HILL : Equivariant computations + the gap theorem

Note Title 2/11/2010

<u>Thm</u> θ_j does not exist for $j \geq 7$

Proof sketch: ① Construct cohom theory Ω such s.t. if $\exists\, \theta_j$, then it is detected in $\pi_* \Omega$. DETECTION

② $\pi_{-2}\Omega = 0$ GAP

③ $\pi_{k+256}\Omega = \pi_k \Omega$ PERIODICITY

We need equivariant htpy theory

Ω is built out of MU.

$MU_{\mathbb{R}}$ = "Real manifolds"

= honest C_2-equivariant spectrum
studied by Landweber, Araki, Hu-Kriz

$$MU^{(4)} = MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$$ ~ acted on by C_8.

$$\Omega = \left(D^{-1} MU^{(4)}\right)^{C_8}$$

Equivariant homotopy groups

M = G-spectrum

$$[S^n, M]_G = [S^n, M^G] = \pi_n(M^G)$$

trivial action

$$[G \times_H S^n, M]_G = [S^n, M]_H$$

H acts trivially on S^n

$H \subset G$ any subgp

$$\| $$

$$[(G/H)_+ \wedge S^n, M] = \pi_n(M^H)$$

The map $G/H \to G/G = *$ induces

$$\begin{tikzcd}
{[(G/H)_+ \wedge S^n, M]} \arrow[r, phantom, "="] & \pi_n M^H \\
{[(G/G)_+ \wedge S^n, M]} \arrow[u] \arrow[r, phantom, "="] & \pi_n M^G \arrow[u]
\end{tikzcd}$$

$$\mathrm{aut}_G(G/H) = N(H) = \text{normalizer of } H$$

$\Rightarrow$ $N(H)$ acts on $\pi_n(M^H)$ s.t. H acts trivially

$\Rightarrow$ $W(H) = N(H)/H$ acts on $\pi_n(M^H)$

$\Rightarrow$ $\underline{\pi_n}(M)$ is a Mackey functor with $\underline{\pi_n}(M)(G/H) = \pi_n(M^H)$. (The above properties characterize Mackey functors)

Mackey functors are to equivar. htpy thy as Abelian gps " ordinary "

Equiv H^* has coeffs in a Mackey functor

For any Mackey functor $\underline{M}$ there is an EM spectrum $H\underline{M}$

$$\pi_n(H\underline{M}) = \begin{cases} 0 & n \neq 0 \\ \underline{M} & n = 0 \end{cases}$$

2^{nd} Flavor: Let $\underset{\underset{RO(G)}{\cap}}{\overset{V-W}{\overset{\shortparallel}{\alpha}}}$ be a real virtual rep of G

$$[S^\alpha, M]_G = [S^V, S^W \wedge M]_G$$
$$\shortparallel$$
$$[S^0, S^{-\alpha} \wedge M]_G$$

$\Rightarrow$ $\underline{\pi}_0(S^{-\alpha} \wedge M) = \underline{\pi}_\alpha(M)$ $RO(G)$ graded hty

[What are the equivar cohomology operations]

<u>Thm</u> There is an equiv filtration of $MU^{((4))}$ with assoc. graded

$$\bigvee_{p \in I} H\underline{\mathbb{Z}} \wedge \left(C_8/G(p)_+ \wedge S^{n_p \rho_{G(p)}}\right)$$

reg rep of $G(p)$

$$\underline{\pi}_*\left(MU^{((4))}\right)(C_8) = \pi_*\left(MU^{((4))}\right),$$ which has a C_8-action

$$= \mathbb{Z}\left[r_i, \gamma r_i, \gamma^2 r_i, \gamma^3 r_i \ : i > 0\right]$$

$$\gamma^4(r_i) = (-1)^i r_i$$

C_8 acts on set of monomials up to deg n

I = set of orbits. For $p \in I$,

$|p| = n_p \, |G_p|$ where G_p = stabilizer

e.g. one orbit $p = \eta_1$ in degree 2 with $G_p = C_2$

$n_p = 1$

in degree 4 we have orbits of

p	G_p	n_p
η_1^2	C_2	2
$\eta_1 \gamma(\eta_1)$	C_2	2
$\eta_1 \gamma^2(\eta_1)$	C_4	1
η_2	C_2	2

Computing $\underline{\pi}_\star^H (H\underline{\mathbb{Z}} \wedge S^{kP_H})$

$$\cong \underline{\pi}_\star^G \left(H\underline{\mathbb{Z}} \wedge \left((G/H)_+ \wedge S^{kP_H}\right)\right)$$

$H\underline{\mathbb{Z}}$ reps equiv homology and tells us how to build things out of cells of the form $(G/H)_+ \wedge S^n$ and $(G/H)_+ \wedge D^{n+1}$

$P_2 = \mathbb{C}$ with conjugation

$S^{P_2} = S^2$ with reflection thru equator

$$(S^{P_2})^{C_2} = S^1$$

Cellular chains

$\rho_g := \rho_{C_g}$

$$\begin{array}{ll} 2 & \mathbb{Z}\oplus\mathbb{Z} = \mathbb{Z}[C_2] \\ & \quad\downarrow \text{fold} \\ 1 & \mathbb{Z} = \mathbb{Z}[C_2/C_2] \end{array}$$

If we forget C_2-action, we are building a cell structure on S^2.

Another example $\mathbb{Z}\rho_2 = 2 \oplus 2\sigma$

$$S^{2\rho_2} = S^2 \wedge S^{2\sigma} \quad \text{and} \quad \left(S^{2\rho_2}\right)^{C_2} = S^2$$

$$\left(S^{2\rho_2}\right)^{C_1} = S^4$$

Cell complex has the form

$$
\begin{array}{cccc}
4 & \mathbb{Z}[C_2] & & C_{2+}\wedge D^4 \\
 & \Big\downarrow{\scriptstyle 1-\gamma} & & \\
3 & \mathbb{Z}[C_2] & & C_{2+}\wedge D^3 \\
 & \Big\downarrow{\scriptstyle \varepsilon} & & \\
2 & \mathbb{Z} & & S^2
\end{array}
$$

γ = gp generator

ε = fold map
= augmentation

Example $S^{\rho_4 - 1}$ $\quad \rho_4 = 1 \oplus \sigma \oplus \lambda$

λ = rotation of $\mathbb{R}^2$ by $\pi/2$

$$\left(S^{\rho_4 - 1}\right)^{C_4} = S^0 \qquad \left(S^{\rho_4 - 1}\right)^{C_1} = S^3$$

$$\left(S^{\rho_4 - 1}\right)^{C_2} = S^1 = (C_4/C_2)_+ \wedge D^1$$

Chain complex

$$\begin{array}{lcl}
3 & \mathbb{Z}^4 & = \mathbb{Z}[C_4] \\
 & \downarrow{\scriptstyle 1+\gamma} & \\
2 & \mathbb{Z}^4 & = \mathbb{Z}[C_4] \\
 & \downarrow{\scriptstyle 1-\gamma} & \\
1 & \mathbb{Z}^2 & = \mathbb{Z}[C_4/C_2] \\
 & \downarrow{\scriptstyle \varepsilon} & \\
0 & \mathbb{Z} & = \mathbb{Z}[C_4/C_4]
\end{array}$$

Thm $\pi_*\left((H\underline{\mathbb{Z}} \wedge S^{k\rho_G})^G\right) = 0$ for $-4 < * < 0$ for all k.

Pf Obvious for $k \geq 0$. These are computed by applying fixed pt functor to chain complex. For $k < 0$, $S^{k\rho_G}$ is SW dual of $S^{-k\rho_G}$. If $k < -2$ we have nothing in dim -2. for $k=-1$ or -2

we have $\mathbb{Z} \xrightarrow{\text{diag}} \mathbb{Z}[C_2] \xrightarrow{1-\gamma}$

-1 or -2 $\qquad -2$ or -3

fixing to $\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0}$

so $\pi_{-2} = 0$

D is in $\pi_{19\rho_8} MU^{(4)}$.