

MILLER NORM FUNCTOR

Note Title

10/27/2010

BAG | MUG

Let (\underline{M}, \otimes) be a symm. monoidal category
 A a finite set

$$M: A \rightarrow \text{ob}(\underline{M}) \Rightarrow \bigotimes_{\alpha \in A} M_{\alpha} =: \bigotimes_A M$$

Suppose a finite gp G acts on A . Does G
act on $\bigotimes_A M$?

Tensor product is natural in A so

$$\begin{array}{ccc} A \xrightarrow{\theta} A' & \text{induces} & \bigotimes_{A'} M_{\theta} \longrightarrow \bigotimes_A M \\ \downarrow & & \downarrow \\ M & & M \end{array}$$

There is a 1-cocycle condition

This does not give us what we want

Naturality in M : $M \xrightarrow{\theta} M'$ gives

$$\bigotimes_A M \longrightarrow \bigotimes_{A'} M'$$

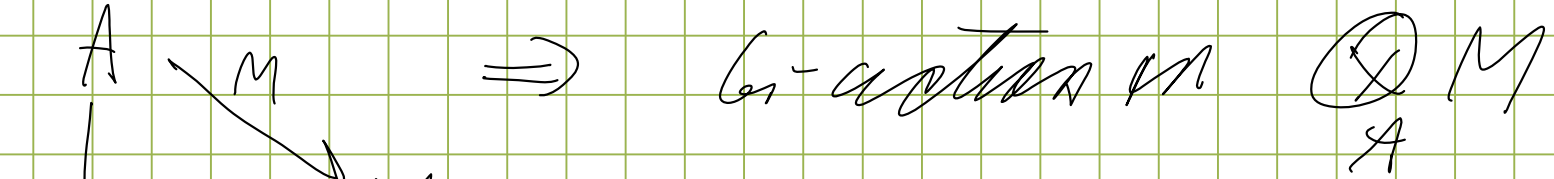
Let $M' = M_g$: For each $a \in A$, $g \in G$ we want

$$M_a \longrightarrow M_{ga} \Rightarrow \bigotimes_A M \xrightarrow{\theta} \bigotimes_A M$$

if certain coherence conditions are met.

category $G \cdot A = \mathcal{B}_A G$ Translation category

$$\text{ob}(G \cdot A) = A \quad G \cdot A(a, a') = \{g \in G : ga = a'\}$$



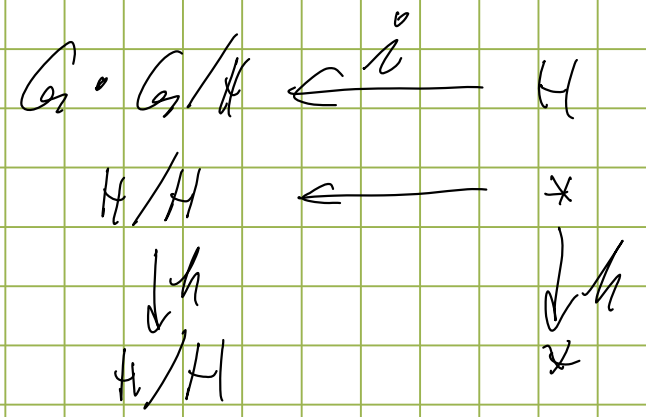
$$\mathbb{P}_A G := G \cdot A \xrightarrow{\bar{M}} M \quad \mathbb{P}_*^{\otimes}: M^{G \cdot A} \rightarrow M^G$$

$$\mathbb{P}G := G$$

$$(\mathbb{p} \circ \mathbb{q})_*^{\otimes} = \mathbb{p}_*^{\otimes} \circ \mathbb{q}_*^{\otimes}$$

$$\mathbb{p}_*^{\otimes} (M \otimes N) \cong \mathbb{p}_*^{\otimes} M \otimes \mathbb{p}_*^{\otimes} N$$

Example Let $A = G/H$



$$\begin{array}{ccc}
 M^{G \cdot G/H} & \xrightarrow{i^*} & M^H \\
 \downarrow \text{①} & & \\
 M^G & \xleftarrow{N_{G/H}^{\text{Norm}} \text{ (Functor)}} &
 \end{array}$$

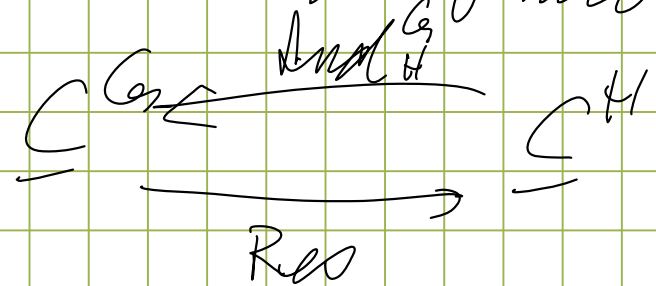
\Rightarrow

i^* is an equivalence
of categories
 i^* equiv

$N_{G/H}^{\text{Norm}}$ depends on choice of quasi-inverse of i^* ,
but any two such are canonically isom

Norm and induction , I

\underline{C} cat with finite \mathbb{H}

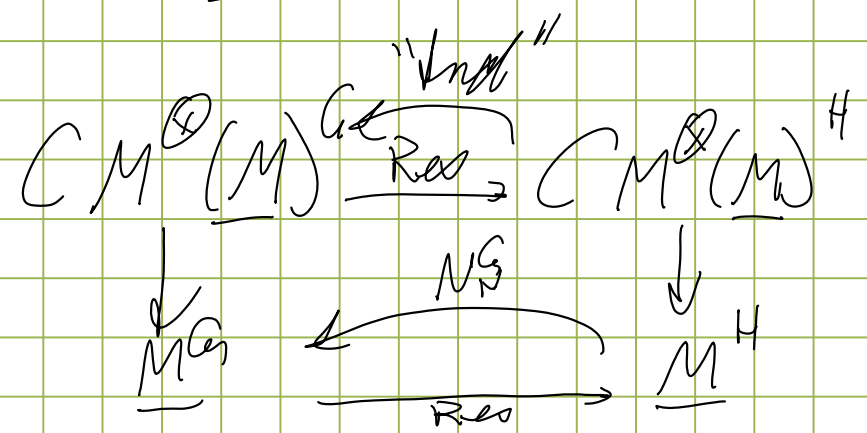


$\mathbb{H} \subset G_1$ finite grp

$\text{Ind}_{\mathbb{H}}^{G_1} = \text{left adjoint}$

$$\text{Ind}_{\mathbb{H}}^{G_1} X = \coprod_{G_1/\mathbb{H}} X$$

e.g. $\underline{C} = \text{CM}^{\otimes}(\underline{M})$



There are adjunction morphisms

$$\underline{P}: N_H^G \text{Res}_H^G R \longrightarrow R \quad \text{"power operation"}$$

$$\underline{i}: X \longrightarrow \text{Res}_H^G N_H^G X$$

Example $\underline{M} = R$ -modules with \otimes

$$CM^{\otimes}(M) = \text{comm } R\text{-algebras}$$

$$N_H^G \text{Sym}(V) = \text{Sym}(\text{Ind}_H^G V)$$

Example $\underline{M} = \underline{hA} =$ stable category with \wedge

$\text{CMP } \underline{M} =$ comm category ring spectra

\downarrow If H (a subgroup of G) acts on X with ring spectrum $H_x(X)$ torsion free then

$$H_x(N_H^G X) \cong N_H^G H_x(X)$$

Example $\underline{M} = \underline{A} =$ category of orth spectra with \wedge

$$N_H^G : \underline{A}^H \longrightarrow \underline{A}^G$$

Hesselholt-Hovey result (also due to Mandell-May)

says a G -spectrum is the same as a G -object in \underline{A}

Fig. $N_H^G S^{-V} = S^{-\text{Ind}_H^G V}$ for an H -rep V

$$X = \text{hocolim}_{\rightarrow} S^{-V} \wedge X_V$$

$$N_H^G(X) = \text{hocolim}_{\rightarrow} S^{-\text{Ind}_H^G V} \wedge N_H^G(X_V)$$

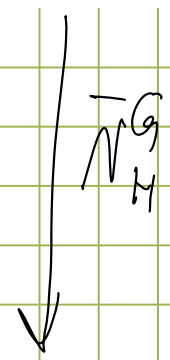
Internal norm

$V = H$ -rep

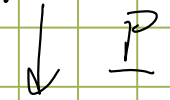
$R = G$ -comm ring spectrum

$$\Pi_V^H(\text{Res}_H^G R) = [S^V, \text{Res}_H^G R]^H$$

(Quotient
Mod)



$$\left[S \xrightarrow{\text{Ind}_H^G V}, \text{NG}_H \text{ Res}_H^G R \right]_{G_H}$$



$$\text{NG}_H \xrightarrow{\text{Ind}_H^G V} (R)$$

$$= [S \xrightarrow{\text{Ind}_H^G V}, R]_{G_H}$$

$MU^{(G)}$

$$H = C_2$$

$$MU_{\mathbb{R}} = MR$$

Araki
Sandhu
Hu-Krueger

$MU_{\mathbb{R}}$ is C_2 -comm ring spectrum

We will denote it by MU

$$H_* (MU) = \mathbb{Z} [x_1, x_2, \dots] \quad (x_i | = 2i)$$

C_2 -action

$$x_i \mapsto \bar{x}_i = (-1)^i x_i$$

$\sigma = \text{sign rep.}$

$$H_* MU = \text{sym} \left(\bigoplus_{i \geq 0} \sigma^i [2i] \right)$$

For $C_2 \leq G$ we have $N_{C_2}^G(MU)$, a G -comm ring spectrum

In particular $G_1 = G_m$

$$H_x(N_2^{2n} MV) = \text{Sym} \left(\bigoplus_{j \geq 0} \text{Ind}_2^{2n} \otimes^j [2j] \right)$$

For a generator γ of G , $\gamma^n \gamma_j = (-1)^j \gamma_j$

$$\rightarrow = \mathbb{Z} [G \cdot \gamma_1, G \cdot \gamma_2, \dots]$$

Norm + induction II

Suppose (M, \otimes) has finite co-products distributed across \otimes

$$N_H^G \left(\coprod_{i \in I} M_i \right) = ?$$

assume H is central in G

G is not acting on indexing set I

G acts on $\text{Map}(G/H, I)$

Pick one elt in each orbit, a set F

$$N_H^G \left(\coprod_{i \in I} M_i \right) \cong \prod_{f \in F} \text{Ind}_{\text{Stab}(f)}^G \left(N_H^{\text{Stab}(f)} \left(\bigotimes_{g \in \text{Stab}(f)} M_{f(g)} \right) \right)$$

~~Eg~~ $[G:H]=2 : \text{Map}(G/H, I) = \{(i, j) : i, j \in I\}$

Totally ordered I and let

$$F = \{(i, j) : i \leq j\}$$

$$\begin{aligned} N_H^G \left(\coprod_i M_i \right) &= \coprod_i N_H^G(M_i) \quad \text{if } i=j, \text{Stab} = G \\ &\quad \coprod_{i < j} \text{Ind}_H^G(M_i \otimes M_j) \quad \text{if } i < j, \text{Stab} = \{ \end{aligned}$$

Example $[S^{iP_H}] = \bigvee_{i \geq 0} S^{ijP_H}$

is an associative H -ring structure, but not a comm ring spectrum (despite appearances because the quadratic construction behaves badly)

Assume again that H is central in G

$$N_H^G \left(\bigvee_{i \geq 0} S^{ijP_H} \right) = \bigvee_{\beta \in F} \text{Ind}_{\text{Stab}(q)}^{G_\beta} \left(N_H^{\text{Stab}(q)} \left(\bigwedge_{g \in G_\beta / \text{Stab}(q)} S^{\beta(g) i P_H} \right) \right)$$

~~u~~

$$X_H = N_H^{\text{Stab}(g)} \sum |B| \cdot P_H = \sum |B| \cdot P_{\text{Stab}(g)}$$

$$\text{where } |B| = \sum_{g \in G / \text{Stab}(g)} b(g)$$

= wedge of regular star cells,
isotropic if $H \neq I$.

HRM Baulinski mess is on his website.
it neglects to require H central in G