ALGEBRAS FOR ENRICHED ∞-OPERADS

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ABSTRACT. Using the description of enriched ∞-operads as associative algebras in symmetric sequences, we define algebras for enriched ∞-operads as certain modules in symmetric sequences. For $\mathbf{V}$ a symmetric monoidal model category and $\mathbf{O}$ a Σ-cofibrant operad in $\mathbf{V}$ for which the model structure on $\mathbf{V}$ can be lifted to one on $\mathbf{O}$-algebras, we then prove that strict algebras in $\mathbf{V}$ are equivalent to ∞-categorical algebras in the symmetric monoidal ∞-category associated to $\mathbf{V}$. We also show that for an ∞-operad $\mathcal{O}$ enriched in a suitable closed symmetric monoidal ∞-category $\mathcal{V}$, we can equivalently describe $\mathcal{O}$-algebras in $\mathcal{V}$ as morphisms of ∞-operads from $\mathcal{O}$ to a self-enrichment of $\mathcal{V}$.

CONTENTS

1. INTRODUCTION

If $\mathbf{V}$ is a symmetric monoidal category whose tensor product is compatible with colimits, then (one-object[1]) *operads* enriched in $\mathbf{V}$ can be described as associative algebras in $\mathrm{Fun}(\mathbb{F}^{\simeq}, \mathbf{V})$, the category of *symmetric sequences* in $\mathbf{V}$. Here $\mathbb{F}^{\simeq}$ denotes the groupoid $\coprod_n B\Sigma_n$ of finite sets and bijections, and the monoidal structure on symmetric sequences is the *composition product*, which is a monoidal structure given by the formula

$$(X \odot Y)(n) \cong \coprod_{k=0}^{\infty} \left(\coprod_{i_1+\cdots+i_k=n} (Y(i_1) \otimes \cdots \otimes Y(i_k)) \times_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_k}} \Sigma_n \right) \otimes_{\Sigma_k} X(k).$$

In a previous paper [Hau22] we proved that (one-object) ∞-operads enriched in a suitable symmetric monoidal ∞-category $\mathcal{V}$ admit a similar description, as associative algebras in $\mathrm{Fun}(\mathbb{F}^{\simeq}, \mathcal{V})$ using a monoidal structure given by the same formula.

Our goal in this short paper is to use this description of ∞-operads to study *algebras* for enriched ∞-operads. Classically, if $\mathbf{O}$ is a (one-object) $\mathbf{V}$-operad, then an $\mathbf{O}$-*algebra* in $\mathbf{V}$ consists of an object $A \in \mathbf{V}$ and Σ_n-equivariant morphisms

$$A^{\otimes n} \otimes \mathbf{O}(n) \to A$$

compatible with the composition and unit of $\mathbf{O}$. This data can be packaged in a convenient way using the composition product: an $\mathbf{O}$-algebra is the same thing as a

[1]We focus on the one-object case for simplicity, but similar descriptions apply to (∞-)operads with any fixed set (space) of objects.

right[2] $\mathbf{O}$-module M in $\mathrm{Fun}(\mathbb{F}^{\simeq}, \mathbf{V})$ that is concentrated in degree zero, i.e. $M(n)$ is the initial object $\emptyset$ for $n \neq 0$. Indeed, such a right $\mathbf{O}$-module is given by a morphism

$$M \odot \mathbf{O} \to M,$$

and expanding out the composition product we see that (since $M(n)$ vanishes for $n \neq 0$) this is precisely given by a map

$$\coprod_k M(0)^{\otimes k} \otimes_{\Sigma_k} \mathbf{O}(k) \to M(0).$$

Here we take the corresponding modules in the ∞-categorical setting (and their analogues for many-object operads) as a *definition* of algebras for enriched ∞-operads. For a symmetric monoidal ∞-category $\mathcal{V}$ (whose tensor product is compatible with colimits indexed by ∞-groupoids) and a $\mathcal{V}$-enriched ∞-operad $\mathcal{O}$, this results in an ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ with several pleasant properties, including the expected formula for free $\mathcal{O}$-algebras, as we will see in §3 after reviewing the results of [Hau22] in §2.

We then prove two main results about this ∞-categorical notion of $\mathcal{O}$-algebras. First, in §4 we prove a rectification result for algebras over operads enriched in a symmetric monoidal model category:

Theorem 1.1 (See Theorem 4.10)**.** *Let $\mathbf{V}$ be a symmetric monoidal model category (with cofibrant unit) and $\mathbf{O}$ an S-coloured Σ-cofibrant $\mathbf{V}$-operad such that the category $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})$ admits a model structure whose weak equivalences and fibrations are detected by the forgetful functor to $\mathrm{Fun}(S, \mathbf{V})$. Then this model category describes the ∞-category of algebras for this operad in the ∞-categorical localization of $\mathbf{V}$. That is, there is an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})[W_{\mathbf{O}}^{-1}] \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$$

where on the left $W_{\mathbf{O}}$ denotes the collection of weak equivalences between $\mathbf{O}$-algebras, and on the right $\mathcal{V} := \mathbf{V}[W^{-1}]$ is the localization of $\mathbf{V}$ at its weak equivalences and $\mathcal{O}$ denotes $\mathbf{O}$ viewed an ∞-operad enriched in $\mathcal{V}$ via the localization functor.

The comparison applies, for instance, to *all* Σ-cofibrant operads in chain complexes over a field of characteristic zero or in simplicial sets.[3] We in fact prove a slightly more general result that avoids the assumption that the unit is cofibrant, which applies to all Σ-cofibrant operads in symmetric spectra. The proof boils down to a combination of model-categorical results of Pavlov–Scholbach and our formula for free algebras, using the same strategy as [Lur17, Theorems 4.1.4.4] and [PS18a, Theorem 7.10] to prove that both sides are ∞-categories of algebras for equivalent monads.

Another classical description of algebras over (one-object) $\mathbf{V}$-operads uses *endomorphism operads*: For v an object of $\mathbf{V}$ there is an operad $\mathrm{End}_{\mathbf{V}}(v)$ with n-ary operations given by the internal Hom $\mathrm{HOM}_{\mathbf{V}}(v^{\otimes n}, v)$ (where the Σ_n-action permutes the factors in $v^{\otimes n}$). If $\mathbf{O}$ is a (one-object) $\mathbf{V}$-operad then we can describe $\mathbf{O}$-algebras in $\mathbf{V}$ with underlying object v as morphisms of one-object operads $\mathbf{O} \to \mathrm{End}_{\mathbf{V}}(v)$. More generally, we can consider an S-coloured endomorphism operad $\mathrm{End}_{\mathbf{V}}(f)$ for any map of sets $S \to \mathrm{ob}\,\mathbf{V}$, where operations from $(s_1, \ldots, s_n)$ to s' are given by $\mathrm{HOM}_{\mathbf{V}}(f(s_1) \otimes \cdots \otimes f(s_n), f(s'))$; for an S-coloured operad $\mathbf{O}$,

[2]This is correct under our convention for the ordering of the composition product, chosen to be compatible with our construction of the ∞-categorical version; in most references on ordinary operads the reverse ordering is used, so that $\mathbf{O}$-algebras are certain *left* $\mathbf{O}$-modules.

[3]For more general model categories, including chain complexes in positive characteristic, there is typically only a *semi*-model structure on algebras over a Σ-cofibrant operad; see Remark 4.14 for more discussion of this case.

we can then identify $\mathbf{O}$-algebras in $\mathbf{V}$ given by f on objects with morphisms of S-coloured operads from $\mathbf{O}$ to $\mathrm{End}_{\mathbf{V}}(f)$.

In §5 we will use Lurie's construction of endomorphism algebras [Lur17, §4.7.1], following work of Hinich [Hin20] in the case of enriched ∞-categories, to construct endomorphism ∞-operads $\mathrm{End}_{\mathcal{V}}(f)$ for any map of ∞-groupoids $f\colon X \to \mathcal{V}^{\simeq}$, where $\mathcal{V}$ is a closed symmetric monoidal ∞-category compatible with colimits indexed by small ∞-groupoids. Moreover, we show that these endomorphism ∞-operads can be combined into a self-enrichment of $\mathcal{V}$, which gives our second main result:

Theorem 1.2 (See Theorem 5.12)**.** *Let $\mathcal{V}$ be a closed symmetric monoidal ∞-category compatible with colimits indexed by small ∞-groupoids. Then there exists a $\mathcal{V}$-∞-operad $\overline{\mathcal{V}}$, whose object of multimorphisms from $(v_1,\ldots,v_n)$ to w is the internal Hom $\mathrm{MAP}_{\mathcal{V}}(v_1 \otimes \cdots \otimes v_n, w)$, such that for a $\mathcal{V}$-∞-operad $\mathcal{O}$ there is a natural equivalence of ∞-groupoids*

$$\{\mathcal{O}\text{-algebras in } \mathcal{V}\} \simeq \{\text{morphisms of } \mathcal{V}\text{-}\infty\text{-operads } \mathcal{O} \to \overline{\mathcal{V}}\}.$$

Warning 1.3. Throughout this paper we use the term "$\mathcal{V}$-∞-operad" to refer to the *algebraic* notion of an ∞-operad, given by objects of multimorphisms in $\mathcal{V}$ with homotopy-coherently associative and unital composition operations. Thus we have a class of fully faithful and essentially surjective morphisms between enriched ∞-operads that we would have to invert to get the "correct" ∞-category of $\mathcal{V}$-∞-operads. In terms of the description of $\mathcal{V}$-∞-operads we use, this means we are not requiring these to be "complete" (see [CH20, §3]). In the terminology of [AF18], our enriched ∞-operads can be thought of as being *flagged* enriched ∞-operads, meaning a (complete) enriched ∞-operad equipped with an essentially surjective morphism of ∞-groupoids to its space of objects. Note that, as we will see in Remark 3.10, the ∞-categories of algebras we will study are in fact invariant under fully faithful and essentially surjective maps of enriched ∞-operads, so it does not really make a difference whether we use complete objects or not.

1.1. **Related Work.** Much of our work here is not particularly reliant on the specific construction of the composition product from [Hau22]. An alternative construction, using the description of symmetric sequences in $\mathcal{V}$ as the free presentably symmetric monoidal ∞-category on $\mathcal{V}$ and generalizing the approach to 1-categorical operads due to Trimble [Tri] and Carboni, has been worked out by Brantner [Bra17]; however, this construction of ∞-operads has not yet been compared to any of the other approaches. In the setting of dendroidal sets, Heuts describes algebras valued in spaces and ∞-categories in terms of dendroidal versions of left and cocartesian fibrations in [Heu11].

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2. ∞-Operads as Algebras

In this section we will review the main results on enriched ∞-operads from [Hau22], where we showed that enriched ∞-operads can be viewed as associative algebras in a double ∞-category of symmetric collections. For this the relevant notion of enriched ∞-operads is an enriched variant of Barwick's definition of ∞-operads [Bar18] as presheaves on a category $\mathbf{\Delta}_{\mathbb{F}}$ satisfying Segal (and completeness) conditions; this definition was first introduced in [CH20]. We will now briefly review this definition, as well as a slight generalization considered in [Hau22] that we will use to define modules over enriched ∞-operads; we start by recalling the definition of Barwick's category $\mathbf{\Delta}_{\mathbb{F}}$:

Definition 2.1. Let $\mathbb{F}$ denote a skeleton of the category of finite sets, with objects $\mathbf{k} := \{1, \ldots, k\}$, $k = 0, 1, \ldots$. We write $\mathbf{\Delta}_{\mathbb{F}}$ for the category whose objects are pairs $([n], f : [n] \to \mathbb{F})$, with a morphism $([n], f) \to ([m], g)$ given by a morphism $\phi : [n] \to [m]$ in $\mathbf{\Delta}$ and a natural transformation $\eta : f \to g \circ \phi$ such that

(i) the map $\eta_i : f(i) \to g(\phi(i))$ is injective for all $i = 0, \ldots, m$,

(ii) the commutative square

$$\begin{array}{ccc} f(i) & \xrightarrow{\eta_i} & g(\phi(i)) \\ \downarrow & & \downarrow \\ f(j) & \xrightarrow{\eta_j} & g(\phi(j)) \end{array}$$

is cartesian for all $0 \leq i \leq j \leq m$.

Note that there is an obvious projection $\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$; this is a double ∞-category. There is also a functor $V : \mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}} \to \mathbb{F}_*$ which takes $([n], f)$ to $(\coprod_{i=1}^{n} f(i))_+$; see [CH20, Definition 2.2.11] for a complete definition.

Remark 2.2. An object of $\mathbf{\Delta}_{\mathbb{F}}$ is a sequence of maps of finite sets

$$\mathbf{a}_0 \xrightarrow{f_1} \mathbf{a}_1 \to \cdots \xrightarrow{f_n} \mathbf{a}_n,$$

which we can think of as a forest of $|\mathbf{a}_n|$ oriented trees with n levels: the edges are the elements of all the sets $\mathbf{a}_i$, and since each vertex has a unique outgoing edge we can also think of the elements of $\mathbf{a}_i$ for $i > 0$ as the vertices; the function f_i assigns to each edge in level $i-1$ the vertex of which it is an incoming edge. Note that this means the functor V takes each forest to its set of vertices (with a disjoint base point). The morphisms in $\mathbf{\Delta}_{\mathbb{F}}$ are defined so that a vertex in the source is mapped to a subtree of the target with the same number of incoming edges.

Definition 2.3. For $X \in \mathcal{S}$, we write $\mathbf{\Delta}_{\mathbb{F},X}^{\mathrm{op}} \to \mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$ for the left fibration corresponding to the functor $\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}} \to \mathcal{S}$ obtained as the right Kan extension of the functor $* \to \mathcal{S}$ with value X along the inclusion $\{([0], \mathbf{1})\} \hookrightarrow \mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$.

Remark 2.4. The fibre of $\mathbf{\Delta}_{\mathbb{F},X}^{\mathrm{op}} \to \mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$ at an object F is equivalent to a product of copies of X indexed by the number of edges in the forest F; we can thus think of an object of $\mathbf{\Delta}_{\mathbb{F},X}^{\mathrm{op}}$ as a forest whose edges are labelled by points of X.

Notation 2.5. If $\mathcal{O}$ is a non-symmetric ∞-operad[4], we write $\mathcal{O}_{\mathbb{F}} := \mathcal{O} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$ and $\mathcal{O}_{\mathbb{F},X} := \mathcal{O} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}_{\mathbb{F},X}^{\mathrm{op}}$.

Definition 2.6. We say a morphism in $\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$ is *operadic inert* if it lies over an inert morphism in $\mathbf{\Delta}^{\mathrm{op}}$. We then call a morphism in $\mathbf{\Delta}_{\mathbb{F},X}^{\mathrm{op}}$ *operadic inert* if it is a (necessarily cocartesian) morphism over an operadic inert morphism in $\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$. If $\mathcal{O}$ is a non-symmetric ∞-operad, we similarly say a morphism in $\mathcal{O}_{\mathbb{F},X}$ is *operadic inert* if it maps to an inert morphism in $\mathcal{O}$ and an operadic inert morphism in $\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$. The functor $V : \mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}} \to \mathbb{F}_*$ takes operadic inert morphisms to inert morphisms, hence if $\mathcal{V}$ is a symmetric monoidal ∞-category we can define an *operadic algebra* for $\mathcal{O}_{\mathbb{F},X}$

[4]We do not review the definition here, as it will not really play a role in this paper: the only examples we will encounter are $\mathbf{\Delta}^{\mathrm{op}}$ and the non-symmetric operad for right modules, which we describe explicitly in Definition 3.1. We refer the reader to [Hau22, §2.1] for a brief review, or [GH15, §2.2] for more motivation and discussion of the definition.

in $\mathcal{V}$ to be a commutative square

$$\begin{tikzcd} \mathcal{O}_{\mathbb{F},X} \arrow[r, "A"] \arrow[d] & \mathcal{V}^{\otimes} \arrow[d] \\ \mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}} \arrow[r, "V"] & \mathbb{F}_*, \end{tikzcd}$$

such that A takes operadic inert morphisms to inert morphisms in $\mathcal{V}^{\otimes}$. We write $\mathrm{Alg}^{\mathrm{opd}}_{\mathcal{O}_{\mathbb{F},X}}(\mathcal{V})$ for the full subcategory of $\mathrm{Fun}_{/\mathbb{F}_*}(\mathcal{O}_{\mathbb{F},X}, \mathcal{V}^{\otimes})$ spanned by the operadic algebras. We also write $\mathrm{Algd}^{\mathrm{opd}}_{\mathcal{O}_{\mathbb{F}}}(\mathcal{V}) \to \mathcal{S}$ for the cartesian fibration corresponding to the functor $X \mapsto \mathrm{Alg}^{\mathrm{opd}}_{\mathcal{O}_{\mathbb{F},X}}(\mathcal{V})$ and refer to its objects as *operadic $\mathcal{O}_{\mathbb{F}}$-algebroids* in $\mathcal{V}$.

Remark 2.7. The condition for a commutative triangle

$$\begin{tikzcd} \mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F},X} \arrow[rr, "F"] \arrow[dr] & & \mathcal{V}^{\otimes} \arrow[dl] \\ & \mathbb{F}_* & \end{tikzcd}$$

to be an operadic algebra is essentially that the value of F at a forest whose edges are labelled by points of X consists of a list of the values of F at the corollas (one-vertex subtrees) of the forest. The latter should be thought of as the objects of multimorphisms in a $\mathcal{V}$-enriched ∞-operad, whose homotopy-coherent composition is encoded by the rest of the data in F. Indeed, operadic algebras for $\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F},X}$ gives one of the notions of enriched ∞-operads introduced in [CH20], which justifies the following notation:

Notation 2.8. For a space X, we write $\mathrm{Opd}_X(\mathcal{V}) := \mathrm{Alg}^{\mathrm{opd}}_{\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F},X}}(\mathcal{V})$. We also write $\mathrm{Opd}(\mathcal{V}) := \mathrm{Algd}^{\mathrm{opd}}_{\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}}(\mathcal{V})$, so that we have a cartesian fibration $\mathrm{Opd}(\mathcal{V}) \to \mathcal{S}$ whose fibre at X is $\mathrm{Opd}_X(\mathcal{V})$.

Notation 2.9. For $X, Y \in \mathcal{S}$, we write $\mathbb{F}^{\simeq}_{X,Y}$ for the ∞-groupoid $\coprod_{n=0}^{\infty} X^{\times n}_{h\Sigma_n} \times Y$. For a functor $\Phi\colon \mathbb{F}^{\simeq}_{X,Y} \to \mathcal{V}$ we will denote its value at $((x_1, \dots, x_n), y)$ by $\Phi\binom{x_1,\dots,x_n}{y}$. We also abbreviate $\mathbb{F}^{\simeq}_{X} := \mathbb{F}^{\simeq}_{X,X}$ and write $\mathrm{Coll}_X(\mathcal{V}) := \mathrm{Fun}(\mathbb{F}^{\simeq}_{X}, \mathcal{V})$; we refer to the objects of this ∞-category as (symmetric) *X-collections* in $\mathcal{V}$.

To state the main result we will use from [Hau22], we need to recall one further definition:

Definition 2.10. A *double ∞-category* is a cocartesian fibration $\mathcal{F} \to \mathbf{\Delta}^{\mathrm{op}}$ that corresponds to a functor $F\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$ that satisfies the *Segal condition*: for every n, the functor

$$F([n]) \to F([1]) \times_{F([0])} \cdots \times_{F([0])} F([1]),$$

induced by the value of F at the inert maps $[0], [1] \to [n]$ in $\mathbf{\Delta}$, is an equivalence. We say a double ∞-category is *framed* if the functor $F([1]) \to F([0]) \times F([0])$, induced by the two face maps $[0] \to [1]$, is a cocartesian fibration. If $\mathcal{O}$ is a non-symmetric ∞-operad, then an *$\mathcal{O}$-algebra* in $\mathcal{F}$ is a commutative triangle

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "A"] \arrow[dr] & & \mathcal{F} \arrow[dl] \\ & \mathbf{\Delta}^{\mathrm{op}} & \end{tikzcd}$$

such that A preserves cocartesian morphisms over inert maps in $\mathbf{\Delta}^{\mathrm{op}}$.

Remark 2.11. If $\mathcal{F} \to \mathbf{\Delta}^{\mathrm{op}}$ is a double ∞-category, then we think of this as an ∞-categorical version of a double category where

- the objects are the objects of $\mathcal{F}_{[0]}$,
- the vertical morphisms are the morphisms in $\mathcal{F}_{[0]}$,
- the horizontal morphisms are the objects of $\mathcal{F}_{[1]}$,
- the squares are the morphisms in $\mathcal{F}_{[1]}$.

Theorem 2.12 (See [Hau22, Corollary 4.2.8])**.** *Suppose $\mathcal{V}$ is a symmetric monoidal ∞-category compatible with colimits indexed by small ∞-groupoids. Then there exists a framed double ∞-category $\mathrm{COLL}(\mathcal{V})$ such that:*

(i) $\mathrm{COLL}(\mathcal{V})_0 \simeq \mathcal{S}$*, i.e. the objects of* $\mathrm{COLL}(\mathcal{V})$ *are small ∞-groupoids and the vertical morphisms are morphisms thereof.*
(ii) A horizontal morphism from X to Y is a functor $\mathbb{F}^{\simeq}_{X,Y} \to \mathcal{V}$.
(iii) If Φ is a horizontal morphism from X to Y and Ψ is one from Y to Z then their composite $\Phi \odot_Y \Psi$ is given by the formula

$$\Phi \odot_Y \Psi\begin{pmatrix} x_1, \dots, x_n \\ z \end{pmatrix} \simeq \operatorname*{colim}_{\mathbf{n}\to\mathbf{m}\to\mathbf{1}} \operatorname*{colim}_{(y_i)\in Y^{\times m}} \bigotimes_{i\in\mathbf{m}} \Phi\begin{pmatrix} x_k : k \in \mathbf{n}_i \\ y_i \end{pmatrix} \otimes \Psi\begin{pmatrix} y_1, \dots, y_k \\ z \end{pmatrix}.$$

(iv) If $\mathcal{O}$ is any non-symmetric ∞-operad then there is a natural equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{COLL}(\mathcal{V})) \simeq \mathrm{Algd}^{\mathrm{opd}}_{\mathcal{O}_{\mathbb{F}}}(\mathcal{V}).$$

(v) If $F \colon \mathcal{V} \to \mathcal{W}$ is a symmetric monoidal functor that preserves colimits indexed by small ∞-groupoids, then composition with F induces a morphism of double ∞-categories $\mathrm{COLL}(\mathcal{V}) \to \mathrm{COLL}(\mathcal{W})$.

Remark 2.13. In (iii), the outer colimit is more precisely over the groupoid $\mathrm{Fact}(\mathbf{n} \to \mathbf{1})$ of factorizations $\mathbf{n} \to \mathbf{m} \to \mathbf{1}$, with morphisms given by diagrams

$$\begin{tikzcd} & \mathbf{m} \arrow[dd, "\sim"] \arrow[dr] & \\ \mathbf{n} \arrow[ur] \arrow[dr] & & \mathbf{1}. \\ & \mathbf{m}' \arrow[ur] & \end{tikzcd}$$

Remark 2.14. In particular, associative algebras in $\mathrm{COLL}(\mathcal{V})$ are equivalent to ∞-operads enriched in $\mathcal{V}$:

$$\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{COLL}(\mathcal{V})) \simeq \mathrm{Algd}^{\mathrm{opd}}_{\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}}(\mathcal{V}) \simeq \mathrm{Opd}(\mathcal{V}).$$

For $X \in \mathcal{V}$, the ∞-category

$$\mathrm{COLL}(\mathcal{V})(X, X) \simeq \mathrm{Coll}_X(\mathcal{V}) \simeq \mathrm{Fun}(\mathbb{F}^{\simeq}_X, \mathcal{V})$$

of horizontal endomorphisms of X has a monoidal structure given by composition. Moreover, by [Hau22, Proposition 3.4.8] a morphism $f \colon X \to Y$ induces a natural lax monoidal functor $f^* \colon \mathrm{Coll}_Y(\mathcal{V}) \to \mathrm{Coll}_X(\mathcal{V})$, given by composition with the induced map $\mathbb{F}^{\simeq}_X \to \mathbb{F}^{\simeq}_Y$. By [Hau22, Corollary 3.4.10] we also have:

Corollary 2.15. *Let $\mathcal{O}$ be a weakly contractible non-symmetric ∞-operad. Then the functor*

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{COLL}(\mathcal{V})) \to \mathcal{S}$$

given by evaluation at $* \in \mathcal{O}_0$ is a cartesian fibration corresponding to the functor $\mathcal{S} \to \mathrm{Cat}_\infty$ that takes X to $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_X(\mathcal{V}))$ and a morphism $f \colon X \to Y$ to the functor given by composition with the lax monoidal functor $f^* \colon \mathrm{Coll}_Y(\mathcal{V}) \to \mathrm{Coll}_X(\mathcal{V})$.

Remark 2.16. This corollary applies in particular to the weakly contractible non-symmetric ∞-operad $\mathbf{\Delta}^{\mathrm{op}}$, so that by Remark 2.14, enriched ∞-operads with X as space of objects are given by associative algebras in $\mathrm{Coll}_X(\mathcal{V})$, i.e.

$$\mathrm{Opd}_X(\mathcal{V}) \simeq \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Coll}_X(\mathcal{V})).$$

Remark 2.17. For $f\colon X \to Y$, the lax monoidal functor $f^*\colon \mathrm{Coll}_Y(\mathcal{V}) \to \mathrm{Coll}_X(\mathcal{V})$ is given by composition with a morphism of ∞-groupoids $f_{\mathbb{F}^{\simeq}}\colon \mathbb{F}^{\simeq}_X \to \mathbb{F}^{\simeq}_Y$. Since $\mathcal{V}$ has colimits indexed by ∞-groupoids, this functor has a left adjoint $f_!$, given by left Kan extension along $f_{\mathbb{F}^{\simeq}}$. Moreover, since $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_X(\mathcal{V})) \to \mathrm{Coll}_X(\mathcal{V})$ detects limits and sifted colimits for any non-symmetric ∞-operad $\mathcal{O}$, the functor $f^*\colon \mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_Y(\mathcal{V})) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_X(\mathcal{V}))$ preserves limits and sifted colimits, since this is true for $f^*\colon \mathrm{Coll}_Y(\mathcal{V}) \to \mathrm{Coll}_X(\mathcal{V})$. If $\mathcal{V}$ is presentably symmetric monoidal, then the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_Y(\mathcal{V}))$ is presentable, since it is equivalent to $\mathrm{Alg}^{\mathrm{opd}}_{\mathcal{O}_{\mathbb{F},X}}(\mathcal{V})$, which in turn is equivalent to the ∞-category of algebras in $\mathcal{V}$ for some symmetric ∞-operad. It then follows from the adjoint functor theorem that $f^*\colon \mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_Y(\mathcal{V})) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_X(\mathcal{V}))$ has a left adjoint. This implies:

Corollary 2.18. *Let $\mathcal{O}$ be a weakly contractible non-symmetric ∞-operad and $\mathcal{V}$ a presentably symmetric monoidal ∞-category. Then the functor*

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{COLL}(\mathcal{V})) \to \mathcal{S}$$

given by evaluation at $* \in \mathcal{O}_0$ is also a cocartesian fibration. □

In general the cocartesian morphisms over f are not easily described in terms of the left Kan extension along the map $f_{\mathbb{F}^{\simeq}}\colon \mathbb{F}^{\simeq}_X \to \mathbb{F}^{\simeq}_Y$. However, we can derive a simple description in the case of monomorphisms of ∞-groupoids:

Proposition 2.19. *Suppose $i\colon X \hookrightarrow Y$ is a monomorphism of ∞-groupoids. Then the left adjoint $i_!\colon \mathrm{Coll}_X(\mathcal{V}) \to \mathrm{Coll}_Y(\mathcal{V})$ has a canonical monoidal structure, such that composition with $i_!$ and i^* gives for any ∞-operad $\mathcal{O}$ an adjunction*

$$i_! : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_X(\mathcal{V})) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_Y(\mathcal{V})) : i^*.$$

Proof. We will prove this by applying [Lur17, Corollary 7.3.2.12], which requires us to show that for $\Phi, \Psi \in \mathrm{Coll}_X(\mathcal{V})$, the canonical map

$$i_!(\Phi \odot_X \Psi) \to i_!\Phi \odot_Y i_!\Psi$$

is an equivalence.

We first describe $i_!\Phi$ more explicitly: For $(y_1, \dots, y_n)$ in $Y^n_{h\Sigma_n}$, we can identify the fibre of $X^n_{h\Sigma_n}$ over this point as $X_{y_1} \times \cdots \times X_{y_n}$ using the commutative diagram

$$\begin{tikzcd}
X_{y_1} \times \cdots \times X_{y_n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^n_{h\Sigma_n} \arrow[d] \\
\{(y_1, \dots, y_n)\} \arrow[r] & Y^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^n_{h\Sigma_n} \arrow[d] \\
 & * \arrow[r] & B\Sigma_n,
\end{tikzcd}$$

where all three squares are cartesian. Hence the fibre of $\mathbb{F}^{\simeq}_X \to \mathbb{F}^{\simeq}_Y$ at $\binom{y_1,\dots,y_n}{y}$ is equivalent to $\prod_i X_{y_i} \times X_y$, giving

$$i_!\Phi\binom{y_1,\dots,y_n}{y} \simeq \operatorname*{colim}_{(x_1,\dots,x_n,x) \in \prod_i X_{y_i} \times X_y} \Phi\binom{x_1,\dots,x_n}{x}.$$

We can then rewrite the formula for $i_!(\Phi \odot_X \Psi)\binom{y_1,\ldots,y_n}{y}$ as

$$\begin{aligned} i_!(\Phi \odot_X \Psi)\binom{y_1,\ldots,y_n}{y} &\simeq \operatorname*{colim}_{(x_1,\ldots,x_n,x)\in\prod_i X_{y_i}\times X_y} (\Phi \odot_X \Psi)\binom{x_1,\ldots,x_n}{x} \\ &\simeq \operatorname*{colim}_{(x_1,\ldots,x_n,x)\in\prod_i X_{y_i}\times X_y} \operatorname*{colim}_{\mathbf{n}\to\mathbf{m}\to\mathbf{1}} \operatorname*{colim}_{(x'_j)\in X^m} \bigotimes_j \Phi\binom{x_i : i\in\mathbf{n}_j}{x'_j} \otimes \Psi\binom{x'_1,\ldots,x'_m}{x} \\ &\simeq \operatorname*{colim}_{\mathbf{n}\to\mathbf{m}\to\mathbf{1}} \operatorname*{colim}_{(y'_j)\in Y^m} \operatorname*{colim}_{(x_i)\in\prod_i X_{y_i}} \operatorname*{colim}_{(x'_j)\in\prod_j X_{y'_j}} \operatorname*{colim}_{x\in X_y} \bigotimes_j \Phi\binom{x_i : i\in\mathbf{n}_j}{x'_j} \otimes \Psi\binom{x'_1,\ldots,x'_m}{x}. \end{aligned}$$

On the other hand $(i_!\Phi \odot_Y i_!\Psi)\binom{y_1,\ldots,y_n}{y}$ is equivalent to

$$\operatorname*{colim}_{\mathbf{n}\to\mathbf{m}\to\mathbf{1}} \operatorname*{colim}_{(y'_j)\in Y^m} \operatorname*{colim}_{(x_i)\in\prod_i X_{y_i}} \operatorname*{colim}_{(x'_j)\in\prod_j X_{y'_j}} \operatorname*{colim}_{(x''_j)\in\prod_j X_{y'_j}} \operatorname*{colim}_{x\in X_y} \bigotimes_j \Phi\binom{x_i : i\in\mathbf{n}_j}{x'_j} \otimes \Psi\binom{x''_1,\ldots,x''_m}{x},$$

and the canonical map corresponds under these equivalences to the map of colimits arising from the diagonal map $\prod_j X_{y'_j} \to \prod_j X_{y'_j} \times \prod_j X_{y'_j}$. Since these are ∞-groupoids, this map is cofinal if and only if it is an equivalence, which holds if and only if the spaces X_y for $y \in Y$ are either contractible or empty, i.e. if and only if i is a monomorphism. □

Corollary 2.20. *Let $i\colon X \to Y$ be a monomorphism of ∞-groupoids and $\mathcal{O}$ a weakly contractible non-symmetric ∞-operad.*

(i) For every $A \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_X(\mathcal{V}))$, the unit morphism $A \to i^*i_!A$ is an equivalence.

(ii) The functor

$$i_!\colon \mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_X(\mathcal{V})) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Coll}_Y(\mathcal{V}))$$

is fully faithful. □

Remark 2.21. We will also need a more general version of Theorem 2.12, which follows by using part (iii) of [Hau22, Proposition 3.5.6] instead of (vi): If $F\colon \mathcal{V} \to \mathcal{W}$ is a symmetric monoidal functor then composition with F induces a morphism of generalized non-symmetric ∞-operads $F_*\colon \mathrm{COLL}(\mathcal{V}) \to \mathrm{COLL}(\mathcal{W})$, which restricts to lax monoidal functors $F_*\colon \mathrm{Coll}_X(\mathcal{V}) \to \mathrm{Coll}_X(\mathcal{W})$. These are compatible with the lax monoidal functors f^* coming from maps of spaces $f\colon X \to Y$: A priori the square

$$\begin{tikzcd} \mathrm{Coll}_Y(\mathcal{V}) \arrow[r, "F_*"] \arrow[d, "f^*"] & \mathrm{Coll}_Y(\mathcal{W}) \arrow[d, "f^*"] \\ \mathrm{Coll}_X(\mathcal{V}) \arrow[r, "F_*"] & \mathrm{Coll}_X(\mathcal{W}) \end{tikzcd}$$

only commutes up to a natural transformation, but this is clearly a natural equivalence since both functors are given by composition.

3. Algebras for ∞-Operads as Modules

In this section we define algebras for an enriched ∞-operad $\mathcal{O}$ as certain right $\mathcal{O}$-modules in $\mathrm{COLL}(\mathcal{V})$. We first recall the definition of the non-symmetric ∞-operad for right modules, and prove that this is weakly contractible, allowing us to apply Corollary 2.15:

Definition 3.1. Let **rm** denote the non-symmetric operad for right modules. This has two objects, a and m, and there is a unique multimorphism $(x_1,\ldots,x_n) \to y$ if $x_1 = \cdots = x_n = y = a$ ($n = 0$ allowed) or $x_1 = y = m$ and $x_2 = \cdots = x_n = a$, and no multimorphisms otherwise. We write $\mathrm{RM} \to \mathbf{\Delta}^{\mathrm{op}}$ for the corresponding non-symmetric ∞-operad, or in other words the category of operators of **rm**. This has objects sequences $(x_1,\ldots,x_n)$ with each x_i being either a or m, and a morphism

$(x_1, \ldots, x_n) \to (y_1, \ldots, y_m)$ is given by a map $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$ and multimorphisms $(x_{\phi(i-1)+1}, \ldots, x_{\phi(i)}) \to y_i$ in $\mathbf{rm}$.

Proposition 3.2. *The category* RM *is weakly contractible.*

Proof. In this proof it is convenient to use the notation $(i_0, \ldots, i_n)_{\mathrm{RM}}$ for the object of RM given by the sequence $(a, \ldots, a, m, \cdots, m, a, \ldots, a)$ where there are n copies of m and i_t copies of a between the tth and $(t+1)$th copy of m (and i_0 before the first and i_n after the last). Define a functor $\mu\colon \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}} \to \mathrm{RM}$ over $\mathbf{\Delta}^{\mathrm{op}}$ by taking $[n]$ to the unique object of the form $[0, \ldots, 0]_{\mathrm{RM}} = (m, \ldots, m)$ over $[n]$, and determined on morphisms by the inert morphisms between these objects. We claim that μ is coinitial, and so in particular a weak homotopy equivalence. To see this, it suffices by [Lur09, Theorem 4.1.3.1] to show that for every object $X \in \mathrm{RM}$ the category $(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}})_{/X}$ is weakly contractible. But this category has a terminal object: if $X = (i_0, \ldots, i_n)_{\mathrm{RM}}$ then any morphism $(0, \ldots, 0)_{\mathrm{RM}} \to X$ factors as an inert morphism followed by the (unique) degeneracy $\mu([n]) \to X$. Since $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}$ is weakly contractible (for example, because the inclusion $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ is cofinal and $\mathbf{\Delta}^{\mathrm{op}}$ has an initial object), this implies that RM is also weakly contractible. □

Corollary 3.3. *The functor*

$$\mathrm{Algd}^{\mathrm{opd}}_{\mathrm{RM}_{\mathbb{F}}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathrm{RM}}(\mathrm{COLL}(\mathcal{V})) \to \mathcal{S},$$

given by evaluation at $() \in \mathrm{RM}_0$*, is a cartesian fibration corresponding to the functor* $\mathcal{S} \to \mathrm{Cat}_\infty$ *that takes* X *to* $\mathrm{Alg}_{\mathrm{RM}}(\mathrm{Coll}_X(\mathcal{V}))$ *and a morphism* $f\colon X \to Y$ *to the functor given by composition with the lax monoidal functor* $f^*\colon \mathrm{Coll}_Y(\mathcal{V}) \to \mathrm{Coll}_X(\mathcal{V})$*.* □

To define algebras we want to restrict to those modules that are concentrated in degree 0, which will be justified by the next proposition.

Definition 3.4. We say that $\Phi \in \mathrm{Coll}_X(\mathcal{V})$ is *concentrated in degree* 0 if

$$\Phi\binom{x_1, \ldots, x_n}{y} \simeq \emptyset$$

whenever $n > 0$, where $\emptyset$ denotes the initial object in $\mathcal{V}$.

Proposition 3.5. *Let* $\mathcal{V}$ *be a symmetric monoidal* ∞*-category compatible with colimits indexed by small* ∞*-groupoids.*

(i) The functor $Z\colon \mathrm{Coll}_X(\mathcal{V}) \to \mathrm{Fun}(X, \mathcal{V})$ *given by composition with* $X \hookrightarrow \mathbb{F}^{\simeq}_X$ *has a fully faithful left adjoint, which identifies* $\mathrm{Fun}(X, \mathcal{V})$ *with the collections that are concentrated in degree* 0*.*

(ii) If $M\colon \mathbb{F}^{\simeq}_X \to \mathcal{V}$ *is concentrated in degree* 0*, then so is* $M \odot_X N$ *for any* $N \in \mathrm{Coll}_X(\mathcal{V})$*.*

(iii) The composition product induces a right $\mathrm{Coll}_X(\mathcal{V})$*-module structure on the* ∞*-category* $\mathrm{Fun}(X, \mathcal{V})$*.*

(iv) For $f\colon X \to Y$*, composition with* f *and the induced functor* $\mathbb{F}^{\simeq}_X \to \mathbb{F}^{\simeq}_Y$ *gives a lax* RM*-monoidal functor*

$$f^*\colon (\mathrm{Fun}(Y, \mathcal{V}), \mathrm{Coll}_Y(\mathcal{V})) \to (\mathrm{Fun}(X, \mathcal{V}), \mathrm{Coll}_X(\mathcal{V}))$$

(v) Composition with a symmetric monoidal functor $F\colon \mathcal{V} \to \mathcal{W}$ *gives a lax* RM*-monoidal functor*

$$F_*\colon (\mathrm{Fun}(X, \mathcal{V}), \mathrm{Coll}_X(\mathcal{V})) \to (\mathrm{Fun}(X, \mathcal{W}), \mathrm{Coll}_X(\mathcal{W})).$$

If F *preserves colimits indexed by small* ∞*-groupoids, then* F_* *is an* RM*-monoidal functor.*

Proof. Part (i) is obvious from the description of $\mathbb{F}_X^{\simeq}$ as $\coprod_n X_{h\Sigma_n}^{\times n} \times X$ and the formula for pointwise left Kan extensions, while part (ii) follows immediately from the description of composition of horizontal morphisms in $\mathrm{COLL}(\mathcal{V})$ in Theorem 2.12. Part (iii) then holds by combining parts (i) and (ii), and parts (iv) and (v) follow by restricting the lax monoidal functors discussed in §2. □

Definition 3.6. Let $\mathcal{O}$ be a $\mathcal{V}$-∞-operad with space of objects X, viewed as an associative algebra in $\mathrm{Coll}_X(\mathcal{V})$. An $\mathcal{O}$*-algebra in* $\mathcal{V}$ is a right $\mathcal{O}$-module in $\mathrm{Fun}(X, \mathcal{V})$. We write $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ for the ∞-category $\mathrm{RMod}_{\mathcal{O}}(\mathrm{Fun}(X, \mathcal{V}))$ of these right modules.

Remark 3.7. By Proposition 3.5(iv) we see that for $\mathcal{O} \in \mathrm{Opd}_Y(\mathcal{V})$, composition with $f\colon X \to Y$ gives a functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \to \mathrm{Alg}_{f^*\mathcal{O}}(\mathcal{V})$, while composition with a symmetric monoidal functor $F\colon \mathcal{V} \to \mathcal{W}$ gives a functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \to \mathrm{Alg}_{F_*\mathcal{O}}(\mathcal{W})$.

Since there is always a formula for free modules, with this definition we immediately get a formula for free algebras over enriched ∞-operads:

Proposition 3.8. *The forgetful functor* $U_{\mathcal{O}}\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \to \mathrm{Fun}(X, \mathcal{V})$ *has a left adjoint* $F_{\mathcal{O}}$*, and the endofunctor* $U_{\mathcal{O}}F_{\mathcal{O}}$ *satisfies*

$$U_{\mathcal{O}}F_{\mathcal{O}}M(x) \simeq \coprod_n \operatorname*{colim}_{(x_1,\ldots,x_n)\in X^n_{h\Sigma_n}} M(x_1) \otimes \cdots \otimes M(x_n) \otimes \mathcal{O}\binom{x_1,\ldots,x_n}{x}.$$

Moreover, $U_{\mathcal{O}}$ *preserves sifted colimits and the adjunction is monadic.*

Proof. By [Lur17, Corollary 4.2.4.8] the left adjoint $F_{\mathcal{O}}$ exists, and $U_{\mathcal{O}}F_{\mathcal{O}}(M)$ is given by the composition product $M \odot \mathcal{O}$ (with M viewed as a symmetric sequence concentrated in degree 0). Expanding out this composition product now gives the formula.

It follows from [Lur17, Proposition 4.2.3.1] that $U_{\mathcal{O}}$ detects equivalences and from [Lur17, Corollary 4.2.3.5] that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ has sifted colimits and $U_{\mathcal{O}}$ preserves these, since the composition product preserves sifted colimits in each variable. The adjunction is therefore monadic by the monadicity theorem for ∞-categories, [Lur17, Theorem 4.7.3.5]. □

Applying [GH15, Proposition A.5.9], we get:

Corollary 3.9. *If* $\mathcal{V}$ *is a presentably symmetric monoidal* ∞*-category and* $\mathcal{O}$ *is a* $\mathcal{V}$*-enriched* ∞*-operad, then the* ∞*-category* $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ *is presentable.* □

Remark 3.10. Let $F\colon \mathcal{O} \to \mathcal{O}'$ be a morphism of $\mathcal{V}$-∞-operads given on spaces of objects by $f\colon X \to Y$, and suppose f is surjective on π_0 and F is fully faithful in the sense that all the maps

$$\mathcal{O}\tbinom{x_1,\ldots,x_n}{y} \to \mathcal{O}'\tbinom{f(x_1),\ldots,f(x_n)}{f(y)}$$

are equivalences in $\mathcal{V}$. Then we have a commutative square

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}'}(\mathcal{V}) \arrow[r, "F^*"] \arrow[d] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \arrow[d] \\ \mathrm{Fun}(Y, \mathcal{V}) \arrow[r, "f^*"'] & \mathrm{Fun}(X, \mathcal{V}), \end{tikzcd}$$

where the surjectivity of f implies that the composite functor $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{V}) \to \mathrm{Fun}(X, \mathcal{V})$ is a monadic right adjoint. Using the formula from Proposition 3.8 it is easy to see that F^* gives an equivalence of monads on $\mathrm{Fun}(X, \mathcal{V})$ and so gives an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ by [Lur17, Corollary 4.7.3.16]. This applies in particular if $\mathcal{O}'$ is the completion of $\mathcal{O}$, so by a 2-out-of-3 argument it follows that any fully faithful and essentially surjective morphism of $\mathcal{V}$-∞-operads $F\colon \mathcal{O} \to \mathcal{P}$ induces an equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{P}}(\mathcal{V})$$

on ∞-categories of algebras in $\mathcal{V}$.

We end this section by showing that the nullary operations of a $\mathcal{V}$-∞-operad $\mathcal{O}$ give a canonical $\mathcal{O}$-algebra, using the next observation:

Proposition 3.11. *$Z\colon \mathrm{Coll}_X(\mathcal{V}) \to \mathrm{Fun}(X, \mathcal{V})$ is a functor of $\mathrm{Coll}_X(\mathcal{V})$-modules.*

Proof. By definition of the $\mathrm{Coll}_X(\mathcal{V})$-module structure on $\mathrm{Fun}(X, \mathcal{V})$, the inclusion $\mathrm{Fun}(X, \mathcal{V}) \to \mathrm{Coll}_X(\mathcal{V})$ is a functor of $\mathrm{Coll}_X(\mathcal{V})$-modules. Using [Lur17, Corollary 7.3.2.7], this implies that its right adjoint Z is a lax RM-monoidal functor. Thus for $M, N \in \mathrm{Coll}_X(\mathcal{V})$ there are natural maps

$$Z(M) \odot_X N \to Z(M \odot_X N);$$

by the formula for $\odot_X$ these maps are equivalences, and so Z is an RM-monoidal functor. $\square$

Corollary 3.12. *If $\mathcal{O}$ is an associative algebra in $\mathrm{Coll}_X(\mathcal{V})$ and $M \in \mathrm{Coll}_X(\mathcal{V})$ is a right $\mathcal{O}$-module, then the restriction $Z(M) \in \mathrm{Fun}(Y, \mathcal{V})$ is also a right $\mathcal{O}$-module.* $\square$

Since an algebra is canonically a right module over itself, this specializes to:

Corollary 3.13. *Suppose $\mathcal{O}$ is an algebra in $\mathrm{Coll}_X(\mathcal{V})$, i.e. a $\mathcal{V}$-∞-operad with X as space of objects. Then the functor $Z(\mathcal{O})\colon X \to \mathcal{V}$ picking out the nullary operations is canonically a right $\mathcal{O}$-module.* $\square$

4. Comparison with Model Categories of Operad Algebras

Let $\mathbf{V}$ be a symmetric monoidal model category (with cofibrant unit). Then by [Lur17, Proposition 4.1.7.4] the localization $\mathbf{V}[W^{-1}]$ (with W the class of weak equivalences) is a symmetric monoidal ∞-category, and the localization functor $\mathbf{V} \to \mathbf{V}[W^{-1}]$ is symmetric monoidal when restricted to the cofibrant objects. If $\mathbf{O}$ is a (levelwise cofibrant) operad in $\mathbf{V}$ then this means we can also view $\mathbf{O}$ as an operad in $\mathbf{V}[W^{-1}]$. Moreover, in good cases there is a model structure on the category $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})$ of $\mathbf{O}$-algebras in $\mathbf{V}$. In this section we will give conditions under which the corresponding ∞-category $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})[W_{\mathbf{O}}^{-1}]$ (with $W_{\mathbf{O}}$ the class of weak equivalences of $\mathbf{O}$-algebras) is equivalent to the ∞-category $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V}[W^{-1}])$, defined as in the previous section. In order to do the comparison in sufficient generality to cover examples such as symmetric spectra, we do not want to assume that the unit of the monoidal structure is cofibrant. Instead we consider model categories with a subcategory of *flat* objects in the following sense:

Definition 4.1. Let $\mathbf{V}$ be a symmetric monoidal model category.[5] A *subcategory of flat objects* is a full subcategory $\mathbf{V}^\flat$ that satisfies the following conditions:

- $\mathbf{V}^\flat$ is a symmetric monoidal subcategory, i.e. the unit is flat and the tensor product of two flat objects is flat,
- If X is flat and $Y \to Y'$ is a weak equivalence between flat objects, then $X \otimes Y \to X \otimes Y'$ is again a weak equivalence.
- All cofibrant objects are flat.

Example 4.2. If the unit of $\mathbf{V}$ is cofibrant, then the subcategory $\mathbf{V}^c$ of cofibrant objects is a subcategory of flat objects.

[5]We assume that model categories have functorial factorizations.

Proposition 4.3. *Let* $\mathbf{V}$ *be a symmetric monoidal model category and* $\mathbf{V}^\flat$ *a subcategory of flat objects. Then the inclusions* $\mathbf{V}^c \hookrightarrow \mathbf{V}^\flat \hookrightarrow \mathbf{V}$ *induce equivalences of localizations*

$$\mathbf{V}^c[W^{-1}] \xrightarrow{\sim} \mathbf{V}^\flat[W^{-1}] \xrightarrow{\sim} \mathbf{V}[W^{-1}],$$

where we denote the collections of weak equivalences in the subcategories by W *in all cases.*

Proof. Let $Q\colon \mathbf{V} \to \mathbf{V}$ be a cofibrant replacement functor, with a natural weak equivalence $\eta\colon Q \to \mathrm{id}$. If i denotes the inclusion $\mathbf{V}^c \hookrightarrow \mathbf{V}$ then we may view Q as a functor $\mathbf{V} \to \mathbf{V}^c$ and η as a natural transformation $iQ \to \mathrm{id}_{\mathbf{V}}$. If X is cofibrant, then $\eta_X : QX \to X$ is a morphism in $\mathbf{V}^c$, so we may view $\eta i\colon iQi \to i$ as a natural transformation $\eta^c\colon Qi \to \mathrm{id}_{\mathbf{V}^c}$. The functor Q preserves weak equivalences, and both η and η^c are natural weak equivalences. It follows that Q induces a functor $\mathbf{V}[W^{-1}] \to \mathbf{V}^c[W^{-1}]$ and the transformations η and η^c induce transformations that exhibit this as an inverse of the functor $\mathbf{V}^c[W^{-1}] \to \mathbf{V}[W^{-1}]$ induced by i. The same argument applies to Q restricted to the full subcategory $\mathbf{V}^\flat$; the functor $\mathbf{V}^\flat[W^{-1}] \to \mathbf{V}[W^{-1}]$ is therefore an equivalence by the 2-of-3 property of equivalences. □

Corollary 4.4. *Let* $\mathbf{V}$ *be a symmetric monoidal model category and* $\mathbf{V}^\flat$ *a subcategory of flat objects. Then the* ∞*-category* $\mathbf{V}[W^{-1}]$ *inherits a symmetric monoidal structure such that the functor* $\mathbf{V}^\flat \to \mathbf{V}[W^{-1}]$ *is symmetric monoidal.*

Proof. By assumption, in $\mathbf{V}^\flat$ the tensor product is compatible with weak equivalences, and so the ∞-category $\mathbf{V}[W^{-1}] \simeq \mathbf{V}^\flat[W^{-1}]$ inherits a symmetric monoidal structure with this property by [Lur17, Proposition 4.1.7.4]. □

Using Remark 3.7, composition with the symmetric monoidal functor $\mathbf{V}^\flat \to \mathbf{V}[W^{-1}]$ gives a natural functor

$$\mathrm{Alg}_{\mathbf{O}}(\mathbf{V}^\flat) \to \mathrm{Alg}_{\mathbf{O}}(\mathbf{V}[W^{-1}]),$$

if $\mathbf{O}$ is a levelwise flat $\mathbf{V}$-operad. Here we can interpret $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V}^\flat)$ as the classical ordinary category of $\mathbf{O}$-algebras in $\mathbf{V}^\flat$.

Definition 4.5. An S-coloured operad $\mathbf{O}$ in a symmetric monoidal model category $\mathbf{V}$ is called *admissible* if there exists a model structure on $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})$ where a morphism is a weak equivalence or a fibration precisely if its underlying morphism in $\mathrm{Fun}(S, \mathbf{V})$ is one (i.e. it is a weak equivalence or fibration in $\mathbf{V}$ for each element of S).

Definition 4.6. An S-coloured $\mathbf{V}$-operad $\mathbf{O}$ is called Σ-*cofibrant* if the unit map $\mathbb{1}_S \to U(\mathbf{O})$ is a cofibration in the projective model structure on $\mathrm{Fun}(\mathbb{F}_S^{\simeq}, \mathbf{V})$, where U denotes the forgetful functor from operads to collections and $\mathbb{1}_S$ is the monoidal unit for the composition product, given by

$$\mathbb{1}_S\binom{s_1,\ldots,s_n}{s'} = \begin{cases} \mathbb{1}, & n = 1, s_1 = s', \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $\mathbb{1}$ is the monoidal unit in $\mathbf{V}$.

Example 4.7. A one-coloured $\mathbf{V}$-operad $\mathbf{O}$ is Σ-cofibrant precisely if $\mathbb{1} \to \mathbf{O}(1)$ is a cofibration, and the object $\mathbf{O}(n)$ is projectively cofibrant in $\mathrm{Fun}(B\Sigma_n, \mathbf{V})$ for all $n \neq 1$.

Definition 4.8. Let $\mathbf{V}$ be a symmetric monoidal model category and $\mathbf{V}^\flat$ a subcategory of flat objects. We will say that a $\mathbf{V}$-operad $\mathbf{O}$ is *flat* if it is enriched in the full subcategory $\mathbf{V}^\flat$.

Remark 4.9. Since cofibrant objects are flat, if $\mathbf{O}$ is Σ-cofibrant then it is flat precisely if in addition the objects of (unary) endomorphisms $\mathbf{O}(x,x) \in \mathbf{V}$ are all flat.

By [PS18a, Proposition 6.2], if $\mathbf{O}$ is an admissible Σ-cofibrant $\mathbf{V}$-operad, then cofibrant $\mathbf{O}$-algebras have cofibrant underlying objects in $\mathbf{V}$. Since cofibrant objects are in particular flat, if $\mathbf{O}$ is flat, admissible and Σ-cofibrant we have a functor

$$\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})^c \to \mathrm{Alg}_{\mathbf{O}}(\mathbf{V}^\flat) \to \mathrm{Alg}_{\mathbf{O}}(\mathbf{V}[W^{-1}]).$$

This takes weak equivalences in $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})^c$ to equivalences in $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V}[W^{-1}])$, since the weak equivalences are lifted from the weak equivalences in $\mathbf{V}$, and so induces a functor of ∞-categories

$$\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})^c[W_{\mathbf{O}}^{-1}] \to \mathrm{Alg}_{\mathbf{O}}(\mathbf{V}[W^{-1}]),$$

where $W_{\mathbf{O}}$ denotes the collection of weak equivalences between $\mathbf{O}$-algebras.

Theorem 4.10. *Let $\mathbf{V}$ be a symmetric monoidal model category equipped with a subcategory $\mathbf{V}^\flat$ of flat objects. If $\mathbf{O}$ is a flat admissible Σ-cofibrant $\mathbf{V}$-operad, then the functor*

$$\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})^c[W_{\mathbf{O}}^{-1}] \to \mathrm{Alg}_{\mathbf{O}}(\mathbf{V}[W^{-1}])$$

is an equivalence of ∞-categories.

Proof. We follow the proof of [PS18a, Theorem 7.10], which in turn is a variant of those of [Lur17, Theorems 4.1.4.4, 4.5.4.7]. Let S be the set of objects of $\mathbf{O}$. The right Quillen functor $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V}) \to \mathrm{Fun}(S,\mathbf{V})$ induces a functor of ∞-categories $U\colon \mathrm{Alg}_{\mathbf{O}}(\mathbf{V})^c[W_{\mathbf{O}}^{-1}] \to \mathrm{Fun}(S,\mathbf{V}[W^{-1}])$, which is a right adjoint by [MG16, Theorem 2.1]. As $\mathbf{O}$ is Σ-cofibrant, the forgetful functor preserves sifted homotopy colimits by [PS18a, Proposition 7.8]. Since it also detects weak equivalences, it follows by [Lur17, Theorem 4.7.3.5] (the monadicity theorem for ∞-categories) that U is a monadic right adjoint. The same holds for the forgetful functor $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V}[W^{-1}]) \to \mathrm{Fun}(S,\mathbf{V}[W^{-1}])$ by Proposition 3.8, so using [Lur17, Corollary 4.7.3.16] we see that to show that the functor $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})^c[W_{\mathbf{O}}^{-1}] \to \mathrm{Alg}_{\mathbf{O}}(\mathbf{V}[W^{-1}])$ is an equivalence it suffices to show that the two associated monads on $\mathrm{Fun}(S,\mathbf{V}[W^{-1}])$ have equivalent underlying endofunctors. This follows from the formula in Proposition 3.8, since the Σ_n-orbits that appear in the formula for free strict $\mathbf{O}$-algebras are homotopy orbits when $\mathbf{O}$ is Σ-cofibrant. □

The cases to which this result applies are primarily those where *all* operads are admissible, as more generally we only have semi-model structure on algebras over Σ-cofibrant operads. This includes the following examples, as discussed in [PS18b, §7]:

(i) the category Set_Δ of simplicial sets, equipped with the Kan–Quillen model structure,
(ii) the category Top of compactly generated weak Hausdorff spaces, equipped with the usual model structure,
(iii) the category Ch_k of chain complexes of k-vector spaces, where k is a field of characteristic 0 (or more generally a ring containing $\mathbb{Q}$), equipped with the projective model structure,
(iv) the category Sp^Σ of symmetric spectra, equipped with the positive stable model structure,

In the first three examples the unit is cofibrant, and in the positive stable model structure on symmetric spectra a suitable subcategory of flat objects is supplied by the S-*cofibrant* objects of [Shi04] (see also [Sch07, Chapter 5], where these are called *flat* objects). Note that a Σ-cofibrant operad in symmetric spectra is necessarily flat, since the flat objects are the cofibrant objects in a model structure whose cofibrations include the usual cofibrations.

Specializing to these cases, we have:

Corollary 4.11.

(i) Let $\mathbf{O}$ *be a* Σ*-cofibrant simplicial operad, then*

$$\mathrm{Alg}_{\mathbf{O}}(\mathrm{Set}_{\Delta})[W_{\mathbf{O}}^{-1}] \simeq \mathrm{Alg}_{\mathbf{O}}(\mathcal{S}).$$

(ii) Let $\mathbf{O}$ *be a* Σ*-cofibrant topological operad, then*

$$\mathrm{Alg}_{\mathbf{O}}(\mathrm{Top})[W_{\mathbf{O}}^{-1}] \simeq \mathrm{Alg}_{\mathbf{O}}(\mathcal{S}).$$

(iii) Let $\mathbf{O}$ *be a* Σ*-cofibrant dg-operad over a field* k *of characteristic zero, then*

$$\mathrm{Alg}_{\mathbf{O}}(\mathrm{Ch}_k)[W_{\mathbf{O}}^{-1}] \simeq \mathrm{Alg}_{\mathbf{O}}(\mathcal{D}(k)),$$

where $\mathcal{D}(k)$ *is the derived* ∞*-category of* k*-modules.*

(iv) Let $\mathbf{O}$ *be a* Σ*-cofibrant operad in symmetric spectra, then*

$$\mathrm{Alg}_{\mathbf{O}}(\mathrm{Sp}^{\Sigma})[W_{\mathbf{O}}^{-1}] \simeq \mathrm{Alg}_{\mathbf{O}}(\mathrm{Sp}),$$

where Sp *is the* ∞*-category of spectra.*

Remark 4.12. The case of simplicial operads was already proved as [PS18a, Theorem 7.10].

Remark 4.13. According to Spitzweck's thesis [Spi01, Theorem 4], a $\mathbf{V}$-operad with a single object that is cofibrant in the semi-model structure on one-object operads in $\mathbf{V}$ is admissible without further assumptions on $\mathbf{V}$. A version for coloured operads does not yet seem to appear in the literature, but if this is correct then the comparison of Theorem 4.10 would apply in general for such cofibrant operads.

Remark 4.14. If $\mathbf{O}$ is a Σ-cofibrant operad in a symmetric monoidal model category $\mathbf{V}$, then under much weaker assumptions on $\mathbf{V}$ there exists a *semi-model* structure on the category $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})$, by a result of Spitzweck [Spi01, Theorem 5] in the one-object case and White–Yau for coloured operads [WY18, Theorem 6.3.1]. Using results of Cisinski [Cis19], White and Yau have recently extended the results relating structures in model categories to their analogues in ∞-categories needed to carry out the proof of Theorem 4.10 in the setting of semi-model categories, and thereby extended the comparison with ∞-operad algebras to the case where there is only a semi-model structure on algebras over a Σ-cofibrant operad; see [WY24, §7.3].

5. Endomorphism ∞-Operads

The first goal of this subsection is to prove that for any morphism of ∞-groupoids $f\colon X \to \mathcal{V}^{\simeq}$ there exists a corresponding *endomorphism* ∞*-operad* $\mathrm{End}_{\mathcal{V}}(f)$, where $\mathcal{V}$ denotes a closed symmetric monoidal ∞-category compatible with small ∞-groupoid-indexed colimits. Our strategy for obtaining these objects is taken from [Hin20, §6.3] and uses the construction of endomorphism algebras from [Lur17, §4.7.1], which we first briefly recall:[6]

Suppose $\mathcal{A}$ is a monoidal ∞-category and $\mathcal{M}$ is right-tensored over $\mathcal{A}$. An *endomorphism algebra* for an object $M \in \mathcal{M}$ is an associative algebra $\mathrm{End}(M)$ in $\mathcal{A}$ and a right $\mathrm{End}(M)$-module structure on M with the universal property that for any associative algebra A in $\mathcal{A}$, right A-module structures on M are naturally equivalent to morphisms of associative algebras $A \to \mathrm{End}(M)$.

By [Lur17, Proposition 4.7.1.30, Theorem 4.7.1.34] there exists a monoidal ∞-category $\mathcal{A}[M]$ whose objects are pairs $(X \in \mathcal{A}, M \otimes X \to M \text{ in } \mathcal{M})$, with the

[6]We restate it for right instead of left modules.

property that an associative algebra in $\mathcal{A}[M]$ corresponds to an assocative algebra $A \in \mathcal{A}$ together with a right A-module structure on M. An endomorphism algebra for M is thus precisely a terminal object in $\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{A}[M])$. Since the terminal object of $\mathcal{A}[M]$ has a unique algebra structure if it exists, we have:

Proposition 5.1 ([Lur17, Corollary 4.7.1.40])**.** *If $\mathcal{A}[M]$ has a terminal object $(A, M \otimes A \to M)$ then A is the underlying object of an endomorphism algebra for M.* □

We also note that by construction the forgetful functor $\mathcal{A}[M] \to \mathcal{A}$ is a right fibration, corresponding to the functor

$$A \mapsto \mathrm{Map}_{\mathcal{M}}(M \otimes A, M).$$

In the case of $\mathrm{Coll}_X(\mathcal{V})$ and its right module $\mathrm{Fun}(X, \mathcal{V})$ we can explitly identify this functor:

Proposition 5.2. *For $M \in \mathrm{Fun}(X, \mathcal{V})$ and $S \in \mathrm{Coll}_X(\mathcal{V})$ there is a natural equivalence*

$$\mathrm{Map}_{\mathrm{Fun}(X,\mathcal{V})}(M \odot S, M) \simeq \mathrm{Map}_{\mathrm{Coll}_X(\mathcal{V})}(S, \mathrm{End}_{\mathcal{V}}(M)),$$

where $\mathrm{End}_{\mathcal{V}}(M)\colon \mathbb{F}_X^{\simeq} \to \mathcal{V}$ is the functor given by

$$\mathrm{End}_{\mathcal{V}}(M)\binom{x_1,\ldots,x_n}{x} \simeq \mathrm{MAP}_{\mathcal{V}}(M(x_1) \otimes \cdots \otimes M(x_n), M(x)),$$

with $\mathrm{MAP}_{\mathcal{V}}$ denoting the internal Hom in $\mathcal{V}$.

Proof. Since X is an ∞-groupoid, the twisted arrow ∞-category $\mathrm{Tw}(X)$ is equivalent to X, and so [GHN17, Proposition 5.1] yields a natural equivalence

$$\mathrm{Map}_{\mathrm{Fun}(X,\mathcal{V})}(M \odot S, M) \simeq \lim_{x \in X} \mathrm{Map}_{\mathcal{V}}((M \odot S)(x), M(x)).$$

Now the description of $M \odot S$ from Proposition 3.8 shows that this is naturally equivalent to

$$\lim_{x\in X} \mathrm{Map}_{\mathcal{V}}\left(\coprod_n \operatorname*{colim}_{(x_1,\ldots,x_n)\in X^n_{h\Sigma_n}} M(x_1)\otimes\cdots\otimes M(x_n)\otimes S\binom{x_1,\ldots,x_n}{x}, M(x)\right).$$

Taking the limit out and applying the universal property of MAP, this becomes

$$\lim_{x\in X}\left(\prod_n \lim_{(x_1,\ldots,x_n)\in X^n_{h\Sigma_n}} \mathrm{Map}_{\mathcal{V}}(S\tbinom{x_1,\ldots,x_n}{x}, \mathrm{End}(M)\tbinom{x_1,\ldots,x_n}{x})\right).$$

We can now combine the limits to get a limit over $\coprod_n X \times X^n_{h\Sigma_n} \simeq \mathbb{F}_X^{\simeq}$, i.e.

$$\lim_{\xi \in \mathbb{F}_X^{\simeq}} \mathrm{Map}_{\mathcal{V}}(S(\xi), \mathrm{End}_{\mathcal{V}}(M)(\xi)).$$

Applying [GHN17, Proposition 5.1] once more now identifies this limit (since $\mathbb{F}_X^{\simeq}$ is again an ∞-groupoid) with $\mathrm{Map}_{\mathrm{Fun}(\mathbb{F}_X^{\simeq},\mathcal{V})}(S, \mathrm{End}_{\mathcal{V}}(M))$, as required. □

Corollary 5.3. *For any $M\colon X \to \mathcal{V}$, the ∞-category $\mathrm{Coll}_X(\mathcal{V})[M]$ has a terminal object.*

Proof. By Proposition 5.2, the functor $\mathrm{Coll}_X(\mathcal{V})^{\mathrm{op}} \to \mathcal{S}$ corresponding to the right fibration

$$\mathrm{Coll}_X(\mathcal{V})[M] \to \mathrm{Coll}_X(\mathcal{V})$$

is represented by the object $\mathrm{End}_{\mathcal{V}}(M)$. This implies that we have an equivalence

$$\mathrm{Coll}_X(\mathcal{V})[M] \simeq \mathrm{Coll}_X(\mathcal{V})_{/\mathrm{End}_{\mathcal{V}}(M)}.$$

Since the right-hand side clearly has a terminal object, this completes the proof. □

Applying Proposition 5.1, we get:

Corollary 5.4. *For any $M \in \mathrm{Fun}(X, \mathcal{V})$ there exists an endomorphism ∞-operad $\mathrm{End}_{\mathcal{V}}(M)$ in $\mathrm{Opd}_X(\mathcal{V}) \simeq \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Coll}_X(\mathcal{V}))$ whose underlying X-collection is*

$$\mathrm{End}_{\mathcal{V}}(M)\binom{x_1,\ldots,x_n}{y} \simeq \mathrm{MAP}(M(x_1) \otimes \cdots \otimes M(x_n), M(y)).$$

This has the universal property that, for any $\mathcal{O} \in \mathrm{Opd}_X(\mathcal{V})$, morphisms $\mathcal{O} \to \mathrm{End}_{\mathcal{V}}(M)$ in $\mathrm{Opd}_X(\mathcal{V})$ correspond to $\mathcal{O}$-algebra structures on M, i.e. there is natural equivalence

$$\mathrm{Map}_{\mathrm{Opd}_X(\mathcal{V})}(\mathcal{O}, \mathrm{End}_{\mathcal{V}}(M)) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})^{\simeq}_M,$$

where the right-hand side denotes the underlying ∞-groupoid of the fibre of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \to \mathrm{Fun}(X, \mathcal{V})$ at M.

Remark 5.5. For $X \simeq *$, so that the functor $* \to \mathcal{V}$ picks out an object v of $\mathcal{V}$, we get an ∞-categorical analogue of the classical endomorphism operad: $\mathrm{End}_{\mathcal{V}}(v)$ is a one-object $\mathcal{V}$-∞-operad with underlying symmetric sequence

$$\mathrm{End}_{\mathcal{V}}(v)(n) \simeq \mathrm{MAP}_{\mathcal{V}}(v^{\otimes n}, v).$$

If $\mathcal{O}$ is a one-object $\mathcal{V}$-∞-operad, the universal property says that an $\mathcal{O}$-algebra structure on v is equivalent to a morphism of one-object ∞-operads $\mathcal{O} \to \mathrm{End}_{\mathcal{V}}(v)$.

Example 5.6. By Corollary 3.13, if $\mathcal{O}$ is any $\mathcal{V}$-∞-operad with space of objects X, then the functor $Z(\mathcal{O})\colon X \to \mathcal{V}$ picking out the nullary operations is canonically a right $\mathcal{O}$-module. This corresponds to a canonical morphism of $\mathcal{V}$-∞-operads $\mathcal{O} \to \mathrm{End}(Z(\mathcal{O}))$, given by maps

$$\mathcal{O}\binom{x_1,\ldots,x_n}{y} \to \mathrm{MAP}_{\mathcal{V}}(Z(\mathcal{O})(x_1) \otimes \cdots \otimes Z(\mathcal{O})(x_n), Z(\mathcal{O})(y)),$$

adjoint to the composition maps

$$\mathcal{O}\binom{}{x_1} \otimes \cdots \otimes \mathcal{O}\binom{}{x_n} \otimes \mathcal{O}\binom{x_1,\ldots,x_n}{y} \to \mathcal{O}\binom{}{y}$$

for $\mathcal{O}$.

We now observe that the endomorphism algebras are compatible with the lax monoidal functors $f^*\colon \mathrm{Coll}_Y(\mathcal{V}) \to \mathrm{Coll}_X(\mathcal{V})$ induced by morphisms of ∞-groupoids $f\colon X \to Y$:

Proposition 5.7. *For $f\colon X \to Y$ a morphism in $\mathcal{S}$ and $M\colon Y \to \mathcal{V}$, there is a natural equivalence of* RM*-algebras*

$$f^*(M, \mathrm{End}_{\mathcal{V}}(M)) \xrightarrow{\sim} (f^*M, \mathrm{End}_{\mathcal{V}}(f^*M)).$$

Proof. If $\mathcal{O}$ is a $\mathcal{V}$-∞-operad with Y as space of objects, the lax monoidal functor f^* induces a a natural functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \to \mathrm{Alg}_{f^*\mathcal{O}}(\mathcal{V}),$$

given on the underlying functors to $\mathcal{V}$ by composition with f. Applying this to the $\mathcal{V}$-∞-operad $\mathrm{End}_{\mathcal{V}}(M)$ and the canonical $\mathrm{End}_{\mathcal{V}}(M)$-algebra structure on M, we obtain an $f^*\mathrm{End}_{\mathcal{V}}(M)$-algebra structure on $f^*M = M \circ f$. By the universal property of endomorphism ∞-operads this corresponds to a morphism of ∞-operads $f^*\mathrm{End}_{\mathcal{V}}(M) \to \mathrm{End}_{\mathcal{V}}(f^*M)$. Using the explicit description of the underlying collection of $\mathrm{End}_{\mathcal{V}}(M)$ in terms of internal Homs we see that this is an equivalence. □

There exists a universal functor from an ∞-groupoid to $\mathcal{V}$, namely the inclusion $\mathcal{V}^{\simeq} \to \mathcal{V}$ of the underlying ∞-groupoid of $\mathcal{V}$. Our construction does not apply directly to this, since the ∞-groupoid $\mathcal{V}^{\simeq}$ is not small. However, by passing to a larger universe we can define a universal endomorphism ∞-operad for $\mathcal{V}$:

Definition 5.8. Let $\mathcal{V}$ be a large closed symmetric monoidal ∞-category compatible with colimits indexed by small ∞-groupoids. By [Lur17, Proposition 4.8.1.10] there is a very large presentable ∞-category $\widehat{\mathcal{V}}$ compatible with large colimits, with a fully faithful symmetric monoidal functor $\mathcal{V} \hookrightarrow \widehat{\mathcal{V}}$ that preserves colimits over small ∞-groupoids. Let $\mathrm{Opd}(\widehat{\mathcal{V}})$ be the ∞-category of $\widehat{\mathcal{V}}$-enriched ∞-operads with potentially large spaces of objects.

Remark 5.9. Since $\mathcal{V}$ is a symmetric monoidal full subcategory of $\widehat{\mathcal{V}}$, we can regard $\mathrm{Opd}(\mathcal{V})$ as a full subcategory of $\mathrm{Opd}(\widehat{\mathcal{V}})$, containing precisely those $\widehat{\mathcal{V}}$-∞-operads whose spaces of objects are small and whose objects of multimorphisms all lie in the full subcategory $\mathcal{V}$. Similarly, for any $\mathcal{O} \in \mathrm{Opd}_X(\mathcal{V})$ we have a pullback square

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \arrow[r, hook] \arrow[d] & \mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{V}}) \arrow[d] \\
\mathrm{Fun}(X, \mathcal{V}) \arrow[r, hook] & \mathrm{Fun}(X, \widehat{\mathcal{V}})
\end{tikzcd}$$

where the horizontal maps are fully faithful.

Definition 5.10. Let $i\colon \mathcal{V}^{\simeq} \to \widehat{\mathcal{V}}$ denote the inclusion of the space of objects in the full subcategory $\mathcal{V}$. Applying Corollary 5.4 in the enlarged universe to i, we get an endomorphism $\widehat{\mathcal{V}}$-∞-operad

$$\overline{\mathcal{V}} := \mathrm{End}_{\widehat{\mathcal{V}}}(i).$$

The formula for its multimorphism objects implies that we can regard $\overline{\mathcal{V}}$ as a (large) $\mathcal{V}$-∞-operad (i.e. an object in the ∞-category $\mathrm{Opd}_{\mathcal{V}^{\simeq}}(\mathcal{V}) \subseteq \mathrm{Opd}_{\mathcal{V}^{\simeq}}(\widehat{\mathcal{V}})$, which makes sense also when the ∞-groupoid of objects is large). Moreover, for any map of ∞-groupoids $M\colon X \to \mathcal{V}^{\simeq}$ where X is small, we can regard

$$\mathrm{End}_{\widehat{\mathcal{V}}}(i \circ M) \simeq M^*\overline{\mathcal{V}}$$

as an object of $\mathrm{Opd}_X(\mathcal{V})$.

Lemma 5.11. *For any map $M\colon X \to \mathcal{V}^{\simeq}$ where X is a small ∞-groupoid, we have a canonical equivalence*

$$M^*\overline{\mathcal{V}} \simeq \mathrm{End}_{\mathcal{V}}(M).$$

Proof. For $\mathcal{O} \in \mathrm{Opd}_X(\mathcal{V})$ we have a natural equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})_M \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{V}})_{iM}.$$

Using the universal property of the endomorphism objects this corresponds to a natural equivalence

$$\mathrm{Map}(\mathcal{O}, M^*\overline{\mathcal{V}}) \xrightarrow{\sim} \mathrm{Map}(\mathcal{O}, \mathrm{End}_{\mathcal{V}}(M)),$$

and so an equivalence $M^*\overline{\mathcal{V}} \xrightarrow{\sim} \mathrm{End}_{\mathcal{V}}(M)$, as required. □

Let U denote the canonical $\overline{\mathcal{V}}$-algebra structure on i, which we can regard as an object of $\mathrm{Alg}_{\overline{\mathcal{V}}}(\mathcal{V})$. For every map $M\colon X \to \mathcal{V}^{\simeq}$ with X small, the pullback M^*U is then the canonical $\mathrm{End}_{\mathcal{V}}(M)$-algebra structure on M, which leads to the following:

Theorem 5.12. *For any small $\mathcal{V}$-∞-operad $\mathcal{O}$, the morphism of ∞-groupoids*

$$\mathrm{Map}_{\mathrm{Opd}(\widehat{\mathcal{V}})}(\mathcal{O}, \overline{\mathcal{V}}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})^{\simeq},$$

which takes $\phi\colon \mathcal{O} \to \overline{\mathcal{V}}$ to $\phi^*U \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})^{\simeq}$, is an equivalence.

Proof. Let X be the space of objects of $\mathcal{O}$. Then we have a commutative triangle of ∞-groupoids

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Opd}(\widehat{\mathcal{V}})}(\mathcal{O}, \overline{\mathcal{V}}) \arrow[rr] \arrow[dr] & & \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})^{\simeq} \arrow[dl] \\ & \mathrm{Map}(X, \mathcal{V}). \end{tikzcd}$$

It suffices to show that we have an equivalence on the fibres over each map $M\colon X \to \mathcal{V}$. But we have an equivalence between $\mathrm{Map}_{\mathrm{Opd}(\widehat{\mathcal{V}})}(\mathcal{O}, \overline{\mathcal{V}})_M$ and

$$\mathrm{Map}_{\mathrm{Opd}_X(\mathcal{V})}(\mathcal{O}, M^*\overline{\mathcal{V}}) \simeq \mathrm{Map}_{\mathrm{Opd}_X(\mathcal{V})}(\mathcal{O}, \mathrm{End}_{\mathcal{V}}(M)),$$

under which the map to $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})^{\simeq}_M$ is equivalent to that taking $\phi\colon \mathcal{O} \to \mathrm{End}_{\mathcal{V}}(M)$ to ϕ^* applied to the canonical $\mathrm{End}_{\mathcal{V}}(M)$-algebra structure on M. This is an equivalence by the universal property of the endomorphism algebra. □

Remark 5.13. In [CH20] we constructed a natural tensoring of $\mathcal{V}$-∞-operads over ∞-categories. This induces an enrichment in ∞-categories, given by

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})) \simeq \mathrm{Map}_{\mathrm{Opd}^{\mathcal{V}}_\infty}(\mathcal{C} \otimes \mathcal{O}, \mathcal{P}).$$

For $\mathcal{P} = \overline{\mathcal{V}}$, we can use Theorem 5.12 to identify the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\overline{\mathcal{V}})$ with the Segal space $\mathrm{Alg}_{\Delta^\bullet \otimes \mathcal{O}}(\mathcal{V})^{\simeq}$. We expect that this should in fact be equivalent to the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$, but to prove we need a better understanding of the tensoring of $\mathcal{V}$-∞-operads and ∞-categories. As this was defined rather inexplicitly in [CH20], we suspect that this requires setting up a new definition of enriched ∞-operads where the tensoring can be described more concretely.

In the case where $\mathcal{V}$ is the ∞-category $\mathcal{S}$ of spaces, we can identify $\overline{\mathcal{S}}$ explicitly:

Proposition 5.14. *Let $\mathcal{S}^\times$ denote the symmetric monoidal ∞-category given by the cartesian product in $\mathcal{S}$, viewed as an $\mathcal{S}$-enriched ∞-operad. There is an equivalence $\mathcal{S}^\times \xrightarrow{\sim} \overline{\mathcal{S}}$.*

Proof. For $X \in \mathcal{S}$, we have $Z(\mathcal{S}^\times)(X) \simeq \mathrm{Map}_{\mathcal{S}}(*, X) \simeq X$, and the functor $Z(\mathcal{S}^\times)\colon \mathcal{S}^{\simeq} \to \mathcal{S}$ is the inclusion of the underlying ∞-groupoid. Hence, thinking of $\overline{\mathcal{S}}$ as an endomorphism object for ∞-operads enriched in large spaces, by Example 5.6 there is a canonical morphism $\mathcal{S}^\times \to \overline{\mathcal{S}}$. This is given by equivalences

$$\mathcal{S}^\times\tbinom{X_1,\ldots,X_n}{Y} \xrightarrow{\sim} \mathrm{Map}_{\mathcal{S}}(X_1 \times \cdots \times X_n, Y),$$

and so it is an equivalence of $\mathcal{S}$-∞-operads. □

Remark 5.15. It follows that for $\mathcal{O}$ an $\mathcal{S}$-∞-operad, the ∞-groupoid $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S})^{\simeq}$ in our sense is equivalent to $\mathrm{Map}_{\mathrm{Opd}(\widehat{\mathcal{S}})}(\mathcal{O}, \mathcal{S}^\times)$. This is the underlying ∞-groupoid of the ∞-category of $\mathcal{O}$-algebras in $\mathcal{S}$ defined in [Lur17], so for $\mathcal{S}$-enriched ∞-operads our notion of $\mathcal{O}$-algebras agrees with that of [Lur17], at least on the level of ∞-groupoids.

References

[AF18] David Ayala and John Francis, *Flagged higher categories*, Topology and quantum theory in interaction, Contemp. Math., vol. 718, Amer. Math. Soc., Providence, RI, 2018, pp. 137–173, available at arXiv:1801.08973.

[Bar18] Clark Barwick, *From operator categories to higher operads*, Geom. Topol. **22** (2018), no. 4, 1893–1959, available at arXiv:1302.5756.

[Bra17] Lukas Brantner, *The Lubin–Tate theory of spectral Lie algebras* (2017). Available from `https://people.maths.ox.ac.uk/brantner/brantnerthesis.pdf`.

[CH20] Hongyi Chu and Rune Haugseng, *Enriched ∞-operads*, Adv. Math. **361** (2020), 106913, 85, available at arXiv:1707.08049.

[Cis19] Denis-Charles Cisinski, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, Cambridge, 2019. Available from https://cisinski.app.uni-regensburg.de/CatLR.pdf. MR3931682

[GH15] David Gepner and Rune Haugseng, *Enriched ∞-categories via non-symmetric ∞-operads*, Adv. Math. **279** (2015), 575–716, available at arXiv:1312.3178.

[GHN17] David Gepner, Rune Haugseng, and Thomas Nikolaus, *Lax colimits and free fibrations in ∞-categories*, Doc. Math. **22** (2017), 1225–1266, available at arXiv:1501.02161.

[Hau22] Rune Haugseng, ∞-*Operads via symmetric sequences*, Math. Z. **301** (2022), no. 1, 115–171, available at arXiv:1708.09632.

[Heu11] Gijs Heuts, *Algebras over ∞-operads* (2011), available at arXiv:1110.1776.

[Hin20] Vladimir Hinich, *Yoneda lemma for enriched ∞-categories*, Adv. Math. **367** (2020), 107129, available at arXiv:1805.07635.

[Lur09] Jacob Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. Available from http://math.ias.edu/~lurie/.

[Lur17] ———, *Higher Algebra*, 2017. Available at http://math.ias.edu/~lurie/.

[MG16] Aaron Mazel-Gee, *Quillen adjunctions induce adjunctions of quasicategories*, New York J. Math. **22** (2016), 57–93, available at arXiv:1501.03146.

[PS18a] Dmitri Pavlov and Jakob Scholbach, *Admissibility and rectification of colored symmetric operads*, J. Topol. **11** (2018), no. 3, 559–601, available at arXiv:1410.5675.

[PS18b] ———, *Homotopy theory of symmetric powers*, Homology Homotopy Appl. **20** (2018), no. 1, 359–397, available at arXiv:1510.04969.

[Sch07] Stefan Schwede, *An untitled book project about symmetric spectra*, 2007. Available from http://www.math.uni-bonn.de/~schwede/.

[Shi04] Brooke Shipley, *A convenient model category for commutative ring spectra*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 473–483.

[Spi01] Markus Spitzweck, *Operads, algebras and modules in general model categories* (2001), available at arXiv:math/0101102.

[Tri] Todd H. Trimble, *Notes on the Lie operad*. Available from http://math.ucr.edu/home/baez/trimble/.

[WY18] David White and Donald Yau, *Bousfield localization and algebras over colored operads*, Appl. Categ. Structures **26** (2018), no. 1, 153–203.

[WY24] ———, *Smith ideals of operadic algebras in monoidal model categories*, Algebr. Geom. Topol. **24** (2024), no. 1, 341–392.

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