# KRIZ SLICE SS

Note Title 10/26/2010

$M\mathbb{R} = MU_{\mathbb{R}} = MU_{(C_2)}$

$= MU$ with complex conjugation

$$\Sigma^{\mathbb{C}} BU(n)^{\gamma_n^{\mathbb{C}}} \longrightarrow BU(n+1)^{\gamma_{n+1}^{\mathbb{C}}}$$

$\mathbb{C} = 1+\alpha$ $\quad (\alpha = \text{sign rep} = \sigma)$

There is a theory of real oriented spectra

i.e. $C_2$-equiv comm ring spectrum $E$

with

$$\begin{array}{ccc} S^{1+\alpha} & \longrightarrow & \Sigma^{1+\alpha} E \\ \cap & \nearrow & \\ \mathbb{C}P^{\infty} & & \end{array}$$

$\mathbb{C}P^{\infty}$ with involution

There is always a map $M\mathbb{R} \longrightarrow E$

This was studied by Araki.

Can construct $BP_{\mathbb{R}}$

$M\mathbb{R}$ is a $C_2$ $E_\infty$-ring spectrum

There are real analogs of $K(n)$, $E(n)$, $BP\langle n\rangle$.

What is $M\mathbb{R}_\star = \pi_\star^{C_2}(M\mathbb{R})$?

Tate diagram

$$\begin{tikzcd}
M\mathbb{R} \arrow[r] \arrow[d] & \widetilde{EC_2} \wedge M\mathbb{R} \arrow[d] \\
F(EC_{2+}, M\mathbb{R}) \arrow[r] & \widetilde{EC_2} \wedge F(EC_{2+}, M\mathbb{R}) =: \widehat{M\mathbb{R}}
\end{tikzcd}$$

= Tate spectrum

Thm: The vertical maps are equivs

There is a $RO(C_2)$ graded SS (Tate SS)

$$E^2 = H^*(C_2, MU_*) \Rightarrow \pi_* MR^{hC_2}$$

$$E^2 = \hat{H}^{-*}(C_2, MU_*) \Rightarrow \pi_* \widetilde{MR}$$

The former is obtained from the latter by cutting off. The latter is simpler

$$E^1 = MU_*[\sigma, \sigma^{-1}][a, a^{-1}] \Rightarrow \widetilde{MR}_*$$

$$|x_i| = i(1+\alpha) \qquad |\sigma| = \alpha - 1 \qquad u = \sigma^{-2}$$

$$a : S^0 \longrightarrow S^\alpha$$

$E^1$ term is assoc. but not comm. $\qquad \rho = 1 + \alpha$

<u>Thm</u> $d_{2^{k+1}-1}\, \sigma^{-2^k} = v_k\, a^{2^{k+1}-1}$ $\qquad v_k \in \pi_{(2^k-1)\rho}(BP_{\mathbb{R}})$

There is a proof using the $C_2$-equiv Steenrod algebra.

We know $\Phi^{C_2} M\mathbb{R} = MO \longrightarrow \widetilde{M\mathbb{R}}$

$\Phi^{C_2} BP\mathbb{R} = H\mathbb{Z}/2 \longrightarrow \widetilde{BP\mathbb{R}}$

map of comm ring spectra

so $\widetilde{BP\mathbb{R}}$ is a $H\mathbb{Z}/2$-module and its FGL has $[2](x) = 0$. Hence $v_n$ maps to $0$ in $\pi_* \widetilde{BP\mathbb{R}}$

If $\mu$ is a monomial in $v_n$'s and $\sigma, \sigma^{-1}$ of dim $k+l\alpha$

defect $= \delta(\mu) := k+l$

$\delta(v_n) = 2(2^n-1)$

$\delta(\sigma) = \delta(\sigma^{-1}) = 0$

The $v_n$'s are perm. cycles because $\pi_* MU$ is generated by "real manifolds", i.e. cx pts of real proj alg varieties

If $d_i(m) = \sum u_k a^2$

Observe $\delta(u_k) - \delta(m) = i+1$

Assume inductively that the stated diffs are correct for $k=0, \cdots, n$. Then

$$E_{\star\star}^{2^n} = \mathbb{Z}/2\,[v_n, v_{n+1}, \cdots][\sigma^{\pm 2^n}][a^{\pm 1}]$$

$$\delta(v_n) = (2^n - 1)2$$

Which monomials have lower defect?

Only $\sigma^{-2^n}$ can support a diff killing $v_n$.

$$E_{\star\star}^{\infty} = \mathbb{Z}/2\,[a, a^{-1}] = \pi_\star\, H\mathbb{Z}/2 \quad \text{so} \quad \widetilde{BPR}^{\mathbb{Z}/2} = H\mathbb{Z}/2$$

$$\widetilde{MR}^{\mathbb{Z}/2} = MO$$

# Slices of $M\mathbb{R}$

$$\pi^{C_2}_{k+\ell} H\underline{\mathbb{Z}} = \begin{cases} (H\underline{\mathbb{Z}})^{c} & k \geq 0 \\ (H\underline{\mathbb{Z}})^{h} & k \leq -2 \\ 0 & k = -1 \end{cases}$$

Borel $H^{*}$

Borel $H_{*}$

We know $\pi_{*}(H\underline{\mathbb{Z}})^{\mathbb{Z}/p^n}$

$$\hat{H\mathbb{Z}}_{*} \quad \cdots \; \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \; \cdots$$

For $a$-periodic spectra

(meaning $S^0 \wedge E \xrightarrow{\sim} S^{\alpha} \wedge E$ so $C_{2+} \wedge E \simeq *$

## Reduction Thm

$$M\mathbb{R}/(x_1, x_2, \dots) \xrightarrow{\simeq} H\underline{\mathbb{Z}}$$

real orientation

Can compute Borel cohom of

$$M\mathbb{R}/(x_1, x_2, \dots)$$

and geometric fixed pts.

$$\Phi^{C_2} BP\mathbb{R}/(v_1, v_2, \dots)$$

We find its $\pi_*$ is $\mathbb{Z}/2$ in every even dim.

The RT can be reduced.

$M\mathbb{R}$ is an $E_\infty$-ring spectrum

$\rightsquigarrow$ candidates for slices as wedges of $H\underline{\mathbb{Z}}$

$$BP\mathbb{R} = \underset{\longleftarrow}{\mathrm{holim}}\; BP\mathbb{R}/(\bar{v}_1^{n_1})$$

$$BP\mathbb{R}/\bar{v}_1^{n_1} = \underset{\overleftarrow{n_2}}{\mathrm{holim}}\; BP\mathbb{R}/(\bar{v}_1^{n_1}, \bar{v}_2^{n_2})$$

etc.