \cong F_V A$$

are π_*-isomorphisms.

Recall the operation $\Box$ from (2.12).

Definition 2.47. Let $M\lambda_{V,W}$ be the mapping cylinder of $\lambda_{V,W}$. Then $\lambda_{V,W}$ factors as the composite of a q-cofibration $k_{V,W} : F_{V\oplus W}S^W \longrightarrow M\lambda_{V,W}$ and a deformation retraction $r_{V,W} : M\lambda_{V,W} \longrightarrow F_V S^0$. Let $j_{V,W} : F_V S^0 \longrightarrow M\lambda_{V,W}$ be the evident homotopy inverse of $r_{V,W}$. Restricting to V and W in $sk\mathscr{I}_G$, let K be the union of FJ and the set of all maps of the form $i\Box k_{V,W}$, $i \in I$.

We need a characterization of the maps that satisfy the RLP with respect to K. It is the equivariant analogue of [13, 9.5] and we delete the proof, but this is the place where we need the notion of a G-topological model category.

Definition 2.48. A commutative diagram of based G-spaces

$$\begin{array}{ccc} D & \xrightarrow{g} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[f]{} & B \end{array}$$

in which p and q are Serre fibrations is a *homotopy pullback* if the induced map $D \longrightarrow A \times_B E$ is a weak equivalence of G-spaces.

Proposition 2.49. *A map $p : E \longrightarrow B$ satisfies the RLP with respect to K if and only if p is a level fibration and the diagram*

$$\begin{array}{ccc} EV & \xrightarrow{\tilde{\sigma}} & \Omega^W E(V \oplus W) \\ {\scriptstyle pV}\downarrow & & \downarrow{\scriptstyle \Omega^W p(V\oplus W)} \\ BV & \xrightarrow[\tilde{\sigma}]{} & \Omega^W B(V \oplus W) \end{array} \tag{2.50}$$

is a homotopy pullback for all V and W.

From here, the proof of Theorem 2.43 is virtually identical to that of its nonequivariant version in [13, §9]. We record the main steps of the argument since they give the order of proof and encode useful information about the q-fibrations and q-cofibrations. Rather than repeat the proofs, we point out the main input. The following corollary is immediate.

Corollary 2.51. *The trivial map $F \longrightarrow *$ satisfies the RLP with respect to K if and only if F is an Ω-G-spectrum.*

It is at this point that the key result, Theorem 2.34, comes into play. It implies that $p^{-1}(*) \longrightarrow *$ is a π_*-isomorphism in the following analogue of [13, 9.8].

Corollary 2.52. *If $p : E \longrightarrow B$ is a π_*-isomorphism that satisfies the RLP with respect to K, then p is a level acyclic fibration.*

Arguing as in [13, 9.9], Lemma 2.46 and Theorem 2.35 imply the following characterizations.

Proposition 2.53. *Let $f : X \longrightarrow Y$ be a map of orthogonal G-spectra.*

(i) *f is an acyclic q-cofibration if and only if it is a retract of a relative K-cell complex.*
(ii) *f is a q-fibration if and only if it satisfies the RLP with respect to K, and X is fibrant if and only if it is an orthogonal Ω-G-spectrum.*
(iii) *f is an acyclic q-fibration if and only if it is a level acyclic fibration.*

The proof of the model axioms is completed as in [13, §9]. The properness of the model structure is implied by the following more general statements.

Lemma 2.54. *Consider the following commutative diagram:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[g]{} & Y. \end{array}$$

(i) *If the diagram is a pushout in which i is an h-cofibration and f is a π_*-isomorphism, then g is a π_*-isomorphism.*
(ii) *If the diagram is a pullback in which j is a level fibration and g is a π_*-isomorphism, then f is a π_*-isomorphism.*

The following consequence of Propositions 2.49 and 2.53 leads to the proof of Theorem 2.40, which uses cofibrant approximation in the level model structure and Lemma 2.41.

Lemma 2.55. *If A is a based G-CW complex, then $(- \wedge A, F(A, -))$ is a Quillen adjoint pair on $G\mathscr{I}\mathscr{S}$ or $G\mathscr{P}$ with its stable model structure.*

The following result, which is immediate from Lemmas 2.55 and 2.38, implies that the homotopy category with respect to the stable model structure really is an "equivariant stable homotopy category", in the sense that the functors Σ^V and Ω^V on it are inverse equivalences of categories for $V \in \mathscr{V}$.

Theorem 2.56. *For $V \in \mathscr{V}$, the pair (Σ^V, Ω^V) is a Quillen equivalence.*

Finally, as in [13, 10.3] we have the following promised comparison theorem.

Theorem 2.57. *The pair $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence between the categories $G\mathscr{P}$ and $G\mathscr{I}\mathscr{S}$ with their stable model structures.*

2.5. **The positive stable model structure.** In §2, we explained the positive level model structure, and we need the concomitant positive stable model structure, as in [13, §14].

Definition 2.58. A G-prespectrum or orthogonal G-spectrum X is a positive Ω-G-spectrum if $\tilde{\sigma} : X(V) \longrightarrow \Omega^{W-V}X(W)$ is a weak equivalence for $V^G \neq 0$.

Definition 2.59. Define *acyclic positive q-cofibrations*, *positive q-fibrations*, and *acyclic positive q-fibrations* as in Definition 2.42, but starting with the positive level classes of maps specified in Definition 2.26.

Theorem 2.60. *The categories $G\mathscr{I}\mathscr{S}$ and $G\mathscr{P}$ are compactly generated proper G-topological model categories with respect to the π_*-isomorphisms, positive q-fibrations, and positive q-cofibrations. The positive fibrant objects are the positive Ω-G-spectra.*

The set of generating positive q-fibrations is the set F^+I specified in Definition 2.27. The set of generating acyclic positive q-cofibrations is the union, K^+, of the set F^+J specified there and the set of maps of the form $i\square k_{V,W}$ with $i \in I$ and $V^G \neq 0$ from Definition 2.47.

The proof of Theorem 2.60 depends on the positive analogue of Theorem 2.34.

Theorem 2.61. *A π_*-isomorphism between positive Ω-G-spectra is a positive level equivalence.*

This can be shown by restricting the proof of Theorem 2.34 to positive Ω-G-spectra and positive levels. For orthogonal Ω-G-spectra, there is an illuminating alternative argument. Indeed, for $V \in \mathscr{V}$ and an orthogonal G-spectrum X, the map $\lambda = \lambda_{0,V} : F_V S^V \longrightarrow F_0 S^0 = S$ induces a map

$$\lambda^* : X \cong F(S, X) \longrightarrow F(F_V S^V, X). \tag{2.62}$$

Standard adjunctions imply that

$$F(F_V S^V, X)(W) \cong \Omega^V X(V \oplus W),$$

and this leads to the following relationship between orthogonal Ω-G-spectra and orthogonal positive Ω-G-spectra.

Lemma 2.63. *If E is a positive orthogonal Ω-G-spectrum, then $F(F_1S^1, E)$ is an orthogonal Ω-G-spectrum and λ^* is a positive level equivalence.*

Therefore, for orthogonal G-spectra, Theorem 2.61 can be proven by applying Theorem 2.34 to $F(F_1S^1, -)$. From here, Theorem 2.60 is proven by the same arguments as for the stable model structure, but with everything restricted to positive levels. Similarly, the proof of the following comparison result is the same as the proof of Theorem 2.57.

Theorem 2.64. *The pair $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence between the categories $G\mathscr{P}$ and $G\mathscr{I}\mathscr{S}$ with their positive stable model structures.*

The relationship between the stable model structure and the positive stable model structure is given by the following equivariant analogue of [13, 14.6].

Proposition 2.65. *The identity functor from $G\mathscr{I}\mathscr{S}$ with its positive stable model structure to $G\mathscr{I}\mathscr{S}$ with its stable model structure is the left adjoint of a Quillen equivalence, and similarly for $G\mathscr{P}$.*

2.6. **Model categories of ring and module G-spectra.** As we summarize here, the categories of orthogonal ring spectra and of modules over an orthogonal ring spectrum are Quillen model categories. The proofs are essentially the same as those in the nonequivariant case given in [13, §§12, 14]. but the inclusion of the case of symmetric spectra there dictated a more complicated line of argument than is necessary here. We give an outline.

In the language of [19], we show that the monoid and pushout-product axioms hold for orthogonal G-spectra. As in [11, 11.2], the following elementary complement to Lemmas 2.23 and 2.24 is used repeatedly.

Lemma 2.66. *If $i : X \longrightarrow Y$ is an h-cofibration of orthogonal G-spectra and Z is any orthogonal G-spectrum, then $i \wedge \mathrm{id} : X \wedge Z \longrightarrow Y \wedge Z$ is an h-cofibration.*

The following lemma is the key step in the proof of the cited axioms. Its nonequivariant analogue is part of the proof of [13, 12.3].

Lemma 2.67. *Let Y be an orthogonal G-spectrum such that $\pi_*(Y) = 0$. Then $\pi_*(F_V S^V \wedge Y) = 0$ for any $V \in \mathscr{V}$.*

Proposition 2.68. *If X is a cofibrant orthogonal G-spectrum, then the functor $X \wedge (-)$ preserves π_*-isomorphisms.*

Proof. When $X = F_V S^V$, this is implied by Lemma 2.67, as we see by using the usual mapping cylinder construction to factor a given π_*-isomorphism as a composite of an h-cofibration and a G-homotopy equivalence and comparing long exact sequences given by Lemma 2.66 and Theorem 2.35(vi). As in the proof of [13, 12.3], the general case follows by use of Theorems 2.35, 2.36, and 2.40. □

As in [13, 12.5 and 12.6], this together with other results already proven implies the monoid and pushout-product axioms. These apply to $G\mathscr{I}\mathscr{S}$ with both its stable and its positive stable model structures.

Proposition 2.69 (Monoid axiom)**.** *For any acyclic (positive) q-cofibration $i : X \longrightarrow Y$ of orthogonal G-spectra and any orthogonal G-spectrum Z, the map $i \wedge \mathrm{id} : X \wedge Z \longrightarrow Y \wedge Z$ is a π_*-isomorphism and an h-cofibration. Moreover, cobase changes and sequential colimits of such maps are also π_*-isomorphisms and h-cofibrations.*

Proposition 2.70 (Pushout-product axiom)**.** *If $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ are (positive) q-cofibrations of orthogonal G-spectra and i is a π_*-isomorphism, then the (positive) q-cofibration $i \Box j : (Y \wedge W) \cup_{X \wedge W} (X \wedge Z) \longrightarrow Y \wedge Z$ is a π_*-isomorphism.*

As in [13, §§12, 14], the methods and results of [19], together with Proposition 2.65, entitle us to the following conclusions. More explicitly, [13, 5.13] specifies conditions for the category of algebras over a monad in a compactly generated topological model category $\mathscr{C}$ to inherit a structure of topological model category, and that result generalizes to G-topological model categories. The pushout-product and monoid axioms allow the verification of the conditions in the cases on hand.

Theorem 2.71. *Let R be an orthogonal ring G-spectrum, and consider the stable model structure on $G\mathscr{I}\mathscr{S}$.*

(i) *The category of left R-modules is a compactly generated proper G-topological model category with weak equivalences and q-fibrations created in $G\mathscr{I}\mathscr{S}$.*

(ii) *If R is cofibrant as an orthogonal G-spectrum, then the forgetful functor from R-modules to orthogonal G-spectra preserves q-cofibrations, hence every cofibrant R-module is cofibrant as an orthogonal G-spectrum.*
(iii) *If R is commutative, the symmetric monoidal category $G\mathscr{I}\mathscr{S}_R$ of R-modules also satisfies the pushout-product and monoid axioms.*
(iv) *If R is commutative, the category of R-algebras is a compactly generated right proper G-topological model category with weak equivalences and q-fibrations created in $G\mathscr{I}\mathscr{S}$.*
(v) *If R is commutative, every q-cofibration of R-algebras whose source is cofibrant as an R-module is a q-cofibration of R-modules, hence every cofibrant R-algebra is cofibrant as an R-module.*
(vi) *If $f : Q \longrightarrow R$ is a weak equivalence of orthogonal ring G-spectra, then restriction and extension of scalars define a Quillen equivalence between the categories of Q-modules and of R-modules.*
(vii) *If $f : Q \longrightarrow R$ is a weak equivalence of commutative orthogonal ring G-spectra, then restriction and extension of scalars define a Quillen equivalence between the categories of Q-algebras and of R-algebras.*

Parts (i), (iii), (iv), (vi), and (vii) also hold for the positive stable model structure.

Parts (ii) and (v) do not hold for the positive stable model structure, in which S_G is not cofibrant. As in [13, 12.7], we have the following generalization of Proposition 2.68, which is needed in the proofs of parts (vi) and (vii) of the theorem.

Proposition 2.72. *For a cofibrant right R-module M, the functor $M \wedge_R N$ of N preserves π_*-isomorphisms.*

2.7. **The model category of commutative ring G-spectra.** Let $\mathbb{C}$ be the monad on orthogonal G-spectra that defines commutative orthogonal ring G-spectra. Thus $\mathbb{C}X = \bigvee_{i\geq 0} X^{(i)}/\Sigma_i$, where $X^{(i)}$ is the i^{th} smash power, with $X^{(0)} = S_G$.

Theorem 2.73. *The category of commutative orthogonal ring G-spectra is a compactly generated proper G-topological model category with q-fibrations and weak equivalences created in the positive stable model category of orthogonal G-spectra. The sets $\mathbb{C}F^+I$ and $\mathbb{C}K^+$ are the generating sets of q-cofibrations and acyclic q-cofibrations.*

This is a consequence of the following two results, which (together with two general results on colimits [2, I.7.2, VII.2.10]) verify the criteria for inheritance of a model structure given in [13, 5.13].

Lemma 2.74. *The sets $\mathbb{C}F^+I$ and $\mathbb{C}K^+$ satisfy the Cofibration Hypothesis.*

Lemma 2.75. *Every relative $\mathbb{C}K^+$-cell complex is a π_*-isomorphism.*

Remark 2.76 (Correction). As in [13, §15], the proof of the previous lemma reduces to use of the second statement of the following result, which is [12, III.8.4]. It is the analogue of the nonequivariant results [2, III.5.1] and [2, 15.5]. Its proof in [12] is incorrect, as Mandell and I realized some time ago. Mike Hopkins rediscovered the error, and [9, B.52] fixes it. The culprit is mainly a typo in [12], noted in [9, B.52] and corrected below: $E\Sigma_i$ in [12] should read $E_G\Sigma_i$, the total space of the universal (G, Σ_j)-bundle. It is a $G \times \Sigma_i$-space characterized by the property that, for a subgroup $H \subset G \times \Sigma_i$, $(E_G\Sigma_i)^H$ is empty unless $H \cap \Sigma_i = \{e\}$, when it is contractible. Correcting the typo corrects the proof, as the details of [9, B.52] make clear. As a matter of detail, there is a related misstatement immediately above [9, B.51], where $H \subset \Sigma_i$ should read $H \cap \Sigma_i \neq \{e\}$, but that does not effect the proof.

Lemma 2.77. *Let A be a based G-CW complex, X be an orthogonal G-spectrum, and $V^G \neq 0$. Then the quotient map*

$$q : (E\Sigma_{i+} \wedge_{\Sigma_i} (F_V A)^{(i)}) \wedge X \longrightarrow ((F_V A)^{(i)}/\Sigma_i) \wedge X$$

is a π_*-isomorphism. If X is a positive cofibrant orthogonal G-spectrum, then

$$q : E_G\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)} \longrightarrow X^{(i)}/\Sigma_i$$

is a π_*-isomorphism.

2.8. **Families, cofamilies, and isotropy separation.** We discuss in model theoretical terms the familiar idea of concentrating G-spaces or G-spectra at or away from a family of subgroups. Write $[X, Y]_G$ for maps $X \longrightarrow Y$ in the homotopy category Ho$G\mathscr{I}\mathscr{S}$ with respect to the stable model structure.

We call weak equivalences G-equivalences, and we drop the adjective orthogonal from orthogonal G-spectra. We will not use any other kind. For the moment, they can be indexed on any universe. For $H \subset G$, we say that a G-map is an H-equivalence if it is a weak equivalence when regarded as an H-map; we will treat restriction to subgroups systematically shortly. Let $\mathscr{F}$ be a family of subgroups of G, namely a set of subgroups closed under passage to conjugates and subgroups. There is a universal $\mathscr{F}$-space $E\mathscr{F}$. It is a G-CW complex such that $(E\mathscr{F})^H$ is contractible for $H \in \mathscr{F}$ and empty for $H \notin \mathscr{F}$. Of course, its cells must be of orbit type G/H with $H \in \mathscr{F}$. The following definitions make sense for G-spaces, orthogonal G-spectra, or objects in virtually any category of G-objects.

Definition 2.78. Consider G-spectra and G-maps

(i) A map $f : X \longrightarrow Y$ is a $\mathscr{F}$-equivalence if it is an H-equivalence for all $H \in \mathscr{F}$.
(ii) X is an $\mathscr{F}$-spectrum if the map $\pi : E\mathscr{F}_+ \wedge X \longrightarrow X$ induced by the projection $E\mathscr{F}_+ \longrightarrow S^0$ is a G-equivalence.

In the model structures on $G\mathscr{I}\mathscr{S}$, the generating q-cofibrations and generating acyclic q-cofibrations are obtained by applying the functors F_V to certain maps $G/H_+ \wedge A \longrightarrow G/H_+ \wedge B$ of G-spaces. We can restrict attention to those $H \in \mathscr{F}$ in all of these definitions. We refer to $\mathscr{F}$-cofibrations rather than q-cofibrations for the retracts of the resulting relative $\mathscr{F}$-cell complexes. The following is a generic result that applies starting from any given q-type model structure in sight.

Theorem 2.79. *The category $G\mathscr{I}\mathscr{S}$ is a compactly generated proper G-topological model category with weak equivalences the $\mathscr{F}$-equivalences and with generating $\mathscr{F}$-cofibrations and generating acyclic $\mathscr{F}$-cofibrations obtained from the original generating q-cofibrations and generating acyclic q-cofibrations by restricting to orbits G/H with $H \in \mathscr{F}$.*

A similar result holds for $G\mathscr{T}$, $G\mathscr{P}$ and $G\mathscr{S}$. We refer to these as *$\mathscr{F}$-model structures.* The following result is a straightforward inspection of the G/H appearing in cells, but we warn the reader that the underlying spaces of $\mathscr{F}$-cell spectra are generally not $\mathscr{F}$-spaces, so that the conclusion is not as obvious as its space level analogue.

Lemma 2.80. *If A is a based $\mathscr{F}$-CW complex and X is a cell G-spectrum, then $A \wedge X$ is an $\mathscr{F}$-cell object.*

This and fibrant approximation are used to prove that (i) implies (iii) in the following result.

Proposition 2.81. *The following conditions on a map $f : X \longrightarrow Y$ are equivalent.*

(i) *f is an $\mathscr{F}$-equivalence.*
(ii) *$f_* : \pi_*^H(X) \longrightarrow \pi_*^H(Y)$ is an isomorphism for $H \in \mathscr{F}$.*
(iii) $\mathrm{id} \wedge f$: *is an $E\mathscr{F}_+ \wedge X \longrightarrow E\mathscr{F}_+ \wedge Y$ is a G-equivalence.*

Remark 2.82. When X is fibrant, $\pi_*^H(X) = \pi_*(X^H)$, just as for G-spaces, as we will see shortly. Thus, when X and Y are fibrant, $f : X \longrightarrow Y$ is an $\mathscr{F}$-equivalence if and only $f^H : X^H \longrightarrow Y^H$ is a weak equivalence for all $H \in \mathscr{F}$.

While the model structure above is the most natural one for us here, there is another very important model structure with the same weak equivalences.

Definition 2.83. Let E be cofibrant in the stable model structure.

(i) A map $f : X \longrightarrow Y$ is an E-equivalence if $\mathrm{id} \wedge f : E \wedge X \longrightarrow E \wedge Y$ is a G-equivalence.
(ii) Z is E-local if $f^* : [Y, Z]_G \longrightarrow [X, Z]_G$ is an isomorphism for all E-equivalences $f : X \longrightarrow Y$.
(iii) An E-localization of X is an E-equivalence $\lambda : X \longrightarrow Y$ from X to an E-local object Y.

Theorem 2.84 (Bousfield localization)**.** *Let E be cofibrant. Then $G\mathscr{I}\mathscr{S}$ has an E-model structure whose equivalences are the E-equivalences and whose E-cofibrations are the q-cofibrations. The E-fibrant objects are the E-local objects, and E-fibrant approximation constructs a Bousfield localization $\lambda : X \longrightarrow L_E X$ of X at E.*

Taking $E = E\mathscr{F}_+$, we call the resulting model structures *Bousfield $\mathscr{F}$-model structures*. Here Bousfield localization takes the following elementary form.

Proposition 2.85. *The map $\xi : X \longrightarrow F(E\mathscr{F}_+, X)$ induced by the projection $E\mathscr{F}_+ \longrightarrow S^0$ is an $E\mathscr{F}_+$-localization of X.*

Proof. The map ξ is an $E\mathscr{F}_+$-equivalence by [7, 17.2], and it is immediate by adjunction that $F(E\mathscr{F}_+, X)$ is $E\mathscr{F}_+$-local. □

Completion theorems in equivariant stable homotopy theory are concerned with the comparison of this Bousfield localization at G-spectra such as S_G, K_G, and MU_G with another, more algebraically computable, Bousfield localization. See for example [4, 4.1], [5]. This is an extremely important strand of equivariant stable homotopy theory, but it does not enter directly into the Kervaire invariant problem. Study of this for $E = MU_G$ in [8] led to the first introduction of norm maps.

Now return to the notion of an $\mathscr{F}$-spectrum in Definition 2.78. While that is an intrinsic notion, independent of any model structure, it has the following characterization.

Theorem 2.86. *A G-spectrum X is an $\mathscr{F}$-spectrum if and only if its $\mathscr{F}$-cofibrant approximation $\gamma : \Gamma X \longrightarrow X$ is a G-equivalence.*

Let $\mathrm{Ho}\mathscr{F}\mathscr{I}\mathscr{S}$ denote the homotopy category associated to the $\mathscr{F}$-model structure on $G\mathscr{I}\mathscr{S}$, or, equivalently, the Bousfield $\mathscr{F}$-model structure, and write $[X,Y]_{\mathscr{F}}$ for the set of maps $X \longrightarrow Y$ in this category. The results above imply the following description of $\mathrm{Ho}\mathscr{F}\mathscr{I}\mathscr{S}$.

Theorem 2.87. *Smashing with $E\mathscr{F}_+$ defines an isomorphism*

$$[X,Y]_{\mathscr{F}} \cong [E\mathscr{F}_+ \wedge X, E\mathscr{F}_+ \wedge Y]_G$$

and thus gives an equivalence of categories from $Ho\mathscr{F}\mathscr{I}\mathscr{S}$ to the full subcategory of objects $E\mathscr{F}_+ \wedge X$ in $HoG\mathscr{I}\mathscr{S}$.

There is an analogous theory for cofamilies, namely complements $\mathscr{F}'$ of families. Thus $\mathscr{F}'$ is the set of subgroups of G not in $\mathscr{F}$. We define $\tilde{E}\mathscr{F}$ to be the cofiber of $E\mathscr{F}_+ \longrightarrow S^0$. Then $(\tilde{E}\mathscr{F})^H$ is contractible if $H \in \mathscr{F}$ and is S^0 if $H \notin \mathscr{F}$. In contrast to the situation for G-spaces, the evident analogue of Proposition 2.81 is false for G-spectra. This motivates the following variant of Definition 2.78.

Definition 2.88. A map $f : X \longrightarrow Y$ is an $\mathscr{F}'$-equivalence if it is an $\tilde{E}\mathscr{F}$-equivalence. A G-spectrum X is an $\mathscr{F}'$-spectrum if the map $\lambda : X \longrightarrow \tilde{E}\mathscr{F} \wedge X$ induced by the inclusion $S^0 \longrightarrow \tilde{E}\mathscr{F}$ is a G-equivalence.

Remark 2.89. We do not obtain model structures by restricting attention to orbits G/H with $H \in \mathscr{F}'$, but we do still have the Bousfield $\mathscr{F}'$-model structure whose weak equivalences are the $\mathscr{F}'$-equivalences and whose cofibrations are the q-cofibrations.

We have the following analogue of Theorem 2.86.

Theorem 2.90. *A G-spectrum X is an $\mathscr{F}'$-spectrum if and only if the $\tilde{E}\mathscr{F}$-fibrant approximation $\lambda : X \longrightarrow L_{\tilde{E}\mathscr{F}}X$ is a G-equivalence, or equivalently, X is $\tilde{E}\mathscr{F}$-local.*

Let $\mathrm{Ho}\mathscr{F}'\mathscr{I}\mathscr{S}$ denote the homotopy category associated to the Bousfield $\mathscr{F}'$-model structure and write $[X,Y]_{\mathscr{F}'}$ for the set of maps $X \longrightarrow Y$ in this category. The previous theorem implies the following one.

Theorem 2.91. *Smashing with $\tilde{E}\mathscr{F}$ defines an isomorphism*

$$[X,Y]_{\mathscr{F}'} \cong [\tilde{E}\mathscr{F} \wedge X, \tilde{E}\mathscr{F} \wedge Y]_G$$

and thus gives an equivalence of categories from $Ho\mathscr{F}'\mathscr{C}$ to the full subcategory of objects $\tilde{E}\mathscr{F} \wedge X$ in $HoG\mathscr{C}$.

For a G-spectrum X, smashing with the cofiber sequence that defines $\tilde{E}\mathscr{F}$ gives the "isotropy separation cofiber sequence"

$$E\mathscr{F}_+ \wedge X \longrightarrow X \longrightarrow \tilde{E}\mathscr{F} \wedge X.$$

Smashing it with the map $\xi : X \longrightarrow F(E\mathscr{F}_+, X)$ gives the $\mathscr{F}$-Tate diagram

$$\begin{array}{ccccc} E\mathscr{F}_+ \wedge X & \longrightarrow & X & \longrightarrow & \tilde{E}\mathscr{F} \wedge X \\ \downarrow & & \downarrow & & \downarrow \\ E\mathscr{F}_+ \wedge F(E\mathscr{F}_+, X) & \longrightarrow & F(E\mathscr{F}_+, X) & \longrightarrow & \tilde{E}\mathscr{F} \wedge F(E\mathscr{F}_+, X) \end{array}$$

The left horizontal arrow is an equivalence and the bottom right corner is called the Tate $\mathscr{F}$-spectrum of X. See [7]. In the most important special case, $\mathscr{F} = \{e\}$ and the diagram is written as

$$\begin{array}{ccccc} EG_+ \wedge X & \longrightarrow & X & \longrightarrow & \tilde{E}G \wedge X \\ \downarrow & & \downarrow & & \downarrow \\ EG_+ \wedge F(EG_+, X) & \longrightarrow & F(EG_+, X) & \longrightarrow & \tilde{E}G \wedge F(EG_+, X). \end{array}$$

Isotropy separation is used in the Kervaire invariant proof and everywhere else in equivariant stable homotopy theory.

3. "Change" functors for orthogonal G-spectra

We develop the versions for orthogonal G-spectra of the central structural features of equivariant stable homotopy theory: change of universe, change of group, fixed point and orbit spectra, and geometric fixed point spectra. The last notion is very important in the Kervaire invariant problem, and the geometric fixed point functor on orthogonal spectra is far more satisfactory than it analogs in earlier constructions of the equivariant stable category.

3.1. **Change of universe.** Change of universe plays a fundamental role in the homotopy level theory of [11]. The theory for orthogonal G-spectra takes a precise point-set level form, which is used in [9]. For brevity, we focus on inclusions of universes. A key fact is the following non-obvious implication of the definition of an orthogonal G-spectrum, which really justifies the name.

Lemma 3.1. *Let V and W be G-inner product spaces in $\mathscr{V}$ of the same dimension. Then, for orthogonal G-spectra X, the evaluation G-map*

$$\mathscr{J}_G(V,W) \wedge X(V) \longrightarrow X(W)$$

of the G-functor X induces a G-homeomorphism

$$\alpha : \mathscr{J}_G(V,W) \wedge_{O(V)} X(V) \longrightarrow X(W).$$

Its domain is homeomorphic, but not necessarily G-homeomorphic, to $X(V)$.

Change of universe appears in several equivalent guises. We could apply the general theory of prolongation functors left adjoint to forgetful functors, but we can be more explicit, mimicking the analogous theory for EKMM spectra [3].

Definition 3.2. Let $\mathscr{V}$ and $\mathscr{V}'$ be collections of representations as in 1.1 and 2.1. Thus both collections contain all trivial representations. Define a G-functor $I^{\mathscr{V}}_{\mathscr{V}'} : \mathscr{I}^{\mathscr{V}'}_G \mathscr{S} \longrightarrow \mathscr{I}^{\mathscr{V}}_G \mathscr{S}$ by letting

$$(I^{\mathscr{V}}_{\mathscr{V}'} X)(V) = \mathscr{J}^{\mathscr{V}}_G(\mathbb{R}^n, V) \wedge_{O(n)} X(\mathbb{R}^n)$$

for $X \in \mathscr{I}^{\mathscr{V}'}_G \mathscr{S}$ and $V \in \mathscr{V}$ with $\dim V = n$. We omit specification of the morphism or, equivalently, evaluation G-maps here.

If $\mathscr{V} = \mathscr{V}(U)$ and $\mathscr{V}' = \mathscr{V}(U')$ for universes U and U', then $\mathscr{V} \subset \mathscr{V}'$ if and only if there is a G-linear isometry $U \longrightarrow U'$. This is the starting point for the change of universe functors in [11]. By inspection or [13, §3], the inclusion $\mathscr{V} \subset \mathscr{V}'$ induces a full and faithful strong symmetric monoidal functor $\mathscr{J}^{\mathscr{V}}_G \longrightarrow \mathscr{J}^{\mathscr{V}'}_G$. Then the theory of prolongation of functors of [12, I.2.10] or [13, §3] applies. Here we have a more natural looking but equivalent form of the definition of $I^{\mathscr{V}}_{\mathscr{V}'}$, namely

$$(I^{\mathscr{V}}_{\mathscr{V}'}X)(V) = X(V) \tag{3.3}$$

for $X \in \mathscr{I}^{\mathscr{V}'}_G\mathscr{S}$ and $V \in \mathscr{V}$. However, Definition 3.2 also gives a functor $I^{\mathscr{V}'}_{\mathscr{V}}$. By the following theorem, it is an inverse isomorphism to $I^{\mathscr{V}}_{\mathscr{V}'}$, hence is left (and right) adjoint to $I^{\mathscr{V}'}_{\mathscr{V}}$ and therefore coincides with the prolongation functor. Writing $F^{\mathscr{V}}_V A$ to indicate the universe of shift desuspension functors, it follows by inspection of right adjoints and use of the inverse isomorphism property that

$$I^{\mathscr{V}}_{\mathscr{V}'}F^{\mathscr{V}'}_V A \cong F^{\mathscr{V}}_V A \quad \text{and} \quad I^{\mathscr{V}'}_{\mathscr{V}}F^{\mathscr{V}}_V A \cong F^{\mathscr{V}'}_V A \tag{3.4}$$

for $V \in \mathscr{V}$ and any based G-space A.

Returning to general collections, write $\Sigma^{\mathscr{V}} : \mathscr{T}_G \longrightarrow \mathscr{I}^{\mathscr{V}}_G\mathscr{S}$ for the suspension G-spectrum functor. The following result is analogous to [3, 2.3, 2.4].

Theorem 3.5. *Consider collections $\mathscr{V}$, $\mathscr{V}'$ and $\mathscr{V}''$.*

(i) *$I^{\mathscr{V}}_{\mathscr{V}'} \circ \Sigma^{\mathscr{V}'}$ is naturally isomorphic to $\Sigma^{\mathscr{V}}$.*
(ii) *$I^{\mathscr{V}}_{\mathscr{V}'} \circ I^{\mathscr{V}'}_{\mathscr{V}''}$ is naturally isomorphic to $I^{\mathscr{V}}_{\mathscr{V}''}$.*
(iii) *$I^{\mathscr{V}}_{\mathscr{V}}$ is naturally isomorphic to the identity functor.*
(iv) *The functor $I^{\mathscr{V}}_{\mathscr{V}'}$ commutes with smash products with based G-spaces.*
(v) *The functor $I^{\mathscr{V}}_{\mathscr{V}'}$ is strong symmetric monoidal.*

Therefore $I^{\mathscr{V}}_{\mathscr{V}'}$ is an equivalence of categories with inverse $I^{\mathscr{V}'}_{\mathscr{V}}$.[2] *Moreover, $I^{\mathscr{V}}_{\mathscr{V}'}$ is homotopy preserving, hence $I^{\mathscr{V}}_{\mathscr{V}'}$ and $I^{\mathscr{V}'}_{\mathscr{V}}$ induce inverse equivalences of the homotopy categories obtained by passing to homotopy classes of maps.*

We turn to the relationship with model structures. It is important to realize what Lemma 3.1 does not imply: a map $f : X \longrightarrow Y$ can be a weak equivalence at level $\mathbb{R}^n$ for all n but still not be a level equivalence. The point is that the H-fixed point functors do not commute with passage to orbits over $O(n)$.

Similarly, it is important to realize what the last statement of Theorem 3.5 does not imply: the functors $I^{\mathscr{V}}_{\mathscr{V}'}$ do not preserve either level equivalences or π_*-isomorphisms in general. Therefore, there is no reason to expect the homotopy categories associated to the model structures to be equivalent. However, (3.3) and the characterization of q-fibrations and acyclic q-fibrations given in 2.53 imply the following result.

Theorem 3.6. *If $\mathscr{V} \subset \mathscr{V}'$, then the functor $I^{\mathscr{V}}_{\mathscr{V}'} : G\mathscr{I}^{\mathscr{V}'}_G \longrightarrow G\mathscr{I}^{\mathscr{V}}_G$ preserves level equivalences, level fibrations, q-fibrations, and acyclic q-fibrations, and similarly for the positive analogues of these classes of maps. Therefore $(I^{\mathscr{V}'}_{\mathscr{V}}, I^{\mathscr{V}}_{\mathscr{V}'})$ is a Quillen adjoint pair relating the respective level, positive level, stable, and positive stable model structures.*

There is another way to think about change of universe. For $\mathscr{V} \subset \mathscr{V}'$, we can define new $\mathscr{V}$-model structures on the category of orthogonal G-spectra indexed on $\mathscr{I}^{\mathscr{V}'}_G$. For the $\mathscr{V}$-level model structure (or positive $\mathscr{V}$-level model structure), we define weak equivalences and fibrations by restricting attention to levels in $\mathscr{V}$; equivalently, the $\mathscr{V}$-level equivalences and fibrations are created by the forgetful functor $I^{\mathscr{V}}_{\mathscr{V}'}$. We define the $\mathscr{V}$-cofibrations of $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$ to be the G-maps that

[2] In [9], the authors overlooked that this result was already in [12]; they learned it from Hesselholt and Hovey.

satisfy the LLP with respect to the $\mathscr{V}$-level acyclic fibrations. Compare [13, 6.10]. We then let the $\mathscr{V}$-stable equivalences and the $\mathscr{V}$-fibrations be created by $I^{\mathscr{V}}_{\mathscr{V}'}$. Thus the $\mathscr{V}$-stable equivalences are the $\mathscr{V}$ π_*-isomorphisms, that is, the maps that induce isomorphisms of the homotopy groups defined using only those $V \in \mathscr{V}$ in the relevant colimits. Arguing as for the stable model structure, we obtain the following result.

Theorem 3.7. *For $\mathscr{V} \subset \mathscr{V}'$, the category $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$ of orthogonal G-spectra indexed on $\mathscr{I}_G^{\mathscr{V}'}$ and natural G-maps has a $\mathscr{V}$-stable model structure in which the functor $I^{\mathscr{V}}_{\mathscr{V}'}$ creates the $\mathscr{V}$-stable equivalences and the $\mathscr{V}$-fibrations. The acyclic $\mathscr{V}$-fibrations coincide with the $\mathscr{V}$-level acyclic fibrations, and the $\mathscr{V}$-cofibrations are the maps that satisfy the LLP with respect to the acyclic $\mathscr{V}$-fibrations. The pair $(I^{\mathscr{V}'}_{\mathscr{V}}, I^{\mathscr{V}}_{\mathscr{V}'})$ is a Quillen equivalence between $G\mathscr{I}^{\mathscr{V}}\mathscr{S}$ with its stable model structure and $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$ with its $\mathscr{V}$-stable model structure. The analogous statements for positive $\mathscr{V}$-stable model structures hold.*

Here the Quillen equivalence is easily proven using the usual characterization [13, A.2]. It is a rare example of an interesting "Quillen equivalence" of model categories that is an actual equivalence of underlying categories. There is another observation to make along the same lines.

Corollary 3.8. *For $\mathscr{V} \subset \mathscr{V}'$, the identity functor $\mathrm{Id} : G\mathscr{I}^{\mathscr{V}'}\mathscr{S} \longrightarrow G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$ is the right adjoint of a Quillen adjoint pair relating the (positive) stable model structure on $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$ to the (positive) $\mathscr{V}$-stable model structure on $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$.*

Thus the forgetful functor $I^{\mathscr{V}}_{\mathscr{V}'} : G\mathscr{I}^{\mathscr{V}'}\mathscr{S} \longrightarrow G\mathscr{I}^{\mathscr{V}}\mathscr{S}$ relating the (original) stable model structures factors through the $\mathscr{V}$-stable model structure on $G\mathscr{I}^{\mathscr{V}'}\mathscr{S}$. That is, the Quillen adjoint pair of Theorem 3.6 is the composite of the Quillen adjoint pair of Corollary 3.8 and the Quillen adjoint equivalence of Theorem 3.7.

Remark 3.9. There is yet another way to think about change of universe. Fix $\mathscr{I}_G = \mathscr{I}_G^{\mathscr{A}\ell\ell}$. Then, for any $\mathscr{V}$, the (positive) $\mathscr{V}$-stable model structure on the category $G\mathscr{I}\mathscr{S}$ is Quillen equivalent to the (positive) stable model structure on $G\mathscr{I}^{\mathscr{V}}\mathscr{S}$, and similarly for the various categories of rings and modules. However, to make sense of some of the constructions in the following sections, we must work with original categories of orthogonal $G\mathscr{I}_G^{\mathscr{V}}$-spectra indexed on $\mathscr{V}$, with their intrinsic model structures.

Remark 3.10. In addition to changes of $\mathscr{V}$, we must deal with changes of the choice of "indexing G-spaces" within a given $\mathscr{V}$, as in 1.5. Thus let $\mathscr{W} \subset \mathscr{V}$ be a cofinal set of G-inner product spaces that is closed under finite direct sums and contains the $\mathbb{R}^n$. We have a forgetful functor $I^{\mathscr{W}}_{\mathscr{V}} : \mathscr{I}_G^{\mathscr{V}}\mathscr{S} \longrightarrow \mathscr{I}_G^{\mathscr{W}}\mathscr{S}$ specified as in (3.3). It can also be specified as in Definition 3.2 and, arguing as in that definition and Theorem 3.5, $I^{\mathscr{W}}_{\mathscr{V}}$ is an equivalence of categories with inverse equivalence $I^{\mathscr{V}}_{\mathscr{W}}$. We can carry out all of our model category theory in the more general context. The functor $I^{\mathscr{W}}_{\mathscr{V}} : G\mathscr{I}^{\mathscr{V}}\mathscr{S} \longrightarrow G\mathscr{I}^{\mathscr{W}}\mathscr{S}$ preserves q-fibrations, and cofinality ensures that $I^{\mathscr{W}}_{\mathscr{V}}$ creates the stable equivalences in $G\mathscr{I}^{\mathscr{V}}\mathscr{S}$. We conclude that $(I^{\mathscr{V}}_{\mathscr{W}}, I^{\mathscr{W}}_{\mathscr{V}})$ is a Quillen equivalence.

3.2. **Change of groups.** Let H be a subgroup of G and write $\iota : H \longrightarrow G$ for the inclusion. For a G-space A, let ι^*A denote A regarded as an H-space via ι.

We want analogues for G-spectra of such space level observations as (2.2) – (2.5). Spectra mean orthogonal spectra throughout.

This involves change of universe as well as change of groups. If $\mathscr{V} = \{V\}$ is a collection of representations of G, then $\iota^*\mathscr{V} = \{\iota^*V\}$ is a collection of representations of H. According to our conventions in 2.1, G-summands of representations in $\mathscr{V}$ are in $\mathscr{V}$, but this need not be true of H-summands of representations in $\iota^*\mathscr{V}$. For example, not all H-representations are in $\iota^*\mathscr{A}\ell\ell(G)$. However, we can let $\mathscr{W}$ be the collection of H-representations that are isomorphic to summands of representations in $\iota^*\mathscr{V}$. Since $\iota^*\mathscr{V}$ is cofinal in $\mathscr{W}$ and closed under finite direct sums, Remark 3.10 applies. For example, if $\mathscr{V} = \mathscr{A}\ell\ell(G)$, then $\mathscr{W} = \mathscr{A}\ell\ell(H)$ since any H-representation is a summand of a G-representation.

To fix ideas and simplify notation, we work with $\mathscr{A}\ell\ell(H)$ when defining H-spectra, and we do not introduce notation for the change of universe functor that passes from H-spectra indexed on $\iota^*\mathscr{A}\ell\ell(G)$ to H-spectra indexed on $\mathscr{A}\ell\ell(H)$.

Definition 3.11. For a G-spectrum X, let ι^*X be the H-spectrum that is specified by $(\iota^*X)(\iota^*V) = \iota^*X(V)$ for representations V of G and is then extended to all representations of H by Remark 3.10.

Except that the statement about q-cofibrations requires inspection of cells, the following result is clear from the definitions.

Lemma 3.12. *The functor ι^* preserves level fibrations, level equivalences, q-cofibrations, π_*-isomorphisms, and q-fibrations.*

We claim that the functor ι^* has both a left and a right adjoint. On the space level, for H-spaces B, the left adjoint of ι^* is given by $G_+ \wedge_H B$ and the right adjoint is given by the G-space of H-maps $F_H(G_+, B)$. For G-spaces A and A', we have obvious identifications of H-spaces

$$\iota^*F(A, A') = F(\iota^*A, \iota^*A') \quad \text{and} \quad \iota^*(A \wedge A') = \iota^*A \wedge \iota^*A'.$$

On passage to left and right adjoints, respectively, these formally imply natural isomorphisms of G-spaces

$$(G_+ \wedge_H B) \wedge A \cong G_+ \wedge_H (B \wedge \iota^*A)$$

and

$$F(A, F_H(G_+, B)) \cong F_H(G_+, F(\iota^*A, B)),$$

and it is easy to write down explicit isomorphisms.

Proposition 3.13. *Let X be an orthogonal G-spectrum and Y be an orthogonal H-spectrum. Let $G_+ \wedge_H Y$ be the orthogonal G-spectrum specified by*

$$(G_+ \wedge_H Y)(V) = G_+ \wedge_H Y(\iota^*V)$$

for representations V of G. Then there is an adjunction

$$G\mathscr{I}\mathscr{S}(G_+ \wedge_H Y, X) \cong H\mathscr{I}\mathscr{S}(Y, \iota^*X),$$

which is a Quillen adjoint pair relating the respective (positive) level and stable model structures. Moreover, there is a natural isomorphism

$$(G_+ \wedge_H Y) \wedge X \cong G_+ \wedge_H (Y \wedge \iota^*X).$$

In particular,

$$G/H_+ \wedge X \cong G_+ \wedge_H \iota^*X.$$

Proposition 3.14. *Let X be a G-spectrum and Y be an H-spectrum. Let $F_H(G_+, Y)$ be the G-spectrum specified by*

$$F_H(G_+, Y)(V) = F_H(G_+, Y(\iota^* V))$$

for representations V of G. Then there is an adjunction

$$G\mathscr{I}\mathscr{S}(X, F_H(G_+, Y)) \cong H\mathscr{I}\mathscr{S}(\iota^* X, Y),$$

which is a Quillen adjoint pair relating the respective (positive) level and stable model structures. Moreover, there is a natural isomorphism

$$F(X, F_H(G_+, Y)) \cong F_H(G_+, F(\iota^* X, Y)).$$

In particular,

$$F(G/H_+, X) \cong F_H(G_+, \iota^* X).$$

The following fundamental result is called the Wirthmüller isomorphism, after its precursor [21], and is not in [12]. Its homotopical proof in [11] applies verbatim, and a more modern exposition is given in [17].

Theorem 3.15 (The Wirthmüller isomorphism)**.** *In the stable homotopy category of G-spectra, the canonical natural map*

$$G_+ \wedge_H Y \longrightarrow F_H(G_+, Y)$$

is an isomorphism for H-spectra Y. That is, the left adjoint of ι^* is isomorphic to its right adjoint.

Corollary 3.16. *Orbit G-spectra $\Sigma^\infty_G G/H_+$ are self-dual in the stable category.*

3.3. **Fixed point and orbit spectra.** We relate (orthogonal) G-spectra to (orthogonal) spectra via fixed point and orbit functors, just as for G-spaces.

Definition 3.17. Let $G\mathscr{I}^{\text{triv}}\mathscr{S}$ denote the category of G-spectra indexed only on trivial G-representations. We call the objects of $G\mathscr{I}^{\text{triv}}\mathscr{S}$ "naive" G-spectra, in contrast to the genuine G-spectra of $G\mathscr{I}\mathscr{S}$. The G-fixed point functor is the composite of the change of universe functor

$$G\mathscr{I}\mathscr{S} = G\mathscr{I}^{\mathscr{A}\ell\ell}\mathscr{S} \longrightarrow G\mathscr{I}^{\text{triv}}\mathscr{S}$$

and the G-fixed point functor

$$G\mathscr{I}^{\text{triv}}\mathscr{S} \longrightarrow \mathscr{I}\mathscr{S}.$$

More generally, for $H \subset G$, define $X^H = (\iota^* X)^H$.

The following fundamental result relating equivariant and nonequivariant homotopy groups is immediate from the definitions.

Proposition 3.18. *Let E be an Ω-G-spectrum. Then*

$$\pi_*^H(E) \cong \pi_*(E^H).$$

For any G-spectrum X, $\pi_*^H(X) \cong \pi_*^H(RX)$, where RX is a fibrant approximation of X in the stable or positive stable model structure.

Giving spaces trivial G-action, we obtain a functor

$$\varepsilon^* : \mathscr{I}\mathscr{S} \longrightarrow G\mathscr{I}^{triv}\mathscr{S}. \tag{3.19}$$

We then have the following fixed-point adjunction and its composite with the evident change of universe adjunction.

Proposition 3.20. *Let X be a naive G-spectrum and Y be a nonequivariant spectrum. There is a natural isomorphism*

$$G\mathscr{I}^{triv}\mathscr{S}(\varepsilon^* Y, X) \cong \mathscr{I}\mathscr{S}(Y, X^G).$$

For (genuine) G-spectra X, there is a natural isomorphism

$$G\mathscr{I}\mathscr{S}(i_*\varepsilon^* Y, X) \cong \mathscr{I}\mathscr{S}(Y, (i^* X)^G),$$

where $i_* = I_{triv}^{\mathscr{A}\ell\ell}$ and $i^* = I_{\mathscr{A}\ell\ell}^{triv}$. Both of these adjunctions are Quillen adjoint pairs relating the respective (positive) level and stable model structures.

The last statement means that passage to fixed points preserves q-fibrations and acyclic q-fibrations. We have the following observation about q-cofibrations. In the following two results, we agree to be less pedantic and to write $(-)^G$ for the composite of i^* and passage to G-fixed points. With this notation, the counit of the second adjunction is a natural G-map $i_*\varepsilon^* X^G \longrightarrow X$.

Proposition 3.21. *For a representation V and a G-space A, $(F_V A)^G = *$ unless G acts trivially on V, when $(F_V A)^G \cong F_V(A^G)$ as a nonequivariant spectrum. The functor $(-)^G$ preserves q-cofibrations, but* not *acyclic q-cofibrations.*

For the last statement, for non-trivial representations V of G, the maps $k_{0,V}$ of 2.47 are acyclic q-cofibrations, whereas $k_{0,V}^G$ is equivalent to $* \longrightarrow S$.

Warning 3.22. The last statement implies that the functor $(-)^G$ is not a Quillen left adjoint. This functor does not behave homotopically as one might expect from the results of [11]. The reason is that it does not commute with fibrant replacement, whereas all objects are fibrant in the context of [11], and we must replace G-spectra by weakly equivalent Ω-G-spectra before passing to fixed points in order to obtain the correct homotopy groups.

The following two results are in marked contrast to the situation in [11, 16], where the (categorical) fixed point functor does *not* satisfy analogous commutation relations. The point is that these results do not imply corresponding commutation results on passage to homotopy categories, in view of Warning 3.22.

Taking $V = 0$, Proposition 3.21 has the following implication.

Corollary 3.23. *For based G-spaces A,*

$$(\Sigma^\infty A)^G \cong \Sigma^\infty(A^G).$$

(This isomorphism of spectra does not *imply an isomorphism in $Ho\mathscr{I}\mathscr{S}$).*

Note that the functors i_* and ε^* are strong symmetric monoidal.

Proposition 3.24. *For G-spectra X and Y, there is a natural map of (nonequivariant) spectra*

$$\alpha : X^G \wedge Y^G \longrightarrow (X \wedge Y)^G,$$

and α is an isomorphism if X and Y are cofibrant. (This isomorphism of spectra does not *imply an isomorphism in $Ho\mathscr{I}\mathscr{S}$).*

We can obtain a more general and detailed version of Proposition 3.20. Let NH denote the normalizer of H in G and let $WH = NH/H$. We can obtain an H-fixed point functor from G-spectra to WH-spectra. It factors as a composite

$$G\mathscr{I}\mathscr{S} \longrightarrow NH\mathscr{I}\mathscr{S} \longrightarrow NH\mathscr{I}^{H\text{-triv}}\mathscr{S} \longrightarrow WH\mathscr{I}\mathscr{S}$$

of a change of group functor as in Definition 3.11, a change of universe functor, and a fixed point functor, all three of which are right adjoints.

It is useful to be more general about the last two functors. Thus let N be any normal subgroup of G, let $J = G/N$, and let $\varepsilon : G \longrightarrow J$ be the quotient homomorphism. In the situation above, we are thinking of the normal subgroup H of NH with quotient group WH.

Definition 3.25. Let $G\mathscr{I}^{N\text{-triv}}\mathscr{S}$ be the category of G-spectra indexed on N-trivial representations of G. Define

$$\varepsilon^* : J\mathscr{I}\mathscr{S} \longrightarrow G\mathscr{I}^{N\text{-triv}}\mathscr{S}$$

by regarding J-spaces (and spectra) as N-trivial G-spaces (and spectra). Define

$$(-)^N : G\mathscr{I}^{N\text{-triv}}\mathscr{S} \longrightarrow J\mathscr{I}\mathscr{S}$$

by passage to N-fixed points spacewise, $(X^N)(V) = X(V)^N$ for a J-representation V regarded as an N-trivial G-representation.

Proposition 3.26. *Let $X \in G\mathscr{I}^{N\text{-triv}}\mathscr{S}$ and $Y \in J\mathscr{I}\mathscr{S}$. There is a natural isomorphism*

$$G\mathscr{I}^{N\text{-triv}}\mathscr{S}(\varepsilon^*Y, X) \cong J\mathscr{I}\mathscr{S}(Y, X^N).$$

For (genuine) G-spectra X, there is a natural isomorphism

$$G\mathscr{I}\mathscr{S}(i_*\varepsilon^*Y, X) \cong J\mathscr{I}\mathscr{S}(Y, (i^*X)^N),$$

where $i_* = I^{\mathscr{A}\ell\ell}_{N\text{-triv}}$ and $i^* = I^{N\text{-triv}}_{\mathscr{A}\ell\ell}$. Both of these adjunctions are Quillen adjoint pairs relating the respective (positive) level and stable model structures.

Similarly, we can define orbit spectra. Here again, we must first restrict to trivial representations. However, since this change of universe functor is a right adjoint and passage to orbits is a left adjoint, the composite functor appears to be of no practical value (just as in [11]).

Definition 3.27. For $X \in G\mathscr{I}^{\text{triv}}\mathscr{S}$, define X/G by $(X/G)(V) = X(V)/G$ for an inner product space V. More generally, for $X \in G\mathscr{I}^{N\text{-triv}}\mathscr{S}$, define $X/N \in J\mathscr{I}\mathscr{S}$ by $(X/N)(V) = X(V)/N$ for a J-representation V regarded as an N-trivial G-representation.

Proposition 3.28. *Let $X \in G\mathscr{I}^{N\text{-triv}}\mathscr{S}$ and $Y \in J\mathscr{I}\mathscr{S}$. There is a natural isomorphism*

$$G\mathscr{I}^{N\text{-triv}}\mathscr{S}(X, \varepsilon^*Y) \cong J\mathscr{I}\mathscr{S}(X/N, Y).$$

This adjunction is a Quillen adjoint pair relating the respective (positive) level and stable model structures.

Remark 3.29. The left and right adjoints of ε^* in this section and of ι^* in the previous section can be regarded as special cases of a composite construction that applies to an arbitrary homomorphism $\alpha : H \longrightarrow G$ of compact Lie groups. Let $N = \text{Ker}(\alpha)$ and $K = H/N$. We have a quotient homomorphism $\varepsilon : H \longrightarrow K$ and an inclusion $\iota : K \longrightarrow G$ induced by α. Since $\alpha = \iota \circ \varepsilon$, $\alpha^* = \varepsilon^* \circ \iota^*$. Therefore, if $X \in G\mathscr{I}\mathscr{S}$ and $Y \in H\mathscr{I}^{N\text{-triv}}\mathscr{S}$, we have the composite adjunctions

$$G\mathscr{I}\mathscr{S}(G_+ \wedge_K Y/N, X) \cong H\mathscr{I}^{N\text{-triv}}\mathscr{S}(Y, \alpha^*X)$$

and

$$G\mathscr{I}\mathscr{S}(X, F_K(G_+, Y^N)) \cong H\mathscr{I}^{N\text{-triv}}\mathscr{S}(\alpha^*X, Y).$$

3.4. **Geometric fixed point spectra.** There are two G-fixed point functors on (orthogonal) G-spectra, namely the "categorical" one already defined and another "geometric" one. Because the categorical fixed point functor seems to enjoy some of the basic properties that motivated the introduction of the geometric fixed point functor in the classical setting, the discussion requires some care. We want a version of the G-fixed point functor for which the commutation relations of Corollary 3.23 and Proposition 3.24 are true, but which also preserves acyclic q-cofibrations, so that these properties remain true after passage to homotopy categories.

To establish context, we record the tom Dieck splitting, which is proven in [11]. (As an aside, there is an informative new proof due to Guillou and May).

Theorem 3.30 (tom Dieck splitting)**.** *Let A be a based G-space. Then there is a natural isomorphism*

$$(\Sigma_G^\infty A)^G \cong \bigvee_{(H)} \Sigma^\infty (EWH_+ \wedge_{WH} A^H)$$

in the stable homotopy category, where H ranges over the conjugacy classes of subgroups of the (finite) group G.

Let $\mathscr{P}$ denote the family of proper subgroups of G. Using an easy commutation relation with suspension and the observation that $(A \wedge \tilde{E}\mathscr{P})^H = *$ unless $H = G$, this implies that

$$((\Sigma_G^\infty A) \wedge \tilde{E}\mathscr{P})^G \cong (\Sigma_G^\infty (A \wedge \tilde{E}\mathscr{P}))^G \cong \Sigma^\infty A^G.$$

This suggests a standard homotopical definition of the geometric fixed point functor:

$$\Phi^G(X) \equiv (X \wedge \tilde{E}\mathscr{P})^G.$$

Any definition must give this answer in the stable homotopy category. However, [12] gives a more precise construction (denoted Φ_M^G in [9]). The essential property needed there is that Φ^G be a symmetric monoidal functor, before passage to homotopy, and the more precise construction gives that. The details of proof are less obvious than in previous sections and may be found in [12].

In this section, we work from the beginning in the general context of a normal subgroup N of G with quotient group J. The reader may wish to focus on the special case $N = G$, in which case J is the trivial group. However, G plays two quite different roles in that case, and the general case is valuable in other applications and clarifies some issues of equivariance. We need some categorical preliminaries which generalize Definition 1.21. As there, we can think in terms of Thom complexes. (Some misleading misprints are corrected in the following definition.)

Definition 3.31. Let E denote the extension

$$e \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\varepsilon} J \longrightarrow e.$$

We define a category $\mathscr{J}_E$ enriched over the category $J\mathscr{T}$ of (based) J-spaces. The objects of $\mathscr{J}_E$ are the G-representations V. We have the J-space $\mathscr{I}_E(V,W)$ of N-linear isometries $\iota^*V \longrightarrow \iota^*W$ and can form a bundle and Thom J-space as in Definition 1.21. A non-basepoint arrow $(f,x) : V \longrightarrow W$ of $\mathscr{J}_E(V,W)$ is an N-linear isometry $f : V \longrightarrow W$ together with a point $x \in W^N - f(V^N)$; composition and identity arrows are defined as in Definition 1.21. Then $\mathscr{J}_E = G\mathscr{J}$ when $N = G$ and $\mathscr{J}_E = \mathscr{J}_G$ when $N = e$. Let

$$\phi : \mathscr{J}_E \longrightarrow \mathscr{J}_J$$

be the N-fixed point J-functor. It sends the G-representation V to the J-representation V^N and sends an arrow $(f,x) : V \longrightarrow W$ to the N-fixed point arrow $(f^N, x) \in \mathscr{J}_J(V^N, W^N)$. Let

$$\nu : \mathscr{J}_J \longrightarrow \mathscr{J}_E$$

be the J-functor that sends a J-representation V to V regarded as a G-representation by pullback along ε and is given on morphism spaces $\mathscr{J}_J(V,W)$ by identity maps; this makes sense since every linear isometry $V \longrightarrow W$ is an N-map. Then

$$\phi \circ \nu = \mathrm{Id} : \mathscr{J}_J \longrightarrow \mathscr{J}_J.$$

Definition 3.32. Let $\mathscr{J}_E\mathscr{T}$ denote the category of $\mathscr{J}_E$-spaces, namely (continuous) J-functors $\mathscr{J}_E \longrightarrow \mathscr{T}_J$. Note that a $\mathscr{J}_E$-space Y has structural J-maps

$$Y(V) \wedge S^{W^N - V^N} \longrightarrow Y(W)$$

for $V \subset W$. Let

$$\mathbb{U}_\phi : \mathscr{J}_J\mathscr{T} \longrightarrow \mathscr{J}_E\mathscr{T} \quad \text{and} \quad \mathbb{U}_\nu : \mathscr{J}_E\mathscr{T} \longrightarrow \mathscr{J}_J\mathscr{T}$$

be the forgetful functors induced by ϕ and ν. Left Kan extension along ϕ and ν gives prolongation functors

$$\mathbb{P}_\phi : \mathscr{J}_E\mathscr{T} \longrightarrow \mathscr{J}_J\mathscr{T} \quad \text{and} \quad \mathbb{P}_\nu : \mathscr{J}_J\mathscr{T} \longrightarrow \mathscr{J}_E\mathscr{T}$$

left adjoint to $\mathbb{U}_\phi$ and $\mathbb{U}_\nu$. Since $\phi \circ \nu = \mathrm{Id}$, $\mathbb{U}_\nu \circ \mathbb{U}_\phi = \mathrm{Id}$ and therefore $\mathbb{P}_\phi \circ \mathbb{P}_\nu \cong \mathrm{Id}$.

With these definitions in place, we can define the geometric fixed point functors.

Definition 3.33. Define a fixed point functor $\mathrm{Fix}^N : \mathscr{J}_G\mathscr{T} \longrightarrow \mathscr{J}_E\mathscr{T}$ by sending a G-spectrum X to the $\mathscr{J}_E$-space $\mathrm{Fix}^N X$ with

$$(\mathrm{Fix}^N X)(V) = X(V)^N$$

and with evaluation J-maps

$$X(V)^N \wedge \mathscr{J}_G(V,W)^N \longrightarrow X(W)^N$$

obtained by passage to N-fixed points from the evaluation G-maps of X. Define the geometric fixed point functor

$$\Phi^N : \mathscr{J}_G\mathscr{T} \longrightarrow \mathscr{J}_J\mathscr{T}$$

to be the composite $\mathbb{P}_\phi \circ \mathrm{Fix}^N$. Define a natural J-map $\gamma : X^N \longrightarrow \Phi^N X$ of J-spectra by observing that the categorical fixed point functor can be reinterpreted as $X^N = \mathbb{U}_\nu \mathrm{Fix}^N X$ and letting γ be the map

$$\mathbb{U}_\nu \eta : X^N = \mathbb{U}_\nu \mathrm{Fix}^N X \longrightarrow \mathbb{U}_\nu \mathbb{U}_\phi \mathbb{P}_\phi \mathrm{Fix}^N X = \mathbb{P}_\phi \mathrm{Fix}^N X = \Phi^N X, \tag{3.34}$$

where $\eta : \mathrm{Id} \longrightarrow \mathbb{U}_\phi \mathbb{P}_\phi$ is the unit of the prolongation adjunction.

We have the following analogue of Proposition 3.21.

Proposition 3.35. *For a representation V of G and a G-space A,*

$$\Phi^N(F_V A) \cong F_{V^N} A^N.$$

The functor Φ^N preserves q-cofibrations and acyclic q-cofibrations.

Analogues of Corollary 3.23 and Proposition 3.24 follow readily.

Corollary 3.36. *For based G-spaces A,*

$$\Phi^N \Sigma^\infty_G A \cong \Sigma^\infty_J (A^N).$$

The functors $\mathbb{P}_\phi$ and $\mathbb{P}_\nu$ are strong symmetric monoidal, by the elementary categorical properties of prolongation [13, 3.3].

Proposition 3.37. *For G-spectra X and Y, there is a natural J-map*

$$\alpha : \Phi^N X \wedge \Phi^N Y \longrightarrow \Phi^N(X \wedge Y)$$

of J-spectra, and α is an isomorphism if X and Y are cofibrant.

In the previous section, we interpreted the homotopy groups of the categorical fixed points of a fibrant approximation of X as the homotopy groups of X. We now interpret the homotopy groups of the geometric fixed points of a cofibrant approximation of X as a different kind of homotopy groups of X. For this, we introduce homotopy groups of $\mathscr{J}_E$-spaces.

Definition 3.38. Let Y be a $\mathscr{J}_E$-space and X be a G-spectrum. Let $K \subset J$ and write $K = H/N$, where $N \subset H \subset G$.

(i) Define
$$\pi_q^K(Y) = \operatorname{colim}_V \pi_q^K \Omega^{V^N} Y(V) \ \text{ if } \ q \geq 0,$$
where V runs over the indexing G-spaces in the universe U, and
$$\pi_{-q}^K(Y) = \operatorname{colim}_{V \supset \mathbb{R}^q} \pi_0^K \Omega^{V^N - \mathbb{R}^q} Y(V) \ \text{ if } \ q > 0.$$

(ii) Define a natural homomorphism
$$\zeta : \pi_*^K(\mathbb{U}_\nu Y) \longrightarrow \pi_*^K(Y)$$
by restricting colimit systems to N-fixed indexing G-spaces.

(iii) Define
$$\rho_q^K(X) = \pi_q^K(\mathrm{Fix}^N X),$$
so that $\rho_q^K(X) = \operatorname{colim}_V \pi_q^K \Omega^{V^N} X(V)^N$ for $q \geq 0$, and similarly for $q < 0$.

(iv) Define a natural homomorphism
$$\psi : \pi_*^K(X^N) \longrightarrow \pi_*^H(X)$$
by restricting colimit systems to N-fixed indexing G-spaces W, using
$$(\Omega^W X(W)^N)^K \cong (\Omega^W X(W))^H.$$

(v) Define a natural homomorphism
$$\omega : \pi_*^H(X) \longrightarrow \rho_*^K(X)$$
by sending an element of $\pi_q^H(X)$, $q \geq 0$, that is represented by an H-map $f : S^q \wedge S^V \longrightarrow X(V)$ to the element of $\rho_q^K(X)$ that is represented by the K-map $f^N : S^q \wedge S^{V^N} \longrightarrow X(V)^N$, and similarly for $q < 0$.

Define π_*-isomorphisms of $\mathscr{J}_E$-spaces and ρ_*-isomorphisms of G-spectra in the evident way.

If X is an Ω-G-spectrum, then ψ is a natural isomorphism. In this case, we may identify ζ and ω in view of the following immediate observation.

Lemma 3.39. *The homomorphism*

$$\zeta : \pi_*^K(X^N) = \pi_*^K(\mathbb{U}_\nu \mathrm{Fix}^N X) \longrightarrow \pi_*^K(\mathrm{Fix}^N X) = \rho_*^K(X)$$

is the composite of $\psi : \pi_*^K(X^N) \longrightarrow \pi_*^H(X)$ and $\omega : \pi_*^H(X) \longrightarrow \rho_*^K(X)$.

The following complementary observation is proven by examination of colimits.

Lemma 3.40. *For J-spectra Z, the homomorphism*

$$\zeta : \pi_*^K(Z) = \pi_*^K(\mathbb{U}_\nu \mathbb{U}_\phi Z) \longrightarrow \pi_*^K(\mathbb{U}_\phi Z)$$

is an isomorphism.

Via the naturality of ζ, this leads to the following identification of γ_*, where $\gamma = \mathbb{U}_\nu \eta$ as in (3.34). Observe that the unit η of the prolongation adjunction for ϕ induces a natural map

$$\eta_* : \rho_*^K(X) = \pi_*^K(\mathrm{Fix}^N X) \longrightarrow \pi_*^K(\mathbb{U}_\phi \mathbb{P}_\phi \mathrm{Fix}^N X) \xrightarrow{\zeta^{-1}} \pi_*^K(\mathbb{P}_\phi \mathrm{Fix}^N X) = \pi_*^K(\Phi^N X).$$

Lemma 3.41. *Let $K = H/N$, where $N \subset H$. For Ω-G-spectra X, the map $\gamma_* : \pi_*^K(X^N) \longrightarrow \pi_*^K(\Phi^N X)$ is the composite*

$$\pi_*^K(X^N) \cong \pi_*^H(X) \xrightarrow{\omega} \rho_*^K(X) \xrightarrow{\eta_*} \pi_*^K(\Phi^N X).$$

We have the following basic identification of homotopy groups.

Proposition 3.42. *The map $\eta_* : \rho_*^K(X) \longrightarrow \pi_*^K(\Phi^N X)$ is an isomorphism for cofibrant G-spectra X.*

Corollary 3.43. *If $f : X \longrightarrow Y$ is a π_*-isomorphism of G-spectra, then f is a ρ_*-isomorphism.*

To see that the geometric fixed point functor bears the homotopical relationship to the categorical fixed point functor suggested by the tom Dieck isomorphism (or [11, II§3]), we need the following notations and lemmas.

Notations 3.44. Let $\mathscr{F} = \mathscr{F}[N]$ be the family of subgroups of G that do not contain N; when $N = G$, this is the family of proper subgroups of G. let $E\mathscr{F}$ be the universal $\mathscr{F}$-space, and let $\tilde{E}\mathscr{F}$ be the cofiber of the quotient map $E\mathscr{F}_+ \longrightarrow S^0$ that collapses $E\mathscr{F}$ to the non-basepoint. Then $(\tilde{E}\mathscr{F})^H = S^0$ if $H \supset N$ and $(\tilde{E}\mathscr{F})^H$ is contractible if $H \in \mathscr{F}$. The map $S^0 \longrightarrow \tilde{E}\mathscr{F}$ induces a natural map $\lambda : X \longrightarrow X \wedge \tilde{E}\mathscr{F}$ of G-spectra.

Although trivial to prove, the following lemma is surprisingly precise.

Lemma 3.45. *For G-spectra X, the map*

$$\Phi^N \lambda : \Phi^N X \longrightarrow \Phi^N(X \wedge \tilde{E}\mathscr{F})$$

is a natural isomorphism of J-spectra.

Proof. For G-spaces A, $\mathrm{Fix}^N(X \wedge A) \cong (\mathrm{Fix}^N X) \wedge A^N$. Since $(\tilde{E}\mathscr{F})^N = S^0$, the conclusion follows. □

Lemma 3.46. *Let $K = H/N$, where $N \subset H$. For cofibrant G-spectra X, the map $\omega : \pi_*^H(X \wedge \tilde{E}\mathscr{F}) \longrightarrow \rho_*^K(X \wedge \tilde{E}\mathscr{F})$ is an isomorphism.*

The following analogue of [11, II.9.8] gives an isomorphism in the homotopy category Ho$J\mathscr{S}$ between the geometric N-fixed point functor and the composite of the categorical N-fixed point functor with the smash product with $\tilde{E}\mathscr{F}$. Let $\xi : X \longrightarrow RX$ be a fibrant replacement functor on G-spectra, so that ξ is an acyclic cofibration and RX is an Ω-G-spectrum.

Proposition 3.47. *For cofibrant G-spectra X, the diagram*

$$R(X \wedge \tilde{E}\mathscr{F})^N \xrightarrow{\gamma} \Phi^N R(X \wedge \tilde{E}\mathscr{F}) \xleftarrow{\Phi^N(\xi\lambda)} \Phi^N(X)$$

displays a pair of natural π_*-isomorphisms of J-spectra.

Therefore, in the stable homotopy category of J-spectra, we have an equivalence

$$\Phi^N X \simeq (X \wedge \tilde{E}\mathscr{F}[N])^N \tag{3.48}$$

for G-spectra X. We have given a natural geometric definition of Φ^N and have derived (3.48) from that definition.

An important role of the original geometric fixed point functor was to prove an equivalence between the homotopy category of J-spectra indexed on U^N and the homotopy category of G-spectra indexed on U that are concentrated over N, namely G-spectra X such that $\pi_*^H(X) = 0$ unless H contains N.

Theorem 3.49. *A G-spectrum X is concentrated over N if and only if the map $\lambda : X \longrightarrow X \wedge \tilde{E}\mathscr{F}[N]$ is a weak equivalence. Smashing with $\tilde{E}\mathscr{F}[N]$ defines an equivalence of categories from $Ho\mathscr{F}'[N]\mathscr{I}\mathscr{S}$ to the full subcategory of G-spectra concentrated over N in $HoG\mathscr{I}\mathscr{S}$.*

Theorem 3.50. *There is an adjoint equivalence from $HoJ\mathscr{I}\mathscr{S}$ to the full subcategory of G-spectra concentrated over N in $HoG\mathscr{I}\mathscr{S}$.*

By the last statement of Theorem 2.90, for a G-spectrum X concentrated over N and any J-spectrum Y,

$$\lambda^* : [\tilde{E}\mathscr{F}[N] \wedge \varepsilon^\# Y, X]_G \longrightarrow [\varepsilon^\# Y, X]_G \cong [Y, X^N]_J \tag{3.51}$$

is an isomorphism. This gives the required adjunction, and its unit and counit are proven to be equivalences in [11, II.§9].

3.5. ***N*-free *G*-spectra and the Adams isomorphism.** Following [11, II§2], which is clarified by the model theoretic framework, we relate families to change of universe and use this relation to describe N-free (orthogonal) G-spectra and state the Adams isomorphism, which is one of the deeper foundational results in equivariant stable homotopy theory.

Theorem 3.52. *Let $i : U' \longrightarrow U$ be an inclusion of G-universes and consider the family $\mathscr{F} = \mathscr{F}(U, U')$ of subgroups H of G such that there exists an H-linear isometry $U \longrightarrow U'$.*

(i) *$H \in \mathscr{F}$ if and only if U is H-isomorphic to U'.*
(ii) *$\mathscr{I}(U, U')$ is a universal $\mathscr{F}$-space.*
(iii) *$i_* : Ho\mathscr{F}\mathscr{S}^{U'} \longrightarrow Ho\mathscr{F}\mathscr{S}^{U}$ is an equivalence of categories.*

Now return to the consideration of a normal subgroup N of G with quotient group J. Let U be a complete G-universe and let $U' = U^N$. Using these universes, the results of §3.1 allow us to transport the conclusion of the previous theorem to G-spectra.

Definition 3.53. Define $\mathscr{F}(N)$ to be the family of subgroups H of G such that $H \cap N = e$. (By contrast, $\mathscr{F}[N]$ is the family of subgroups H such that H does not contain N. Clearly $\mathscr{F}(N) \subset \mathscr{F}[N]$, with equality only if $N = e$). An $\mathscr{F}(N)$-spectrum indexed on either U or U^N is called an N-free G-object, and an $\mathscr{F}(N)$-cell complex is called an N-free G-cell complex.

Thus an N-free G-cell complex is built up out of cells of orbit types G/H such that $H \cap N = e$. This correctly captures the intuition. The following elementary observation [11, II.2.4] ties things together.

Lemma 3.54. *The families $\mathscr{F}(U, U^N)$ and $\mathscr{F}(N)$ are the same.*

Theorem 3.55. *For a normal subgroup N of G,*

$$i_* : Ho\mathscr{F}(N)\mathscr{I}\mathscr{S}^{U^N} \longrightarrow Ho\mathscr{F}(N)\mathscr{I}\mathscr{S}^{U}$$

is an equivalence of categories.

In either universe, we can identify $\mathrm{Ho}\mathscr{F}(N)\mathscr{I}\mathscr{S}$ with the full subcategory of N-free G-spectra in $\mathrm{Ho}G\mathscr{I}\mathscr{S}$. The previous result is summarized by the slogan that "N-free G-spectra live in the N-trivial universe". It gives

$$[X/N, Y]_J \cong [X, \varepsilon^* Y]_G \cong [i_* X, \varepsilon^\# Y]_G \tag{3.56}$$

for an N-free G-spectrum X and any G-spectrum Y, both indexed on U^N. We can ask about the behavior with the order of variables reversed, and the Adams isomorphism ([1, 11]) relating the orbit and fixed point functors gives the answer. On homotopy categories (with G finite) there is a natural isomorphism

$$X/N \cong (i_* X)^N \tag{3.57}$$

for an N-free G-spectrum X indexed on U^N. Use of i_* to pass to the complete universe before taking fixed points is essential. This result is proven for LMS G-spectra X in [11, II§7], but the conclusion carries over to orthogonal G-spectra. Using standard adjunctions, this implies that

$$[Y, X/N]_J \cong [\varepsilon^\# Y, i_* X]_G. \tag{3.58}$$

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