The Kervaire Invariant

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Abstract

- Just over a year ago Hill, Hopkins and Ravenel announced that they have a solution to the Kervaire invariant one problem – the only dimensions in which there are framed manifolds with Kervaire invariant one are $2, 6, 14, 30, 62$ and possibly 126.
- The hunt to find examples in these six special cases has begun!
- Summarize what is known about constructing examples.
- Bokstedt's approach involving exceptional Lie groups in an essential way.
- There is a relation with the (differential) geometry of the Gromoll – Meyer sphere.

The Kervaire sphere

- Start with the complex polynomial

$$f_d(z_1, \ldots, z_{d+1}) = z_1^2 + \cdots + z_d^2 + z_{d+1}^3$$

The Kervaire sphere is the link of the singular point of f_{2n+1}

$$K^{4n+1} = f_{2n+1}^{-1}(0) \cap S^{4n+3} \subset \mathbb{C}^{2n+2}$$

- We know that K^{4n+1} is homeomorphic to S^{4n+1}.
- Problem: When is K^{4n+1} diffeomorphic to S^{4n+1}.
- Answer: When $4n + 1 = 1, 5, 13, 29, 61$ and possibly 125

The Kervaire Invariant

- A framing of a manifold M is an isomorphism F of the stable normal bundle of M with a trivial bundle.
- Suppose the dimension of M is $2n$. Use a framing F to construct a quadratic function

$$q = q_F : H^n(M; \mathbb{Z}/2) \to \mathbb{Z}/2.$$

$$q(x + y) = q(x) + q(y) + \langle x, y \rangle$$

where $\langle x, y \rangle$ is the mod 2 intersection number of x and y.
- Since q is quadratic $|q^{-1}(1)| \neq |q^{-1}(0)|$
- q has a $\mathbb{Z}/2$ invariant (its Arf invariant)

$$A(q) = 1 \iff |q^{-1}(1)| > |q^{-1}(0)|.$$

- The Kervaire invariant $K(M, F)$ is the Arf invariant of q_F.
- The Kervaire invariant in dimensions $4k + 2$ can be thought of as the analogue of the signature in dimensions $4k$.

The Kervaire Invariant

- K^{4n+1} is the boundary of a framed $4n+2$ manifold P_0^{4n+2}.
- If K^{4n+1} is diffeomorphic to S^{4n+1} we can glue a disc onto P_0^{4n+2} to form a smooth manifold P^{4n+2}.
- P^{4n+2} can be framed and there is a framing F of P^{4n+2} such that $K(P^{4n+2}, F) = 1$
- Kervaire goes on to prove that

$$K(M^{10}, F) = 0, \quad \text{for all} \qquad (M^{10}, F)$$

- It follows that K^9 cannot be diffeomorphic to S^9.
- Problem: In what dimensions is the Kervaire invariant of framed manifolds non-zero.
- Answer: It is zero except in dimensions $2, 6, 14, 30, 62$ and possibly 126.

Browder plus Hill, Hopkins, and Ravenel

- Browder (1969): The Kervaire invariant of framed manifolds is zero except in dimensions of the form $2^{k+1} - 2$.
- Browder (continued): There is a framed manifold with Kervaire invariant one in dimension $2^{k+1} - 2$ if and only if h_k^2 in the E_2-term of the classical mod 2 Adams spectral sequence is an infinite cycle.
- Hill, Hopkins and Ravenel (2009): h_k^2 is not an infinite cycle if $k \geq 7$

Existence Results

- h_1^2, h_2^2 and h_3^2 are infinite cycles – this follows from the existence of elements of Hopf invariant one.
- h_4^2 is an infinite cycle – this is due to Barratt, Mahowald, and Tangora (1970)
- h_5^2 is an infinite cycle – this is due to Barratt, Jones and Mahowald (1983)
- It is not known whether h_6^2 is an infinite cycle or not

Dimensions 2, 6 and 14

$$K(S^1 \times S^1, F_1 \times F_1) = 1$$

where F_1 is the complex framing of S^1

$$K(S^3 \times S^3, F_3 \times F_3) = 1$$

where F_3 is the quaternionic framing of S^3

$$K(S^7 \times S^7, F_7 \times F_7) = 1$$

where F_7 is the octonionic framing of S^7

Dimension 30

- The dihedral group D_8 acts freely on a closed orientable surface Y^2 of genus 5 with quotient $RP^2 + (S^1 \times S^1)$.
- Via its usual permutation representation in Σ_4 it also acts on $(S^7)^4$.
- Now form

$$M^{30} = Y^2 \times_{D_8} (S^7 \times S^7 \times S^7 \times S^7).$$

- Any framing of S^7 induces a framing of M^{30}.
- Let F be the framing of M^{30} induced by the octonionic framing of S^7. Then

$$K(M, F) = 1.$$

- This seems to be the only known explicit example in dimension 30.
- There is no explicit example in dimension 62.

An attempt to generalise

- We can consider 62 dimensional manifolds of the form

$$M^{62} = Y^6 \times_G (S^7)^8$$

  where G is a subgroup of Σ_8.
- We can choose Y^6 and G so that framings of S^7 induce framings of M^{62}
- However if we equip such a manifold with a framing induced by a framing of S^7 then it has Kervaire invariant zero.

A second attempt

- We could try to replace $S^7 \times S^7$ by a 30 dimensional framed manifold P with Kervaire invariant one and an involution.
- The involution gives an action of D_8 on $P \times P$ and we can form
$$Y^2 \times_{D_8} (P \times P)$$
where Y^2 is the surface of genus 5 used in the 30 dimensional example.
- It is not true that any framing of P induces a framing of M.
- If F is a framing of P which does induce a framing of $Y^2 \times_{D_8} (P \times P)$, then $K(P, F) = 0$.

Bokstedt

- We have the following list of six special examples of homogeneous spaces with dimensions $4, 8, 16, 32, 64, 128$

$$U(3)/(U(2) \times U(1)) \qquad Sp(3)/(Sp(2) \times Sp(1))$$

$$F_4/Spin(9) \qquad E_6/((Spin(10) \times U(1))/\mathbb{Z}_4)$$

$$E_7/((Spin(12) \times Sp(1))/\mathbb{Z}_2) \qquad E_8/((Spin(16)/\mathbb{Z}_2)$$

- The first three are CP^2, HP^2 and OP^2.
- The last three are sometimes called the projective planes of the bio-ctonions $C \otimes O$, the quater-octonions $H \otimes O$ and the octo-octonions $O \otimes O$
- In each case there is a middle dimensional cohomology class u such that u^2 is the fundamental class in the top dimensional cohomology.

Some Homotopy Theory

- Let X be a space and

$$f : S^{2n+m} \to X, \quad g : X \to S^m$$

be two maps. Form the mapping cones

$$Y = X \cup_f D^{2n+m+1}, \quad Z = S^m \cup_f C(X).$$

- Assume f and g are both zero in mod 2 cohomology. Then there are isomorphisms

$$H^j(Y) \to H^j(X) \to H^{j+1}(Z) \quad \text{for } m < j < 2n + m + 1$$

- $a \in H^m(Z)$ is the mod 2 cohomology class corresponding to S^m
- $b \in H^{2n+m+1}(Y)$ is the cohomology class corresponding to the $2n + m + 1$ disc
- $\phi : H^j(Y) \to H^{j+1}(Z)$ is the above isomorphism

More homotopy theory

- Are there triples (X, f, g) as above such that

$$Sq^{n+1}(a) = \phi(x), \quad Sq^{n+1}(x) = b$$

- Easy to show that if so $n+1$ must be of the form 2^k
- When $n+1 = 2^k$ such a triple exists if and only if h_k^2 is an infinite cycle.
- When $k = 1, 2, 3$ then by Hopf invariant one we can take X to be the sphere S^{2n-1}.
- When $k = 4$ there are examples where X has two cells.
- When $k = 5$ in the only known example X has 9 cells

Back to Bokstedt

- P is the homogeneous space and $2n+2 = 2^{k+1}$ is its dimension.
- X is the $2n+1$ skeleton of P and $f : S^{2n+1} \to X$ is the attaching map of the $2n+2$ cell.
- Now suppose X is (stably $2n+2$ Spanier Whitehead) self dual. Then for some (large) m we can form the triple

$$\Sigma^{m-1} f : S^{2n+m} \to \Sigma^{m-1} X, \quad g : \Sigma^{m-1} X \to S^m$$

  where g is the appropriate Spanier Whitehead dual of f.
- This triple then satisfies the conditions required to show that h_k^2 is an infinite cycle.

Bokstedt

- In the first three cases X is

$$S^2 = CP^1, \quad S^4 = HP^1, \quad S^8 = OP^1$$

  so it is self dual.
- In the next case X is not self-dual.
- However, by using a combination of Morse theory and some computations in homotopy theory, Bokstedt manages to find a self-dual subcomplex of X and to compress f to this self dual subcomplex.
- It is not known whether this approach can be made to work in dimensions 62 and 126.

The Gromoll – Meyer sphere following Duran and Puttmann

- The Gromoll – Meyer sphere is a Riemannian manifold whose underlying smooth manifold is Milnor's exotic 7 sphere W^7.
- In other words it is a metric on W^7.
- We can identify W^7 with the subspace of $\mathbb{C}^5$ defined by the equations

$$z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^5 = 0$$

$$|z_1|^2 + |z_2|''^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1.$$

- Notice that it contains the Kervaire 5 sphere.
- It is a quotient of a free S^3 action on $SP(2)$ and this defines the metric.

The Gromoll – Meyer sphere

- S^7 is a quotient of a (different) S^3 action on $Sp(2)$ and this quotient also defines the usual metric on S^7.
- Let $\pi_S : Sp(2) \to S^7$, and $\pi_W : Sp(2) \to W^7$ be the two projections.
- It is not true in general that fibres of π_S are fibres of π_W. However if a fibre of π_S contains a matrix A whose entries are real then it is also a fibre of π_W.
- Choose a real matrix $A \in Sp(2)$ with determinant 1.
- Let $Q \in S^7$ be the point $\pi_S(A)$ and let γ be a great circle in S^7 passing through Q.
- Duran and Puttmann show how to lift γ to a smooth (not necessarily closed) curve $\tilde{\gamma} \in Sp(2)$ such that $\pi_W \tilde{\gamma}$ is a geodesic in W which starts and ends at the point $\pi_W(A)$.
- However the closed curve $\pi_W \tilde{\gamma}$ is not smooth.

The Gromoll – Meyer sphere

- From these geometric facts Duran and Puttmann construct an explicit homeomorphism of S^7 with W^7 which is smooth in the complement of a point.
- This diffeomorphism maps a copy of S^5 contained in S^7 diffeomorphically onto the Kervaire sphere $K^5 \subset W^7$.
- They then write down a formula for this diffeomorphism which uses quaternionic multiplication.
- Their formula with quaternionic multiplication replaced by octonionic multiplication gives a diffeomorphism of S^{13} with Σ^{13}
- Their diffeomorphism is G_2 invariant.

One final point

- Using the general theory of Browder and Brown it is possible to define a quadratic form q on $H^n(M^{2n})$ using a weaker structure than a framing.
- However, this quadratic form may not be defined for all values of n.
- When it is defined it will in general take values in $\mathbb{Z}_4$ and quadratic will mean

$$q(x+y) = q(x) + q(y) + 2\langle x, y\rangle.$$

- This $\mathbb{Z}_4$ valued quadratic from has a generalized Kervaire invariant $B(q) \in \mathbb{Z}_8$.
- If q takes values in $\{0, 2\} \subset \mathbb{Z}_4$ we can identify q with a $\mathbb{Z}_2$ valued quadratic form.
- In this case $B(q) \in \{0, 4\} \subset \mathbb{Z}_8$ and $B(q) \neq 0$ if and only if the Arf invariant of the corresponding $\mathbb{Z}_2$ valued quadratic form is non-zero

Codimension 1 immersions

- For example this more general theory applies if the manifold M comes equipped with an isomorphism of $TM \oplus L$ with the trivial bundle where L is a line bundle. If L is trivial this is the same as a framing.
- Geometrically such a structure corresponds (via Smale's immersion theory) to an immersion of M in codimension 1.
- In this context the invariant is defined in all the dimensions $2^{k+1} - 2$ and it is non-zero in all these dimensions.
- In fact in dimensions $2, 6$ this generalized Kervaire invariant can take any value in $\mathbb{Z}_8$. In the other dimnsions of the form $2^{k+1} - 2$ it can take any values in $\{0, 2, 4, 6\} \subset \mathbb{Z}_8$.

Oriented codimension 2 immersions

- The more general theory also applies if the manifold M comes equipped with an isomorphism of $TM \oplus P$ with the trivial bundle where P is an oriented 2-plane bundle.
- This time immersion theory shows that this structure corresponds to an orientation of M and an oriented immersion in codimension 2.
- The quadratic form is defined for all dimensions of the form $2^{k+1} - 2$
- In these dimensions the quadratic from is always $\mathbb{Z}_2$ valued so the invariant is the Kervaire invariant of a $\mathbb{Z}_2$ valued quadratic form.
- In each dimension of the form $2^{k+1} - 2$ there is an oriented codimension 2 immersion with Kervaire invariant one.