

PURITY IN CHROMATICALLY LOCALIZED ALGEBRAIC K -THEORY

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ABSTRACT. We prove a purity property in telescopically localized algebraic K -theory of ring spectra: For $n \geq 1$, the $T(n)$ -localization of $K(R)$ only depends on the $T(0) \oplus \cdots \oplus T(n)$ -localization of R . This complements a classical result of Waldhausen in rational K -theory. Combining our result with work of Clausen–Mathew–Naumann–Noel, one finds that $L_{T(n)}K(R)$ in fact only depends on the $T(n-1) \oplus T(n)$ -localization of R , again for $n \geq 1$. As consequences, we deduce several vanishing results for telescopically localized K -theory, as well as an equivalence between $K(R)$ and $\mathrm{TC}(\tau_{\geq 0}R)$ after $T(n)$ -localization for $n \geq 2$.

CONTENTS

1. Introduction and results	2
2. Preliminaries	4
2.1. Preliminaries from chromatic homotopy theory	4
2.2. Some localization functors	6
2.3. Localizing invariants and K -theory	8
3. Proof of Theorem A	9
3.1. The highly connective case	9
3.2. A truncating property	9
3.3. The general case	11
4. Consequences and examples	14
4.1. Direct consequences	14
4.2. Examples of vanishing results	15
4.3. Examples of purity	18
4.4. Consequences for $K(1)$ -local K -theory	19
4.5. Consequences for $T(n)$ -local K -theory for $n \geq 2$	21
References	22

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1. INTRODUCTION AND RESULTS

It is an interesting question how algebraic K -theory interacts with the chromatic filtration on the ∞ -category of spectra, which arises through the height filtration on the moduli stack \mathcal{M}_{FG} of formal groups. This paper is concerned with precisely such interactions.

The paradigmatic starting point is rationalization. Waldhausen [Wal78] proved that a rational equivalence between connective ring spectra which is a π_0 -isomorphism induces an equivalence in rational algebraic K -theory. An important example is the map $\mathbb{S} \rightarrow \mathbb{Z}$ from the sphere spectrum to the integers, which hence induces an equivalence $K(\mathbb{S})_{\mathbb{Q}} \simeq K(\mathbb{Z})_{\mathbb{Q}}$ of rational K -theory spectra. The parametrized h-cobordism theorem [WJR13] implies that $K(\mathbb{S})$, which is canonically equivalent to Waldhausen's A -theory of a point $A(*)$, controls significant information about high-dimensional manifold topology. The above equivalence of rational K -theory spectra then allows one to import Borel's computation of $K(\mathbb{Z})_{\mathbb{Q}}$ into geometric topology, see for instance [FH78].

From the viewpoint of chromatic homotopy theory, rationalization is the first in a whole sequence of localizations. On the category of p -local spectra, rationalization is the Bousfield localization with respect to $T(0) \stackrel{\text{def}}{=} \mathbb{S}[\frac{1}{p}]$. By the periodicity theorem of Hopkins–Smith [HS98], one can iteratively define higher analogues of multiplication by p : A self-map v_1 of \mathbb{S}/p , a self-map v_2 on $\mathbb{S}/(p, v_1)$, and in general a self map v_{n+1} on $\mathbb{S}/(p, v_1, \dots, v_n)$. We denote by $T(n) = \mathbb{S}/(p, v_1, \dots, v_{n-1})[v_n^{-1}]$ the colimit of a v_n -self map. This defines the class of $T(n)$ -acyclic spectra, and one obtains a corresponding Bousfield localization functor $L_{T(n)}$. The $T(n)$ -acyclic spectra in turn can be related to the acyclic spectra for the better-known Morava K -theories $K(n)$ at the prime p : Every $T(n)$ -acyclic spectrum is $K(n)$ -acyclic. The converse implication is known as the telescope conjecture. This conjecture is known for $n = 1$, for MU -modules, and for ring spectra. We remark that the localizations at the $K(n)$ describe the layers of the chromatic filtration. As these layers are related to representations of the Morava stabilizer group, a p -adic Lie group, they are also accessible by arithmetic means.

The main result, proved jointly between this paper and the related work [CMNN20a], concerns the behavior of algebraic K -theory with respect to $T(n)$ -localizations, and yields the following purity statement.

Theorem A. *Let $n \geq 1$, and let $A \rightarrow B$ be a map of ring spectra which is a $T(n-1) \oplus T(n)$ -equivalence. Then $K(A) \rightarrow K(B)$ is a $T(n)$ -equivalence.*

In this paper, we prove, as Theorem 3.8, that if $A \rightarrow B$ is a $T(0) \oplus T(1) \oplus \dots \oplus T(n)$ -equivalence, then $K(A) \rightarrow K(B)$ is a $T(n)$ -equivalence;¹ this answers a question from an earlier version of this paper, and suffices for many, but not all applications explored here. Using Theorem 3.8, the work [CMNN20a] proves the complementary statement that $L_{T(i)}K(L_{T(j)}\mathbb{S}) = 0$ for $i \geq j + 2$. Theorem A then follows by combining these two assertions, as we explain in the body of the text.

For $n = 1$ and a $K(1)$ -acyclic ring spectrum A , Theorem A (or Theorem 3.8) implies that the natural map induces an equivalence

$$L_{K(1)}K(A) \xrightarrow{\simeq} L_{K(1)}K(A[\frac{1}{p}]).$$

In the case where A is an $H\mathbb{Z}$ -algebra, this was previously shown by Bhatt–Clausen–Mathew [BCM20] using arithmetic techniques, in particular the theory of prismatic cohomology and

¹Moreover, if $n \geq 2$ then one can drop $T(0)$.

its relationship with topological cyclic homology, [BMS19, BS19]. Our results give a different proof of this fact, purely relying on tools from K -theory and homotopy theory, and a generalization to higher chromatic heights.

To put Theorem A into further context, we recall what is known about telescopically localized algebraic K -theory. By work initiated by Thomason [Tho85, TT90] in the case of schemes and generalized and amplified by Clausen, Mathew, Naumann, and Noel [CMNN20b, CM19], $T(n)$ -local K -theory satisfies étale descent on \mathbb{E}_2 -spectral algebraic spaces for every n . In fact, the $T(1)$ -local K -theory of discrete rings is closely related to p -adic étale K -theory (the p -complete étale sheafification of K -theory). Hence, by the proven Quillen–Lichtenbaum conjecture, it is often isomorphic to p -adic K -theory in high enough degrees; see [CM19] for precise statements. In contrast, Mitchell [Mit90] proved that the K -theory of schemes and discrete rings (which are $T(1)$ -acyclic) vanishes $T(n)$ -locally for every $n \geq 2$ and every prime p . For ring spectra or spectral schemes, conjectures of Ausoni–Rognes [AR02, AR08] predict higher chromatic analogs of these statements and suggest to study the $T(n+1)$ -local K -theory of $K(n)$ -local ring spectra.

Many results follow quite quickly from Theorem A. We list a few of them here, and refer to the body of the text for more applications and explanations:

- (1) Let A be an \mathbb{E}_∞ -ring and B be an A -algebra. For $m \geq n + 1$, $L_{T(n)}A = 0$ implies $L_{T(m)}K(B) = 0$; see [CMNN20a, Theorem A].
- (2) For positive integers $m \neq n, n + 1$, the spectrum $K(K(n))$ is $T(m)$ -acyclic.
- (3) The map $K(BP\langle n \rangle) \rightarrow K(E(n))$ is a $T(m)$ -equivalence for $m \geq n + 1$, and both terms vanish $T(m)$ -locally for $m \geq n + 2$.
- (4) For any $n \geq 0$ and $m \geq 2$, the spectrum $K(\tau_{\leq n}\mathbb{S})$ is $T(m)$ -acyclic.
- (5) $K(1)$ -local K -theory of discrete rings satisfies excision, nilinvariance and cdh-descent. In addition, it is \mathbb{A}^1 -homotopy invariant and thus identifies with the $K(1)$ -localization of Weibel’s homotopy K -theory.²
- (6) For any ring spectrum A and $n \geq 2$, there are $T(n)$ -local equivalences between $K(A)$, $K(\tau_{\geq 0}A)$, and $\mathrm{TC}(\tau_{\geq 0}A)$.
- (7) The assembly map in algebraic K -theory for the family of cyclic subgroups

$$\mathrm{colim}_{G/H \in \mathcal{O}_e(G)} K(RH) \longrightarrow K(RG)$$

is a $T(n)$ -local equivalence for $n \geq 2$, any ring spectrum R , and any group G .

Statements (1)–(3) use the full strength of Theorem A, whereas (4)–(7) rely only on Theorem 3.8.

We remark that some cases of (1) in the above list appear in work of Ausoni–Rognes and Angelini-Knoll–Salch [AR02, AR12, AKS20], and that (7) is an immediate consequence of (6) and results of [LRRV17, CMM18].

A special case of (1) in the above list is that the K -theory of any $L_n^{p,f}$ -local ring spectrum is $T(m)$ -acyclic for $m \geq n + 2$. This, however, is used in order to deduce Theorem A from Theorem 3.8, and is due to [CMNN20a]. We indicate their argument in Remark 3.9 for the reader’s convenience.

CONVENTIONS. We fix a prime number p which will be the implicit prime in all Morava K -theories $K(i)$ and $T(i)$ below. We adopt the convention that $K(0) = H\mathbb{Q}$. Whenever we speak of a ring spectrum, we mean an \mathbb{E}_1 -ring spectrum, i.e. an algebra in the symmetric

²From this fact, excision, nilinvariance, and cdh-descent also follow.

monoidal ∞ -category Sp of spectra. By a module over a ring spectrum we mean a right module. Given an \mathbb{E}_k -ring spectrum R for $k \geq 2$, an R -algebra is an algebra in the monoidal ∞ -category $\mathrm{RMod}(R)$ of R -modules. For a spectrum E , we denote by L_E the Bousfield localization functor at E . For a spectrum X and a pointed space Y , we write $X \otimes Y$ for the smash product $X \otimes \Sigma^\infty Y$.

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2. PRELIMINARIES

2.1. Preliminaries from chromatic homotopy theory. For an integer $n \geq 1$, we denote by V_n a type n -complex, i.e. a pointed finite CW-complex with $K(i) \otimes V_n = 0$ for $i < n$ and $K(n) \otimes V_n \neq 0$. We denote by v_n a v_n -self map of V_n , i.e. a map $\Sigma^d V_n \rightarrow V_n$ for some positive integer d inducing an isomorphism on $K(n)$ -homology and nilpotent maps on $K(i)$ -homology for $i \neq n$. Such maps exist by [HS98].

If X is a pointed space or a spectrum, we define its v_n -periodic homotopy groups $v_n^{-1}\pi_*(X; V_n)$ by the formula

$$v_n^{-1}\pi_*(X; V_n) = \mathbb{Z}[v_n^{\pm 1}] \otimes_{\mathbb{Z}[v_n]} \pi_* \mathrm{Map}_*(V_n, X).$$

Definition 2.1. We call a map of pointed spaces or spectra a v_n -periodic equivalence (with $n \geq 1$) if it induces an isomorphism on v_n -periodic homotopy groups. A map of spectra or simple spaces is a v_0 -periodic equivalence if it becomes an equivalence after inverting p , and the v_0 -periodic homotopy groups are by definition the homotopy groups with p inverted. For a spectrum E , we say that another spectrum X is E -acyclic if $E \otimes X = 0$ and say that a map is an E -equivalence if its fibre is E -acyclic.

For a fixed pair (V_n, v_n) we denote by $T(n) = \Sigma^\infty V_n[v_n^{-1}]$ the telescope of v_n . We adopt the convention that $T(0) = \mathbb{S}[\frac{1}{p}]$. We recall that the Bousfield class of a spectrum E is the full subcategory of Sp consisting of the E -acyclic spectra. For the convenience of the reader not familiar with chromatic homotopy theory, we note the following well-known properties.

Lemma 2.2. *Let X be a spectrum and Y be a pointed space.*

- (i) *We have $v_n^{-1}\pi_*(X; V_n) \cong v_n^{-1}\pi_*(\Omega^\infty X; V_n)$.*
- (ii) *The maps $\tau_{\geq k} X \rightarrow X$ and $\tau_{\geq k} Y \rightarrow Y$ are v_n -periodic equivalences for all k and all $n \geq 1$.*
- (iii) *The spectra $K(m)$ are $T(n)$ -acyclic for $n \neq m$.*
- (iv) *Any $T(n)$ -acyclic spectrum is $K(n)$ -acyclic.*
- (v) *A spectrum which is \mathbb{S}/p -acyclic is also $T(n)$ -acyclic for all $n \geq 1$.*
- (vi) *The map $X \rightarrow X_p^\wedge$ is a $T(n)$ -equivalence for all $n \geq 1$.*
- (vii) *The Bousfield class of $T(n)$ does not depend on the choice of (V_n, v_n) .*
- (viii) *A spectrum is $T(n)$ -acyclic if and only if its v_n -periodic homotopy groups vanish.*

Proof. Part (i) follows immediately from the definitions and the equivalence $\mathrm{Map}_*(V_n, X) \simeq \mathrm{Map}_*(V_n, \Omega^\infty X)$. Assertion (ii) follows from the observation that the v_n -periodic homotopy groups of a bounded above spectrum or space vanish. This in turn follows from the fact that the degree d of the self map v_n is positive. Claim (iii) follows from the fact that $K(m) \otimes T(n) \simeq$

$(K(m) \otimes V_n)[v_n^{-1}]$ which vanishes as v_n is nilpotent on Morava K -homology if n is different from m . To see (iv), assume that X is $T(n)$ -acyclic. Then we have $0 = K(n) \otimes T(n) \otimes X$. But $K(n) \otimes T(n) \neq 0$, so, since $K(n)$ is a field spectrum (any module is a direct sum of shifted copies of $K(n)$), we must have $K(n) \otimes X = 0$.

For part (v) observe that some power of p , say p^k , is zero on V_n and hence on $T(n)$. Given an \mathbb{S}/p -acyclic spectrum X , we have $X/p^k = 0$. Thus we see that

$$0 = X/p^k \otimes T(n) \simeq X \otimes T(n)/p^k \simeq X \otimes (T(n) \oplus \Sigma T(n)),$$

and the latter term has $X \otimes T(n)$ as a retract. Thus X is $T(n)$ -acyclic. Statement (vi) follows from (v), since the fibre of $X \rightarrow X_p^\wedge$ is \mathbb{S}/p -acyclic.

For (vii), as in [MS95, Lemma 2.1], we fix a pair (V_n, v_n) and consider the full subcategory of finite p -local spectra consisting of those Y which admit a v_n -self map y and such that $T(n) \otimes Z = 0$ implies that $Y[y^{-1}] \otimes Z = 0$ as well. This is a thick subcategory, as follows from [HS98, Corollary 3.8]. Since it contains V_n , this thick subcategory is given by the thick subcategory of finite spectra of type at least n , see [HS98, Theorem 7]. Hence if $T(n) \otimes Z = 0$ and (V'_n, v'_n) is another choice, also $T(n)' \otimes Z = 0$. Running the same argument also with V'_n instead of V_n gives the claim.

To see (viii), consider a spectrum X and observe that we may calculate its v_n -periodic homotopy groups using the mapping spectrum $\text{map}(V_n, X)$ instead of the mapping space $\text{Map}(V_n, X)$ due to the positivity of the degree of the self-map v_n . Thus, the v_n -periodic homotopy groups of X are isomorphic to the homotopy groups of the spectrum $(DV_n \otimes X)[Dv_n^{-1}]$, where DV_n denotes the dual of the finite spectrum $\Sigma^\infty V_n$ (which is again of type n). This spectrum is equivalent to $T(n) \otimes X$ where $T(n)$ is the telescope of Dv_n . The claim then follows from (vii). \square

We remark that the converse of statement (iv) (for $n \geq 1$) is known as the telescope conjecture. It is known [Mah81, Mil81] to be true in height $n = 1$ but is open in general, see e.g. [Bar19] for a survey. We thank Dustin Clausen for help with the following lemma, which is a consequence of the nilpotence theorem.

Lemma 2.3. *Let R be a ring spectrum, and $n \geq 1$ an integer. Then R is $K(n)$ -acyclic if and only if it is $T(n)$ -acyclic.*

Proof. The “if”-part follows from Lemma 2.2(iv). To see the “only if” statement, we argue first that one can assume that $T(n)$ is a ring spectrum. Indeed, by replacing V_n by $W_n = V_n \otimes DV_n \simeq \text{End}(\Sigma^\infty V_n)$, we can assume that the suspension spectrum of our type n -complex is an \mathbb{E}_1 -ring spectrum. Moreover, the v_n -self-map of V_n defines an element $w \in \pi_*(W_n)$, multiplication with which is a v_n -self map again. By [HS98, Theorem 11] a power of w lies in the center of $\pi_*(W_n)$. Thus the localization $W_n[w^{-1}]$ admits the structure of an \mathbb{E}_1 -ring spectrum. As the Bousfield class of $T(n)$ does not depend on the choice of the type n complex, we can thus indeed assume that $T(n)$ is a ring spectrum. We then observe that a ring spectrum like $T(n) \otimes R$ is zero if and only if its unit is nilpotent. By the nilpotence theorem [HS98, Theorem 3], this is the case if and only if $K(m) \otimes (T(n) \otimes R) = 0$ for all $0 \leq m \leq \infty$. If $m \neq n$ then $K(m) \otimes T(n) \otimes R = 0$ as $K(m) \otimes T(n) = 0$ by Lemma 2.2(iii). Since R is $K(n)$ -acyclic, also $K(n) \otimes T(n) \otimes R = 0$. We thus conclude that $T(n) \otimes R$ vanishes. \square

Remark 2.4. In the proof above, it was not used that R is an algebra in the ∞ -category of spectra. It suffices that R is a unital magma in the homotopy category of spectra.

2.2. Some localization functors. We recall that the functor L_n^f on spectra is defined as Bousfield localization at the spectrum $H\mathbb{Q} \oplus T(1) \oplus \cdots \oplus T(n)$. To formulate our main result, we will use the following variant.

Definition 2.5. We denote by $L_n^{p,f}$ the Bousfield localization at $T(0) \oplus T(1) \oplus \cdots \oplus T(n)$.

Recall that a Bousfield localization functor $L: \mathrm{Sp} \rightarrow \mathrm{Sp}$ is called smashing if it preserves colimits or, equivalently, if it is of the form $LX \simeq X \otimes LS$. If every L -acyclic spectrum is a colimit of compact, L -acyclic spectra, then L is called finite and is in particular smashing, see [Mil92] or [Lur10, Lecture 20, Example 12]. For example, L_n^f is smashing. The same proof also shows that $L_n^{p,f}$ is smashing. For convenience, we recall this proof below. Write $\mathcal{C}_{>n}$ for the ∞ -category of p -local, finite spectra which are of type greater than n , i.e. which are $K(0) \oplus \cdots \oplus K(n)$ -acyclic.

Lemma 2.6. *The category of $L_n^{p,f}$ -acyclic spectra coincides with $\mathrm{Ind}(\mathcal{C}_{>n})$. In particular, $L_n^{p,f}$ is a smashing localization.*

Proof. The Bousfield class $\langle T(0) \oplus T(1) \oplus \cdots \oplus T(n) \rangle$ has as complement the Bousfield class $\langle \Sigma^\infty V_{n+1} \rangle$ of a type $(n+1)$ -spectrum: every spectrum is acyclic for $(T(0) \oplus T(1) \oplus \cdots \oplus T(n)) \otimes V_{n+1}$ and 0 is the only spectrum which is $T(0) \oplus T(1) \oplus \cdots \oplus T(n) \oplus \Sigma^\infty V_{n+1}$ -acyclic. Indeed, this follows easily from the inductive construction of a type $(k+1)$ -complex as V_k/v , where V_k is a type k -complex with v_k -self map v . It follows from [MS95, Proposition 3.3] that every $L_n^{p,f}$ -acyclic spectrum is a colimit of finite $L_n^{p,f}$ -acyclic spectra. The thick subcategory theorem implies that the category of finite $L_n^{p,f}$ -acyclic spectra is precisely $\mathcal{C}_{>n}$. \square

Lemma 2.7. *For integers $0 \leq m < n$ and a spectrum X there is a pullback diagram*

$$\begin{array}{ccc} L_n^{p,f} X & \longrightarrow & L_{T(m+1) \oplus \cdots \oplus T(n)} X \\ \downarrow & & \downarrow \\ L_m^{p,f} X & \longrightarrow & L_m^{p,f} L_{T(m+1) \oplus \cdots \oplus T(n)} X \end{array}$$

natural in X .

Proof. This is a special case of the following well known lemma. \square

Lemma 2.8. *Let E and F be spectra. Assume that L_E preserves F -acyclic spectra. Then there is a pullback diagram*

$$\begin{array}{ccc} L_{E \oplus F} X & \longrightarrow & L_F X \\ \downarrow & & \downarrow \\ L_E X & \longrightarrow & L_E L_F X \end{array}$$

natural in X .

We note that the assumptions of the lemma are for instance satisfied if L_E is smashing, or if L_F annihilates E -local objects.

Proof. Denote the pullback of the diagram $L_E X \rightarrow L_E L_F X \leftarrow L_F X$ by $P(X)$. There is a canonical map $X \rightarrow P(X)$; it is easy to show that this map is an $(E \oplus F)$ -local equivalence, and that $P(X)$ is $(E \oplus F)$ -local. \square

As a consequence of Lemma 2.8 we note that

- (i) for p -local spectra X , the canonical map $L_n^{p,f} X \rightarrow L_n^f X$ is an equivalence, and
- (ii) for $T(1)$ -acyclic spectra X , the canonical map $L_1^{p,f} X \rightarrow X[\frac{1}{p}]$ is an equivalence.

We will use the following criterion to detect $T(i)$ -local equivalences, which was indicated to us by Gijs Heuts.

Proposition 2.9. *Let $f: X \rightarrow Y$ be a map of spectra, and let $i \geq 1$ be an integer. If $\Sigma^\infty \Omega^\infty f$ is a $T(i)$ -local equivalence, then so is f . In other words, the functor $\Sigma^\infty \Omega^\infty: \mathrm{Sp} \rightarrow \mathrm{Sp}$ detects $T(i)$ -local equivalences.*

Proof. We note that the canonical composite

$$\Omega^\infty \longrightarrow \Omega^\infty \Sigma^\infty \Omega^\infty \longrightarrow \Omega^\infty$$

is an equivalence. It is an insight of Bousfield and Kuhn that the $T(i)$ -localization functor $L_{T(i)}$ factors through Ω^∞ via the Bousfield–Kuhn functor Φ_i from pointed spaces to spectra. There is thus an equivalence $\Phi_i \circ \Omega^\infty \simeq L_{T(i)}$; see [Kuh08, Theorem 1.1]. Applying Φ_i to the above composite shows that the composite

$$L_{T(i)} \longrightarrow L_{T(i)} \Sigma^\infty \Omega^\infty \longrightarrow L_{T(i)}$$

is also an equivalence. This implies that $L_{T(i)}(f)$ is a retract of $L_{T(i)}(\Sigma^\infty \Omega^\infty f)$ which proves the lemma. \square

Remark 2.10. Restricted to connective spectra, the functor $\Sigma^\infty \Omega^\infty$ also detects $T(0)$ -local equivalences.

It is, however, not true that the functor $\Sigma^\infty \Omega^\infty$ preserves $T(i)$ -local equivalences. For example, $H\mathbb{Z}$ is $T(i)$ -acyclic, whereas $\Sigma^\infty \mathbb{Z}$ is not: It contains the sphere spectrum as a summand. Nevertheless, $\Sigma^\infty \Omega^\infty$ preserves suitably connective $L_n^{p,f}$ -equivalences, as the following result shows. We say that a space is m -connective if it has trivial homotopy groups in degrees less than m , i.e., is $(m-1)$ -connected. We call a map m -connective if its fibre is m -connective, i.e. if it induces an isomorphism on π_k for $k < m$ and a surjection on π_m .

Proposition 2.11. *Let $n \geq 1$ be an integer. Then there exists $m \geq 2$ such that the following hold:*

- (i) *Let F be an m -connective pointed space whose v_i -periodic homotopy groups vanish for $0 \leq i \leq n$. Then F is $T(i)$ -acyclic for $0 \leq i \leq n$.*
- (ii) *Let $f: X \rightarrow Y$ be an m -connective map between spaces. If f is a v_i -periodic equivalence for $0 \leq i \leq n$, then $\Sigma^\infty f: \Sigma^\infty X \rightarrow \Sigma^\infty Y$ is an $L_n^{p,f}$ -equivalence.*
- (iii) *The functor $\Sigma^\infty \Omega^\infty$ preserves m -connective $L_n^{p,f}$ -equivalences.*

Proof. Part (i) follows from a result of Bousfield ([Bou01, Corollary 4.8], [BHM18, Theorem 3.1]) together with [BHM18, Lemma 3.3], which gives an integer m such that any m -connective space with trivial v_i -periodic homotopy groups for $0 \leq i \leq n$ has trivial $T(i)$ -homology for $0 \leq i \leq n$. Note that in Bousfield’s convention $T(0)$ is $H\mathbb{Q}$, but it is well-known that a simply connected space such that $(\pi_* X)[\frac{1}{p}]$ vanishes is also acyclic for $H\mathbb{Z}[\frac{1}{p}]$ and hence for $T(0) = \mathbb{S}[\frac{1}{p}]$.

To prove (ii), let F be the fibre of f . By assumption, F is an m -connective space whose v_i -periodic homotopy groups vanish for $i \leq n$. Thus F is $T(i)$ -acyclic for $0 \leq i \leq n$ by (i). The Serre spectral sequence in $T(i)$ -homology associated to the fibre sequence $F \rightarrow X \rightarrow Y$

then implies that the map $\Sigma^\infty X \rightarrow \Sigma^\infty Y$ is a $T(i)$ -local equivalence for $0 \leq i \leq n$ and thus an $L_n^{p,f}$ -equivalence.

Finally, to see (iii) one applies Ω^∞ to an m -connective $L_n^{p,f}$ -equivalence. The resulting map of spaces satisfies the assumptions of (ii), so the proposition follows. \square

The following remark will not be used in the sequel.

Remark 2.12. In fact, we can take $m = n + 1$ in Proposition 2.11. Indeed, using what we have proven already, it suffices to show that any $(n + 1)$ -connective space F is $T(i)$ -acyclic for $0 \leq i \leq n$ if its homotopy groups are p -primary torsion and vanish in sufficiently high degrees. By an induction over the Postnikov tower, it hence suffices to show that $T(i) \otimes K(\pi, r) = 0$ if $r > i$ and π is a finite group, as $T(i)$ -homology commutes with filtered colimits and every torsion group is the filtered colimit of its finite subgroups. It is shown in [CSY18, Theorem E] that for a p -local ring spectrum R the following two conditions are equivalent: (1) $R \otimes K(\pi, r) = 0$ for all $r > i$ and (2) $R \otimes K(r) = 0$ for all $r > i$. Statement (2) applies to $R = T(i)$ by Lemma 2.2(iii), so part (i) and hence also (ii) and (iii) of the proposition follow with $m = n + 1$.

2.3. Localizing invariants and K -theory. In this short subsection we recall some notions and facts about algebraic K -theory which we will use throughout this paper.

A localizing invariant is a functor $\text{Cat}_\infty^{\text{perf}} \rightarrow \text{Sp}$ which sends exact sequences to fibre sequences. Here $\text{Cat}_\infty^{\text{perf}}$ refers to the ∞ -category of small, idempotent complete, and stable ∞ -categories and exact sequences are those sequences which are both fibre and cofibre sequences in $\text{Cat}_\infty^{\text{perf}}$, see [BGT13, §5] for details.³ Examples of localizing invariants are non-connective K -theory [BGT13, §9], topological Hochschild homology, topological cyclic homology, etc. For a localizing invariant E and a ring spectrum A , we will write $E(A)$ for $E(\text{Perf}(A))$, where $\text{Perf}(A)$ denotes the ∞ -category of perfect A -modules, which coincides with the compact objects of $\text{RMod}(A)$.

For a connective ring spectrum A , the space $\Omega^\infty \tau_{\geq 1} K(A)$ can be described as a plus-construction [BGT13, Lemma 9.39]: We denote by $\text{GL}(A)$ the \mathbb{E}_1 -space $\text{GL}(A) = \text{colim } \text{GL}_n(A)$, where $\text{GL}_n(A)$ denotes the invertible components in the \mathbb{E}_1 -space $\Omega^\infty \text{End}(A^n)$. In particular, $\pi_0(\text{GL}(A)) \cong \text{GL}(\pi_0(A))$, where the right-hand side denotes the group of invertible matrices over the discrete ring $\pi_0(A)$. There is a canonical map $\text{BGL}(A) \rightarrow \Omega^\infty \tau_{\geq 1} K(A)$ which exhibits the target as the plus construction $\text{BGL}(A)^+$. In particular, this map is a homology equivalence, and hence the map of spectra $\Sigma^\infty \text{BGL}(A) \rightarrow \Sigma^\infty \Omega^\infty \tau_{\geq 1} K(A)$ is an equivalence.

For an arbitrary $\mathcal{C} \in \text{Cat}_\infty^{\text{perf}}$, we will also need the explicit description of the K -theory space $\Omega^\infty K(\mathcal{C})$, which arises via the Waldhausen S_\bullet -construction, cf. [BGT13, Sec. 7.1]. The S_\bullet -construction produces a simplicial object $S_\bullet \mathcal{C} \in \text{Fun}(\Delta^{\text{op}}, \text{Cat}_\infty^{\text{perf}})$ such that there is a natural equivalence of spaces

$$(1) \quad \Omega^\infty K(\mathcal{C}) \simeq \Omega |S_\bullet(\mathcal{C})^\simeq|,$$

where $(-)^{\simeq}$ denotes the underlying space of an ∞ -category; moreover, we have for each $n \geq 0$ a natural equivalence $S_n(\mathcal{C}) \simeq \text{Fun}(\Delta^{n-1}, \mathcal{C})$. Note that both sides have the canonical structure of \mathbb{E}_∞ -spaces since $\text{Cat}_\infty^{\text{perf}}$ is semiadditive, i.e. it has finite biproducts as in [GGN15, Definition 2.1]; In fact, the equivalence 1 is one of \mathbb{E}_∞ -spaces, therefore, we can deloop both sides to obtain

$$(2) \quad \Omega^\infty(\tau_{\geq 0} K(\mathcal{C})[1]) \simeq |S_\bullet(\mathcal{C})^\simeq|.$$

³In [BGT13] localizing invariants are further required to preserve filtered colimits.

3. PROOF OF THEOREM A

In this section, we will prove Theorem A in several steps, each of which will give a special case of the result. Our first step, which we treat in the following subsection, will involve only highly connective maps of connective ring spectra.

3.1. The highly connective case.

Proposition 3.1. *Let $n \geq 1$. There exists $N \geq 1$ such that the following holds: let $A \rightarrow B$ be an N -connective $L_n^{p,f}$ -equivalence between connective ring spectra. Then the induced map $K(A) \rightarrow K(B)$ is again an $L_n^{p,f}$ -equivalence.*

Proof. We take $N = m - 1$, where m is as in Proposition 2.11. By Waldhausen's result (see [Wal78, Propositions 1.1, 2.2] or [LT19, Lemma 2.4]) the map $K(A) \rightarrow K(B)$ is a $T(0)$ -equivalence. It hence remains to prove that the map $K(A) \rightarrow K(B)$ is a $T(i)$ -local equivalence for $1 \leq i \leq n$. By Lemma 2.2(ii), it suffices to show that $\tau_{\geq 1}K(A) \rightarrow \tau_{\geq 1}K(B)$ is a $T(i)$ -local equivalence for $1 \leq i \leq n$.

We consider the following commutative diagram, where we use the plus-construction description of algebraic K -theory for connective ring spectra as recalled in Section 2.3.

$$\begin{array}{ccc} \Sigma^\infty \mathrm{BGL}(A) & \longrightarrow & \Sigma^\infty \mathrm{BGL}(B) \\ \simeq \downarrow & & \downarrow \simeq \\ \Sigma^\infty \Omega^\infty \tau_{\geq 1}K(A) & \longrightarrow & \Sigma^\infty \Omega^\infty \tau_{\geq 1}K(B) \end{array}$$

By Proposition 2.9, it suffices to show that the lower horizontal map is a $T(i)$ -local equivalence for $1 \leq i \leq n$. Since the vertical maps in the above diagram are equivalences, this is the case if the top horizontal map is a $T(i)$ -local equivalence. This and thus the proposition will follow from Proposition 2.11(ii) once we have shown the following: The map $\mathrm{BGL}(A) \rightarrow \mathrm{BGL}(B)$ is m -connective and induces an isomorphism on v_i -periodic homotopy groups for $i \leq n$.

To show this claim, we observe that the classifying space construction B increases the connectivity of a map by 1 and preserves v_i -periodic equivalences. Thus it suffices to see that $\mathrm{GL}(A) \rightarrow \mathrm{GL}(B)$ is an $(m - 1)$ -connective v_i -periodic equivalence for $1 \leq i \leq n$. Observe that by definition the map $A \rightarrow B$ induces an isomorphism between $\pi_0(\mathrm{GL}(A)) = \mathrm{GL}(\pi_0(A))$ and $\mathrm{GL}(\pi_0(B)) = \pi_0(\mathrm{GL}(B))$. Furthermore, $\tau_{\geq 1}\mathrm{GL}(A) \simeq \tau_{\geq 1}\mathrm{M}(A)$, where $\mathrm{M}(A)$ is the space of matrices $\mathrm{colim}_r \Omega^\infty \mathrm{End}(A^r) \simeq \mathrm{colim}_r \Omega^\infty A^{r \times r}$, and similarly for B . As $A \rightarrow B$ is $(m - 1)$ -connective and as $m \geq 2$, we thus see that $\mathrm{GL}(A) \rightarrow \mathrm{GL}(B)$ is $(m - 1)$ -connective. Further, as $A \rightarrow B$ is an L_n^f -equivalence, it induces isomorphisms on v_i -periodic homotopy groups for $i \leq n$. By Lemma 2.2(i) also $\mathrm{M}(A) \rightarrow \mathrm{M}(B)$ is a v_i -periodic equivalence, and, finally, by Lemma 2.2(ii) also $\mathrm{GL}(A) \rightarrow \mathrm{GL}(B)$ is a v_i -periodic equivalence for $i \leq n$, as desired. \square

3.2. A truncating property. Our next goal is to prove a version of Proposition 3.1 with weaker connectivity hypotheses; in fact, we will only need a special case (Proposition 3.4) below, formulated using the language of truncating invariants. To do this, we will need some further preliminaries.

Lemma 3.2. *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

be a pullback square of ring spectra in which A is connective. If $A \rightarrow A'$ is n -connective and $A \rightarrow B$ is m -connective, then the induced map $A' \odot_A^{B'} B \rightarrow B'$ is $(m+n+2)$ -connective.

Here $A' \odot_A^{B'} B$ denotes the ring spectrum associated to the displayed pullback square by [LT19, Main Theorem].

Proof. Denote by I the common fibre of the vertical maps, by J that of the horizontal maps. From [LT19, Remark 1.16] we have an equivalence

$$\mathrm{fib}(A' \odot_A^{B'} B \rightarrow B') \simeq \Sigma \mathrm{fib}(I \otimes_A B \xrightarrow{\mu} I)$$

where the map μ is induced by the right B -module structure on I . Now μ has a section $\sigma: I \rightarrow I \otimes_A B$ induced by the map $A \rightarrow B$. The fibre of σ identifies with $I \otimes_A J$, which is $(n+m)$ -connective as A is connective. In other words, σ is an isomorphism in degrees $\leq m+n-1$ and surjective in degree $m+n$. Since $\mu \circ \sigma \simeq \mathrm{id}_I$ and so μ is surjective in every degree, it then follows that μ is an isomorphism in degrees $\leq m+n$ and surjective in degree $m+n+1$, i.e. μ is $(m+n+1)$ -connective. By the above equivalence, $A' \odot_A^{B'} B \rightarrow B'$ is then $(m+n+2)$ -connective, as desired. \square

Let M be a spectrum. We say that a localizing invariant E is *truncating on M -acyclic ring spectra* if for every M -acyclic connective ring spectrum R , we have $E(R) \xrightarrow{\sim} E(\pi_0 R)$. Note that if a ring spectrum R is M -acyclic, then also $\tau_{\leq k} R$ is M -acyclic for all k as follows by consideration of the ring map $L_M R \rightarrow L_M \tau_{\leq k} R$. The following lemma also appears in similar form in [Mat20, Lemma 3.11].

Lemma 3.3. *Let E be a localizing invariant. Suppose that there exists $k \geq 0$ such that the map $E(R) \xrightarrow{\sim} E(\tau_{\leq k} R)$ is an equivalence for any M -acyclic connective ring spectrum R . Then E is truncating on M -acyclic ring spectra.*

Proof. It suffices to show that if E and $k > 0$ are as in the statement of the lemma, then we have in fact $E(\tau_{\leq k} R) \xrightarrow{\sim} E(\tau_{\leq k-1} R)$ for all M -acyclic connective ring spectra R ; the result then follows by induction on k .

To this end, recall that $\tau_{\leq k} R \rightarrow \tau_{\leq k-1} R$ is a square-zero extension, so there is a pullback square of ring spectra (cf. [Lur17, 7.4.1.29]),

$$\begin{array}{ccc} \tau_{\leq k} R & \longrightarrow & H\pi_0 R \\ \downarrow & & \downarrow \\ \tau_{\leq k-1} R & \longrightarrow & H\pi_0 R \oplus (H\pi_k R)[k+1]. \end{array}$$

All ring spectra in this square are connective and M -acyclic. It follows from [LT19, Main Theorem] that we have a pullback square of spectra

$$\begin{array}{ccc} E(\tau_{\leq k} R) & \longrightarrow & E(H\pi_0 R) \\ \downarrow & & \downarrow \\ E(\tau_{\leq k-1} R) & \longrightarrow & E\left(\tau_{\leq k-1} R \odot_{\tau_{\leq k} R}^{H\pi_0 R \oplus (H\pi_k R)[k+1]} H\pi_0 R\right). \end{array}$$

It thus suffices to show that the right vertical map is an equivalence. But this follows because, by Lemma 3.2, the map of connective, M -acyclic ring spectra

$$H\pi_0 R \rightarrow \tau_{\leq k-1} R \odot_{\tau_{\leq k} R}^{H\pi_0 R \oplus (H\pi_k R)[k+1]} H\pi_0 R$$

induces an equivalence on $\tau_{\leq k}$ and hence on $E(-)$. \square

Proposition 3.4. *For $n \geq 1$, $L_{T(n)}K(-)$ is truncating on $L_n^{p,f}$ -acyclic ring spectra.*

Proof. This follows from Proposition 3.1 and Lemma 3.3. \square

Corollary 3.5 (Cf. also [BCM20]). *For any $n \geq 1$, we have $L_{T(i)}K(\mathbb{Z}/p^n) = 0$ for $i \geq 1$.*

Proof. This follows from Proposition 3.4 since truncating invariants are nilinvariant, [LT19, Corollary 3.5], and Quillen's computation that $K(\mathbb{F}_p)_p^\wedge = H\mathbb{Z}_p$. \square

3.3. The general case. Now we extend the results to nonconnective ring spectra, and then complete the proof of Theorem A. Our strategy of proof is to reduce the nonconnective case to the connective case using the S_\bullet -construction. In this we will work with a not-necessarily stable, but additive ∞ -category \mathcal{A} , about which we make two remarks:

- We can view \mathcal{A} as a symmetric monoidal ∞ -category under \oplus and denote by $K^{\text{add}}(\mathcal{A})$ its group-completion K -theory, cf. [GGN15] for a modern account. If R is a connective ring spectrum and \mathcal{A} is the ∞ -category $\text{Proj}^\omega(R)$ of finitely generated projective R -modules, there is an equivalence $K^{\text{add}}(\mathcal{A}) \simeq \tau_{\geq 0}K(R)$.
- Given two objects X and Y in an additive ∞ -category \mathcal{A} , their mapping space has the canonical structure of a grouplike \mathbb{E}_∞ -space, giving rise to a connective spectrum $\text{hom}_{\mathcal{A}}(X, Y)$. If \mathcal{A} is stable, this is the connective cover of the homomorphism spectrum $\text{Hom}_{\mathcal{A}}(X, X)$. We remark that $\text{Hom}_{\mathcal{A}}(X, X)$ is $L_n^{p,f}$ -acyclic if and only if $\text{hom}_{\mathcal{A}}(X, X)$ is $L_n^{p,f}$ -acyclic.

In the following, we will assume that all ∞ -categories of which we consider the K -theory are idempotent-complete.

Proposition 3.6. *Let \mathcal{C} be an additive ∞ -category. Suppose for each object $X \in \mathcal{C}$, the ring spectrum $\text{hom}_{\mathcal{C}}(X, X)$ is annihilated by $L_n^{p,f}$. Then*

- (i) $L_{T(i)}K^{\text{add}}(\mathcal{C}) = 0$ for $1 \leq i \leq n$, and
- (ii) $L_{T(i)}K(\mathcal{C}) = 0$ for $1 \leq i \leq n$ if \mathcal{C} is stable.

Proof. For the first part, we observe that \mathcal{C} can be written as a filtered colimit of its full subcategories generated by finite direct sums and retracts by finitely many objects. Passing to the direct sum of the generators, and using that K -theory commutes with filtered colimits, we may assume that \mathcal{C} is generated under finite direct sums and retracts by a single object X . Hence, by the additive version of the Schwede–Shipley theorem, $\mathcal{C} \simeq \text{Proj}^\omega(\text{hom}_{\mathcal{C}}(X, X))$ is the ∞ -category of finitely generated projective modules over $\text{hom}_{\mathcal{C}}(X, X)$, which is $L_n^{p,f}$ -acyclic by assumption. Therefore, $K^{\text{add}}(\mathcal{C}) \simeq \tau_{\geq 0}K(\text{hom}_{\mathcal{C}}(X, X))$ is $T(i)$ -acyclic for $1 \leq i \leq n$ by Proposition 3.4 (together with the fact that the $T(1)$ -local K -theory of a p -power torsion discrete ring vanishes by Corollary 3.5).

For the second part, we assume that \mathcal{C} is stable. The Waldhausen S_\bullet -construction gives a simplicial stable ∞ -category $S_\bullet\mathcal{C}$ and a natural equivalence of spaces (as in (2)):

$$\Omega^\infty((\tau_{\geq 0}K(\mathcal{C}))[1]) = |S_\bullet(\mathcal{C})^\simeq|.$$

Note that both sides have the structure of \mathbb{E}_∞ -monoids, using the direct sum on \mathcal{C} (which also gives the canonical \mathbb{E}_∞ -monoid structure on the left arising from Ω^∞), and the map is an

equivalence of \mathbb{E}_∞ -spaces. Therefore, we may group-complete the terms inside the geometric realization on the right-hand-side to obtain an equivalence of connective spectra

$$(3) \quad (\tau_{\geq 0}K(\mathcal{C}))[1] \simeq |K^{\text{add}}(S_\bullet(\mathcal{C}))|,$$

where on the right we consider the additive (group-complete) K -theory as above. The above $L_n^{p,f}$ -local vanishing condition on the stable ∞ -category \mathcal{C} is stable under passage to $\text{Fun}(\Delta^j, \mathcal{C})$ for any $j \geq 0$. Therefore, by the first paragraph of the proof, we find that the right-hand-side of (3) is $T(i)$ -acyclic for $1 \leq i \leq n$, hence the result. \square

Lemma 3.7. *For any ring spectrum A , there is an exact sequence*

$$\mathcal{C}_{>n} \otimes \text{Perf}(A) \longrightarrow \text{Perf}(A) \longrightarrow \text{Perf}(L_n^{p,f}A),$$

and the endomorphism spectrum of every object in $\mathcal{C}_{>n} \otimes \text{Perf}(A)$ is $L_n^{p,f}$ -acyclic.

Proof. Lemma 2.6 and the Thomason–Neeman localization theorem [Nee92, Theorem 2.1] imply that the sequence of small stable ∞ -categories

$$\mathcal{C}_{>n} \longrightarrow \text{Perf}(\mathbb{S}) \longrightarrow \text{Perf}(L_n^{p,f}\mathbb{S})$$

is exact. Tensoring the above exact sequence with $\text{Perf}(A)$, using the fact that $L_n^{p,f}$ is smashing, we obtain the exact sequence

$$\mathcal{C}_{>n} \otimes \text{Perf}(A) \longrightarrow \text{Perf}(A) \longrightarrow \text{Perf}(L_n^{p,f}A).$$

The ∞ -category $\mathcal{C}_{>n} \otimes \text{Perf}(A)$ is generated by A -modules of the form $A \otimes F$ with F being a finite p -local $L_n^{p,f}$ -acyclic spectrum. Their endomorphism spectra $DF \otimes F \otimes A$ are $L_n^{p,f}$ -acyclic as well and so are thus the endomorphism spectra of all objects of $\mathcal{C}_{>n} \otimes \text{Perf}(A)$. \square

Theorem 3.8. *Let A be a ring spectrum. Then the map $A \rightarrow L_n^{p,f}A$ induces an equivalence on $L_{T(i)}K(-)$ for $1 \leq i \leq n$. If $n \geq 2$, the map $A \rightarrow L_{T(1) \oplus \dots \oplus T(n)}A$ induces an equivalence on $L_{T(n)}K(-)$.*

Proof. As K -theory is localizing, the homotopy fibre of $K(A) \rightarrow K(L_n^{p,f}A)$ coincides with $K(\mathcal{C}_{>n} \otimes \text{Perf}(A))$ by the preceding lemma. This is $T(i)$ -acyclic for $1 \leq i \leq n$ by Proposition 3.6. For the last assertion, we consider the pullback diagram

$$\begin{array}{ccc} L_n^{p,f}A & \longrightarrow & L_{T(1) \oplus \dots \oplus T(n)}A \\ \downarrow & & \downarrow \\ A[\frac{1}{p}] & \longrightarrow & (L_{T(1) \oplus \dots \oplus T(n)}A)[\frac{1}{p}] \end{array}$$

from Lemma 2.7 (note that $L_0^{p,f}A = A[\frac{1}{p}]$). By [Tam18] or [LT19] and the fact that $L_0^{p,f}$ is smashing, we deduce that the diagram

$$\begin{array}{ccc} K(L_n^{p,f}A) & \longrightarrow & K(L_{T(1) \oplus \dots \oplus T(n)}A) \\ \downarrow & & \downarrow \\ K(A[\frac{1}{p}]) & \longrightarrow & K((L_{T(1) \oplus \dots \oplus T(n)}A)[\frac{1}{p}]) \end{array}$$

is a pullback.⁴ By the first part, it suffices to prove that the top horizontal map is an equivalence after $T(i)$ -localization for $i \geq 2$. Now each term in the bottom row is a module over

⁴This also follows more classically from Lemma 3.7.

$K(\mathbb{S}[\frac{1}{p}])$, which is p -adically equivalent to $K(\mathbb{Z}[\frac{1}{p}])$ and hence vanishes after $T(i)$ -localization for $i \geq 2$ by Mitchell's result. \square

We now prove Theorem A. The result is a direct consequence of Theorem 3.8 (which proves “one half” of the result) and the results of [CMNN20a] (which proves the “other half”). We note that the results of *loc. cit.* also rely on Theorem 3.8 (but not on Theorem A, so there is no circularity).

Proof of Theorem A. It suffices to show that the map $A \rightarrow L_{T(n-1) \oplus T(n)} A$ induces an equivalence on $L_{T(n)} K(-)$. As in the proof of Theorem 3.8, using that $L_{n-2}^{p,f}$ is a smashing localization (or Lemma 3.7), the diagram

$$\begin{array}{ccc} K(L_n^{p,f} A) & \longrightarrow & K(L_{T(n-1) \oplus T(n)} A) \\ \downarrow & & \downarrow \\ K(L_{n-2}^{p,f} A) & \longrightarrow & K(L_{n-2}^{p,f} (L_{T(n-1) \oplus T(n)} A)) \end{array}$$

is a pullback. By Theorem 3.8, it suffices to prove that the top horizontal map is an equivalence after $T(n)$ -localization. Now each term in the bottom row is a module over $K(L_{n-2}^{p,f} \mathbb{S})$, which vanishes $T(n)$ -locally by [CMNN20a, Theorem C], so we deduce the claim. \square

Remark 3.9. In [CMNN20a], the vanishing of $L_{T(n+2)} K(L_n^{p,f} \mathbb{S})$ is deduced as a special case of a more general result. For the convenience of the reader, we summarize their argument for the exact vanishing that we use here. We wish to show the claim by induction over n . The case $n = 0$ follows from Mitchell's theorem, as in the proof of the second part of Theorem 3.8. By the strengthening of Hahn's result [Hah16] obtained in [CMNN20a, Lemma 4.5], it suffices to show that $K(L_n^{p,f} \mathbb{S})^{tC_p}$ is $T(n+1)$ -acyclic. Now, given any commutative algebra in genuine C_p -spectra whose underlying spectrum with C_p -action is $K(L_n^{p,f} \mathbb{S})$ with trivial C_p -action, there is a ring map from its geometric fixed points to $K(L_n^{p,f} \mathbb{S})^{tC_p}$. An example is the K -theory of the Borel complete categorical Mackey functor, where the genuine fixed points are $K(\text{Fun}(BC_p, \text{Perf}(L_n^{p,f} \mathbb{S})))$. The transfer for this genuine C_p -spectrum is the composite

$$(4) \quad K(L_n^{p,f} \mathbb{S})_{hC_p} \longrightarrow K(L_n^{p,f} \mathbb{S}[C_p]) \longrightarrow K(\text{Fun}(BC_p, \text{Perf}(L_n^{p,f} \mathbb{S}))),$$

and by definition the geometric fixed points are the cofibre of this composite. It hence suffices to show that each of the above two maps is a $T(n+1)$ -local equivalence. For the second, one uses that the Verdier quotient $\text{Fun}(BC_p, \text{Perf}(L_n^{p,f} \mathbb{S})) / \text{Perf}(L_n^{p,f} \mathbb{S}[C_p])$ is linear over $(L_n^{p,f} \mathbb{S})^{tC_p}$. Indeed, calling this quotient \mathcal{Q} and writing $R = L_n^{p,f} \mathbb{S}$, Theorem I.3.3ii and an analogue of Lemma I.3.8iii from [NS18] imply that $\text{End}_{\mathcal{Q}}(R) \simeq R^{tC_p}$. By Theorem I.3.6 in *op. cit.*, \mathcal{Q} has a canonical symmetric monoidal structure and we obtain a symmetric monoidal functor $\text{Perf}(R^{tC_p}) \rightarrow \mathcal{Q}$. The spectrum $(L_n^{p,f} \mathbb{S})^{tC_p}$ is itself an algebra over $L_{n-1}^{p,f} \mathbb{S}$ by Kuhn's blueshift result [Kuh04]. Thus, by induction, the second map in (4) is a $T(n+1)$ -local equivalence.

For the first map one uses Corollary 4.29 (whose proof relies only on Theorem 3.8) to obtain a diagram which is cartesian after $T(i)$ -localization, $i \geq 2$,

$$\begin{array}{ccc} K(L_n^{p,f}\mathbb{S})_{hC_p} & \longrightarrow & K(L_n^{p,f}\mathbb{S}[C_p]) \\ \downarrow & & \downarrow \\ \mathrm{TC}(\tau_{\geq 0}(L_n^{p,f}\mathbb{S}))_{hC_p} & \longrightarrow & \mathrm{TC}(\tau_{\geq 0}(L_n^{p,f}\mathbb{S})[C_p]). \end{array}$$

Now, by [HN19, Theorem 1.4.1], the cofibre of the lower horizontal map belongs to the localizing subcategory of spectra generated by $\tau_{\geq 0}(L_n^{p,f}\mathbb{S})$, and hence vanishes $T(n+1)$ -locally as well.

Question 3.10. *For a ring spectrum A and for $n \geq 2$, does the map $A \rightarrow L_{K(n-1) \oplus K(n)}A$ induce an equivalence on $K(n)$ -local K -theory?*

The above question reduces to proving an analog of Theorem 3.8 for L_n -localization: that is, for $n \geq 2$, it would suffice to show that $A \rightarrow L_n A$ induces an equivalence on $L_{K(n)}K(-)$.

4. CONSEQUENCES AND EXAMPLES

In this section we discuss some consequences and examples of our main result.

4.1. Direct consequences. To begin with, we record some immediate corollaries of Theorem A.

Corollary 4.1. *Let R be a ring spectrum which is $T(n) \oplus T(n-1)$ -acyclic for some $n \geq 1$. Then $L_{T(n)}K(R) = 0$. \square*

Corollary 4.2. *Let $n \geq 2$. Then for any ring spectrum R , we have that the canonical map $L_{T(n)}K(\tau_{\geq 0}R) \rightarrow L_{T(n)}K(R)$ is an equivalence. \square*

In the case of \mathbb{E}_∞ -rings we furthermore find the following redshift phenomenon:

Corollary 4.3. *Let A be an \mathbb{E}_∞ -ring spectrum, B an A -algebra, and let $n \geq 0$. Then $L_{T(n)}A = 0$ implies $L_{T(n+i)}K(B) = 0$ for every integer $i \geq 1$.*

Proof. If A is $T(n)$ -acyclic, then A , and hence also B , is $T(n+j)$ -acyclic for all $j \geq 0$ by [Hah16] and Lemma 2.3. Thus, the result follows from Theorem A. \square

Next, we include the following slight variants of Theorem A in the connective case, and an analog for topological cyclic homology.

Corollary 4.4. *Let $A \rightarrow B$ be a $T(1) \oplus \cdots \oplus T(n)$ -equivalence between connective ring spectra which induces a surjection on π_0 whose kernel is nilpotent. Then $K(A) \rightarrow K(B)$ is again a $T(1) \oplus \cdots \oplus T(n)$ -equivalence.*

Proof. By Theorem A, only the case $n = 1$ requires a further argument, but the argument for it works equally well in the general case: Consider the pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & B \\ \downarrow & & \downarrow \\ A[\frac{1}{p}] & \longrightarrow & B[\frac{1}{p}] \end{array}$$

Since $P \rightarrow A[\frac{1}{p}]$ is a $T(0)$ -localization, applying K -theory to the diagram yields again a pullback, e.g. by [LT19, Main Theorem]. Furthermore, the map $K(A[\frac{1}{p}]) \rightarrow K(B[\frac{1}{p}])$ is a p -adic equivalence, as p -adic K -theory is truncating on $\mathbb{S}[\frac{1}{p}]$ -algebras [LT19, Lemma 2.4] and hence also nilinvariant [LT19, Corollary 3.5]. Hence, $K(P) \rightarrow K(B)$ is a $T(i)$ -equivalence for all $i \geq 1$. Furthermore, $A \rightarrow P$ is a $T(0) \oplus T(1) \oplus \dots \oplus T(n)$ -equivalence. Theorem 3.8 together with the already established results thus implies that $K(A) \rightarrow K(B)$ is also a $T(1) \oplus \dots \oplus T(n)$ -equivalence. \square

Corollary 4.5. *Let $A \rightarrow B$ be a $T(1) \oplus \dots \oplus T(n)$ -equivalence between connective ring spectra which induces a surjection on π_0 whose kernel is nilpotent. Then $\mathrm{TC}(A) \rightarrow \mathrm{TC}(B)$ is again a $T(1) \oplus \dots \oplus T(n)$ -equivalence.*

Proof. By the Dundas–Goodwillie–McCarthy theorem [DGM13, Theorem VII.0.0.2], there is a cartesian square

$$\begin{array}{ccc} K(A) & \longrightarrow & \mathrm{TC}(A) \\ \downarrow & & \downarrow \\ K(B) & \longrightarrow & \mathrm{TC}(B). \end{array}$$

So we deduce the corollary from Corollary 4.4. \square

Remark 4.6. If $A \rightarrow B$ is a $T(0)$ -equivalence between connective ring spectra inducing a surjection on π_0 whose kernel is nilpotent, then it is also true that the map $K(A) \rightarrow K(B)$ is a $T(0)$ -equivalence. Thus, the same also holds true for $\mathrm{TC}(A) \rightarrow \mathrm{TC}(B)$.

Remark 4.7. One can also deduce some consequences for i -fold iterated algebraic K -theory $K^{(i)}$. For example, the canonical maps $K^{(i)}(\mathbb{Z}/p^k) \rightarrow K^{(i-1)}(\mathbb{Z})$, where the latter is induced by the truncation map $K(\mathbb{Z}/p^k) \rightarrow \mathbb{Z}$, are $H\mathbb{Q} \oplus T(1) \oplus T(2) \oplus \dots$ -equivalences for all $i \geq 1$.

Indeed, the case $i = 1$ follows directly from Corollary 4.3 and the fact that $K_j(\mathbb{Z}/p^k) \otimes \mathbb{Q} = 0$ for $j > 0$. We use also that the non-positive K -groups of a ring are invariant under quotients by a nilpotent ideal, as non-positive K -theory is truncating by [BGT13, Theorem 9.53]. In general, we assume inductively that $K^{(i)}(\mathbb{Z}/p^k)$ and $K^{(i-1)}(\mathbb{Z})$ are connective, $\pi_0 K^{(i)}(\mathbb{Z}/p^k) = \pi_0 K^{(i-1)}(\mathbb{Z}) = \mathbb{Z}$, and $K^{(i)}(\mathbb{Z}/p^k) \rightarrow K^{(i-1)}(\mathbb{Z})$ is an equivalence after $H\mathbb{Q} \oplus T(1) \oplus T(2) \oplus \dots$ -localization. We have just seen the case $i = 1$. Given the statement for some $i \geq 1$, we can deduce the statement for $i + 1$, using again [BGT13, Theorem 9.53] and Corollary 4.4 (as well as [LT19, Lemma 2.4] for the rationalization). The result thus follows by induction.

4.2. Examples of vanishing results. We give various examples showing that Theorem A (or Corollary 4.1) implies vanishing statements for suitable telescopic localizations of the K -theory of ring spectra; this recovers a number of existing results in the literature.

First, we begin with the case of $K(n)$, cf. also [AKS20] in the case $n = 2, p = 2, 3$.

Corollary 4.8. *The spectrum $K(K(n))$ vanishes $T(m)$ -locally for every $m \neq 0, n, n + 1$. \square*

Remark 4.9. Using the dévissage result [AGH19, Proposition 4.4] (preceded by [BL14] for connective K -theory) we can also understand the $T(0)$ -localization of $K(K(m))$. There is a fibre sequence

$$(5) \quad K(\mathbb{F}_p) \longrightarrow K(k(m)) \longrightarrow K(K(m))$$

where $k(m)$ is the connective cover of $K(m)$ and the first map is induced by the functor $\mathrm{Perf}(\mathbb{F}_p) \rightarrow \mathrm{Perf}(k(m))$ given by restriction of scalars along the canonical map $k(m) \rightarrow \mathbb{F}_p$.

As $K(-)_{[p]}^{\frac{1}{p}}$ is truncating on $\mathbb{S}_{[p]}^{\frac{1}{p}}$ -acyclic ring spectra, the canonical map $K(k(m)) \rightarrow K(\mathbb{F}_p)$ is a $T(0)$ -equivalence. The composite

$$K(\mathbb{F}_p) \longrightarrow K(k(m)) \longrightarrow K(\mathbb{F}_p)$$

is induced by the functor $\text{Perf}(\mathbb{F}_p) \rightarrow \text{Perf}(k(m)) \rightarrow \text{Perf}(\mathbb{F}_p)$ sending X to $X \otimes_{k(m)} \mathbb{F}_p$, which is equivalent to $\text{id} \oplus \Sigma^{2p^m-1}$ as there is a fibre sequence

$$\Sigma^{2p^m-2} k(m) \xrightarrow{v_m} k(m) \longrightarrow H\mathbb{F}_p.$$

Upon applying any localizing invariant, this gives the zero map. From (5) we thus obtain a fibre sequence

$$K(\mathbb{F}_p)_{[p]}^{\frac{1}{p}} \xrightarrow{0} K(\mathbb{F}_p)_{[p]}^{\frac{1}{p}} \longrightarrow K(K(m))_{[p]}^{\frac{1}{p}}$$

and hence an equivalence

$$K(K(m))_{[p]}^{\frac{1}{p}} \simeq K(\mathbb{F}_p)_{[p]}^{\frac{1}{p}} \oplus \Sigma K(\mathbb{F}_p)_{[p]}^{\frac{1}{p}}.$$

Corollary 4.10. *For any $n \geq 0$, we find that $L_{T(i)}K(\tau_{\leq n}\mathbb{S}) = 0$ for $i \geq 2$.* \square

Ben Antieau has already shown previously that a certain quantitative version of Proposition 3.1 implies $L_{T(n)}K(\tau_{\leq m}\mathbb{S}) = 0$ at least for all n such that $4p - 4 \geq n$, where p is the implicit prime in $T(n)$, and conjectured that Corollary 4.10 is true.

Corollary 4.11. *The map $K(BP\langle n \rangle) \rightarrow K(E(n))$ is a $T(i)$ -equivalence for $i \geq n + 1$. Furthermore, both vanish $T(i)$ -locally for $i \geq n + 2$.* \square

Remark 4.12. The chromatic bound for $K(BP\langle n \rangle)$ has been proved previously by Angelini-Knoll-Salch in the case where $BP\langle n \rangle$ admits an \mathbb{E}_∞ -structure, [AKS20].

The above result implies that the sequence

$$K(BP\langle n-1 \rangle) \longrightarrow K(BP\langle n \rangle) \longrightarrow K(E(n))$$

becomes a fibre sequence after $T(i)$ -localization for $i \geq n + 1$. Whether or not this sequence is a fibre sequence (after replacing the rings with their p -completions) was asked by Rognes, the $n = 1$ case being a theorem of Blumberg-Mandell [BM08], and the $n = 0$ case being a classical theorem of Quillen's. It was then shown by Antieau-Barthel-Gepner that for $n \geq 2$, the above is not a fibre sequence after rationalization, see [ABG18].

The following is an example that arose from a discussion with George Raptis. Recall that for a connected space X its Waldhausen A -theory is given by $A(X) = K(\mathbb{S}[\Omega X]) = K(\Sigma_+^\infty \Omega X)$. In particular, $A(*) = K(\mathbb{S})$. In the following corollary we assume that $n \geq 1$; the case $n = 0$ is due to Waldhausen.

Corollary 4.13. *Let W be a connected space. If $\Sigma^\infty W$ is $T(n) \oplus T(n-1)$ -acyclic, then the canonical maps $A(*) \rightleftarrows A(\Sigma W)$ are mutually inverse $T(n)$ -equivalences.*

For instance, W could be a connected type m -complex for $m > n$.

Proof. The James splitting (see [Ada72, Chapter 10, Theorem 5]) gives an equivalence

$$\Sigma^\infty \Omega \Sigma W \simeq \Sigma^\infty \bigvee_{k \geq 1} W^{\wedge k},$$

which implies that $\Sigma^\infty \Omega \Sigma W$ is $T(n) \oplus T(n-1)$ -acyclic. The claim thus follows from Theorem A. \square

Likewise, we can reprove and extend a recent theorem of Angelini-Knoll and Quigley [AKQ19] about the chromatic localization of the K -theory of certain Thom spectra $y(m)$ considered in [MRS01, Section 3]. To explain the setup, we recall that for a fixed prime p , there is an essentially unique map of \mathbb{E}_2 -spaces

$$\Omega^2 \Sigma^2 S^1 \longrightarrow \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$$

sending a generator of π_1 to the element $1 - p \in \pi_1(\mathrm{BGL}_1(\mathbb{S}_p^\wedge)) \cong \mathbb{Z}_p^\times$. It is a theorem of Mahowald (for $p = 2$) and Hopkins (for odd primes) that its Thom spectrum is $H\mathbb{F}_p$ [Mah79]; see also [ACB19]. We note that $\Omega^2 \Sigma^2 S^1 \simeq \Omega(\Omega S^3)$ and that ΩS^3 has a canonical cell structure with one cell in every even dimension; see [Mil63, Corollary 17.4]. Let us denote by $F_m(\Omega S^3)$ the $2m$ -skeleton of this cell structure. One then obtains maps of \mathbb{E}_1 -spaces

$$\Omega F_{p^{m-1}}(\Omega S^3) \longrightarrow \Omega^2 S^3 \longrightarrow \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$$

whose Thom spectra are denoted by $y(m)$, leaving the prime p implicit as always. One has $y(0) = \mathbb{S}_p^\wedge$ and $y(\infty) = H\mathbb{F}_p$. The above filtration of ΩS^3 can also be described as the James filtration on $\Omega \Sigma S^2$, compare [MRS01, Section 3.1].

Lemma 4.14. *The spectrum $y(m)$ is $L_{m-1}^{p,f}$ -acyclic.*

Proof. We need to show that $y(m)$ is $T(n)$ -acyclic for $n < m$. For $n = 0$ this follows because $\pi_0(y(m)) = \mathbb{F}_p$ (see the paragraph preceding [MRS01, Equation 3.7] for odd p and [AKQ19, Lemma 2.7] for $p = 2$). We now discuss the case where $n > 0$. Again, we distinguish the cases of even and odd primes. For $p = 2$ this follows from [AKQ19, Proposition 2.22] and Lemma 2.3. For odd primes, it is explained in [MRS01] that for a finite type n spectrum V_n , the Adams spectral sequence for the spectrum $V_n \otimes y(m)$ has a vanishing line of slope $\frac{1}{2p^m - 2}$, because this is true for $y(m)$. On the other hand, the element v_n acting on V_n gives an element of slope $\frac{1}{|v_n|}$ for the Adams spectral sequence. Hence, if $n < m$, it follows that the element v_n is nilpotent on $V_n \otimes y(m)$, so that $T(n) \otimes y(m)$ vanishes as claimed. \square

Corollary 4.15. *The map $K(y(m)) \rightarrow K(\mathbb{F}_p)$ is an $L_{m-1}^{p,f}$ -equivalence. In particular, $K(y(m))$ vanishes $T(n)$ -locally for $0 < n < m$.*

Proof. The vanishing follows immediately from Theorem A and Lemma 4.14. The map is also a $T(0)$ -equivalence, as $T(0)$ -local K -theory is truncating on $T(0)$ -acyclic ring spectra by a result of Waldhausen (see also [LT19, Lemma 2.4]). \square

Remark 4.16. This corollary implies the corresponding statement with the $T(i)$ replaced by the Morava K -theories $K(i)$. For $p = 2$, the latter was previously obtained by Angelini-Knoll and Quigley [AKQ19, Theorem 1.3] using trace methods.

We obtain a similar result for the integral versions $z(m)$ of $y(m)$ which appear in [AKQ19] when $p = 2$. Again, there are versions for odd primes, but we refrain from spelling them out here.

Corollary 4.17. *The map $K(z(m)) \rightarrow K(\mathbb{Z}_{(2)})$ is a $T(n)$ -equivalence for $0 < n < m$.*

Proof. By [AKQ19, Proposition 2.22], $z(m)$ is $K(n)$ -acyclic for $1 \leq n < m$, and hence also $T(n)$ -acyclic for $1 \leq n < m$, again by Lemma 2.3. The corollary then follows from Corollary 4.4. \square

4.3. Examples of purity. We list some further examples of purity statements, special cases of which have been studied in the literature before.

Corollary 4.18. *Let A be a ko -algebra. Then the natural map $K(A) \rightarrow K(A[\frac{1}{\beta}])$ is a $T(n)$ -local equivalence for all $n \geq 2$.*

Proof. This follows immediately from Theorem A, as the map $ko \rightarrow ko[\beta^{-1}] = KO$ is a $T(n)$ -equivalence for all $n \geq 1$, hence so is $A \rightarrow A[\frac{1}{\beta}]$ for every ko -algebra. \square

Remark 4.19. By work of Blumberg–Mandell [BM08], there is a fibre sequence of connective K -theory spectra

$$K^{\text{cn}}(\mathbb{Z}) \longrightarrow K^{\text{cn}}(ku) \longrightarrow K^{\text{cn}}(KU)$$

and likewise for ko and KO in place of ku and KU . Together with Mitchell’s result this implies Corollary 4.18 in the case where A is ko or ku . Note also that for ko -algebras A , we have that $K(A)$ is $T(n)$ -acyclic for $n \geq 3$; this follows from Corollary 4.1, but was shown for $A = ku$ and $p \geq 5$ already in [AR02] and in general in [CMNN20a]. Thus Corollary 4.18 is a useful statement only at height 2.

We get a similar result for algebras over the connective spectrum of topological modular forms tmf , see [DFHH14, Beh19] for introductions. Recall that tmf is by definition the connective cover of an \mathbb{E}_∞ -ring spectrum Tmf that arises as the global sections of a sheaf \mathcal{O}^{top} of \mathbb{E}_∞ -ring spectra on the étale site of the compactified moduli stack of elliptic curves $\overline{\mathcal{M}}_{\text{ell}}$. The evaluation of \mathcal{O}^{top} on the uncompactified moduli stack \mathcal{M}_{ell} is the periodic spectrum TMF . Of the following corollary, the first statement was already proven in [CMNN20a].

Corollary 4.20. *Let A be a tmf -algebra.*

- (i) *The spectrum $K(A)$ vanishes $T(n)$ -locally for all $n \geq 4$.*
- (ii) *The map $K(A) \rightarrow K(A \otimes_{tmf} Tmf)$ is a $T(n)$ -equivalence for all $n \geq 2$.*
- (iii) *The map $K(A) \rightarrow K(A \otimes_{tmf} TMF)$ is a $T(3)$ -equivalence.*
- (iv) *At the prime 2, there is a $T(3)$ -local equivalence $K(tmf) \simeq K(TM F) \simeq K(E_2)^{hGL_2(\mathbb{F}_3)}$, where E_2 denotes the Lubin–Tate spectrum for a supersingular elliptic curve over \mathbb{F}_4 . Replacing $GL_2(\mathbb{F}_3)$ by the group of automorphisms over \mathbb{F}_3 of a supersingular elliptic curve over \mathbb{F}_9 , the analogous statement holds at the prime 3 as well.*

Proof. By [Rav84, Theorem 2.1], the spectrum $BP[v_n^{-1}]$ has the same Bousfield class as $K(0) \oplus \cdots \oplus K(n)$ and is thus L_n -local. Thus, every p -local complex oriented ring spectrum whose formal group law has height at most n is L_n -local. Evaluated on any affine, \mathcal{O}^{top} is even and hence complex orientable; moreover its formal group is isomorphic to that of the corresponding generalized elliptic curve and thus has height at most 2 at any prime. We see that $Tmf_{(p)}$ is, as a limit of L_2 -local spectra, itself L_2 -local and thus $T(n)$ -acyclic for $n \geq 3$. As $tmf \rightarrow Tmf$ is a $T(n)$ -equivalence for all $n \geq 1$ by Lemma 2.2, we can deduce moreover that tmf is $T(n)$ -acyclic for all $n \geq 3$. Thus the first two statements follow from our main theorem.

For the third statement, it suffices to show that $Tmf \rightarrow TMF$ is a $T(n)$ -equivalence for $n = 2$ (and hence all $n \geq 2$). As taking global sections of quasi-coherent \mathcal{O}^{top} -modules preserves colimits by [MM15], we have $T(2) \otimes Tmf \simeq \Gamma(T(2) \otimes \mathcal{O}^{\text{top}})$; hence it suffices to show that $T(2) \otimes \mathcal{O}^{\text{top}}(\text{Spec } A) \rightarrow T(2) \otimes \mathcal{O}^{\text{top}}(\text{Spec } A)[\Delta^{-1}]$ is an equivalence for every étale affine $\text{Spec } A \rightarrow \overline{\mathcal{M}}_{\text{ell}}$, where Δ denotes the discriminant. As all generalized elliptic curves of height 2 are actually smooth elliptic curves, inverting v_2 (as we do in $T(2)$) indeed inverts Δ as well.

Note that $K(n)$ -localization and $T(n)$ -localization coincide on L_n -local spectra. Indeed, if X is L_n -local, then the fibre of the map $X \rightarrow L_{K(n)}X$ is L_n -local and $K(n)$ -acyclic, whence L_{n-1} -local and thus $T(n)$ -acyclic. As $L_{K(n)}X$ is also $T(n)$ -local, it follows that $L_{T(n)}X \simeq L_{K(n)}X$. Thus, $tmf \rightarrow TMF \rightarrow L_{K(2)}TMF$ are $T(2)$ -local equivalences and hence induce $T(3)$ -equivalences in K -theory by our main theorem. The faithful $GL_2(\mathbb{F}_3)$ -Galois extension $TMF_{(2)} \rightarrow TMF_{(3)}_{(2)}$ from [MM15, Theorem 7.6] localizes to the Galois extension $L_{K(2)}TMF \rightarrow L_{K(2)}TMF_{(3)} \simeq E_2$ (cf. [Beh19, Proposition 6.6.10], [HMS17, Proposition 3.6]). Thus, the map $K(L_{K(2)}TMF) \rightarrow K(E_2)^{hGL_2(\mathbb{F}_3)}$ is an equivalence after an arbitrary telescopic localization by [CMNN20b, Theorems 5.6, Corollary B.4]. The statement for $p = 3$ is proven analogously using that here $L_{K(2)}TMF \simeq E_2^{hG_{24}}$, where E_2 is the Lubin–Tate spectrum for a supersingular elliptic curve C over \mathbb{F}_9 and G_{24} is its group of automorphisms over \mathbb{F}_3 . \square

Remark 4.21. In [BL14] Barwick and Lawson provide an analog of the Blumberg–Mandell localization sequence (see Remark 4.19) for certain regular ring spectra. In particular, there is a localization sequence of connective K -theory spectra

$$K^{\text{cn}}(\mathbb{Z}) \longrightarrow K^{\text{cn}}(tmf) \longrightarrow K^{\text{cn}}(Tmf),$$

which implies the second part of the previous corollary for $A = tmf$ and certain other regular tmf -algebras. We also remark that in [AGH19] these localization sequences are extended to non-connective K -theory spectra. However, for the present application this is irrelevant as the difference vanishes after telescopic localization.

4.4. Consequences for $K(1)$ -local K -theory. We now record the consequences of Theorem A at height 1 (in fact, we will only use Theorem 3.8, and so the results are independent of [CMNN20b]). Recall also that $K(1)$ and $T(1)$ -localization coincide.

Corollary 4.22. *$K(1)$ -local K -theory is truncating on $K(1)$ -acyclic ring spectra. In fact, for a $K(1)$ -acyclic ring spectrum A , we have $L_{K(1)}K(A) = L_{K(1)}K(A[\frac{1}{p}])$.*

Proof. Let A be a $K(1)$ -acyclic, connective ring spectrum. By Theorem 3.8 we have an equivalence $L_{K(1)}K(A) \simeq L_{K(1)}K(A[\frac{1}{p}])$. The claim follows from this as p -adic K -theory is truncating on \mathbb{S}/p -acyclic ring spectra by a result of Waldhausen (see also [LT19, Lemma 2.4]). \square

In the case of $H\mathbb{Z}$ -algebras, the last assertion of Corollary 4.29 also appears in [BCM20], proved by different methods. From [LT19, Theorems 3.3, A.2] we then get the following. Note that discrete rings are $K(1)$ -acyclic.

Corollary 4.23. *$K(1)$ -local K -theory of discrete rings is nilinvariant and satisfies Milnor excision and cdh-descent.*

We also get the following consequence.

Corollary 4.24. *Let A be a connective and $K(1)$ -acyclic ring spectrum. Then the canonical map $L_{K(1)}K(A) \rightarrow L_{K(1)}K(A[x])$ is an equivalence. In other words, $K(1)$ -local K -theory is homotopy invariant on connective, $K(1)$ -acyclic ring spectra.*

Here, for any ring spectrum A , the symbol $A[x]$ denotes the ring spectrum $A \otimes \Sigma_+^\infty \mathbb{Z}_{\geq 0}$.

Proof. We observe that $A[x] = A \otimes \mathbb{S}[x]$ is also $K(1)$ -acyclic and connective. Hence, by Corollary 4.22 we may assume that A is discrete so that $A[x]$ is the usual discrete polynomial ring $A \otimes_{\mathbb{Z}} \mathbb{Z}[x]$. By Theorem A we may furthermore assume that p is invertible in A . In this case, Weibel [Wei81] has shown that p is also invertible on $NK(A) = \text{fib}(K(A[x]) \rightarrow K(A))$. So the p -completion of $NK(A)$ vanishes, and hence $L_{K(1)}NK(A) = 0$ as well. \square

Remark 4.25. For ring spectra, there are two canonical “affine lines:” The flat affine line $A[x]$ as above, and the smooth affine line $A \otimes \mathbb{S}\{x\}$, where $\mathbb{S}\{x\}$ is the free \mathbb{E}_{∞} -ring on a degree 0 generator. Since the canonical map $\mathbb{S}\{x\} \rightarrow \mathbb{S}[x]$ is a π_0 isomorphism, we also obtain homotopy invariance with respect to the smooth affine line on connective, $K(1)$ -acyclic ring spectra A : Both maps $K(A) \rightarrow K(A\{x\})$ and $K(A\{x\}) \rightarrow K(A[x])$ are $K(1)$ -local equivalences.

Remark 4.26. Recall that Weibel’s homotopy K -theory $KH(A)$ of a discrete ring A is defined as the geometric realization of the simplicial spectrum $K(A[\Delta^{\bullet}])$ with

$$A[\Delta^n] = A[x_0, \dots, x_n]/(x_0 + \dots + x_n - 1) \cong A[x_1, \dots, x_n].$$

It follows from the above corollary that

$$L_{K(1)}K(A) \simeq L_{K(1)}KH(A).$$

By results of Weibel and Cisinski homotopy K -theory satisfies Milnor excision [Wei81] and cdh-descent [Cis13]. In this way we obtain another proof of Corollary 4.23.

Remark 4.27. The analog of Corollary 4.22 for topological cyclic homology does not hold: As $\text{THH}(\mathbb{Z}[\frac{1}{p}])$ is a $\mathbb{Z}[\frac{1}{p}]$ -algebra, it vanishes p -adically. So $\text{TC}(\mathbb{Z}[\frac{1}{p}])$ vanishes p -adically, and a fortiori after $T(1)$ -localization. However, $L_{T(1)}\text{TC}(\mathbb{Z})$ does not vanish: For odd primes p , Bökstedt and Madsen [BM94] computed the connective cover of $\text{TC}(\mathbb{Z})_p^{\wedge} \simeq \text{TC}(\mathbb{Z}_p)_p^{\wedge}$ to be equivalent to $j \oplus \Sigma j \oplus \Sigma^3 ku_p^{\wedge}$ where j is the connective cover of the $K(1)$ -local sphere. In particular, the $T(1)$ - or equivalently $K(1)$ -localization of $\text{TC}(\mathbb{Z})$ is given by

$$(6) \quad L_{K(1)}\text{TC}(\mathbb{Z}) \simeq L_{K(1)}\mathbb{S} \oplus \Sigma L_{K(1)}\mathbb{S} \oplus \Sigma^3 KU_p^{\wedge} \neq 0.$$

For $p = 2$, [Rog99b, Theorem 0.5] and [Rog99a, Formula (0.2)] give a filtration of $L_{K(1)}\text{TC}(\mathbb{Z})$, whose associated graded essentially looks like the summands in (6). Using that KU_p^{\wedge} is rationally non-trivial in infinitely many degrees, while the other two terms are rationally non-trivial only in finitely many degrees, one obtains that $L_{K(1)}\text{TC}(\mathbb{Z})$ is non-trivial at $p = 2$ as well.

Remark 4.28. We point out that, although $K(1)$ -local TC commutes with filtered colimits of rings [CMM18, Theorem G], it does not commute with filtered colimits of categories. In fact, one checks that (cf. also [BCM20, Proposition 2.15]) the filtered colimit $\text{colim}_k \text{Mod}_{\mathbb{Z}/p^k}(\text{Perf}(\mathbb{Z}))$ is the ∞ -category of p -power torsion perfect \mathbb{Z} -modules and we thus obtain an exact sequence

$$\text{colim}_k \text{Mod}_{\mathbb{Z}/(p^k)}(\text{Perf}(\mathbb{Z})) \longrightarrow \text{Perf}(\mathbb{Z}) \longrightarrow \text{Perf}(\mathbb{Z}[\frac{1}{p}]).$$

Assuming that $L_{K(1)}\text{TC}$ commutes with filtered colimits we find that $L_{K(1)}\text{TC}$ of the fibre vanishes, as $L_{K(1)}\text{TC}(\text{Mod}_{\mathbb{Z}/(p^k)}(\text{Perf}(\mathbb{Z})))$ is a module over $L_{K(1)}\text{TC}(\mathbb{Z}/(p^k))$ which vanishes since $L_{K(1)}K(\mathbb{Z}/p^k) = 0$ as above; this is a contradiction.

4.5. Consequences for $T(n)$ -local K -theory for $n \geq 2$. In this subsection, we record some further structural consequences of Theorem A at heights ≥ 2 . Some further structural features in this context are also explored in [CMNN20a].

Corollary 4.29. *Let $n \geq 2$. Then for any ring spectrum A , we have a natural equivalence $L_{T(n)}K(A) = L_{T(n)}K(\tau_{\geq 0}A) = L_{T(n)}\mathrm{TC}(\tau_{\geq 0}A)$.*

Proof. Indeed, this follows because $\tau_{\geq 0}A \rightarrow A$ induces an equivalence on $L_{T(n)}K(-)$ by Theorem 3.8. Now we use the Dundas–Goodwillie–McCarthy theorem [DGM13] combined with Mitchell’s theorem [Mit90] to obtain $L_{T(n)}K(\tau_{\geq 0}A) \xrightarrow{\sim} L_{T(n)}(\mathrm{TC}(\tau_{\geq 0}A))$, hence the result. \square

For a $T(n)$ -acyclic ring spectrum (with $n \geq 2$), we therefore obtain an equivalence

$$L_{T(n)}K(L_{T(n-1)}A) \simeq L_{T(n)}\mathrm{TC}(\tau_{\geq 0}A).$$

This should be compared to the result obtained in [BCM20] that if A is a commutative, p -adically complete ring, then

$$L_{T(1)}K(A[\frac{1}{p}]) \simeq L_{T(1)}\mathrm{TC}(A).$$

Note by contrast that no commutativity or completeness at (p, v_1, \dots, v_{n-1}) is required in Corollary 4.29.

Next, we record a result describing the behavior of group-complete K -theory versus Waldhausen K -theory; this is essentially a restatement of the above in categorical terms, and informally states that for $T(i)$ -local phenomena with $i \geq 2$, it suffices simply to group-complete (rather than split all cofibre sequences) in the definition of K -theory. Given an additive ∞ -category \mathcal{A} , we let $K^{\mathrm{add}}(\mathcal{A})$ denote the group-completion K -theory of \mathcal{A} , which we regard as a connective spectrum.

Corollary 4.30. *Let \mathcal{C} be a stable ∞ -category, and let $\mathcal{A} \subset \mathcal{C}$ be an additive subcategory. Suppose \mathcal{A} generates \mathcal{C} as a thick subcategory. Then the natural map $K^{\mathrm{add}}(\mathcal{A}) \rightarrow K(\mathcal{C})$ induces an equivalence on $T(i)$ -localization, for $i \geq 2$.*

Proof. By passage to filtered colimits, we can assume that \mathcal{A} is generated under coproducts by a single object X (and hence \mathcal{C} is generated as a thick subcategory by X). In particular, we have an equivalence $\tau_{\geq 1}K^{\mathrm{add}}(\mathcal{A}) \simeq \tau_{\geq 1}K(\tau_{\geq 0}\mathrm{End}_{\mathcal{C}}(X))$, while $K(\mathcal{C}) = K(\mathrm{End}_{\mathcal{C}}(X))$. The result then follows from Corollary 4.29. \square

Corollary 4.31. *Let $n \geq 2$. The construction $A \mapsto L_{T(n)}K(A)$, from ring spectra to $T(n)$ -local spectra, preserves sifted colimits. The same holds if we restrict to the subcategory of $T(n-1)$ -local ring spectra.*

Proof. We use here that the construction $R \mapsto \mathrm{TC}(R)/p$, from connective ring spectra to spectra, preserves sifted colimits, cf. [CMM18, Corollary 2.15]. We prove the first claim that $A \mapsto L_{T(n)}K(A)$ preserves sifted colimits as A ranges over all ring spectra. Let $A_i, i \in I$ be a sifted diagram of ring spectra. Then $\tau_{\geq 0}A_i, i \in I$ yields a sifted diagram of connective ring spectra, and using Lemma 2.2 we find that

$$L_{T(n)}\left(\mathrm{colim}_{i \in I} \mathrm{TC}(\tau_{\geq 0}A_i)\right) \xrightarrow{\sim} L_{T(n)}\mathrm{TC}(\mathrm{colim}_{i \in I} \tau_{\geq 0}A_i).$$

Using that $\mathrm{colim}_i \tau_{\geq 0}A_i \rightarrow \tau_{\geq 0}(\mathrm{colim}_i A_i)$ is a $T(1) \oplus \dots \oplus T(n)$ -equivalence and hence induces an equivalence on $L_{T(n)}K(-)$ thanks to Theorem 3.8, we conclude the result from Corollary 4.29.

Finally, suppose A_i is a sifted diagram of $T(n-1)$ -local ring spectra. Then the map $\operatorname{colim}_i A_i \rightarrow L_{T(n-1)}(\operatorname{colim}_i A_i)$ is a $T(n-1) \oplus T(n)$ -equivalence (as $T(n-1)$ -local spectra are $T(n)$ -acyclic), hence the last claim by what has already been proved and by Theorem A. \square

Finally, we record the $T(n)$ -local (for $n \geq 2$) analog of the Farrell–Jones conjecture; the following has been also observed by Marco Varisco for connective ring spectra. We refer to the surveys [RV18, Lüc20] for an introduction to this conjecture and its applications. This will rely on the following result about the assembly map in p -adically completed topological cyclic homology, which follows by combining a result of Lück–Reich–Rognes–Varisco [LRRV17] and finiteness properties of TC from [CMM18]. By contrast, the assembly map from (non- p -completed) p -typical TC need not be an equivalence for the family of cyclic subgroups, cf. [LRRV17, Sec. 6].

Proposition 4.32. *Let R be any connective ring spectrum, and let G be any group. Let $\mathcal{O}_e(G)$ be the subcategory of the orbit category of G spanned by G -sets of the form G/H , with $H \subset G$ cyclic. Then the assembly map*

$$\operatorname{colim}_{G/H \in \mathcal{O}_e(G)} \operatorname{TC}(R[H]) \rightarrow \operatorname{TC}(R[G])$$

is a p -adic equivalence.

Proof. By [LRRV17, Theorem 1.19], the assembly map for the family of cyclic groups for THH is an equivalence. Since TC/p commutes with colimits as a functor from connective cyclotomic spectra to spectra, [CMM18, Theorem 2.7], the result follows. \square

Corollary 4.33. *Let R be any ring spectrum, and let G be any group, and let $\mathcal{O}_e(G)$ be as in Proposition 4.32. Then the assembly map*

$$\operatorname{colim}_{G/H \in \mathcal{O}_e(G)} K(R[H]) \rightarrow K(R[G])$$

is a $T(n)$ -equivalence for $n \geq 2$.

Proof. By Corollary 4.29, we may assume R is connective, and replace K by TC. The result follows from Proposition 4.32. \square

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