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Pushouts of Dwyer maps are $(\infty, 1)$ -categorical

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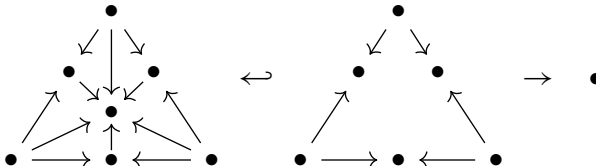
The inclusion of 1-categories into $(\infty, 1)$ -categories fails to preserve colimits in general, and pushouts in particular. We observe that if one functor in a span of categories belongs to a certain previously identified class of functors, then the 1-categorical pushout is preserved under this inclusion. Dwyer maps, a kind of neighborhood deformation retract of categories, were used by Thomason in the construction of his model structure on 1-categories. Thomason previously observed that the nerves of such pushouts have the correct weak homotopy type. We refine this result and show that the weak homotopical equivalence is a weak categorical equivalence. We also identify a more general class of functors along which 1-categorical pushouts are $(\infty, 1)$ -categorical.

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1 Introduction

Classical 1-categories define an important special case of $(\infty, 1)$ -categories. The fact that $(\infty, 1)$ -category theory restricts to ordinary 1-categories can be understood, in part, by the observation that the inclusion of 1-categories into $(\infty, 1)$ -categories is full as an inclusion of $(\infty, 2)$ -categories. This full inclusion is reflective — with the left adjoint given by the functor that sends an $(\infty, 1)$ -category to its quotient “homotopy category” — but not coreflective and as a consequence colimits of ordinary 1-categories need not be preserved by the passage to $(\infty, 1)$ -categories. Indeed there are known examples of colimits of 1-categories that generate nontrivial higher-dimensional structure when the colimit is formed in the category of $(\infty, 1)$ -categories.

For example, consider the span of posets

(1.1) 

The pushout in 1–categories is the arrow category $\bullet \rightarrow \bullet$, while the pushout in $(\infty, 1)$ –categories defines an $(\infty, 1)$ –category which has the homotopy type of the 2–sphere.

As a second example, let M be the monoid with five elements, $e, x_{11}, x_{12}, x_{21}$ and x_{22} , and multiplication rule given by $x_{ij}x_{k\ell} = x_{i\ell}$. Inverting all elements of M yields the trivial group. That is, if one considers M as a 1–category with a single object, then the pushout of the span $M \leftarrow \coprod_M \mathbb{2} \rightarrow \coprod_M \mathbb{I}$ (where $\mathbb{2}$ is the free-living arrow and \mathbb{I} is the free-living isomorphism) in categories is the terminal category $\mathbb{1}$. On the other hand, the pushout of this span in $(\infty, 1)$ –categories is the ∞ –groupoid S^2 as follows from [Fiedorowicz 2002, Lemma]. The results of [McDuff 1979] imply that this example is generalizable to a vast class of monoids.

More generally, the Gabriel–Zisman category of fractions $\mathcal{C}[\mathcal{W}^{-1}]$ is formed by freely inverting the morphisms in a class of arrows \mathcal{W} in a 1–category \mathcal{C} . This can also be constructed as a pushout of 1–categories of the span

$$\mathcal{C} \leftarrow \coprod_{w \in \mathcal{W}} \mathbb{2} \hookrightarrow \coprod_{w \in \mathcal{W}} \mathbb{I}$$

where each arrow in \mathcal{W} is replaced by a free-living isomorphism. By contrast, the $(\infty, 1)$ –category defined by this pushout is modeled by the Dwyer–Kan simplicial localization, which has nontrivial higher dimensional structure in many instances [Dwyer and Kan 1980; Joyal 2008, page 168; Stevenson 2017, Lemma 18]. Indeed, all $(\infty, 1)$ –categories arise in this way [Barwick and Kan 2012].

As the examples above show, pushouts of 1–categories in particular are problematic. Our aim is to prove that a certain class of pushout diagrams of 1–categories are guaranteed to be $(\infty, 1)$ –categorical. The requirement is that one of the two maps in the span that generates the pushout belong to a class of functors between 1–categories first considered by Thomason [1980, Definition 4.1] under the name “Dwyer maps” that feature in a central way in the construction of the Thomason model structure on categories.

Definition 1.2 (Thomason) A full sub-1–category inclusion $I: \mathcal{A} \hookrightarrow \mathcal{B}$ is a *Dwyer map* if the following conditions hold.

- (i) The category \mathcal{A} is a *sieve* in \mathcal{B} , meaning there is a necessarily unique functor $\chi: \mathcal{B} \rightarrow \mathbb{2}$ with $\chi^{-1}(0) = \mathcal{A}$. We write $\mathcal{V} := \chi^{-1}(1)$ for the complementary *cosieve* of \mathcal{A} in \mathcal{B} .
- (ii) The inclusion $I: \mathcal{A} \hookrightarrow \mathcal{W}$ into the *minimal cosieve*¹ $\mathcal{W} \subset \mathcal{B}$ containing \mathcal{A} admits a right adjoint left inverse $R: \mathcal{W} \rightarrow \mathcal{A}$, a right adjoint for which the unit is an identity.

Schwede [2019] describes Dwyer maps as “categorical analogs of the inclusion of a neighborhood deformation retract”. In fact, many examples of Dwyer maps are more like deformation retracts, in that the cosieve \mathcal{W} generated by \mathcal{A} is the full codomain category \mathcal{B} .

¹Explicitly \mathcal{W} is the full subcategory of \mathcal{B} containing every object that arises as the codomain of an arrow with domain in \mathcal{A} .

- Example 1.3** (i) The vertex inclusion $0: \mathbb{1} \rightarrow \mathbb{2}$ is a Dwyer map, with $!: \mathbb{2} \rightarrow \mathbb{1}$ the right adjoint left inverse. The other vertex inclusion $1: \mathbb{1} \rightarrow \mathbb{2}$ is not a Dwyer map.
- (ii) Generalizing the previous example, if \mathcal{A} is a category with a terminal object and $\mathcal{A}^\triangleright$ is the category which formally adds a new terminal object, then the inclusion $\mathcal{A} \hookrightarrow \mathcal{A}^\triangleright$ is a Dwyer map.²

Thomason observed that Dwyer maps are stable under pushouts, as we now recall:

Lemma 1.4 [Thomason 1980, Proposition 4.3] *Any pushout of a Dwyer map I defines a Dwyer map J :*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ I \downarrow & \lrcorner & \downarrow J \\ \mathcal{B} & \xrightarrow{G} & \mathcal{D} \end{array}$$

Note, for example, that Lemma 1.4 explains the Dwyer map of Example 1.3(ii): if \mathcal{A} has a terminal object t , then the pushout

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{t} & \mathcal{A} \\ 0 \downarrow & \lrcorner & \downarrow \\ \mathbb{2} & \longrightarrow & \mathcal{A}^\triangleright \end{array}$$

defines the category $\mathcal{A}^\triangleright$.

Our aim is to show that pushouts of categories involving at least one Dwyer map can also be regarded as pushouts of $(\infty, 1)$ -categories in the sense made precise by considering the nerve embedding from categories into quasicategories:

Theorem 1.5 *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ I \downarrow & \lrcorner & \downarrow J \\ \mathcal{B} & \xrightarrow{G} & \mathcal{D} \end{array}$$

be a pushout of categories, and assume that I is a Dwyer map. Then the induced map of simplicial sets

$$N\mathcal{B} \amalg_{N\mathcal{A}} N\mathcal{C} \rightarrow N\mathcal{D}$$

is a weak categorical equivalence.

By a weak categorical equivalence, we mean a weak equivalence in Joyal’s model structure for quasicategories [Joyal and Tierney 2007, Section 1]. Theorem 1.5 is a refinement of a similar result of Thomason [1980, Proposition 4.3], which proves that the same map is a weak homotopy equivalence.

²If \mathcal{A} does not have a terminal object, then $\mathcal{A} \rightarrow \mathcal{A}^\triangleright$ need not be a Dwyer map. Indeed, if $\mathcal{A} = \mathbb{1} \amalg \mathbb{1}$, the only cosieve containing \mathcal{A} is $\mathcal{A}^\triangleright$ itself, and there cannot be a right adjoint $\mathcal{A}^\triangleright \rightarrow \mathcal{A}$ as \mathcal{A} does not have a terminal object. But see Example 3.5(iii), which explains that this example is *discretely flat*.

While [Theorem 1.5](#) is the natural generalization of Thomason’s result, we prove it by considering instead the embedding of 1–categories as discrete simplicially enriched categories, using Bergner’s model of $(\infty, 1)$ –categories. This tactic was suggested by a referee; for our original argument using the quasicategory model see [\[Hackney et al. 2022\]](#). We show in [Proposition 3.3](#) that a Dwyer map, when considered as a map between discrete simplicial categories, satisfies a certain “flatness” property with respect to the Bergner model structure. Since this discrete embedding of 1–categories into simplicial categories preserves pushouts, unlike the nerve embedding of 1–categories into quasicategories, it is straightforward to prove that:

Theorem 1.6 *The inclusion $\text{Cat}_1 \hookrightarrow \text{Cat}_{(\infty, 1)}$ of the $(\infty, 1)$ –category of 1–categories into the $(\infty, 1)$ –category of $(\infty, 1)$ –categories preserves (homotopy) pushouts along Dwyer maps.*

When then deduce [Theorem 1.5](#) as a corollary of this result.

Though the two previous theorems refer to Dwyer maps, they also hold for the pseudo-Dwyer maps introduced by Cisinski [\[1999\]](#), which are retracts of Dwyer maps. In fact, we prove both [Theorems 1.5](#) and [1.6](#) for more general classes of functors introduced in [Definition 3.4](#) that include the Dwyer maps. The key property of a Dwyer map (or pseudo-Dwyer map) is that it is “discretely flat” as well as a faithful inclusion. For a functor to be *discretely flat* means that pushouts along it, considered as a functor of discrete simplicial categories, preserve Dwyer–Kan equivalences of simplicial categories.

In a companion paper, we give an application of [Theorem 1.5](#) to the theory of $(\infty, 2)$ –categories. There we prove:

Theorem 1.7 [\[Hackney et al. 2023, 4.4.2\]](#) *The space of composites of any pasting diagram in any $(\infty, 2)$ –category is contractible.*

To prove this, we make use of Lurie’s [\[2009b\]](#) model structure of $(\infty, 2)$ –categories as categories enriched over quasicategories. In this model, a *pasting diagram* is a simplicially enriched functor out of the free simplicially enriched category defined by gluing together the objects, atomic 1–cells, and atomic 2–cells of a pasting scheme, while the composites of these cells belong to the homotopy coherent diagram indexed by the nerve of the free 2–category generated by the pasting scheme.

This pair of $(\infty, 2)$ –categories has a common set of objects so the difference lies in their hom-spaces. The essential difference between the procedure of attaching an atomic 2–cell along the bottom of a pasting diagram or along the bottom of the free 2–category it generates is the difference between forming a pushout of hom-categories in the category of $(\infty, 1)$ –categories or in the category of 1–categories. Since one of the functors in the span that defines the pushout under consideration is a Dwyer map, [Theorem 1.5](#) proves that the resulting $(\infty, 2)$ –categories are equivalent.

In [Section 2](#), we analyze 1–categorical pushouts of Dwyer maps. In [Section 3](#), we extend these observations to pushouts of simplicial categories involving a Dwyer map between 1–categories as one leg of the span,

axiomatize the classes of functors that are well-behaved with respect to simplicial pushouts, and prove [Theorem 1.6](#). In [Section 4](#), we deduce [Theorem 1.5](#) as a corollary and consider a further special case [Corollary 4.1](#), which observes that the canonical comparison between the pushout of nerves of categories and the nerve of the pushout is inner anodyne, provided that one of the functors in the span is a Dwyer map and the other is an injective on objects faithful functor.

2 Dwyer pushouts

We now establish some notation that we will freely reference in the remainder of this paper. By [Definition 1.2](#), a Dwyer map $I: \mathcal{A} \hookrightarrow \mathcal{B}$ uniquely determines a functor $\chi: \mathcal{B} \rightarrow \mathbb{2}$ that classifies the sieve $\mathcal{A} := \chi^{-1}(0)$ and its complementary cosieve $\mathcal{V} := \chi^{-1}(1)$

$$\begin{array}{ccccc}
 \mathcal{V} & \hookrightarrow & \mathcal{B} & \hookrightarrow & \mathcal{A} \\
 \downarrow & \lrcorner & \downarrow \chi & \lrcorner & \downarrow \\
 \mathbb{1} & \xrightarrow{1} & \mathbb{2} & \xleftarrow{0} & \mathbb{1}
 \end{array}$$

as well as a right adjoint left inverse adjunction $(I \dashv R, \varepsilon: IR \Rightarrow \text{id}_{\mathcal{W}})$ associated to the inclusion of \mathcal{A} into the minimal cosieve $\mathcal{A} \subset \mathcal{W} \subset \mathcal{B}$. This data may be summarized by the diagram

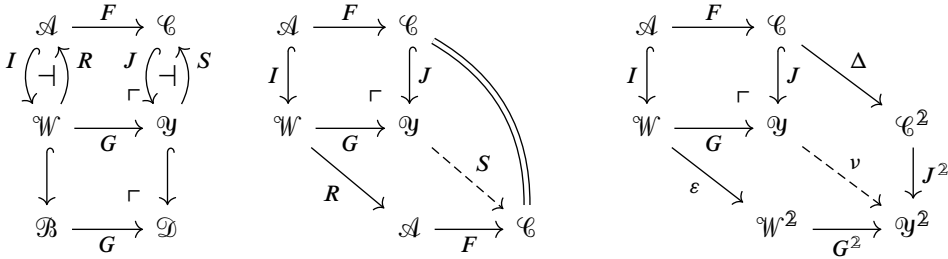
(2.1)

in which $\mathcal{U} := \mathcal{W} \cap \mathcal{V} \cong \mathcal{W} \setminus \mathcal{A}$. Consider the pushout of a Dwyer map along an arbitrary functor $F: \mathcal{A} \rightarrow \mathcal{C}$:

$$\begin{array}{ccccccc}
 & & & \mathcal{A} & \xrightarrow{F} & \mathcal{C} & \\
 & & & \downarrow I & \lrcorner & \downarrow J & \\
 \mathcal{V} & \hookrightarrow & \mathcal{B} & \xrightarrow{G} & \mathcal{D} & \xrightarrow{\pi} & \mathbb{1} \\
 \downarrow & \lrcorner & \downarrow \chi & \downarrow 0 & \downarrow \pi & \downarrow 0 & \\
 \mathbb{1} & \xrightarrow{1} & \mathbb{2} & \xrightarrow{0} & \mathbb{2} & \xleftarrow{0} & \mathbb{1}
 \end{array}$$

The induced functor $\pi: \mathcal{D} \rightarrow \mathbb{2}$ partitions the objects of \mathcal{D} into the two fibers $\text{ob}(\pi^{-1}(0)) \cong \text{ob}\mathcal{C}$ and $\text{ob}(\pi^{-1}(1)) \cong \text{ob}\mathcal{V}$ and prohibits any morphisms from the latter to the former.

The right adjoint left inverse adjunction $(I \dashv R, \varepsilon: IR \Rightarrow \text{id}_{\mathcal{W}})$ associated to the inclusion of \mathcal{A} into the minimal cosieve $\mathcal{A} \subset \mathcal{W} \subset \mathcal{B}$ pushes out to define a right adjoint left inverse $(J \dashv S, \nu: JS \Rightarrow \text{id}_{\mathcal{Y}})$ to the inclusion of \mathcal{C} into the minimal cosieve $\mathcal{C} \subset \mathcal{Y} \subset \mathcal{D}$:



These observations explain the closure of Dwyer maps under pushout and furthermore can be used to explicitly describe the structure of the category \mathcal{D} defined by the pushout of a Dwyer map, as proven in [Bohmann et al. 2015, Proof of Lemma 2.5]; cf also [Schwede 2019, Construction 1.2; Ara and Maltiniotis 2014, Section 7.1].

Proposition 2.2 *The objects in the pushout category \mathcal{D} are given by*

$$\text{ob}\mathcal{C} \amalg \text{ob}\mathcal{V} \xrightarrow{\cong} \text{ob}\mathcal{D}$$

while the hom sets are given by

$$\mathcal{C}(c, c') \cong \mathcal{D}(c, c'), \quad \mathcal{V}(v, v') \cong \mathcal{B}(v, v') \cong \mathcal{D}(v, v'), \quad \mathcal{C}(c, Su) \xrightarrow[\cong]{\nu u \circ (-)} \mathcal{D}(c, u), \quad f \mapsto \hat{f},$$

for all $c, c' \in \mathcal{C}$, $v, v' \in \mathcal{V}$ and $u \in \mathcal{U}$, and are empty otherwise. Functoriality of the inclusions J and G defines the composition on the image of \mathcal{C} and \mathcal{V} . For objects $c, c' \in \mathcal{C}$ and $u, u' \in \mathcal{U}$, the composition map

$$\begin{array}{ccc} \mathcal{D}(u, u') \times \mathcal{D}(c, u) \times \mathcal{D}(c', c) & \xrightarrow{\circ} & \mathcal{D}(c', u') \\ \uparrow \cong & & \uparrow \cong \\ \mathcal{D}(u, u') \times \mathcal{C}(c, Su) \times \mathcal{C}(c', c) & & \\ \downarrow S \times \text{id} & & \\ \mathcal{C}(Su, Su') \times \mathcal{C}(c, Su) \times \mathcal{C}(c', c) & \xrightarrow{\circ} & \mathcal{C}(c', Su') \end{array}$$

is the unique map making the diagram commute.³

To summarize, J and G define fully faithful inclusions

$$\begin{array}{ccccc} \mathcal{V} & \hookrightarrow & \mathcal{D} & \hookleftarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow \pi & \lrcorner & \downarrow \\ \mathbb{1} & \xrightarrow{1} & \mathbb{2} & \xleftarrow{0} & \mathbb{1} \end{array}$$

that are jointly surjective on objects. In particular, we may identify \mathcal{V} with the complementary cosieve of \mathcal{C} in \mathcal{D} .

³Note if $u \in \mathcal{U}$ and $v \in \mathcal{V} \setminus \mathcal{U}$, then $\mathcal{B}(u, v) = \emptyset$.

3 Pushouts in simplicial categories

By *simplicial category* we always mean *simplicially enriched category* as opposed to a simplicial object in Cat . There is a fully faithful inclusion $\text{sCat} \hookrightarrow \text{Cat}^{\Delta^{\text{op}}}$, identifying a simplicial category with an identity-on-objects simplicial object.

The following model structure on sCat is due to Bergner [2007], though Lurie [2009a, A.3.2.4, A.3.2.25] observed that the Bergner model structure is left proper and combinatorial.

Definition 3.1 (Bergner model structure) The category sCat of simplicially enriched categories admits a proper, combinatorial model structure in which:

- A map $f: \mathcal{C} \rightarrow \mathcal{D}$ is a *weak equivalence* just when
 - (W1) for each pair of objects x and y , the map $\mathcal{C}(x, y) \rightarrow \mathcal{D}(fx, fy)$ is a weak homotopy equivalence of simplicial sets, and
 - (W2) the functor $\pi_0 f: \pi_0 \mathcal{C} \rightarrow \pi_0 \mathcal{D}$ is essentially surjective.
- A map $f: \mathcal{C} \rightarrow \mathcal{D}$ is a *fibration* just when
 - (F1) for each pair of objects x and y , the map $\mathcal{C}(x, y) \rightarrow \mathcal{D}(fx, fy)$ is a Kan fibration, and
 - (F2) the functor $\pi_0 f: \pi_0 \mathcal{C} \rightarrow \pi_0 \mathcal{D}$ is an isofibration.

If \mathcal{C} is a simplicial category, then $\pi_0 \mathcal{C}$ is the ordinary category obtained by taking path components of each hom-simplicial set. We call simplicial functors satisfying (W1) *fully faithful* (meaning of course in the homotopical sense), and functors satisfying (W2) *essentially surjective*.

The constant diagram functor $\text{Cat} \rightarrow \text{Cat}^{\Delta^{\text{op}}}$, given by precomposition with $\Delta^{\text{op}} \rightarrow \mathbb{1}$, factors through the full subcategory inclusion $\text{sCat} \hookrightarrow \text{Cat}^{\Delta^{\text{op}}}$. Write $\text{disc}: \text{Cat} \rightarrow \text{sCat}$ for the induced full inclusion, which identifies categories as those simplicial categories with discrete hom-simplicial sets.

Lemma 3.2 The functor $\text{disc}: \text{Cat} \rightarrow \text{sCat}$ preserves limits and colimits.

Proof The precomposition functor $\text{Cat} \rightarrow \text{Cat}^{\Delta^{\text{op}}}$ preserves all limits and colimits, while the full inclusion $\text{sCat} \rightarrow \text{Cat}^{\Delta^{\text{op}}}$ reflects them. The conclusion follows. □

Our key technical result is the following proposition, which observes that when I is a Dwyer map, the functor $\text{disc}(I)$ of simplicial categories is a *flat map* in the terminology of [Hill et al. 2016, B.9] or an *h-cofibration* in the terminology of [Batanin and Berger 2017, 1.1] relative to the Bergner model structure.

Proposition 3.3 If $I: \mathcal{A} \hookrightarrow \mathcal{B}$ is a Dwyer map, then

$$\text{disc } \mathcal{B} \amalg_{\text{disc } \mathcal{A}} (-): \text{disc } \mathcal{A} / \text{sCat} \rightarrow \text{disc } \mathcal{B} / \text{sCat}$$

preserves weak equivalences.

Proof Consider a composable pair of simplicial functors $\text{disc } \mathcal{A} \xrightarrow{F} \mathcal{C}' \xrightarrow{M} \mathcal{C}$ and form the following pushouts:

$$\begin{array}{ccccc} \text{disc } \mathcal{A} & \xrightarrow{F} & \mathcal{C}' & \xrightarrow{M} & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ \text{disc } \mathcal{B} & \longrightarrow & \mathcal{D}' & \xrightarrow{H} & \mathcal{D} \end{array}$$

When M is a weak equivalence in the Bergner model structure, we wish to show that the induced map $H: \mathcal{D}' \rightarrow \mathcal{D}$ between the pushouts is as well.

As in Proposition 2.2, we identify $\text{ob } \mathcal{D}'$ with $\text{ob } \mathcal{C}' \amalg \text{ob } \mathcal{V}$ and similarly $\text{ob } \mathcal{D} = \text{ob } \mathcal{C} \amalg \text{ob } \mathcal{V}$. We regard the simplicial categories \mathcal{D}' and \mathcal{D} as simplicial objects \mathcal{D}'_{\bullet} and \mathcal{D}_{\bullet} via the inclusion $\text{sCat} \hookrightarrow \text{Cat}^{\Delta^{\text{op}}}$. For each n , we have $\mathcal{D}_n = \mathcal{B} \amalg_{\mathcal{A}} \mathcal{C}_n$ and similarly for \mathcal{D}'_n . We have already computed the hom sets of these categories in Proposition 2.2, and the descriptions there are functorial in the \mathcal{C} -variable. Thus,

$$\begin{array}{ccc} \mathcal{C}'(c, c') \xrightarrow{\cong} \mathcal{C}(Mc, Mc') & \mathcal{C}'(c, FRu) \xrightarrow{\cong} \mathcal{C}(Mc, MFRu) \\ \downarrow \cong & \downarrow \cong \\ \mathcal{D}'(c, c') \longrightarrow \mathcal{D}(Hc, Hc') & \mathcal{D}'(c, u) \longrightarrow \mathcal{D}(Hc, Hu) \end{array}$$

for $c, c' \in \mathcal{C}'$ and $u \in \mathcal{U}$. Meanwhile, for $v, v' \in \mathcal{V}$ we have that both $\mathcal{D}(v, v')$ and $\mathcal{D}'(v, v')$ are isomorphic to the discrete simplicial set $\mathcal{V}(v, v')$. Finally, the hom-simplicial sets $\mathcal{D}'(c, v')$, $\mathcal{D}(Hc, v')$, $\mathcal{D}(v, c)$ and $\mathcal{D}(v, Hc)$ are all empty for $c \in \mathcal{C}'$, $v \in \mathcal{V}$ and $v' \in \mathcal{V} \setminus \mathcal{U}$. Thus H is fully faithful.

For essential surjectivity of H , notice that we have a commutative square of functors

$$\begin{array}{ccc} \mathcal{C}' \amalg \text{disc } \mathcal{V} & \xrightarrow{M \amalg \text{id}} & \mathcal{C} \amalg \text{disc } \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{D}' & \xrightarrow{H} & \mathcal{D} \end{array}$$

where the vertical maps are bijective on objects and the top map is essentially surjective. It follows that H is essentially surjective as well. □

Our main results hold not just for Dwyer maps but for arbitrary functors between 1-categories that satisfy the property established in Proposition 3.3 plus some injectivity conditions. The following terminology highlights the required properties.

Definition 3.4 A functor $I: \mathcal{A} \rightarrow \mathcal{B}$ between 1-categories is *discretely flat* if the simplicial functor $\text{disc}(I)$ is flat, ie if

$$\text{disc } \mathcal{B} \amalg_{\text{disc } \mathcal{A}} (-): \text{disc } \mathcal{A} / \text{sCat} \rightarrow \text{disc } \mathcal{B} / \text{sCat}$$

preserves Bergner weak equivalences. If, in addition, I is injective on objects, we call it a *discretely flat cofibration*, and if it is both injective on objects and faithful, we call it a *discretely flat inclusion*.

Dwyer maps are discretely flat inclusions, but such functors aren't the only examples.

- Example 3.5** (i) Since the passage to opposite categories commutes with all of the structures involved in Definition 3.4, *co-Dwyer maps*, whose opposites are Dwyer maps, are also discretely flat inclusions.
- (ii) As flat maps are closed under retracts — see [Hill et al. 2016, B.11] or [Batatin and Berger 2017, Lemma 1.3] — Cisinski’s [1999] *pseudo-Dwyer maps* are also discretely flat inclusions.
- (iii) The inclusion of $\mathbb{1} \amalg \mathbb{1}$ into the cospan category $(\mathbb{1} \amalg \mathbb{1})^\triangleright$ is a discretely flat inclusion. Indeed, the hom-simplicial sets of $(\mathbb{1} \amalg \mathbb{1})^\triangleright \amalg_{(\mathbb{1} \amalg \mathbb{1})} \mathcal{C}$ are readily computed by hand in terms of those for \mathcal{C} , and a variation of the proof of Proposition 3.3 gives the result.

Note not all functors of the form $\mathcal{A} \rightarrow \mathcal{A}^\triangleright$ are discrete flat inclusions. In light of Theorem 1.6, the left-hand map of (1.1) gives a counterexample. An interesting problem is to characterize the class of discretely flat inclusions.

The fact that Dwyer pushouts are homotopy pushouts now follows from a general fact: in a left proper model category, a pushout where one leg is a flat map is automatically a homotopy pushout; see [Batatin and Berger 2017, Section 1.5].

Proposition 3.6 *Suppose $I : \mathcal{A} \rightarrow \mathcal{B}$ is discretely flat. Then for any functor $F : \text{disc } \mathcal{A} \rightarrow \mathcal{C}$ of simplicial categories, the pushout $\text{disc } \mathcal{B} \amalg_{\text{disc } \mathcal{A}} \mathcal{C}$ is a homotopy pushout.*

Proof To form the homotopy pushout of a span in a model category, one replaces it by a cofibrant span as below and then takes the ordinary pushout [Dwyer and Spaliński 1995, 10.4]:

$$\begin{array}{ccccc}
 & & \emptyset & & \\
 & & \downarrow & & \\
 \mathcal{Y} & \longleftarrow & \mathcal{X} & \longrightarrow & \mathcal{Z} \\
 \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
 \text{disc } \mathcal{B} & \xleftarrow{\text{disc}(I)} & \text{disc } \mathcal{A} & \longrightarrow & \mathcal{C}
 \end{array}$$

Thus, we must show that the induced map

$$\mathcal{Y} \amalg_{\mathcal{X}} \mathcal{Z} \rightarrow \text{disc } \mathcal{B} \amalg_{\text{disc } \mathcal{A}} \mathcal{C}$$

is a weak equivalence of simplicial categories. Since $\text{disc}(I)$ is flat by assumption and sCat is left proper, the above map is a weak equivalence by [Hill et al. 2016, B.12]. □

Our model-independent statement, that Dwyer pushouts are $(\infty, 1)$ -categorical, holds generally for discretely flat cofibrations.

Theorem 1.6 *The inclusion $\text{Cat}_1 \hookrightarrow \text{Cat}_{(\infty, 1)}$ of the $(\infty, 1)$ -category of 1-categories into the $(\infty, 1)$ -category of $(\infty, 1)$ -categories preserves (homotopy) pushouts along discretely flat cofibrations.*

Proof The inclusion $\text{Cat}_1 \hookrightarrow \text{Cat}_{(\infty, 1)}$ can be modeled at the point-set level by the right Quillen functor $\text{disc} : \text{Cat} \rightarrow \text{sCat}$. By hypothesis, a discretely flat cofibration $I : \mathcal{A} \rightarrow \mathcal{B}$ is injective on objects, hence

a cofibration in the canonical model structure on Cat . This model structure is left proper, so ordinary pushouts along such maps are homotopy pushouts by [Hirschhorn 2003, 13.5.4]. Since the functor $\text{disc}: \text{Cat} \rightarrow \text{sCat}$ preserves strict pushouts, Proposition 3.6 shows that disc preserves homotopy pushouts along discretely flat cofibrations, so the conclusion follows. \square

4 Pushouts in simplicial sets

In this section, we describe the implications of Theorem 1.6 for the Joyal model structure on simplicial sets, proving the results needed in [Hackney et al. 2023]. We utilize the commutative triangle of right Quillen functors

$$\begin{array}{ccc}
 & \text{Cat} & \\
 N \swarrow & & \searrow \text{disc} \\
 \text{sSet} & \xleftarrow[\cong]{\mathfrak{N}} & \text{sCat}
 \end{array}$$

where \mathfrak{N} is the homotopy coherent nerve, a right Quillen equivalence between the Bergner and Joyal model structures. This diagram commutes up to natural isomorphism since, for any 1–category \mathcal{A} ,

$$(\mathfrak{N} \text{disc } \mathcal{A})_n := \text{hom}(\mathbb{C}[n], \text{disc } \mathcal{A}) \cong \text{hom}([n], \mathcal{A}) =: (N \mathcal{A})_n,$$

which holds because the hom simplicial sets of $\text{disc } \mathcal{A}$ are discrete.

We first explain how to deduce Theorem 1.5 from Proposition 3.6. In fact, we use the terminology of Definition 3.4 to prove a more general version:

Theorem 1.5 *Let*

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\
 I \downarrow & \lrcorner & \downarrow J \\
 \mathcal{B} & \xrightarrow{G} & \mathcal{D}
 \end{array}$$

be a pushout of categories, and assume I is a discretely flat inclusion. Then the induced map of simplicial sets

$$N \mathcal{B} \amalg_{N \mathcal{A}} N \mathcal{C} \rightarrow N \mathcal{D}$$

is a weak categorical equivalence.

Proof We organize the proof into the following commutative square of simplicial sets:

$$\begin{array}{ccc}
 \mathfrak{N} \text{disc } \mathcal{B} \amalg_{\mathfrak{N} \text{disc } \mathcal{A}} \mathfrak{N} \text{disc } \mathcal{C} & \xrightarrow{\cong} & N \mathcal{B} \amalg_{N \mathcal{A}} N \mathcal{C} \\
 \downarrow & & \downarrow \\
 \mathfrak{N}(\text{disc } \mathcal{B} \amalg_{\text{disc } \mathcal{A}} \text{disc } \mathcal{C}) & & \\
 \downarrow \cong & & \\
 \mathfrak{N} \text{disc}(\mathcal{B} \amalg_{\mathcal{A}} \mathcal{C}) & \xrightarrow{\cong} & N(\mathcal{B} \amalg_{\mathcal{A}} \mathcal{C})
 \end{array}$$

The top and bottom isomorphisms are instances of the natural isomorphism $N \cong \mathfrak{N} \text{ disc}$. The vertical maps are the canonical comparison maps induced by the universal property of the pushouts. The bottom left map is an isomorphism since disc preserves pushouts (Lemma 3.2). It remains to show that the upper left map

$$\mathfrak{N} \text{ disc } \mathcal{B} \amalg_{\mathfrak{N} \text{ disc } \mathcal{A}} \mathfrak{N} \text{ disc } \mathcal{C} \rightarrow \mathfrak{N}(\text{disc } \mathcal{B} \amalg_{\text{disc } \mathcal{A}} \text{disc } \mathcal{C})$$

is a weak categorical equivalence, at which point the right map will be a weak categorical equivalence by two-of-three.

Notice that the objects in the top row are homotopy pushouts, since $NI: N\mathcal{A} \rightarrow N\mathcal{B}$ is a cofibration in sSet by the hypothesis that I is faithful and injective on objects. Let $R\mathfrak{N}$ be the right derived functor of \mathfrak{N} . Since discrete simplicial categories are fibrant in the Bergner model structure, the map of simplicial sets above represents the canonical map

$$(R\mathfrak{N}) \text{ disc } \mathcal{B} \amalg_{(R\mathfrak{N}) \text{ disc } \mathcal{A}}^h (R\mathfrak{N}) \text{ disc } \mathcal{C} \rightarrow (R\mathfrak{N})(\text{disc } \mathcal{B} \amalg_{\text{disc } \mathcal{A}}^h \text{disc } \mathcal{C}).$$

This is an equivalence since $R\mathfrak{N}$ is an equivalence of $(\infty, 1)$ -categories, hence preserves homotopy pushouts. We conclude that $N\mathcal{B} \amalg_{N\mathcal{A}} N\mathcal{C} \rightarrow N(\mathcal{B} \amalg_{\mathcal{A}} \mathcal{C})$ is a weak categorical equivalence. \square

In the case where $F: \mathcal{A} \rightarrow \mathcal{C}$ is also injective on objects and faithful, as occurs frequently in [Hackney et al. 2023], we are able to strengthen our conclusion and prove that the canonical comparison map is inner anodyne.

Corollary 4.1 *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ I \downarrow & \lrcorner & \downarrow J \\ \mathcal{B} & \xrightarrow{G} & \mathcal{D} \end{array}$$

be a pushout of categories, in which I is a discretely flat inclusion and F is faithful and injective on objects. Then the induced inclusion of simplicial sets

$$N\mathcal{B} \amalg_{N\mathcal{A}} N\mathcal{C} \hookrightarrow N\mathcal{D}$$

is inner anodyne.

Proof Observe in this case that the canonical map $j: N\mathcal{B} \amalg_{N\mathcal{A}} N\mathcal{C} \hookrightarrow N\mathcal{D}$ is an inclusion and thus, by Theorem 1.5, an acyclic cofibration in the Joyal model structure. This acyclic cofibration is also bijective on 0-simplices and has codomain a quasicategory. By [Stevenson 2018a, 2.19] or [Stevenson 2018b, 5.7] it follows that j is inner anodyne.⁴ \square

⁴See [Campbell 2020] for related discussion and an example of an acyclic cofibration that is bijective on 0-simplices but whose codomain is not a quasicategory that is not inner anodyne.

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