MILLER NORM FUNCTOR

Note Title 10/27/2010

$\boxed{BAG \mid MUG}$

Let $(\underline{M}, \otimes)$ be a symm. monoidal category

A a finite set

$$M : A \to ob(\underline{M}) \Rightarrow \bigotimes_{\alpha \in A} M_\alpha =: \bigotimes_A M$$

Suppose a finite gp G acts on A. Does G act on $\bigotimes_A M$?

Tensor product is natural in A so

$$\begin{array}{l} A \xrightarrow{\theta} A' \\ \qquad\ \downarrow \\ \qquad\ \underline{M} \end{array} \quad \text{induces} \quad \bigotimes_{A'} M_\theta \longrightarrow \bigotimes_{A} M$$

There is a 1-cocycle condition

This does not give us what we want

Naturality in M: $M \xrightarrow{\theta} M'$ gives

$$\bigotimes_A M \longrightarrow \bigotimes_{A'} M'$$

Set $M' = M_g$: For each $a \in A$, $g \in G$, we want

$$M_a \longrightarrow M_{ga} \Rightarrow \bigotimes_A M \xrightarrow{\text{"}g\text{"}} \bigotimes_A M$$

if certain coherence conditions are met.

category $G \cdot A = B_A G$ translation category

$\mathrm{ob}\,(G \cdot A) = A$ $\quad G \cdot A(a, a') = \{ g \in G : ga = a' \}$

$$\begin{tikzcd} A \arrow[d] \arrow[dr, "M"] & \\ B_A G := G \cdot A \arrow[r, dashed, "\overline{M}"'] \arrow[d] & \underline{M} \\ BG := G & \end{tikzcd}$$

$\Longrightarrow$ G-action on $\bigotimes_A M$

$$p_*^{\otimes} : \underline{M}^{G\cdot A} \longrightarrow \underline{M}^{G}$$

$$(p \circ q)_*^{\otimes} = p_*^{\otimes} \circ q_*^{\otimes}$$

$$p_*^{\otimes}(M \otimes N) \cong p_*^{\otimes} M \otimes p_*^{\otimes} N$$

Example Let $A = G/H$

$$\begin{tikzcd} G \cdot G/H & H \arrow[l, "i"'] \\ H/H \arrow[d, "h"] & * \arrow[l] \arrow[d, "h"] \\ H/H & * \end{tikzcd}$$

$$\begin{tikzcd}
M^{G \cdot G/H} \arrow[rr, "i^*"] \arrow[d, "p_*"'] & & M^H \arrow[dll, "N_H^G \text{ (Norm Functor)}"] \\
M^G & &
\end{tikzcd}$$

$\Longrightarrow$ i is an equivalence of categories

i^* equiv

N_H^G depends on choice of quasi-inverse of i^*, but any two such are canonically iso

Norm and induction, I

$\underline{C}$ cat with finite $\coprod$

$$\begin{tikzcd} \underline{C}^G \arrow[r, shift right, "\mathrm{Res}"'] & \underline{C}^H \arrow[l, shift right, "\mathrm{Ind}_H^G"'] \end{tikzcd}$$

$H \subset G$ finite gp

Ind_H^G = left adjoint

$\mathrm{Ind}_H^G X = \coprod_{G/H} X$

e.g. $\underline{C} = CM^{\otimes}(\underline{M})$

$$\begin{tikzcd} CM^{\otimes}(\underline{M})^G \arrow[r, shift right, "\mathrm{Res}"'] \arrow[d] & CM^{\otimes}(\underline{M})^H \arrow[l, shift right, "\text{``}\mathrm{Ind}\text{''}"'] \arrow[d] \\ \underline{M}^G \arrow[r, shift right, "\mathrm{Res}"'] & \underline{M}^H \arrow[l, shift right, "N_H^G"'] \end{tikzcd}$$

There are adjunction morphisms

$$\underline{P} : N_H^G \operatorname{Res}_H^G R \longrightarrow R \quad \text{"power operation"}$$

$$i : X \longrightarrow \operatorname{Res}_H^G N_H^G X$$

<u>Example</u> $\underline{M}$ = R-module with $\otimes$

$$\mathrm{CM}^{\otimes}(\underline{M}) = \text{comm } R\text{-algebras}$$

$$N_H^G \operatorname{Sym}(V) = \operatorname{Sym}(\operatorname{Ind}_H^G V)$$

Example $\underline{M} = h\mathcal{S}$ = stable htpy category with $\wedge$

$CM^{\otimes}\underline{M}$ = comm htpy ring spectra

$\Downarrow$ If H (a subgp of G) acts on ring spectrum X with $H_*(X)$ torsion free then

$$H_*(N_H^G X) \cong N_H^G H_*(X)$$

Example $\underline{M} = \underline{\mathcal{S}}$ = category of orth spectra with $\wedge$

$$N_H^G : \underline{\mathcal{S}}^H \longrightarrow \underline{\mathcal{S}}^G$$

Hesselholt-Hovey hustle (also due to Mandell-May)
says a G-spectrum is the same as a
G-object in $\underline{S}$

Eg. $\quad N_H^G S^{-V} = S^{-\mathrm{Ind}_H^G V}$ for an H-rep V

$$X = \underrightarrow{\mathrm{hocolim}}\ S^{-V} \wedge X_V$$

$$N_H^G(X) = \underrightarrow{\mathrm{hocolim}}\ S^{-\mathrm{Ind}_H^G V} \wedge N_H^G(X_V)$$

Internal norm

$V = H$-rep

$R = G$-comm ring spectrum

$$\pi_V^H(\mathrm{Res}_H^G R) = [S^V, \mathrm{Res}_H^G R]^H$$

$$
\begin{tikzcd}
\begin{pmatrix}\text{Greenlees}\\ \text{May}\end{pmatrix} \arrow[d, "\bar{N}_H^G"] & {\left[S^{\mathrm{Ind}_H^G V}, N_H^G \operatorname{Res}_H^G R\right]^G} \arrow[d, "\underline{P}"] \\
T^G_{\mathrm{Ind}_H^G V}(R) = & {\left[S^{\mathrm{Ind}_H^G V}, R\right]^G}
\end{tikzcd}
$$

$MU^{((G))}$ $\qquad H = C_2$ $\qquad MU_{\mathbb{R}} < M\mathbb{R}$ $\qquad$ { Araki, Landweber, Hu-Kriz

$MU_{\mathbb{R}}$ is C_2-comm ring spectrum

We will denote it by MU

$H_*(MU) = \mathbb{Z}[x_1, x_2 \dots]$ $\qquad |x_i| = 2i$

C_2-action $\qquad x_j \longmapsto \bar{x}_j = (-1)^j x_j$

σ = sign rep.

$$H_* MU = \mathrm{Sym}\left(\bigoplus_{j \geq 0} \sigma^j [2j]\right)$$

For $C_2 \subset G$ we have $N_{C_2}^{G}(MU)$, a G-comm ring spectrum

In particular $G \cong C_{2n}$

$$H_*\left(N_2^{2n} MU\right) = \mathrm{Sym}\left(\bigoplus_{j>0} \mathrm{Ind}_2^{2n} \sigma^j [2j]\right)$$

For a generator γ of G, $\gamma^n x_j = (-1)^j x_j$

$$= \mathbb{Z}[G \cdot x_1, G \cdot x_2 \ldots]$$

Norm + induction II

Suppose $(\underline{M}, \otimes)$ has finite co-products distributing across $\otimes$

$$N_H^G\left(\coprod_{i\in I} M_i\right) = ?$$

assume H is central in G
(G is <u>not</u> acting on indexing set I)

G acts on $\mathrm{Map}(G/H, I)$

Pick one elt in each orbit, a set F

$$N_H^G\left(\coprod_{i\in I} M_i\right) \cong \prod_{f\in F} \mathrm{Ind}_{\mathrm{Stab}(f)}^G\left(N_H^{\mathrm{Stab}(f)} \bigotimes_{g\in \mathrm{Stab}(f)\backslash G} M_{f(g)}\right)$$

Eg. $[G:H]=2$: $\mathrm{Map}(G/H, I) = \{(i,j) : i,j\in I\}$

Totally order I and let

$$F = \{(i,j) : i \leq j\}$$

$$N_H^G\left(\coprod_i M_i\right) = \coprod_i N_H^G(M_i) \quad \text{If } i=j,\ \mathrm{Stab}=G$$

$$\amalg \coprod_{i<j} \mathrm{Ind}_H^G(M_i \otimes M_j) \quad i<j\ \ \mathrm{Stab}=H$$

<u>Example</u> $\mathbb{S}[S^{i\rho_H}] = \bigvee_{i\geq 0} S^{i\rho_H}$

is an associative H-ring structure, but not a comm ring spectrum (despite appearances because the quadratic construction behaves badly)

Assume again that H is central in G

$$N_H^G\Big(\bigvee_{i\geq 0} S^{i\rho_H}\Big) = \bigvee_{f\in F} \mathrm{Ind}_{\mathrm{Stab}(f)}^G \underbrace{\Big(N_H^{\mathrm{Stab}(f)}\Big(\bigwedge_{g\in G/\mathrm{Stab}(f)} S^{f(g)\, i\rho_H}\Big)\Big)}_{X_f^H}$$

$$X_f = N_H^{Stab(f)} S^{|f| \cdot \rho_H} = S^{|f| \rho_{Stab(f)}}$$

$$\text{where } |f| = \sum_{g \in G/Stab(f)} f(g)$$

= wedge of regular slice cells, isotropic if $H \neq 1$.

HRM Bourbaki mss is on his website.
It neglects to require H central in G