Hill – MFO – 9-23-10

Note Title 9/23/2010

Last time we cited $\wedge$ (to be defined below) geometric fixed points $\Phi^{G}(\ )$ with

$$\Phi^{G}(X \wedge Y) = \Phi^{G}(X) \wedge \Phi^{G}(Y)$$

$$\Phi^{G}(G_+ \wedge_H X) = *$$

G is always $C_{2^{n+1}}$

Hence

$$\Phi^{G}\Big(\bigvee_{\substack{\text{monomials}\\ p}} \big(G_+ \wedge_{H_p} S^{\frac{|p|}{|H_p|}\rho_{H_p}}\big) \wedge H\underline{\mathbb{Z}}\Big)$$

$$= \bigvee_{\substack{\text{monomials}\\ \text{fixed by } G}} \Phi^{G}\big(S^{\frac{|p|}{|G|}\rho_G}\big) \wedge \Phi^{G}(H\underline{\mathbb{Z}})$$

$$= \bigvee_{\substack{\text{monomials} \\ \text{fixed by } G}} S^{|P|/|G|} \to \Phi^G(H\underline{\mathbb{Z}})$$

Prop $\Phi^G H\underline{\mathbb{Z}} = H\mathbb{F}_2[b_G]$ with $|b_G| = 2$
(where $G = C_{2^{n+1}}$)

Recall P has $H_P = G$ iff P is a monomial in the $N(m_i) = m_i \circ \gamma m_i \cdots \gamma^{2^n - 1} m_i$

Let

$$\begin{tikzcd} S^i \arrow[r, "a_{i\rho_G}"] \arrow[rr, bend right, "f_i"'] & S^{i\rho} \arrow[r, "N(m_i)"] & MU^{(2^n)} \end{tikzcd}$$

Slice SS for $\Phi^G(\ )$
$E_2 = \mathbb{F}_2[b, f_1, f_2, \ldots]$

$\Rightarrow \pi_* MO = \mathbb{F}_2[x_i : i \neq 2^j - 1]$

The f_i lie on the line of slope $|G| - 1$

and are permanent cycles

b has filtration -2

We must have differentials

$b \to f_1$, $b^2 \to f_3$, $b^4 \to f_7$ etc

length $\quad 1 + |G| \quad 1 + 3|G| \quad 1 + 7|G|$

To define geometric fixed pts we need a space $\tilde{E}P$ with $(\tilde{E}P)^H = \begin{cases} \emptyset & H = G \\ * & H \subsetneq G \end{cases}$

We have a cofiber seq

$$E\mathcal{P}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{P}$$

If $H \subsetneq G$ then the first map is H-equiv

and $S^0 \xrightarrow{\simeq} (\tilde{E}\mathcal{P})^G$

The isotropy separation sequence is

$$E\mathcal{P}_+ \wedge X \longrightarrow X \longrightarrow \tilde{E}\mathcal{P} \wedge X$$

and define $\Phi^G(X) = (\tilde{E}\mathcal{P} \wedge X)^G$

Let V be a rep of G with $V^G = \{0\}$

and $\dim V^H \geq 1$ for H proper

e.g. $V = \sigma$ or $\bar{\rho}$

We can choose $EP = \varinjlim S(nV)$

It has the desired properties, giving

$$\tilde{E}P = \varinjlim S^{nV} = S^{\infty V}$$

$$\tilde{E}P \wedge X = a_\sigma^{-1} X = a_{\bar\rho}^{-1} X = a_V^{-1} X$$

all irred reps of G

We know that $\pi_\bigstar \tilde{E}P \wedge H\underline{\mathbb{Z}} = \pi_\bigstar H\underline{\mathbb{Z}}[a_\sigma^{-1}]$

$$= \mathbb{F}_2[a_\sigma^{\pm 1}, a_V^{\pm 1}, \dots, u_{2\sigma}]$$

Recall $a_V : S^{-V} \longrightarrow H\underline{\mathbb{Z}}$ and $u_{2\sigma} : S^2 \longrightarrow S^{2\sigma} \wedge H\underline{\mathbb{Z}}$

so $\pi_\bigstar^G H\underline{\mathbb{Z}} = \mathbb{F}_2[u_{2\sigma}/a_{2\sigma} =: b]$

Note that b is defined in terms of the sign rep of G, which restricts trivially to any proper subgp.

Our differential now reads

$$d\left(\left(\frac{u_{2\sigma}}{a_\sigma^2}\right)^{2^i}\right) = f_{2^{i+1}-1}$$

so $d\left(u_{2\sigma}^{2^i}\right) = a_\sigma^{2^{i+1}} f_{2^{i+1}-1}$

e.g. $d(u_{2\sigma}) = a_\sigma^2 f_1 = a_\sigma^3 \overline{v}_1$

as in Dugger's spectral sequence.

This does not required Bott periodicity.

If $N(u_{2^{i+1}-1})$ is a unit then

$$d\left(\frac{u_{2\sigma}^{2^{i+1}}}{N(u_{2^{i+1}-1})}\right) = a_\sigma^{2^{i+1}} \, a_{(2^{i+1}-1)\bar{\rho}}$$

This is a unit if the a_σ and a_ρ are invertible.

This means inverting $N(u_{2^{i+1}-1})$ makes

$$\Phi^G\left(N(u_{2^{i+1}-1})^{-1} MU^{(2\mathbb{Z})}\right) = *$$

In the $K_{\mathbb{R}}$-theory case

$$d(u_{2\sigma}) = a_\sigma^3 \bar{v}_1 \quad \text{so} \quad \Phi^G(K_{\mathbb{R}}) = *$$

Homotopy fixed pts

$$X^{hG} = \left(F(EG_+, X)\right)^G$$

Take the G-map $EG_+ \to S^0$ gives

$$X = F(S^0, X) \longrightarrow F(EG_+, X)$$

$$X^G \longrightarrow X^{hG}$$

We will smash this map with isotropy sep. seq.

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$$\begin{tikzcd}
EP_+ \wedge X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \tilde{E}P \wedge X \arrow[d] \\
EP_+ \wedge F(EG_+, X) \arrow[r] & F(EG_+, X) & \tilde{E}P \wedge F(EG_+, X)
\end{tikzcd}$$

Center map is G-equiv iff the other two are

Left map is built on induced cells and can be handled by induction on $|G|$

Right map concerns G-fixed points.

Hence if $\Phi^G(X) = *$, both source and target are $*$.

Let $X = \bar{\Delta}^{-1} MU^{(2^n)}$ where

$\bar{\Delta} : S^{m\rho_G} \longrightarrow MU^{(2^n)}$ is

div by $N(m_{2^j-1})$ for some j

$$\Phi^G X = \Phi^G F(EG_+, X) = *$$

Fixed pt theorem If $\bar{\Delta} : S^{m\rho_G} \to MU^{((2^n))}$ is divisible by $N(y_{2^{i_k}-1,k})$ for all k and some i_k

then $X^{G} = X^{hG}$ for X as above

(Restriction of $MU^{((2^n))}$ to $C_{2^{j+1}}$ for $j < n$) is a smash product of $MU^{((2^j))}$

e.g. $MU^{((2))}$ restricts to $MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$ as C_2

$MU^{((8))}$ restricts to $MU^{((4))} \wedge MU^{((4))}$ over C_4

over C_2 to $MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$

Δ must be divisible by some special $N_{C_2}^{H}(\bar{r}_{2^i-1})$ for each subgp $H \subset G$.

This will make $\Phi^H(X) \simeq *$ for each H.

<u>Remark</u> In (1) for $X = S^0$ we get

$$(EP_+)^G \xrightarrow{\text{tr}} (S^0)^G \longrightarrow (\widetilde{EP})^G = S^0$$

$\pi_0 =$ all elements that are transfers

$\pi_0 = A(G)$

$\pi_0 = A(G)/\text{transfers} = \mathbb{Z}$

Transfer maps is what makes $(S^0)^G$ complicated.

If we invert $\bar{\Delta} : S^{m\rho_G} \longrightarrow MU^{((\mathbb{Z}^n))}$ with $\bar{\Delta}$ div by certain elements then $\pi_*(\ )^{hG}$ is periodic.

$$\bar{\Delta} \cdot u_{m\rho} : S^{m|G|} \longrightarrow MU^{((\mathbb{Z}^n))}$$

If we forget G-action we have map

$$\Sigma^{m|G|} MU^{((\mathbb{Z}^n))} \xrightarrow{\Delta} MU^{((\mathbb{Z}^n))}$$

which is an equiv since Δ is a unit

Prop If $f: X \to Y$ is equivar and an underlying map is an equiv, then $f: X^{hG} \to Y^{hG}$ is an equiv

Thm For $\bar{\Delta}$ as before, we can find a power of u_{mP_G} which is a permanent cycle in the slice SS, so $(\Delta u)^{power}$ is an equivar map and an underlying equiv. Hence we have periodicity.