HOPKINS MAYDAY TALK

Note Title 10/16/2009

[~ 100 people present]

Joint with Hill + me

Thm (HHR) If M is a smooth stably framed manifold with Kervaire invariant one, then $\dim M = 2, 6, 14, 30, 62$ or 126.

History of problem

1930s. Homology + degree had been defined.

Pontryagin: Study $S^{n+k} \xrightarrow{f} S^n$ in terms of $f^{-1}(x_0)$ for a regular value x_0

It is a smooth mfd M_0 of dim k with a framing of its normal bundle in S^{n+k}

=: framed manifold

Choosing a different regular x_1 leads to a regular mfd M_1 which is framed cobordant to M_0.

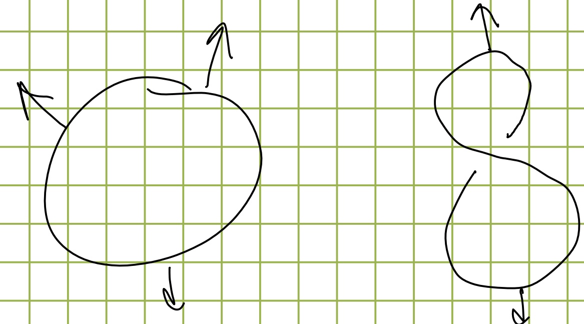
There is a bijection

$$\pi_{n+k} S^n \longleftrightarrow \text{cobordism gp of framed } k\text{-mfds in } \mathbb{R}^{n+k}$$

He computed this gp for small k.

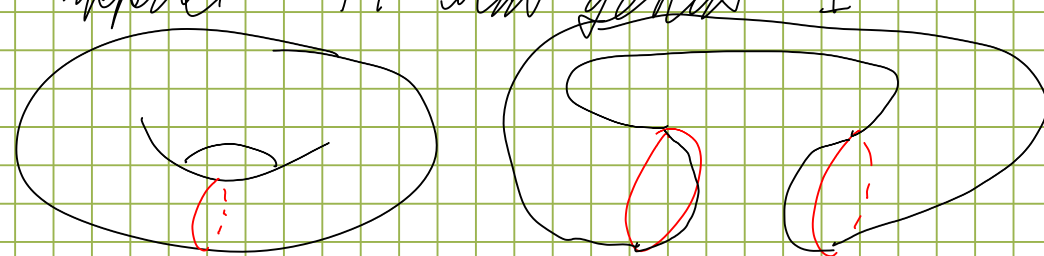
For $k=0$ we get $\pi_n S^n = \mathbb{Z}$

For $k=1$ S^1 has two framings



$$\pi_{n+1} S^n = \mathbb{Z}/2$$

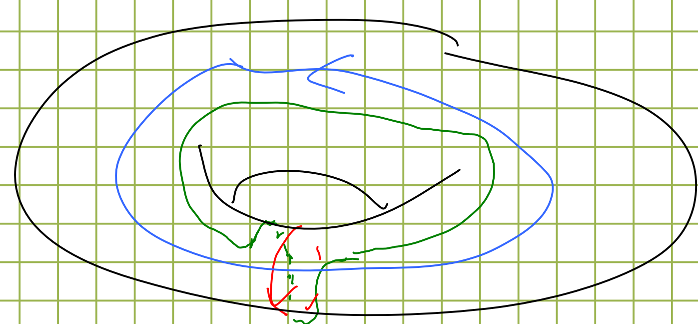
For $k=2$ S^2 has a framing that extends to a disk. Any other framing differs by an element in $\pi_2 SO(n) = 0$.
Hence it represents the trivial elt in $\pi_{n+2} S^n$.
Suppose M has genus 1



Cut along red arc and attach disks
The obstruction is an element

in $\pi_1(SO(n)) = \mathbb{Z}/2$. Thus we get a map $H_1(M;\mathbb{Z}/2) \xrightarrow{\varphi} \mathbb{Z}/2$

If φ is linear then $\ker \varphi$ has nontrivial element and $\pi_{n+2}(S^n) = 0$



Red + blue arcs have trivial framing but the green arc (their sum) does not.

Hence φ is nonlinear

This leads to the definition of $\mathrm{Arf}(\varphi) \in \mathbb{Z}_2$

<u>Question</u>. In which dimensions is every framed mfd cobordant to a topological sphere? Answer is given by Theorem.

Pontryagin's method would work in all dimensions except 2, 6, 14, 30, 62 and 126.

Milnor 1956 Constructs an M homeomorphic but not diffeomorphic to S^7.

Kervaire – Milnor ('58, '63) Θ_k = gp of M^k homeo to S^k under connected sum. (up to h-cobordism). The computed it in terms of $\pi_{n+k}(S^n)$ up to a factor of 2. Our theorem settles the factor 2 (except in dim 126).

Kervaire 1960

(n odd)

Let M^{2n} be a framed $(n-1)$-connected $(2n)$-mfd. He constructs $\varphi : H_n(M;\mathbb{Z}_2) \to \mathbb{Z}/2$

$\Phi(M) := \mathrm{Arf}(\varphi) =$ Kervaire invariant

He constructed a PL mfd M in dim 10 with $\Phi(M) = 1$. Showed any smooth ^{10}-mfd has $\Phi(M) = 0$.

Hence this M is nonsmoothable.

Question: In which dims can $\Phi(M)$ be nonzero? Answered by Theorem.

Browder 1969 If $\Phi(M) = 1$ then $\dim M = 2^{j+1}-2$.

In this dim such an M exists $\iff$

$\exists\, \theta_j \in \pi_{2^{j+1}-2}(S^0)$ representing h_j^2 in

classical Adams SS.

This problem remained open for 40 years.

Thm (Barratt-Jones-Mahowald) $\exists\, \theta_j$ for

$j \leq 5$. [$j=6$ is still open]

Is there a connection to the six

exceptional Lie groups???

EHP sequence (James, ~1958)

(localize at 2)

$$\cdots \to \pi_m(S^n) \xrightarrow{E} \pi_{m+1}(S^{n+1}) \xrightarrow{H} \pi_{m+1}(S^{2n+1}) \xrightarrow{P} \pi_{m-1}(S^n) \to \cdots$$

It leads to an inductive process.

For $m = 2n$ we have

$$\begin{array}{ccc} \pi_{2n+1}(S^{2n+1}) & \longrightarrow & \pi_{2n-1}(S^n) \\ \iota & \longmapsto & \underbrace{[\iota_n, \iota_n]}_{w_n} = \text{Whitehead product} \end{array}$$

Questions

Q1: How far does w_n desuspend?

Q2: is it divisible by 2?

Q1 is equivalent to vector field problem: ω_n desuspends k times $\Longleftrightarrow$ S^n has k vector fields

Q2 n even: Hopf invariant one problem

n odd: Kervaire invariant problem

Sketch of proof

Construct a spectrum $\tilde{\Omega}$ with C_8-action. $\tilde{\Omega} = D^{-1} MU_{\mathbb{R}}^{(4)}$

$$\Omega = \tilde{\Omega}^{C_8} = \tilde{\Omega}^{hC_8} \quad \left(\begin{array}{l}\text{this equality} \\ \text{is easy}\end{array}\right)$$

Ω has 3 properties

<u>Detection Thm</u>: If $\exists\, \theta_j$, its image in $\pi_* \Omega$ is nonzero.

<u>Periodicity Thm</u>: $\pi_k \Omega \cong \pi_{k+256} \Omega$ $\left(\tilde{\Omega}^{hC_8}\right)$

<u>Gap Theorem</u>: $\pi_k \Omega = 0$ for $-4 < k < 0$. $\left(\tilde{\Omega}^{C_8}\right)$

PT has to do with equivariant methods

The use of equivariant theory is essential

It was developed by May.