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# Constructions of elements in Picard groups

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ABSTRACT. We discuss the first author's Picard groups of stable homotopy. We give a detailed description of the calculation of $\mathrm{Pic}_1$, and go on to describe geometric constructions for lifts of the elements of $\mathrm{Pic}_1$. We also construct a 15 cell complex that localizes to what we speculate is an interesting element of $\mathrm{Pic}_2$. For all $n$ we describe an algebraic approximation to $\mathrm{Pic}_n$ using the Adams-Novikov spectral sequence. We also show that the $p$-adic integers embed in the group $\mathrm{Pic}_n$ for all $n$ and $p$.

## 1. Introduction and statement of results

We begin with the basic definition. The functor

$$\Sigma^n : X \mapsto S^n \wedge X$$

is an automorphism of the category of spectra, which preserves cofibration sequences and infinite wedges. If $T$ is another such automorphism, then Brown's representability theorem applied to $\pi_*(TX)$ gives a spectrum $S_T$ with

$$TX = S_T \wedge X$$

and

$$S_{T^{-1}} \wedge S_T = S^0.$$

This motivates the following definition.

DEFINITION 1.1. *A spectrum $Z$ is invertible if and only if there is some spectrum $W$ such that*

$$Z \wedge W = S^0.$$

*Pic is the group of isomorphism classes of invertible spectra, with multiplication given by smash product. Given an isomorphism class $\lambda \in Pic$ we will write $S^\lambda$ for a representative spectrum.*

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Pic itself is not very interesting; it is just the integers and the elements are just spheres. To see this observe that if $Z \in \mathrm{Pic}$ then $H_*(Z) = \mathbf{Z}$ in some dimension, say $k$, and 0 in other dimensions. One can then map $Z$ to $S^k$ by working up the Postnikov tower so that the map induces an isomorphism on homology. By smashing the analogous map for $Z^{-1}$ with $Z$ one gets a map from $S^k$ to $Z$ that is an isomorphism on homology. It follows that $Z = S^k \vee A$ for some $A$. Since the same is true for $Z^{-1}$, it follows that $A = *$.

We can alter the definition to apply to a category of local spectra, but variants of this argument tell us that $\mathrm{Pic} = \mathbf{Z}$ even if we restrict to the $p$-local (or $p$-complete) categories. The situation is more interesting if we look at the category of $K(n)$-local spectra where $K(n)$ is the $n^{\text{th}}$ Morava $K$-theory. We have the following definition.

DEFINITION 1.2. *A $K(n)$ local spectrum $Z$ is invertible in the $K(n)$-local category if and only if there is a spectrum $W$ such that*

$$L_{K(n)}(Z \wedge W) = L_{K(n)}S^0.$$

*$Pic_n$ is the group of isomorphism classes of such spectra, with multiplication given by*

$$(X, Y) \mapsto L_{K(n)}(X \wedge Y).$$

We postpone showing that $\mathrm{Pic}_n$ is a set to section 7.

The following result is useful in identifying elements of $\mathrm{Pic}_n$. We describe the functor $\mathcal{K}_{n,*}(-)$ in section 7. It is essentially a completion of $E(n)_*(-)$.

THEOREM 1.3. *The following are equivalent:*

(i) $L_{K(n)}Z$ *is in* $Pic_n$.

(ii) $dim_{K(n)_*}K(n)_*Z = 1$.

(iii) $\mathcal{K}_{n,*}(Z) \cong \mathcal{K}_{n,*}(S^k)$ *for some* $k$.

PROOF. We reserve the proof of (ii) $\Leftrightarrow$ (iii) for section 7. (i) implies (ii) by the Künneth theorem for Morava $K$-theories.

To show (ii) implies (i), assume $Z$ satisfies (ii). Let $Y = F(Z, L_{K(n)}S^0)$ where $F(A, B)$ is defined to be the representing spectrum for the cohomology theory

$$X \mapsto [X \wedge A, B].$$

There is an "evaluation" map

$$Z \wedge Y \to L_{K(n)}S^0$$

that is the adjoint of the identity in

$$[F(Z, L_{K(n)}S^0), F(Z, L_{K(n)}S^0)] = [Y, F(Z, L_{K(n)}S^0)].$$

We claim that this map is an isomorphism on $K(n)_*$ and hence an equivalence after localizing.



To see this, consider all spectra $X$ such that the evaluation map

$$Z \wedge F(Z, L_{K(n)}X) \to L_{K(n)}X \tag{1}$$

is an isomorphism on $K(n)$. It is easy to see that $X = K(n)$ works. Furthermore, the class of $X$ that makes (1) an isomorphism is closed under cofibrations and wedges.

Hopkins and Ravenel show that if $X$ is a finite type $n$ spectrum, then $L_{K(n)}X$ has a finite filtration where each cofiber is a wedge of $K(n)$'s. This result is unpublished, but can be deduced from [20], Corollary 8.2.7 and Section 8.3. It follows that $X$ finite type $n$ makes (1) an isomorphism on $K(n)$. But if $X$ is finite,

$$Z \wedge F(Z, L_{K(n)}X) \to L_{K(n)}X$$

is the same as

$$Z \wedge F(Z, L_{K(n)}S^0) \wedge X \to L_{K(n)}S^0 \wedge X,$$

and that map is an isomorphism on $K(n)$ if and only if

$$Z \wedge F(Z, L_{K(n)}S^0) \to L_{K(n)}S^0$$

is. □

In the next several sections, we want to give some particular examples from the zoo, $\mathrm{Pic}_1$. We will find spectra which are quite familiar and which represent various classes in $\mathrm{Pic}_1$, and others which are constructed in a simple way from quite natural objects, but have surprising properties.

Before we continue, we would like to illustrate how some of these examples come about. It is a consequence of Thompson's work [23] that the EHP sequence at odd primes is just a Bockstein spectral sequence when one uses $v_1$ periodic homotopy. To be precise, let $M^0(p^j)$ be the $\mathbf{Z}/(p^j)$ Moore spectrum with top cell in dimension 0, and let $J$ be the image of $J$ spectrum at $p$. An interpretation of what is shown in [23] is that there is a map

$$\Omega^{2n+1}S^{2n+1} \to \Omega^\infty(J \wedge M^0(p^n))$$

which is an equivalence in $v_1$ periodic homotopy. Thus the EHP spectral sequence, using $v_1$ periodic homotopy theory is just the spectral sequence associated with the system of spectra

$$M^0(p) \to M^0(p^2) \to \cdots \to M^0(p^n) \to \cdots. \tag{2}$$

The cofibration sequences at the stages of this direct system are

$$M^0(p^n) \to M^0(p^{n+1}) \to M^0(p).$$

The direct limit of this system is $M^0(p^\infty)$. On $K(1)_*$, each map in the system is an isomorphism on the copy of $K(1)_*$ associated to the cell in dimension 0, and

0 on the other copy of $K(1)_*$ and hence $K(1)_*$ of the limit is $K(1)_*(S^0)$. In fact consideration of the cofiber sequence

$$S^{-1} \to L_{\mathbf{Q}}S^{-1} \to M^0(p^\infty)$$

shows that the direct limit is homotopy equivalent to $S^0$ after applying $L_{K(1)}$.

Now if $\lambda$ is a p-adic integer, we can write $\lambda$ as (the limit of) a sequence $a_i = a_{i-1} + \lambda_i p^i$ where $\lambda_i$ is an integer such that $0 \le \lambda_i < p$, and $a_{-1} = 0$. Using $\lambda$ as a template we can describe another system of spectra

$$M^0(p) \xrightarrow{v_1^{\lambda_0}} M^{-qa_0}(p) \to M^{-qa_0}(p^2) \xrightarrow{v_1^{p\lambda_1}} M^{-qa_1}(p^2) \to \cdots. \tag{3}$$

Here $q = 2p-2 = |v_1|$, and $v_1^{p^{i-1}} : \Sigma^{qp^{i-1}} M(p^i) \to M(p^i)$ is an Adams map inducing a $K$-theory isomorphism and multiplication by $v_1^{p^{i-1}}$ on $BP_*$. In this direct system, the $n^{\text{th}}$ stage is $M^{-qa_{n-1}}(p^n)$, and the map from the $n^{\text{th}}$ stage to the $(n+1)^{\text{th}}$ stage is the composite

$$M^{-qa_{n-1}}(p^n) \xrightarrow{v_1^{\lambda_n p^n}} M^{-qa_n}(p^n) \to M^{-qa_n}(p^{n+1}).$$

If $\lambda$ happens to be an ordinary integer, then as in (2) the direct limit of this system is $M^{-\lambda q}(p^\infty)$ which is the same as $S^{-\lambda q}$ after applying $L_{K(1)}$.

It is clear that in $K(1)_*$, the $v_1^k$ maps are isomorphisms, so that $K(1)_*$ of the direct limit is the same as before. Thus the limit has the $K(1)$-theory of $S^0$. This means that for each $\lambda$ we have constructed an element in $\mathrm{Pic}_1$. The homotopy groups will depend on the p-adic integer $\lambda$. One of the main results of this note will be that at odd primes, essentially nothing else happens. We discuss this in detail, from a slightly different perspective, in section 2. At the prime two, we get similar examples and some more.

The rest of the paper is organized as follows. Section 2 calculates $\mathrm{Pic}_1$ at odd primes, and shows that the examples illustrated above contain everything. Section 3 calculates $\mathrm{Pic}_1$ at $p=2$. Section 4 constructs elements of $\mathrm{Pic}_1$ at $p=2$ related to $\mathbf{R}P^\infty$. Section 5 constructs some bizarre elements of $\mathrm{Pic}_1$ at $p=2$. Section 6 contains an interesting example in $\mathrm{Pic}_2$ at $p=2$. Sections 7 and 8 discuss an algebraic approximation for $\mathrm{Pic}_n$ based on the Adams-Novikov spectral sequence, and section 9 generalizes the construction above to prove $\mathbf{Z}_p$ embeds in $\mathrm{Pic}_n$.

This paper was motivated by attempts on the part of the other authors to better understand a talk given by the first author at the Adams symposium in Manchester. The basic definitions and many of the results are due to the first author.

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## 2. Calculation of $\mathrm{Pic}_1$ at $p > 2$

Let $\gamma$ be a topological generator of $(\mathbf{Z}_p)^\times$. For example we can take $\gamma$ to be $(1+p)\xi$ where $\xi$ is a primitive $(p-1)^{\text{st}}$ root of unity in $\mathbf{Z}_p$. $K$ will denote complex $K$-theory completed at $p$, and $K_p$ the Adams summand. We will be considering the Adams operation $\psi^\gamma$. We will need some notation from section 7; we define $\mathcal{K}_{1,*}(X)$ by

$$\mathcal{K}_{1,*}(X) = \varprojlim_n K_*(X \wedge M(p^n)).$$

It is easy to check that $\mathcal{K}_{1,*}(X) = K_*(X)$ if $K_*(X)$ is finitely generated over $K_*$.

We introduce $\mathcal{K}_{1,*}(-)$ because we need an integral invariant so we can conveniently work with Adams operations. Also, because of Theorem 1.3, $\mathcal{K}_{1,*}(-)$ is better adapted for studying $\mathrm{Pic}_1$ than $K_*$ is. In fact, the proof of (ii) $\Leftrightarrow$ (iii) of Theorem 1.3 shows that $\mathcal{K}_{1,*}(X) = 0 \Leftrightarrow K(1)_*(X) = 0$, and hence that $\mathcal{K}_{1,*}(X) = \mathcal{K}_{1,*}(L_{K(n)}X)$. Neither of these facts are true for $K_*(-)$ in place of $\mathcal{K}_{1,*}(-)$ as the example $X = M(p^\infty)$ shows.

The theory $\mathcal{K}_{1,*}(-)$ does have the drawback of not being a homology theory in the usual sense; because of the inverse limit in its definition, it fails to commute with direct limits.

Proposition 2.1. *There is an extension of abelian groups*

$$0 \to M \to Pic_1 \to \mathbf{Z}/(2) \to 0.$$

*The map $Pic_1 \to \mathbf{Z}/(2)$ takes $X$ to the dimension (modulo 2) of the generator of $\mathcal{K}_{1,*}(X)$. $M$ is the kernel of this map, and $ev : M \to (\mathbf{Z}_p)^\times$ is an isomorphism, where $ev(X)$ is the eigenvalue of $\psi^\gamma$ on $\mathcal{K}_{1,0}(X)$.*

Note by Theorem 1.3, $X \in \mathrm{Pic}_1 \Leftrightarrow \mathcal{K}_{1,*}(X) = \mathcal{K}_{1,*}(S^k)$ for some $k$. If $X \in M$, we can take $k=0$. We will prove the statement about $M$ in stages.

Before we embark on the proof, we have to note some number theoretic properties of $\psi^\gamma$. Since it is an Adams operation,

$$\pi_{2n}(\psi^\gamma) = \cdot\gamma^n \tag{4}$$

on $K^{\mathbf{C}}$. The following facts follow since $\gamma$ is a topological generator of $(\mathbf{Z}_p)^\times$.

$$\begin{aligned} \gamma^k &\equiv 1 \pmod p \quad \text{iff} \quad (p-1)|k \\ \gamma^{s(p-1)p^r} &\equiv 1 \pmod{p^{r+1}} \\ \gamma^{s(p-1)p^r} &\not\equiv 1 \pmod{p^{r+2}} \quad \text{if} \quad p \nmid s \end{aligned} \tag{5}$$

Lemma 2.2. *The map ev is a homomorphism.*

Proof. Suppose $X, Y \in M$. Then $\mathcal{K}_{1,*}(X) = \mathcal{K}_{1,*}(Y) = \mathcal{K}_{1,*}(S^0)$, so by the Künneth theorem,

$$\mathcal{K}_{1,*}(X \wedge Y) = \mathcal{K}_{1,*}(X) \otimes_{K_*} \mathcal{K}_{1,*}(Y)$$

as a module over the Adams operations, since the Adams operations are multiplicative. □

LEMMA 2.3.

$$L_{K(1)}X = fiber[(K \wedge X)_p^{\hat{}} \xrightarrow{[\psi^{\gamma}-1]\wedge 1} (K \wedge X)_p^{\hat{}}].$$

PROOF. This follows from Theorem 4.3 of [5] and the fact that $L_{K(1)}X = (L_K X)_p^{\hat{}}$ (Proposition 2.11 of [5]). Taken together these imply that if

$$J = \text{fiber}[K \xrightarrow{\psi^{\gamma}-1} K]$$

then $L_{K(1)}X = (X \wedge J)_p^{\hat{}}$. (Because of (5), it doesn't matter that we are using $K$ where Bousfield uses $KO_{(p)}$.) □

LEMMA 2.4. *ev is injective.*

PROOF. Suppose $ev(X) = 1$. We get a diagram

$$\begin{array}{ccccc} X & \longrightarrow & (K \wedge X)_p^{\hat{}} & \xrightarrow{[\psi^{\gamma}-1]\wedge 1} & (K \wedge X)_p^{\hat{}} \\ & & {\scriptstyle g}\uparrow & & \\ & & S^0 & & \end{array}$$

where $g$ is a generator of $\pi_0(K \wedge X)$. Since $([\psi^{\gamma} - 1] \wedge 1)g = *$, $g$ lifts to $X$, and since $X$ is $K(1)$ local, the lift extends to a map

$$f : L_{K(1)}S^0 \to X.$$

Now $K(1)_*(g)$ is monic, so it follows that $K(1)_*(f)$ is monic. Therefore, $K(1)_*(f)$ is an isomorphism because the source and target are both isomorphic to $K(1)_*$, so since the source and target are $K(1)$-local, $f$ is a homotopy equivalence. □

LEMMA 2.5. *Let $X_{\gamma^n}$ be the fiber of*

$$K \xrightarrow{\psi^{\gamma}-\gamma^n} K.$$

*Then $X_{\gamma^n} = L_{K(1)}S^{2n}$.*

PROOF. Let $f$ below be an inclusion of the Adams summand in that dimension.

$$\begin{array}{ccccc} X_{\gamma^n} & \longrightarrow & K & \longrightarrow & K \\ & & {\scriptstyle f}\uparrow & & \\ L_{K(1)}S^{2n} & \longrightarrow & \Sigma^{2n}K_p & \xrightarrow{\Sigma^{2n}[\psi^{(1+p)}-1]} & \Sigma^{2n}K_p \end{array}$$

Let $i : S^0 \to K_p$ be the unit. By (4), $(\psi^{\gamma} - \gamma^n)fi = *$, so $fi : S^{2n} \to K$ factors through $X_{\gamma^n}$. But $X_{\gamma^n}$ is $K(1)$ local, so the map from $L_{K(1)}S^{2n}$ lifts to $X_{\gamma^n}$.

Using the numbers from (5) it is easy to check that this map $L_{K(1)}S^{2n} \to X_{\gamma^n}$ is an isomorphism on homotopy groups. □

COROLLARY 2.6. *ev is onto.*



PROOF. It suffices to show that the image of $ev$ contains the $\mathbf{Z}_p \subset \mathbf{Z}_p^{\times}$ that is the subgroup of units congruent to 1 modulo $p$. This is because Lemma 2.5 allows us to deduce that the other summand of $\mathbf{Z}_p^{\times}$ is in that image.

Since $\gamma$ is a generator of $\mathbf{Z}_p^{\times}$, we can take $\gamma^{p-1}$ as a generator of the $\mathbf{Z}_p$ summand. Take $\lambda \in \mathbf{Z}_p \subset \mathbf{Z}_p^{\times}$ be a unit congruent to 1 modulo $p$. Let $X_\lambda$ be the fiber of $\psi^{\gamma} - \lambda$. We claim $X_\lambda \in M$. We have checked this above in the case $\lambda = \gamma^{n(p-1)}$. Any $\lambda$ is equal to some power of $\gamma^{p-1}$ modulo $p$, so $K(1)_*(X_\lambda) = K(1)_*(X_{\gamma^n}) = K(1)_*$ for some $n$.

Since $\gamma^{p-1}$ is a topological generator of $\mathbf{Z}_p \subset \mathbf{Z}_p^{\times}$, $\lambda = \gamma^{\pi(p-1)}$ for some $\pi \in \mathbf{Z}_p$. We check that $ev(X_\lambda) = \gamma^{-\pi(p-1)}$. This is true when $\pi$ is an ordinary integer by Lemma 2.5.

For other $\pi$, choose $k \in \mathbf{Z}$ so that $\gamma^k \equiv \gamma^{\pi} \pmod{p^r}$. Then $\lambda \equiv \gamma^{k(p-1)} \pmod{p^r}$, from which one can conclude

$$X_\lambda \wedge M(p^r) \simeq X_{\gamma^{k(p-1)}} \wedge M(p^r).$$

From this we conclude that $ev(X_\lambda) \equiv ev(X_{\gamma^{k(p-1)}}) = \gamma^{-k(p-1)} \equiv \gamma^{-\pi(p-1)}$ where the congruences are modulo $p^r$. Since $r$ was arbitrary, $ev(X_\lambda) = \gamma^{-\pi} = \lambda^{-1}$. □

An easy formula for the homotopy of $X_\lambda$ is given by

$$\begin{aligned} \pi_{2k-1}(X_\lambda) &= \mathbf{Z}_p/(\gamma^k - \lambda) \\ \pi_{2k}(X_\lambda) &= \begin{cases} \mathbf{Z}_p & \lambda = \gamma^k \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Notice that because $\gamma$ is a topological generator for $\mathbf{Z}_p^{\times}$, for any $\lambda$ there are ordinary integers $k$ making $\gamma^k$ arbitrarily close to $\lambda$. This implies that no $X_\lambda$ has a finite homotopy exponent. This is not true when $p = 2$.

We remind the reader that $q = 2p - 2$.

PROPOSITION 2.7. *The extension given above for $Pic_1$ is not split, so*

$$Pic_1 \simeq \mathbf{Z}_p \oplus \mathbf{Z}/q.$$

PROOF. We will leave out $L_{K(1)}$ localizations here. $S^{-1}$ is a lift of the non-zero element of $\mathbf{Z}/2$. If the extension splits then $S^{-1} \wedge X_\mu$ has order 2 for some $\mu$; that is

$$X_{\mu^2} = S^2 = X_\gamma,$$

so $\gamma = \mu^2$. This contradicts the choice of $\gamma$ as a topological generator of $\mathbf{Z}_p^{\times}$, so the extension doesn't split.

Since $\mathbf{Z}_p^{\times} \simeq \mathbf{Z}_p \oplus \mathbf{Z}/(p-1)$, the only non-split extension is the one given in the statement of the proposition. □

Proposition 2.7 implies there is some element of $Z \in \mathrm{Pic}_1$ of order $q$. To find $Z$, note that $Z$ must project to $1 \in \mathbf{Z}/2$, so we can write $Z = S^{-1} \wedge X_\mu$. Since $Z$ has order $q$,

$$S^0 = Z^{\wedge q} = (S^{-1} \wedge X_\mu)^{\wedge q} = S^{-q} \wedge X_{\mu^q}.$$

This means

$$X_{\gamma^{p-1}} = X_{\mu^q}.$$

So $\mu = \sqrt[q]{\gamma^{p-1}}$. By examining the group of units of $\mathbf{Z}/p^r$ it is easy to see that $\mu$ exists, in fact if we take $\gamma = \xi(1+p)$ then $\mu = \sqrt[q]{1+p}$ which exists by Hensel's Lemma.

An explicit isomorphism

$$\mathbf{Z}_p \oplus \mathbf{Z}/q \xrightarrow{F} \mathrm{Pic}_1$$

is given by

$$F(\lambda, n) = X_{(1+p)^\lambda} \wedge Z^n.$$

We now wish to show that the elements of $\mathrm{Pic}_1$ come from geometrically defined spectra. In particular, they are derived from complex vector bundles over $B\Sigma_p$, the classifying space of the symmetric group on $p$ letters.

We abuse notation by calling any map of spaces $\nu : X \to BU$ a complex vector bundle. When $X$ is compact, the map lands in some $BU(n)$ and hence defines a vector bundle (though not uniquely); we define the *Thom spectrum* as usual (see for example [22]) to be the desuspension of the Thom space of that bundle by the (real) dimension of the bundle. Thus the Thom spectrum doesn't depend on the dimension of the bundle. If $X$ is not compact, we define the Thom spectrum by taking limits of the Thom spectra of the compact subspaces of $X$. Recall that a Thom space has a spherical cell in dimension $n$ if $n$ is the dimension of the vector bundle; we call the cofiber of the inclusion of that cell the *reduced* Thom space. This guarantees a spherical 0 cell in a Thom spectrum. The *reduced* Thom spectrum is the cofiber of the inclusion of this 0 cell, and we denote the reduced Thom spectrum of $\nu$ by $X^\nu$.

Let $\xi = \rho - [p]$ be the virtual complex vector bundle over $B\Sigma_p$ given by subtracting the trivial bundle of dimension $p$ from the permutation bundle. Recall that $\widetilde{K}((B\Sigma_p)_{(p)}) = \mathbf{Z}_p$ with topological generator $\xi$. We make the following definition.

Definition 2.8. *For* $\lambda \in \mathbf{Z}_p$

$$P_{q\lambda+(q-1)} = ([B\Sigma_p]^{\lambda\xi})_{(p)}.$$

The transfer $(B\Sigma_p)_{(p)} \to S^0$ is a $K(1)_*$ isomorphism, so $L_{K(1)}B\Sigma_p = L_{K(1)}S^0$. By the Thom isomorphism $L_{K(1)}P_{\lambda q+(q-1)} \in M \subseteq \mathrm{Pic}_1$.

We write $P_n^m$ for the stunted skeleton of $(B\Sigma_p)_{(p)}$:

$$P_n^m = [(B\Sigma_p)_{(p)}]^{(m)}/[(B\Sigma_p)_{(p)}]^{(n-1)}.$$



Here the subscript $(p)$ means localization, and the superscripts indicate skeleta. This explains our notation $P_{q\lambda+(q-1)}$; it has been chosen so that when $\lambda$ is a positive integer,

$$P_{\lambda q+(q-1)} = \Sigma^{-\lambda q}P^\infty_{\lambda q+(q-1)} = \Sigma^{-\lambda q}(B\Sigma_p)_{(p)}/[(B\Sigma_p)_{(p)}]^{(\lambda q)}.$$

Here we use a CW-structure for $(B\Sigma_p)_{(p)}$ having one cell in each dimension congruent to 0 or $-1$ modulo $q$.

We have the following proposition about stunted classifying spaces.

Proposition 2.9. (i) $\Sigma^{kq}P^{mq+nq}_{mq-1} \simeq \Sigma^{mq}P^{kq+nq}_{kq-1}$ *if* $k - m \equiv 0$ *modulo* $p^n$.

(ii) $(B\Sigma_p^{(nq)})^{k\xi}_{(p)} \simeq \Sigma^{-kq}P^{nq+kq}_{(k+1)q-1}$ *where* $B\Sigma_p^{(nq)}$ *is the nq skeleton of* $B\Sigma_p$.

This follows from the analog of the second part of Proposition 2.9 for $BS^1 = \mathbf{C}P^\infty$ (see [2]), which leads to an analog of our Proposition 2.9 for $B\mathbf{Z}/(p)$. Then we use the transfer, and the associated splittings (after localization) to deduce Proposition 2.9 itself.

Write $\lambda$ as the limit of a sequence $a_i = a_{i-1} + \lambda_i p^i \in \mathbf{Z}_p$, with $a_{-1} = 0$. A consequence of the definition of Thom spectra and Proposition 2.9 is that $P_{\lambda q+(q-1)}$ is the following direct limit.

$$P^q_{q-1} \hookrightarrow \Sigma^{-a_0 q}P^{a_0q+2q}_{a_0q+q-1} \hookrightarrow \Sigma^{-a_1 q}P^{a_1q+3q}_{a_1q+q-1} \hookrightarrow \Sigma^{-a_2 q}P^{a_2q+4q}_{a_2q+q-1} \hookrightarrow \cdots$$

If $\lambda$ is an ordinary integer,

$$P_{\lambda q+(q-1)} = \Sigma^{-\lambda q}[(B\Sigma_p)_{(p)}/(B\Sigma_{(p)})_p^{(\lambda q)}] = S^{-q\lambda}$$

where the last equality only holds after localizing with respect to $K(1)$. This can be checked by noting that $L_{K(1)}P_{\lambda q+(q-1)} \in M \subseteq \mathrm{Pic}_1$, and $ev(P_{\lambda q+(q-1)}) = ev(S^{-q\lambda})$.

It follows by taking inverse limits (that is working mod $p^r$ for all $r$) that $L_{K(1)}P_{\lambda q+(q-1)} = X_{\gamma^{\lambda(1-p)}}$. We then have the following proposition.

Proposition 2.10.

$$Pic_1 \equiv \{L_{K(1)}(P_{\lambda q+(q-1)} \wedge S^i) \quad | \quad \lambda \in \mathbf{Z}_p, \quad 0 \le i < q\}.$$

Although this discussion looks somewhat different than the discussion in the introduction around (3), it can be translated into those terms by observing that

$$L_1 P^{nq+rq}_{nq-1} \simeq L_1 M^0(p^{r+1}).$$

Recall as noted above that for $p > 2$, $\mathbf{Z}_p^\times$ is topologically cyclic. If we topologize $M$ be making $ev$ a homeomorphism, we see $L_{K(1)}S^2$ is a topological generator of $M$ since it is taken to a topological generator of $\mathbf{Z}_p^\times$ under $ev$. Now since $L_{K(1)}S^2$ is a generator, every element of $M$ is topologically close to some power of $L_{K(1)}S^2$.

There are two differences we will see when $p = 2$. One is that the localizations of ordinary even dimensional spheres have square roots that are not topologically close to localizations of ordinary spheres. The other difference will be caused by the fact that $\mathbf{Z}_2^\times$ is not topologically cyclic. Because of these facts, for $p = 2$ we will see that the analog of Proposition 2.10 does not hold. In other words, ordinary spheres and the analogs of the $P_{\lambda q+(q-1)}$ are not sufficient to give $\mathrm{Pic}_1$.

## 3. Calculation of $\mathrm{Pic}_1$ for $p = 2$

In order to minimize the technicalities surrounding the difference between the functors $L_1$ and $L_{K(1)}$, in this section we write $KO$ for the real $K$-theory spectrum completed at 2. We use $K$ for the complex $K$-theory spectrum completed at 2. We write

$$\mathcal{K}_{1,*}(X) = \varprojlim_k K_*(X \wedge M(2^k)),$$

as well as

$$\mathcal{K}O_*(X) = \varprojlim_k KO_*(X \wedge M(2^k)).$$

In general $KO_*(X) \neq \mathcal{K}O_*(X)$ unless $KO_*(X)$ is finitely generated over $KO_*$. It is also not necessarily the same as the 2-completion of $KO_*(X)$ unless $KO_*(X)$ is finitely generated over $KO_*$. $\mathcal{K}O_*(-)$ is introduced here for the same reason we introduced $\mathcal{K}_{1,*}(-)$ in section 2; we need an integral invariant so we can work with real Adams operations. The same caution applies – $\mathcal{K}O_*(-)$ is not a representable homology theory since it doesn't satisfy the direct limit axiom.

We need the following specialization of Theorem 1.3 for this calculation.

THEOREM 3.1. *Let $p = 2$. The following are equivalent:*
(i) $L_{K(1)}Z \in Pic_1$.
(ii) $KO_*(Z; \mathbf{Z}/(2)) = KO_*(S^k; \mathbf{Z}/(2))$ *for some* $k$.
(iii) $\mathcal{K}O_*(X) = \mathcal{K}O_*(S^k)$ *for some* $k$.

Note that $\mathcal{K}O_*(S^k) = KO_*(S^k)$, and similarly $\mathcal{K}_{1,*}(S^k) = K_*(S^k)$. Theorem 3.1 follows from Theorem 1.3 and the following lemma.

LEMMA 3.2. *$K(1)_*(X) = K(1)_* \Leftrightarrow KO_*(X; \mathbf{Z}/(2)) = KO_*(S^{2r}; \mathbf{Z}/(2))$ for some $r$. Similarly, $\mathcal{K}_{1,*}(X) = \mathcal{K}_{1,*}(S^{2r}) \Leftrightarrow \mathcal{K}O_*(X) = \mathcal{K}O_*(S^{2r})$ for some $r$.*

PROOF. We start with the second statement. Recall the spectrum

$$Y = \Sigma^{-3}\mathbf{R}P^2 \wedge \mathbf{C}P^2 = M(2) \wedge M(\eta).$$

Note $KO \wedge Y = K(1) = K \wedge M(2)$ and $K = KO \wedge M(\eta)$. These are all easy to prove by calculating the mod 2 cohomology of the appropriate connective theories. Consistently below, when we write $\eta$ as a self map, we will mean to smash $\eta$ on the sphere with the spectrum concerned.



Assume that $\mathcal{K}O_*(X) = \mathcal{K}O_*(S^{2r})$. Then there is an equivalence,

$$\Sigma^{2r}KO \simeq (KO \wedge X)^\wedge_2.$$

Hence we get

$$\begin{array}{ccccc}
S^1 \wedge \Sigma^{2r}KO & \xrightarrow{\eta\wedge 1} & \Sigma^{2r}KO & \longrightarrow & \Sigma^{2r}K \\
\simeq\downarrow & & \simeq\downarrow & & \downarrow\simeq \\
S^1 \wedge (KO \wedge X)^\wedge_2 & \xrightarrow{\eta\wedge 1} & (KO \wedge X)^\wedge_2 & \longrightarrow & (K \wedge X)^\wedge_2
\end{array}$$

so $\mathcal{K}_{1,*}(X) = \mathcal{K}_{1,*}(S^{2r}) = \mathcal{K}_{1,*}(S^0)$.

Now suppose $\mathcal{K}_{1,*}(X) = K_*(S^0)$. We want to show $\mathcal{K}O_*(X) = KO_*(S^{2r})$ for some $r$. We are interested in the following cofibration:

$$\Sigma(KO \wedge X)^\wedge_2 \xrightarrow{\eta} (KO \wedge X)^\wedge_2 \to (K \wedge X)^\wedge_2 \to \Sigma^2(KO \wedge X)^\wedge_2. \tag{6}$$

The sequence is a cofibration before completion because $K = KO \wedge M(\eta)$. Completion is localization with respect to a homology theory, so the sequence is still a cofibration.

We are assuming that $(K \wedge X)^\wedge_2 \simeq K$ and we wish to prove that $(KO \wedge X)^\wedge_2 \simeq \Sigma^{2r}KO$. First note that (6) splits rationally, since $\eta$ is rationally trivial. This shows that $(KO \wedge X)^\wedge_2$ has the right number of $\mathbf{Z}_2$'s in the right places to be an even suspension of $KO^\wedge_2$.

The homotopy of $(KO \wedge X)^\wedge_2$ is a module over $KO_*(S^0)$, and by using this fact (in particular that $\eta^3$ and $2\eta$ are null), the cofibration (6), the value of $\pi_*((K \wedge X)^\wedge_2)$ and a diagram chase, we can deduce that (for some $r$)

$$\pi_*((KO \wedge X)^\wedge_2) = \Sigma^{2r}KO_* \qquad \text{or}$$

$$\pi_*((KO \wedge X)^\wedge_2) = \begin{cases} \mathbf{Z}_2 & * \equiv 0 \pmod{4 + 2r} \\ \mathbf{Z}/(2) & * \equiv 1, 2 \pmod{4 + 2r} \\ 0 & \text{else} \end{cases} \tag{7}$$

with the obvious $KO_*$-module structure.

One can check that (7) is not realizable as a $KO_*$-module. Suppose (7) holds. Map $KO$ to $(KO \wedge X)^\wedge_2$ by using a generator of one of the $\mathbf{Z}_2$'s and the $KO$-module structure. The $KO_*$-module structures force this map to be injective on homotopy, and hence the homotopy of the cofiber would be

$$N = \begin{cases} \mathbf{Z}/(2) & * \equiv 0, 1, 2 \pmod{8 + 2r} \\ 0 & \text{else.} \end{cases}$$

The element $\eta$ acts non-trivially whenever possible. Now we can map $KO \wedge M(2)$ into the spectrum with homotopy $N$ by using a generator of one of the $\mathbf{Z}/(2)$'s in dimension $2r$. If this map commutes with multiplication by $\eta$ (as it must) then it cannot be a map of groups, since $\eta^2$ on the generator of $\pi_0(KO \wedge M(2))$

is a multiple of 2, but its image under our map is supposed to be $\eta^2$ times the generator of $N_0$, which is the generator of $N_2$.

It follows that $(KO \wedge X)^{\hat{}}_2$ has the homotopy of $\Sigma^{2r}KO$, and the fact that it is a $KO$-module spectrum allows one to construct an equivalence with $\Sigma^{2r}KO$.

To prove the first statement, if we know $KO(X;\mathbf{Z}/(2))$ we proceed as in the case where we knew $\mathcal{K}O_*(X)$, and if we know $K(1)_*(X)$ we use Theorem 1.3, the second statement, and reduction modulo 2. □

For the next theorem, note that $\mathbf{Z}_2^\times \simeq \mathbf{Z}_2 \times \mathbf{Z}/(2)$.

THEOREM 3.3. *There is an extension of abelian groups*

$$0 \to M \to Pic_1 \to \mathbf{Z}/8 \to 0. \tag{8}$$

*The map to* $\mathbf{Z}/8$ *takes* $X$ *to the dimension* (mod 8) *of the generator of* $\mathcal{K}O_*(X)$ *for* $X \in Pic_1$. $M$ *is by definition the kernel of this map and there is an isomorphism* $M \xrightarrow{ev} (\mathbf{Z}_2)^\times$ *where* $ev(X)$ *is the eigenvalue of the Adams operation* $\psi^3$ *on* $\mathcal{K}O_0(X)$. *The extension is non-trivial and*

$$Pic_1 \simeq \mathbf{Z}_2^\times \times \mathbf{Z}/4 \simeq \mathbf{Z}/2 \times \mathbf{Z}_2 \times \mathbf{Z}/4.$$

To prove this, our first steps are just as in the odd primary case. The proofs that $ev$ is a homomorphism, that

$$L_{K(1)}X = \text{fiber}[(KO \wedge X)^{\hat{}}_2 \xrightarrow{[\psi^3-1]\wedge 1} (KO \wedge X)^{\hat{}}_2],$$

and that $ev$ is injective are exactly as before, except replacing $K$ with $KO$.

LEMMA 3.4. *ev is onto.*

PROOF. Take $\lambda \in \mathbf{Z}_2^\times$. Let $X_\lambda = \text{fiber}(KO \xrightarrow{\psi^3-\lambda} KO)$. Since $\lambda$ is a unit, $\lambda \equiv 1 \pmod 2$, so $\psi^3 - \lambda$ has the same effect on $K(1)_*$ as $\psi^3 - 1$. Therefore $K(1)_*(X_\lambda) = K(1)_*$, so $X_\lambda \in \text{Pic}_1$. We need to verify that $X_\lambda \in M$, and then compute $ev(X_\lambda)$.

To see that $X_\lambda \in M$, first note that $\lambda \equiv 1$ modulo 2 since $\lambda$ is a unit. So $\psi^3 - \lambda = (\psi^3 - 1) + \mu$ where $\mu$ is an even 2-adic integer. It follows that

$$(\psi^3 - \lambda)^{2^n} \equiv (\psi^3 - 1)^{2^n}$$

modulo $2^n$, and hence if $n > 1$ (since then $1_{M(2^n)}$ has order $2^n$)

$$\text{fiber}[(\psi^3 - \lambda)^{2^n}] \wedge M(2^n) \simeq \text{fiber}[(\psi^3 - 1)^{2^n}] \wedge M(2^n).$$



Now let $X_{\lambda,n}$ be the fiber of $(\psi^3 - \lambda)^{2^n}$. It is easy to check, by induction, using the diagram of cofiber sequences

$$\begin{array}{ccccc}
X_{\lambda,i} & \longrightarrow & X_{\lambda,i+1} & \longrightarrow & X_\lambda \\
\downarrow & & \downarrow & & \downarrow \\
X_{\lambda,i} & \longrightarrow & KO & \xrightarrow{(\psi^3-\lambda)^i} & KO \\
\downarrow & & \downarrow{\scriptstyle (\psi^3-\lambda)^{i+1}} & & \downarrow{\scriptstyle (\psi^3-\lambda)} \\
* & \longrightarrow & KO & \xrightarrow{=} & KO
\end{array}$$

and $\mathcal{K}O_*(X_\lambda) = \Sigma^r KO_*$ that $\mathcal{K}O_*(X_{\lambda,n})$ is an extension of $\Sigma^r KO_*$ by the $KO_*$ module $\mathcal{K}O_*(X_{\lambda,n-1})$ (that is, the long exact sequence in $\mathcal{K}O_*(-)$ is really a short exact sequence). Since $r = 0$ when $\lambda = 1$, and $X_{\lambda,2^n} \wedge M(2^n) = X_{1,2^n} \wedge M(2^n)$, it follows that $r = 0$. This shows that $X_\lambda \in M$.

We want to check that $ev(X_\lambda) = \lambda^{-1}$, i.e. $ev(\psi^3|_{\mathcal{K}O_0(X_\lambda)}) = \lambda^{-1}$. Actually, since the map $KO \to K$ induces an isomorphism

$$\mathcal{K}O_0(X_\lambda) \to \mathcal{K}_{1,0}(X_\lambda),$$

$ev(\psi^3|_{\mathcal{K}O_0(X_\lambda)}) = ev(\psi^3|_{\mathcal{K}_{1,0}(X_\lambda)})$, and it is this second number we will actually compute.

To make our calculation we need to understand $\mathcal{K}_{1,*}(KO)$. We will describe this with the isomorphism

$$\mathcal{K}_{1,*}(K) = \text{maps}_{\text{cont.}}(\mathbf{Z}_2^\times, K_*)$$

via the map

$$f \in \mathcal{K}_{1,r}(K) \mapsto [\nu \in \mathbf{Z}_2^\times \mapsto (\overline{f}(\nu) : S^r \xrightarrow{f} (K \wedge K)^{\hat{}}_2 \xrightarrow{1\wedge\psi^\nu} (K \wedge K)^{\hat{}}_2 \xrightarrow{\mu} K)].$$

We denote this element of $\text{maps}_{\text{cont.}}(\mathbf{Z}_2^\times, K_*)$ by $\overline{f}$. This result appears originally in [1], and is also deducible from [16], though it doesn't appear in exactly this form in either place. It is spelled out in this form in [8].

We note $\mathbf{Z}_2^\times \simeq \mathbf{Z}_2 \times \mathbf{Z}/(2)$ and with this $\mathbf{Z}_2$ in mind as a quotient of the $\mathbf{Z}/(2)$ action,

$$\mathcal{K}_{1,*}(KO) = \text{maps}_{\text{cont.}}(\mathbf{Z}_2, K_*). \tag{9}$$

To see this we start by recalling that $K$ has a $\mathbf{Z}/(2)$ action give by $\psi^{-1}$ (complex conjugation). We have

$$KO \simeq K^{h\mathbf{Z}/(2)} \text{ and } (K \wedge K)^{\hat{}\,h\mathbf{Z}/(2)}_2 \simeq (K \wedge KO)^{\hat{}}_2$$

from Lemma 3.5 below. In the second equivalence above, $\mathbf{Z}/(2)$ acts on the second factor. (The first equivalence is implicit in [3, Corollary 3.8] and is made more explicit in the proof of 3.1 in [**21**], which gives a map inducing

$$KO^* = \varprojlim_k KR^*((E\mathbf{Z}/(2)^{(k)}),$$

where $X^{(k)}$ is the $k$ skeleton of $X$. On the category of $\mathbf{Z}/(2)$-CW complexes, $KR^*$ is represented by the naive $\mathbf{Z}/(2)$ spectrum obtained by letting $\mathbf{Z}/(2)$ act by conjugation on the complex $K$-theory spectrum.)

Since $\psi^{-1}$ gives the non-trivial involution of $K$, the homotopy fixed point spectral sequence for $\pi_*([K\wedge K]_2^{\hat{}\,h\mathbf{Z}/(2)})$ collapses at

$$E_2 = H_c^*(\mathbf{Z}/(2); K_{1,*}(K)) = \begin{cases} K_{1,*}(K)^{\mathbf{Z}/(2)} & * = 0 \\ 0 & \text{else.} \end{cases}$$

Equation (9) follows because an element of $\text{maps}_{\text{cont.}}(\mathbf{Z}_2^\times, K_*)$ is fixed under the $\mathbf{Z}/(2)$ action iff it passes to the quotient $\mathbf{Z}_2 = \mathbf{Z}_2^\times/\mathbf{Z}/(2)$.

Using the multiplicative properties of the Adams operations,

$$\begin{array}{ccc} (K\wedge K)_2^{\hat{}} & \xrightarrow{\mu} & K \\ {\scriptstyle \psi^\nu\wedge\psi^\nu}\downarrow & & \downarrow{\scriptstyle \psi^\nu} \\ (K\wedge K)_2^{\hat{}} & \xrightarrow{\mu} & K \end{array}$$

one can check

$$\mu(1\wedge\psi^x)(\psi^\nu\wedge 1)f \simeq \psi^\nu\mu(1\wedge\psi^{x\nu^{-1}})f,$$

in other words

$$\overline{(\psi^\nu\wedge 1)f}(x) = \psi_*^\nu\overline{f}(x\nu^{-1}).$$

Now we examine the diagram

$$\begin{array}{ccccc} (K\wedge X_\lambda)_2^{\hat{}} & \longrightarrow & (K\wedge KO)_2^{\hat{}} & \xrightarrow{1\wedge(\psi^3-\lambda)} & (K\wedge KO)_2^{\hat{}} \\ {\scriptstyle \psi^3\wedge 1}\downarrow & & {\scriptstyle \psi^3\wedge 1}\downarrow & & {\scriptstyle \psi^3\wedge 1}\downarrow \\ (K\wedge X_\lambda)_2^{\hat{}} & \longrightarrow & (K\wedge KO)_2^{\hat{}} & \xrightarrow{1\wedge(\psi^3-\lambda)} & (K\wedge KO)_2^{\hat{}}. \end{array}$$

We want to know the eigenvalue of the leftmost vertical map on $\pi_0$. Suppose $g \in \pi_0(K\wedge X_\lambda)_2^{\hat{}}$. We abuse notation by identifying $g$ with its image in $\pi_0(K\wedge KO)_2^{\hat{}}$. Then $\overline{g}$ is an element of $\text{maps}_{\text{cont.}}(\mathbf{Z}_2, K_*)$ that is in the kernel of $1\wedge(\psi^3 - \lambda)$. This kernel is the kernel of precomposition by multiplication by 3 minus multiplication by $\lambda$. In other words, $\overline{g}(x3) = \lambda\overline{g}(x)$.

Now

$$\overline{(\psi^3\wedge 1)g}(x) = \psi_*^3\overline{g}(x\cdot 3^{-1}) = \overline{g}(x\cdot 3^{-1}) = \lambda^{-1}\overline{g}(x).$$

The second equality follows because $\overline{g}(x\cdot 3^{-1}) \in \pi_0K$, so $\psi^3$ acts as the identity.

Hence $ev(X_\lambda) = \lambda^{-1}$. □

We need the following lemma for the proof of Lemma 3.4.



LEMMA 3.5. *For any spectrum $E$, where $\mathbf{Z}/(2)$ acts only on the spectrum $K$,*

$$[(E\wedge K)_2^{\hat{}}]^{h\mathbf{Z}/(2)} \simeq (E\wedge KO)_2^{\hat{}}.$$

PROOF. Since completion and homotopy fixed points are both homotopy inverse limits, it suffices to show

$$(E\wedge K\wedge W)^{h\mathbf{Z}/(2)} \simeq (E\wedge KO\wedge W) \tag{10}$$

for $W = M(2^n)$, all $n$.

Let $\mathcal{C}$ be the class of all spectra $W$ satisfying (10). $\mathcal{C}$ is closed under cofibrations and retracts, so if we can show $\mathbf{C}P^2\wedge M(2)$ is in $\mathcal{C}$ then it follows that $M(2^n)$ is in $\mathcal{C}$ for all $n$.

$K^*(\mathbf{C}P^2)$ is free over $\mathbf{Z}/(2)$ since $x$ and $x^2$ are generators over $K^*$, so $x$ and $\psi^{-1}(x) = -x + x^2$ are generators over $K^*$ ($\psi^{-1}(x)$ is determined by recalling that $x = L - 1$ where $L$ is the tautological line bundle).

Hence $K^*(\mathbf{C}P^2\wedge M(2)) = K(1)^*(\mathbf{C}P^2)$ is also free over $\mathbf{Z}/(2)$, and it follows that $K(1)_*(\mathbf{C}P^2) = K_*(\mathbf{C}P^2\wedge M(2))$ is free over $\mathbf{Z}/(2)$. Finally,

$$\pi_*(E\wedge K\wedge \mathbf{C}P^2\wedge M(2)) = K(1)_*(E)\otimes_{K(1)_*}K(1)_*(\mathbf{C}P^2)$$

is free over $\mathbf{Z}/(2)$ since the second factor is.

So the homotopy fixed point spectral sequence for $E\wedge K\wedge\mathbf{C}P^2\wedge M(2)$ collapses at

$$\begin{aligned} E_2 &= \pi_*(E\wedge K\wedge\mathbf{C}P^2\wedge M(2))^{\mathbf{Z}/(2)} = \\ &[K(1)_*(E)\otimes_{K(1)_*}K(1)_*(\mathbf{C}P^2)]^{\mathbf{Z}/(2)} = K(1)_*(E). \end{aligned}$$

Since this is also $\pi_*(E\wedge KO\wedge\mathbf{C}P^2\wedge M(2)) = \pi_*(E\wedge K^{h\mathbf{Z}/(2)}\wedge\mathbf{C}P^2\wedge M(2))$, the natural map $E\wedge K^{h\mathbf{Z}/(2)}\wedge\mathbf{C}P^2\wedge M(2)\to(E\wedge K\wedge\mathbf{C}P^2\wedge M(2))^{h\mathbf{Z}/(2)}$ is an equivalence. □

We have now shown Theorem 3.3 except for identifying the extension. Suppose the extension was trivial. Then we can lift the generator of $\mathbf{Z}/(8)$ to an element $X\wedge S^1$ (where $X\in M$) of order 8. So $(X\wedge S^1)^{\wedge 8}\simeq S^0$ (after appropriate localizations), i.e. $X^{\wedge 8} = S^{-8}$ in $\text{Pic}_1$. So $ev(X^{\wedge 8}) = ev(X)^8 = ev(S^{-8}) = 3^4$. This is a contradiction since $3^4$ does not have an $8^{\text{th}}$ root in $\mathbf{Z}_2$.

To see what the extension is, let $X\in M$ be the element with $ev(X) = 3$. Then $X\wedge S^2$ is a lift of an element of order 4 in $\mathbf{Z}/(8)$, so $(X\wedge S^2)^{\wedge 4}\in M$, and

$$ev((X\wedge S^2)^{\wedge 4}) = ev(X^{\wedge 4})\cdot ev(S^8) = 3^4\cdot 3^{-4} = 1$$

so $X\wedge S^2$ has order 4 in $\text{Pic}_1$. It follows that the extension is as claimed.

## 4. Stunted projective spaces and $\text{Pic}_1$

In this section we will construct most of the elements in $\text{Pic}_1$ at p=2. We use stunted projective spaces and related objects to do this. Our constructions give these elements as localizations of geometrically defined spectra related to $\mathbf{R}P^\infty$, in place of the abstract construction of Lemma 3.4.

Recall the short exact sequence (8) from Theorem 3.3, where $M \xrightarrow{ev} \mathbf{Z}_2^\times$ is an isomorphism. We will use the ideal of virtual vector bundles $4\mathbf{Z}_2 \subseteq \mathbf{Z}_2 \simeq KO^0(\mathbf{R}P^\infty)$ to make Thom spectra.

Now $\overline{KO}^0(\mathbf{R}P^\infty) = \mathbf{Z}_2$ has topological generator $\lambda - [1]$ where $\lambda$ is the usual line bundle over $\mathbf{R}P^\infty$ and [1] is the one dimensional trivial bundle. Call this generator $\xi$. Then

$$(\mathbf{R}P^\infty)^\xi = \Sigma^{-1}\mathbf{R}P^\infty/\mathbf{R}P^1$$

and generally

$$(\mathbf{R}P^m)^{n\xi} = \Sigma^{-n}\mathbf{R}P^{n+m}/\mathbf{R}P^n.$$

To simplify notation we make the following definition,

DEFINITION 4.1. *For $\gamma \in \mathbf{Z}_2$,*

$$P_{\gamma+1} = (\mathbf{R}P^\infty)^{\gamma\xi}.$$

This is convenient notation because when $n$ is a positive integer, $P_n$ is a desuspension of the cofiber of $\mathbf{R}P^{n-1} \to \mathbf{R}P^\infty$.

Thom spectra of even elements of $KO^0(\mathbf{R}P^\infty) = \mathbf{Z}_2$ give elements of $\text{Pic}_1$, because these have complex structures so we can use the Thom isomorphism theorem to show $K(1)_*P_{2\gamma+1} \simeq K(1)_*S^0$. We will restrict to studying $P_{4\gamma+1}$ because their localizations are in the subgroup $M$ of Theorem 3.3 (also, Proposition 4.5 implies this is not a real restriction).

$L_{K(1)}P_{4\gamma+1} \in M$ since bundles divisible by 4 in $KO^0(\mathbf{R}P^\infty)$ satisfy the hypotheses of the $KO$-theory Thom isomorphism so $\mathcal{K}O_*(P_{4\gamma+1}) = \mathcal{K}O_*(\mathbf{R}P^\infty) = KO_*(S^0)$ (the fact that we are using $\mathcal{K}O_*(-)$ instead of $KO_*$ is not an issue, since if the $KO$ Thom isomorphism holds for $\nu$, then so does the $KO \wedge M(2^n)$ Thom isomorphism for all $n > 1$, and $\mathcal{K}O_*(-)$ is just the inverse limit of these theories). We remind the reader that we are taking *reduced* Thom spectra to make these elements of $\text{Pic}_1$.

It is worth making explicit that $P_5$ corresponds to $(0,2) \in \mathbf{Z}/(2) \times \mathbf{Z}_2 = \mathbf{Z}_2^\times$. The point is that

$$\begin{aligned} ev(L_{K(1)}P_5) = ev(\psi^3|_{\Sigma^{-4}L_{K(1)}\mathbf{R}P^\infty/\mathbf{R}P^4}) &= ev(\Sigma^{-4}L_{K(1)}\mathbf{R}P^\infty) \\ &= ev(L_{K(1)}S^{-4}) = 9 = 3^2. \end{aligned}$$

To see the second equality above, observe that the map

$$\mathcal{K}O_0(\mathbf{R}P^\infty) \to \mathcal{K}O_0(\mathbf{R}P^\infty/\mathbf{R}P^4)$$



must be multiplication by some non-zero element of $\mathbf{Z}_2$ (the map can't be zero since $\mathcal{K}O_*(\mathbf{R}P^4)$ is all torsion). It follows by naturality that $\psi^3$ has the same eigenvalue on both $\mathbf{Z}_2$'s concerned. To identify the element of $\mathbf{Z}/(2) \times \mathbf{Z}_2 = \mathbf{Z}_2^\times$, we observe that $3 \in \mathbf{Z}_2^\times = \mathbf{Z}/(2) \times \mathbf{Z}_2$ is a topological generator of the $\mathbf{Z}_2$ summand. We observe that this is unlike the $p > 2$ case since no $S^k$ satisfies both $L_{K(1)}S^k \in M$ and $ev(S^k) = 9$.

It is interesting to see just how one constructs these Thom spectra. As an example we will take $4\lambda + 1 = \sqrt{17}$ and make the Thom spectrum of $4\lambda\xi$, that is, $P_{\sqrt{17}}$. Let $n = 4a + b$ where $0 \le b < 4$. Let $\phi(n) = 8a + 2^b$. James periodicity gives the following equivalences. For a simple proof see [11].

PROPOSITION 4.2. *If $k - m \equiv 0 \pmod{2^n}$ then*

$$\Sigma^k P_m^{m+\phi(n)-1} \simeq \Sigma^m P_k^{k+\phi(n)-1}.$$

To construct the reduced Thom spectrum $(\mathbf{R}P^\infty)^{4\lambda\xi} = (\mathbf{R}P^\infty)^{(\sqrt{17}-1)\xi}$, we do as described above: construct the reduced Thom spectrum for $4\lambda\xi$ restricted to each $\mathbf{R}P^{\phi(n)}$ and fit these together into a direct system. In [2], these are identified with stunted projective spaces $\Sigma^{-4k}P_{4k+1}^{4k+\phi(n)}$.

If we want to include $P_{4k+1}^{4k+\phi(n)}$ into a stunted projective space with $\phi(n+1) - \phi(n)$ more cells, we have two choices

$$P_{4k+1}^{4k+\phi(n)} \to P_{4k+1}^{4k+\phi(n+1)} \text{ and}$$

$$P_{4k+1}^{4n+\phi(n)} = \Sigma^{-2^n}P_{4k+1+2^n}^{4k+\phi(n)+2^n} \to \Sigma^{-2^n}P_{4k+1+2^n}^{4k+\phi(n+1)+2^n}.$$

The next Thom spectrum (for $\mathbf{R}P^{\phi(n+1)}$) tells us which space to use. In terms of the number $\sqrt{17}$, this is a 2-adic integer so there is a sequence of zeroes and ones which represents it. For example we note that $9961^2 \equiv 17 \pmod{2^{16}}$. Thus the first 14 terms of the sequence for $4\lambda = \sqrt{17} - 1$ is 10011011101000. To construct $P_{\sqrt{17}}$ we will use the 1's in the dyadic expansion of $4\lambda$ to guide us. Because $4\lambda$ always starts with two 0's we always begin with $P_1^4$ since $\phi(2) = 4$. The next zero in the expansion for $\sqrt{17}$ tells us to include this in $P_1^8$ since $\phi(3) = 8$. At the next stage, the 1 that is present is an instruction to include $P_1^8$ into $\Sigma^{-8}P_9^{17}$ rather than into $P_1^9$. We get the sequence

$$P_1^4 \to P_1^8 \to \Sigma^{-8}P_9^{16} \to \Sigma^{-8}P_9^{17}.$$

Here we are using the fact that $\phi(4) - \phi(3) = 9 - 8 = 1$. The next 0 gives us the inclusion

$$\Sigma^{-8}P_9^{17} \to \Sigma^{-8}P_9^{18}.$$

To summarize the process, each time there is a 1 in the dyadic expansion we use Proposition 4.2. Each time there is a 0 we just include one stunted projective space into the next. If we continue in this fashion through the digits which represent 9961 we will have $\Sigma^{-9960}P_{9961}^{9988}$. To continue, we need to find the next digits in $\sqrt{17}-1$. By the results of [2] on Thom spaces over projective spaces, this is precisely the construction of the Thom spectrum.

Before we continue, we note the following results.

PROPOSITION 4.3. $L_{K(1)}P_{8n+1} = L_{K(1)}S^{-8n}$.

PROOF. $ev(P_{-8n+1}) = ev(\Sigma^{-8n}\mathbf{R}P^\infty) = ev(S^{-8n})$. □

PROPOSITION 4.4.

$$L_{K(1)}P_{4\lambda+1} \simeq X_{9^\lambda} \in M \subset Pic_1$$

PROOF. See the proof of Lemma 3.4 for the meaning of $X_{9^\lambda}$. This Proposition follows immediately from knowing $L_{K(1)}P_{4\lambda+1} \in M$, since the construction of $P_{4\lambda+1}$ implies that $ev(P_{4\lambda+1}) = 9^{-\lambda}$. □

PROPOSITION 4.5.

$$\Sigma^{-2}L_{K(1)}P_{4\gamma-1} \simeq L_{K(1)}P_{4\gamma+1}$$

PROOF. The results of [12] applied to the finite subcomplexes of the spectra concerned show that $KO_*(\Sigma^{-2}P_{4\gamma-1}) = KO_*(P_{4\gamma+1})$. The evident map $\Sigma^{-2}P_{4\gamma-1} \to P_{4\gamma+1}$ together with naturality shows that both spectra have the same value under the map $ev$. So Theorem 3.3 shows the spectra are homotopy equivalent. □

PROPOSITION 4.6. *$L_{K(1)}P_5 = L_{K(1)}X$ where $X$ is the complex $S^{-4}\cup_\eta e^{-2}\cup_2 e^{-1}$.*

PROOF. The map $\lambda : \mathbf{R}P^\infty \to S^0$ is a $K(1)_*$ isomorphism [17, Theorem 9.1]. Consider the diagram of cofibrations (where $\iota$ is the obvious inclusion)

$$\begin{array}{ccccc}
\mathbf{R}P^2 & \xrightarrow{\lambda\circ\iota} & S^0 & \longrightarrow & \Sigma^4 X \\
{\scriptstyle\iota}\downarrow & & \| & & \downarrow \\
\mathbf{R}P^\infty & \xrightarrow{\lambda} & S^0 & \longrightarrow & A \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^2 P_3 & \longrightarrow & * & \longrightarrow & \Sigma^3 P_3
\end{array}$$

After localization, $A = *$, therefore $\Sigma^3 P_3 = \Sigma^5 X$, but since by Proposition 4.5 $\Sigma^3 P_3 = \Sigma^5 P_5$ after localization, we are done. □

The first author has proposed that Picard groups are an appropriate way of indexing homotopy groups. These projective space examples allow one to make this more precise. Let $J$ be the connective image of $J$ spectrum and let

$$N_i = \{k \in N | 2^i J_k(P_{4\lambda+1}) \neq 0\}.$$

Then $\{N_i\}$ forms a 2-adic neighborhood system which converges to a 2-adic integer, $\bar{\lambda}$. If $\lambda$ is an ordinary integer, $h$, then $\bar{\lambda} = -4h-1$. In general, $(\bar{\lambda}+1)$



can be viewed as giving the "dimension" of the element of $\mathrm{Pic}_1$, and if $h$ happens to be even,

$$[P_{4h+1}, L_{K(1)}X] = [S^{-4h}, L_{K(1)}X].$$

If we do this for the $\sqrt{17}$ example we get $N_i = \{k | k \equiv -\sqrt{17}-1 \pmod{2^i}\}$.

## 5. Interesting examples in $\mathrm{Pic}_1$ at $p=2$

The spectra $\{L_{K(1)}P_{4\lambda+1}\}$ form a subgroup of $\mathrm{Pic}_1$, in fact they give the subgroup $0 \times 2\mathbf{Z}_2 \subseteq \mathbf{Z}/(2) \times \mathbf{Z}_2 = M \subseteq \mathrm{Pic}_1$, so the cokernel of the inclusion into $M$ is $\mathbf{Z}/(2) \times \mathbf{Z}/(2)$. We can try to lift the generators of the quotient. $(1,0)$ lifts to an element of order 2, $X_{-1}$. $X_{-1} \wedge X_{-1} = S^0$ after localization. Similarly, $(0,1)$ lifts to $X_3$, which is a square root of $L_{K(1)}P_5$. $M$ is parametrized by $X_{(-1)^a3^b}$ for $(a,b) \in \mathbf{Z}/(2) \times \mathbf{Z}_2$. Whenever $a$ or $b$ (or both) are odd, we get element of $\mathrm{Pic}_1$ with the following properties:

- $K(1) \wedge X_{(-1)^a3^b} = K(1)$. This is just because $S^{(a,b)} \in \mathrm{Pic}_1$.
- $\pi_*(X_{(-1)^a3^b})$ has exponent 4. In fact the homotopy of this spectrum is "half" of the homotopy of $L_1M(2)$ ($L_1M(4)$ in case both $a$ and $b$ are odd) in the sense that it is generated by the periodicity element and *one* lightning flash.

Let $\lambda = (-1)^a3^b$. If exactly one of $a$ or $b$ is odd, then $\psi^3 - \lambda \equiv 2 \pmod 4$ in dimensions divisible by 4. From this it is clear that 4 is an exponent for the homotopy groups of the fiber, and there is a non-trivial extension in $\pi_{8k+2}(X_\lambda)$ making 4 the actual exponent for that group. Similarly, if $a$ and $b$ are both odd, $\psi^3 - \lambda \equiv 4 \pmod 8$.

Of course $X_\lambda$ can be constructed as in Lemma 3.4, but here we describe "lifts" of these elements when $a$ or $b$ is odd constructed by successively attaching mod 2 (or mod 4 when both $a$ and $b$ are odd) Moore spectra. Our construction will give lifts in the sense that the spectra we construct will be 0-connected spectra that localize to the elements of $\mathrm{Pic}_1$ under consideration.

The spectra we are interested in describing then, are the $X_\lambda$ where $\lambda \equiv 3, 5$, or 7 modulo 8. We describe two of the cases in detail below.

Write $M(2)$ for the mod 2 Moore spectrum with the bottom cell in dimension 0.

THEOREM 5.1. *Take $\Sigma^k X_\lambda \in Pic_1$ with $\lambda \equiv 3(4)$. Then $X_\lambda$ has homotopy exponent 4, and there are 0-connected finite type spectra $X_n$, $0 \le n < \infty$ and a 0-connected spectrum $X_\infty = \varinjlim_n X_n$ such that*

(i) *$L_{K(1)}X_\infty = X$.*
(ii) *$\Sigma^{k_n}M(2) \to X_n \to X_{n+1}$ is a cofibration for some $k_n > 0$ and $f_n : \Sigma^{k_n}M(2) \to X_n$.*
(iii) *$\Sigma^{k_n}M(2) \to X_n \to \Sigma^{k_{n-1}+1}M(2)$ on $K(1)_*$ carries the copy of $K(1)_*$ corresponding to the top cell of the domain to the copy of $K(1)_*$ corresponding to the bottom cell of the range and is 0 on the other copy*

*of $K(1)_*$, and in $\pi_* L_1$ takes $v_1^{4r} i$ to the generator of $\pi_{8s-1} L_1 M(2)$ for appropriate $r, s$.*

PROOF. Without loss of generality, assume $k = 0$. Since $\lambda \equiv 3(4)$, the effect of $\psi^3 - \lambda$ on $\pi_{4m} KO\hat{_2}$ is to take $\mathbf{Z}_2$ to $2\mathbf{Z}_2$ by multiplication by some 2-adic integer congruent to 2 modulo 4. Call that integer $\tau_m$. This is sufficient to establish the claim about the homotopy exponent since $\pi_*(X_\lambda)$ is thus an extension of direct sums of $\mathbf{Z}/(2)$'s.

We get the following diagram:

$$\begin{array}{ccccccc} \Sigma^{-1} M(2) & \longrightarrow & (S^0)\hat{_2} & \xrightarrow{\tau_0} & (S^0)\hat{_2} & \longrightarrow & M(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_\lambda & \longrightarrow & KO\hat{_2} & \xrightarrow{\psi^3 - \lambda} & KO\hat{_2} & \longrightarrow & \Sigma X_\lambda . \end{array}$$

An analysis of the diagram in homotopy shows that after applying $L_{K(1)}$, the map $M(2) \to \Sigma X_\lambda$ is onto in homotopy. Checking the diagram in $K(1)_*$ shows that this map has the same effect in $K(1)_*$ as $M(2) \to S^1$. Let $X_{\lambda,1}$ be the cofiber of $L_{K(1)} \Sigma^{-1} M(2) \to X_\lambda$. It follows from the effect on $K(1)_*$ that $X_{\lambda,1} \in \mathrm{Pic}_1$.

Using the cofibration that defines $X_{\lambda,1}$, we see that

$$ev(\psi^3|_{KO_{4m}(X_\lambda)}) = ev(\psi^3|_{KO_{4m}(X_{\lambda,1})}).$$

The numbers on the left are all congruent to 3 modulo 4, hence so are the numbers on the right. Therefore $X_{\lambda,1}$ satisfies the same hypotheses as $X_\lambda$, so we can iterate this construction.

We get the following tower of spectra:

$$\begin{array}{ccccccc} \Sigma^{-1} L_{K(1)} M(2) & & \Sigma^{-1} L_{K(1)} M(2) & & \Sigma^{-1} L_{K(1)} M(2) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ X_\lambda & \longrightarrow & X_{\lambda,1} & \longrightarrow & X_{\lambda,2} & \longrightarrow & \cdots . \end{array}$$

Note that $\varinjlim_i X_{\lambda,i} = *$ since each map in the direct system is 0 on homotopy.

Next we take the constant tower $X_\lambda$ and map it into the tower constructed above. Call the cofiber at each stage $Y_n$. We get a new tower:

$$\begin{array}{ccccccc} \Sigma^{-1} L_{K(1)} M(2) & & \Sigma^{-1} L_{K(1)} M(2) & & \Sigma^{-1} L_{K(1)} M(2) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ * & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots . \end{array}$$

Here $Y_0 = *$ and $Y_1 = L_{K(1)} M(2)$. The spectrum $\varinjlim_n Y_n$ is obviously $\Sigma X_\lambda$.

We wish to replace the tower we just constructed with a tower of finite spectra. Assume inductively that $Y_n = L_{K(1)} X_n$ where $X_n$ is finite, connective and torsion. Then for $k$ arbitrarily large, we can use the $n = 1$ telescope conjecture to



find a map $\Sigma^{8k-1} M(2) \to X_n$ that localizes to the map in our tower above. Take $X_{n+1}$ to be the cofiber; it follows that $X_{n+1}$ is finite, connective and torsion.

It remains to check that $L_{K(1)} X_\infty = X$ where $X_\infty = \varinjlim_n X_n$. We have just made the tower

$$\begin{array}{ccccccc} \Sigma^{k_0} M(2) & & \Sigma^{k_1} M(2) & & \Sigma^{k_2} M(2) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ * & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots . \end{array}$$

Now if we apply $L_1$ to this tower we get our tower involving the $Y_n$'s, and since $L_1$ is smashing, that implies that $L_1 X_\infty = \varinjlim_n Y_n = X_\lambda$ But $X$ is $K(1)$-local by hypothesis, so $X = L_{K(1)} X = L_{K(1)} L_1 X_\infty = L_{K(1)} X_\infty$. □

In the case of $\Sigma^k X_\lambda$ where $\lambda \equiv 5(8)$, we can prove a theorem similar to Theorem 5.1 involving $\mathbf{Z}/(4)$ Moore spectra instead of $\mathbf{Z}/(2)$ Moore spectra.

Theorem 5.1 shows that we can construct these examples by adjoining Moore spaces. We would like to construct $X_\lambda$ as in Theorem 5.1 explicitly by giving specific $X_i$ and attaching maps to gain some understanding of these examples. The calculations necessary to give the complete description are extensive. In the following we will give a sketch proof of an explicit construction. It should not be viewed as complete. We try to indicate where to do the calculations and what calculations are necessary.

We wish to look at $[M(2), M(2)]_*$. This corresponds to $\pi_*(M(2) \wedge M(2))$ by Spanier-Whitehead duality. Further, we want maps which are essential in $L_{K(1)}$. Thus we need to calculate $J_*(M(2) \wedge M(2))$. Using [12] we can check the following.

LEMMA 5.2. *The map*

$$\pi_{8k-1}(M(2) \wedge M(2)) \to J_{8k-1}(M(2) \wedge M(2))$$

*is onto and $J_{8k-1}(M(2) \wedge M(2)) = \mathbf{Z}/4 \oplus \mathbf{Z}/2$.*

LEMMA 5.3. *The class of order 2 in $J_{8k-1}(M(2) \wedge M(2))$ lifts to give a map*

$$\Sigma^{8k-1} M(2) \to M(2)$$

*which is detected by $K(1)_*$.*

PROOF. A specific choice for the generator involves composing the Bockstein $\delta$ with the periodicity element. This choice is non-trivial in $K(1)$. The other choice involves adding twice the generator of order 4, and the extra term is 0 on $K(1)$ since it is a multiple of 2. □

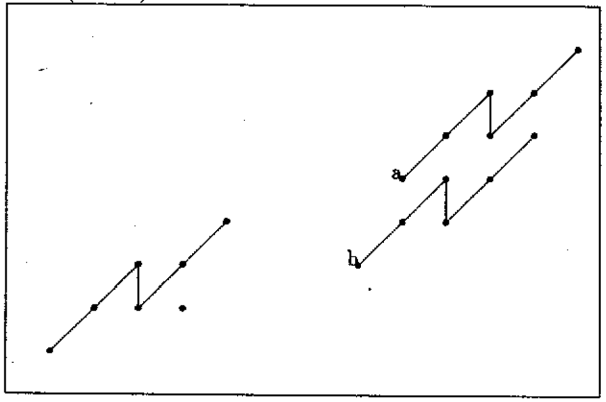
Given any class of order 4, if we add the class of order 2 from the lemma above the result still has order 4, so we get

COROLLARY 5.4. *There is a choice of the class of order 4 in $J_{8k-1}(M(2) \wedge M(2))$ so that a lifted map*

$$f_k : \Sigma^{8k-1} M(2) \to M(2)$$

*is detected by $K(1)_*$.*

The following picture gives the first few groups of $J_*(M(2))$. The pattern beginning in dimension 7 continues periodically every 8 dimensions. The class labeled $a$ in the following picture represents $v_1^4$ and $v_1^{4k}a$ is the corresponding generator in dimension $8k+8$. The class labeled $b$ represents a generator of the image of $J$ on the bottom cell. If $\rho_j \in \pi_{8k-1}(S^0)$ is a generator of the image of $J$, then $\rho_j a = v_1^{4j} b$. In addition, $v_1 a$ $(v_1 b)$ is a name for a generator of the $\mathbf{Z}/4$ in the 10 stem (9 stem).




Next we wish to calculate what $f = f_2$ does in homotopy.

THEOREM 5.5. i. $f_*(v_1^{4k} a) = v_1^{4(k+1)} b$.
ii. $f_*(v_1^{4k} v_1 a) = v_1^{4(k+1)} v_1 b + \eta v_1^{4k+8} a$.
iii. $f_*(v_1^{4k} b) = 0$
iv $f_*(v_1^{4k} v_1 b) = \eta v_1^{4k+8} b$.
v. *The classes $v_1^{4k} b, \eta v_1^{4k} a + v_1^{4k} v_1 b$, and their $\eta$ composites give the kernel of $f_*$.*

The proof of this follows easily from the calculations given in [13].

Now we can begin the construction of $X$. Let $X_1 = M(2)$. Let $X_2 = M(2) \cup_f \Sigma^{16} M(2)$. Let

$$p : X_2 \to \Sigma^{16} M(2)$$

be the pinch map. Theorem 5.5 allows one to define some classes in $\pi_{31}(X_2)$ which map to the kernel of $f$ under $p$. If we define

$$g_2 : \Sigma^{31} M(2) \to X_2$$



so that $g_* p_*$ maps onto the kernel of $p_*$, then Theorem 5.5 can be used to show that $g \cdot p$ induces in homotopy the same map as $\Sigma^{16} f_2$. Thus we can form $X_3 = X_2 \cup_g \Sigma^{32} M(2)$. Calculations with modified Adams spectral sequences can be made that show we can iterate this process and construct $X_\infty$.

THEOREM 5.6. *The homotopy of $L_1(X)$ satisfies*

$$\pi_*(L_1(X)) = KO_*(M(2)).$$

*The $K(1)_*$-theory of $X$ satisfies*

$$K(1)_*(X) = K(1)_*(S^0).$$

This discussions in this chapter have an interesting connection with the generating hypothesis. Let $X_n$ be the nth stage of the construction outlined immediately above, or the spectrum of the same name from Theorem 5.1. This is a finite spectrum with $KO_* X_n$ having order $2^n$. Yet the $E_2$ term of an appropriate modified Adams spectral sequence looks just like the $E_2$ term of the corresponding stage for a construction of $bo \wedge M^0(2)$. This spectrum has order 4. The generating hypothesis would require $\pi_* X_n$ to have elements of order $2^n$. It is hard to see how this might happen.

The $X_n$'s provide counterexamples at $p = 2$ to the most optimistic conjecture one might make as an $L_1$ version of the generating hypothesis, since the map $4 : X_n \to X_n$ is 0 on $\pi_* L_1 X_n$, but is of order $2^{n-2}$. There is an example of this sort in [7, Remark 1.7], our example has the property that 4 is of high order if $n$ is large.

## 6. An example for $\mathrm{Pic}_2$ at $p = 2$

Let $A_1$ be a complex which is free over $A(1)$ on one generator, where $A(1)$ is the sub-algebra of the Steenrod algebra generated by $Sq^1$ and $Sq^2$. Let $M_\nu$ be $\Sigma^{-4} HP^2$. We will prove the following result which gives a novel element in $\mathrm{Pic}_2$. This example is alluded to in [6]. Its importance in this context was certainly not appreciated at that time.

We believe this example gives an element of $\mathrm{Pic}_2$ that is not the localization of any ordinary sphere, but is in the kernel of the map (18) of section 7; that is, this element is not detected by doing algebra over the Morava stabilizer algebra. This is analogous to the role $P_5$ has in $\mathrm{Pic}_1$.

We use the notation $X^{(n)}$ for the $n$ skeleton of $X$.

THEOREM 6.1. *There is a map*

$$v_2 : \Sigma^6((A_1 \wedge M_\nu)^{(9)}) \to A_1 \wedge M_\nu$$

*so that if $B$ is the cofiber, then $H_*(H^*(B); Q_i) = \mathbf{Z}/2$ for $i = 0, 1, 2$. In particular, $L_{K(2)} B \in Pic_2$. In addition, if*

$$S^9 \to (A_1 \wedge M_\nu)^{(9)} \xrightarrow{v_2} A_1 \wedge M_\nu$$

*is a cofiber sequence, then,*

$$S^{15} \to \Sigma^6((A_1 \wedge M_\nu)^{(9)}) \xrightarrow{v_2} A_1 \wedge M_\nu$$

*is* $h_{20}^3 \neq *$.

We begin with some generalities. Let $X\langle 1\rangle$ be the Thom complex of the natural bundle over $\Omega S^2$. Let $X_5$ be the Thom complex of the natural bundle over $\Omega S^5$, and recall that $M_\nu = X_5^{(4)}$. The following is proved in [6]. In particular note that $X\langle 1\rangle = X_2$ and use [6, Theorem 3.4] for i and [6, Theorem 3.7] for ii.

**Theorem 6.2.** i. $CP^2 \wedge \mathbf{R}P^2 \wedge X_5 = \Sigma^3 X\langle 1\rangle$.
ii. *$A_1 \wedge X_5$ is an $X\langle 1\rangle$ module.*

We construct a self map $\Sigma^6 A_1 \wedge X_5 \to A_1 \wedge X_5$ using $v_2 \in \pi_6(X\langle 1\rangle)$ and the module structure over $X\langle 1\rangle$.

Now consider the diagram

$$\begin{array}{ccccc} A_1 \wedge M_\nu & \to & A_1 \wedge X_5 & \xrightarrow{p} & A_1 \wedge X_5/A_1 \wedge M_\nu \\ & & \uparrow{\scriptstyle v_2} & & \\ \Sigma^6 A_1 \wedge M_\nu & \xrightarrow{i} & \Sigma^6 A_1 \wedge X_5 & & \end{array}$$

We want to understand $pv_2i$. If this composite is null, we can make a $v_2$ map on all of $A_1 \wedge M_\nu$, otherwise there is some obstruction. We calculate a little of $Ext_A(H^*(A_1 \wedge X_5), \mathbf{Z}/2)$. This is quite easy and we will just state the result.

**Proposition 6.3.** *$Ext_A(H^*(A_1 \wedge X_5), \mathbf{Z}/2)$ has the following values. For $s = 0$ and $t = 0$ we have $\mathbf{Z}/2$. For $s = 1$ we have the following classes: $h_{20}$, $v_2$, $h_{21}$, $h_{30}$, $v_3, \cdots$. For $s = 2$ we have the following classes: $h_{20}^2$, $v_2^2$, $h_{20}v_2$, $h_{21}h_{20}$. For $s = 3$ we have just $h_{20}^3$ and these are all the classes for $t - s < 16$.*

We now begin the proof of Theorem 6.1. In the following diagram we construct the Adams resolution for $A_1 \wedge X_5$ through the range we are interested in and through level one. This is the left hand side. The right hand side is the corresponding resolution of $A_1 \wedge X_5/A_1 \wedge M_\nu$. The map between them is the map induced by $p$.

(11)
$$\begin{array}{ccc} F_1 & \xrightarrow{h} & \bar{F}_1 \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & \bar{E}_1 \\ \downarrow & & \downarrow \\ K(\mathbf{Z}/2, 0) & \xrightarrow{Sq(8)} & K(\mathbf{Z}/2, 8) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_1} \\ K(\mathbf{Z}/2; 6, 7, 12, 14, 15) & & K(\mathbf{Z}/2, 14, 15, 16) \end{array}$$



First note that $f_1$ can be represented by

$$(Sq(0,2), Sq(0,0,1), Sq(0,4), Sq(0,0,2), Sq(0,0,0,1)).$$

Similarly, $g_1$ can be represented by

$$(Sq(0,2), Sq(0,0,1), Sq(8)).$$

We use the fact that $g_1 Sq(8) = hf_1$ and

$$Sq(0,2)Sq(8) = Sq(8,2) + Sq(7,0,1)$$
$$Sq(8)Sq(0,2) = Sq(8,2) + Sq(4,1,1)$$

to see that

(12) $$h^*(\iota_{13}) = Sq(8)\iota_5 + (Sq(7) + S(4,1))\iota_6 + \iota_{13} + \epsilon_5 Sq(0,0,1)\iota_6 + \epsilon_4 Sq(1,2)\iota_6 + \epsilon_8 Sq(2,2)\iota_5 + \epsilon_9 Sq(1,0,1)\iota_5$$

where we don't know $\epsilon_5$ and we only know $\epsilon_4 + \epsilon_8 + \epsilon_9 = 0$. The projection of $v_2$ to $E_1$ lifts to $F_1$ and we get the factorization

$$\Sigma^6 A_1 \wedge X_5 \to F_1 \to E_1.$$

This map restricted to $F_1$ sends the fundamental class in dimension six to the generator and is zero on the other fundamental classes. Let us call this map $v_2$ also. Then (12) shows that

(13) $$i^* v_2^* h^*(\iota_{13}) = Sq(7) + Sq(4,1)$$

It follows that $pv_2i \neq 0$.

We now need to know if we can alter the map $v_2 i : \Sigma^6 A_1 \wedge M_\nu \to A_1 \wedge X_5$ so that it is null after composing with $p$ (and still the same on the bottom cell of the domain). If we had a map

(14) $$g : \Sigma^6 A_1 \wedge M_\nu \to A_1 \wedge X_5$$

such that $pg = pv_2i$ but $g|_{S^6} = *$ we could add this map to $v_2 i$ to give a map which lifts to a $v_2$ map $\Sigma^6 A_1 \wedge M_\nu \to A_1 \wedge M_\nu$. So we need to determine if such a map exists.

Suppose we have a $g$ as in (14). Using the low dimensional homotopy groups of the domain and range of $p$, it is easy to check that $pg|_{12\text{-skeleton}} = *$. Also, $g$ must be 0 on cohomology, so the continuation of $g$ to $E_1$ lifts to $F_1$:

$$\tilde{g} : \Sigma^6 A_1 \wedge M_\nu \to F_1$$

Since $pg = pv_2i$, $\tilde{g}^*\iota_{13} = Sq^2 Sq^1 Sq^4 \iota_6$ from equations (13) and (12).

We claim that while such a map can exist on the 13 skeleton (induced from taking $\tilde{g}$ as described and 0 on the other fundamental classes), it cannot be extended to all of $\Sigma^6 A_1 \wedge M_\nu$. We can extend to the 14 skeleton because $h_{30}$ which is represented by $\iota_{13}$ has order 2. We can extend to the 15 skeleton because the 15 cell is not attached to the 13 cell that hits the image of $h_{30}$. In order to extend to the last cell though, we need $v_1 h_{30} = *$, but

$$v_1 h_{30} = \langle v_1, h_{20}^2, v_1 \rangle$$

because $d_2 h_{30} = v_1 h_{20}^2$ in the ASS for $X(1)$, and $v_1 h_{30} + h_{30} v_1$ has image $v_1 h_{30}$ in $A_1 \wedge X_5$. From [**14**, Lemma 4.3] it follows that

$$\langle v_1, h_{20}^2, v_1 \rangle = h_{20}^2 v_1 \cup_1 v_1 = h_{20}^3$$

in $\pi_*(X(1))$, so the same is true in $A_1 \wedge X_5$, and $h_{20}^3 \neq *$. This proves that no map as in (14) exists unless we at least restrict to the 15 skeleton.

We will now show that there is such a map if we restrict to the 15 skeleton, i.e. there is a

$$g_3 : (\Sigma^6 A_1 \wedge M_\nu)^{(15)} \to A_1 \wedge X_5$$

with $pg_3 = pv_2 i|_{15\text{-skeleton}}$ and $g_3|_{S^6} = *$.

Let

$$g_1 : (\Sigma^6 A_1 \wedge M_\nu)^{(15)} \to A_1 \wedge X_5$$

be any map such that $pg_1|_{13\text{-skeleton}} = pv_2 i|_{13\text{-skeleton}}$ (such a map can be derived from $\tilde{g}$ as described above). If $p(g_1 - v_2 i)|_{14\text{-skeleton}} \neq *$, we construct a map

$$\varepsilon : \Sigma^6 A_1 \wedge M_\nu \to A_1 \wedge X_5$$

which is trivial on the 13 skeleton and such that $p\varepsilon = p(g_1 - v_2 i)$ when restricted to the 14 skeleton, using that $p_*$ is onto in dimension 14. So if we take $g_2 = g_1 - \varepsilon$ then $pg_2|_{14\text{-skeleton}} = pv_2 i|_{14\text{-skeleton}}$. We can repeat this in dimension 15 if necessary to produce $g_3$ so that $pg_3 = pv_2 i$ on the 15 skeleton. $g|_{15\text{-skeleton}} - g_3$ will be the desired map that lifts to $v_2 : \Sigma^6((A_1 \wedge M_\nu)^{(9)}) \to A_1 \wedge M_\nu$ as in Theorem 6.1.

This completes the proof.

We speculate that this example plays a role in $\mathrm{Pic}_2$ which is similar to the role played by $P_5$. We do not believe it is just a suspension of the sphere. The complete understanding of the role it will play will require substantially more information about $\mathrm{Pic}_2$.

## 7. Algebraic approximations

In this section we discuss an algebraic approximation to $\mathrm{Pic}_n$. The important observation is that we can use the Adams-Novikov spectral sequence to approximate $\pi_*(X)$. Roughly speaking, if $X$ and $Y$ are two elements of $\mathrm{Pic}_n$ and $E(n)_*(X) \neq E(n)_*(Y)$ as comodules, then we know they are different elements of $\mathrm{Pic}_n$. Furthermore, it will be easy to see that in the generic situation ($p$ large with respect to $n$), if $X$ and $Y$ give the same comodule structure, they are equal in $\mathrm{Pic}_n$.

PROPOSITION 7.1 (HOPKINS-RAVENEL). *If $X$ satisfies $v_{n-1}^{-1} BP_*(X) = 0$, then*



i. $BP_*(L_n X) = v_n^{-1} BP_*(X)$.
ii. *The Adams-Novikov spectral sequence for $L_n X$ converges to $\pi_*(L_n X)$, and*

$$E_2 = v_n^{-1} Ext^{*,*}_{BP_* BP}(BP_*, BP_*(X)).$$

The first part is part of [**19**, Theorem 1]. In the second part the identification of the $E_2$ term follows from the first part, and the convergence of the spectral sequence was communicated to us by Ravenel.

Recall that $E(n)_* = \mathbf{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$, where $E(n)$ is the Johnson-Wilson homology theory. From [**17**], $E(n)$ has the same Bousfield type as $K(0) \vee \cdots \vee K(n)$. We define a flat $E(n)_*$ algebra:

$$E_n = \mathbf{W}_{\mathbf{F}_{p^n}}[[u_1, \ldots, u_{n-1}]][u, u^{-1}].$$

Here $\mathbf{W}_{\mathbf{F}_{p^n}}$ is the Witt vectors of the field $\mathbf{F}_{p^n}$. It is isomorphic to $\mathbf{Z}_p[\xi]$ where $\xi$ is a primitive $p^n - 1$ root of 1. The $u_i$'s all have degree 0, and $|u| = -2$. The map from $E(n)_*$ to $E_n$ sends $v_i$ to $u_i u^{1-p^i}$ and $v_n$ to $u^{1-p^n}$.

By composing the map $E(n)_* \to E_n$ with the usual map $BP_* \to E(n)_*$, we get a map from $BP_*$ to $E_n$. Since $E_n$ is flat over $E(n)_*$, this induces a complex oriented homology theory by

$$E_n(X) = E_n \otimes_{BP_*} BP_*(X) = E_n \otimes_{E(n)_*} E(n)_*(X).$$

Morava proves in [**15**]

PROPOSITION 7.2 (MORAVA). *If $BP_*(X)$ is $v_0, \ldots, v_{n-1}$ torsion, then*

$$\mathbf{W}_{\mathbf{F}_{p^n}} \otimes_{\mathbf{Z}_{(p)}} v_n^{-1} Ext^{*,*}_{BP_* BP}(BP_*, BP_*(X)) = H^{*,*}(S_n, E_n(X)).$$

Here $S_n$ is the $n^{\text{th}}$ Morava stabilizer group. This is a $p$-adic Lie group related to the study of height $n$ formal group laws in characteristic $p$.

**Example:** Take $n = 1$, $p > 2$. Then $E_1$ is completed complex $K$-theory, and $S_1$ is the group of units in $\mathbf{Z}_p$ congruent to 1 modulo $p$, which is isomorphic to $\mathbf{Z}_p$. To see how this operates on $E_1(X)$, a generator for this group is the Adams operation $\psi^{p+1}$. Note that $S_1$ acts non-trivially on $E_1(S^0)$.

Now for our purposes, we are less interested in the homology theory $E_n$ than in a theory we define below. This theory will be the same as $E_n$ in case $BP_*(X)$ is finitely generated, but will have the advantage of coming with a more tractable Adams-Novikov spectral sequence. It will *not* be a homology theory; we have an inverse limit in our definition that will prevent this theory from satisfying the direct limit axiom.

We confuse $E_n$ with its representing spectrum and inductively on $k < n$, define $E_n/(p^{i_0}, \ldots, v_k^{i_k})$ to be the cofiber of

$$v_k^{i_k} : \Sigma^{i_k |v_k|} E_n/(p^{i_0}, \ldots, v_{k-1}^{i_{k-1}}) \to E_n/(p^{i_0}, \ldots, v_{k-1}^{i_{k-1}}).$$

(The spectrum $E(n)/(p^{i_0},\dots,v_{k-1}^{i_{k-1}})$ is defined similarly.) In case there is a finite spectrum $M(p^{i_0},\dots,v_k^{i_k})$ satisfying

$$BP_*(M(p^{i_0},\dots,v_k^{i_k})) = BP_*/(p^{i_0},\dots,v_k^{i_k}),$$

$E_n/(p^{i_0},\dots,v_k^{i_k}) = E_n \wedge M(p^{i_0},\dots,v_k^{i_k})$.

We have

$$E_n \simeq \varprojlim_{(i_0,\dots,i_{n-1})} E_n/(p^{i_0},\dots,v_{n-1}^{i_{n-1}})$$

so we view $E_n$ as a "pro-spectrum" and use this to define the theory below.

DEFINITION 7.3. *The $n^{th}$ Morava module of $X$, $\mathcal{K}_{n,*}(X)$ is defined by*

$$\mathcal{K}_{n,*}(X) = \varprojlim_{(i_0,\dots,i_{n-1})} [E_n/(p^{i_0},\dots,v_{n-1}^{i_{n-1}})]_*(X).$$

Since smashing doesn't commute with inverse limits, $\mathcal{K}_{n,*}(X)$ is not, in general, the same as $E_n(X)$ though the two theories coincide when $X$ is finite.

PROPOSITION 7.4. *If $\mathcal{K}_{n,*}(X)$ is finitely generated over $E_n$, then there is a spectral sequence*

$$E_2 = H^{*,*}(S_n;\mathcal{K}_{n,*}(X)) \Longrightarrow \mathbf{W}_{\mathbf{F}_{p^n}} \otimes_{\mathbf{Z}_p} \pi_*(L_{K(n)}X).$$

PROOF. This spectral sequence is constructed by taking the inverse limit of the Adams-Novikov spectral sequences for $L_n(X \wedge M(p^{i_0},\dots,v_{n-1}^{i_{n-1}}))$. Each of these spectral sequences converges and has an $E_2$ term of the given form by Proposition 7.1 and Proposition 7.2. The finiteness condition in the hypotheses guarantees that all relevant $\lim^1$ terms vanish. Also, these spectral sequences are 0 in negative homological degrees. Together these conditions imply that the inverse limit of these spectral sequences converges to the inverse limit of the abutments, which will be the homotopy of the inverse limit of $L_n(X \wedge M(p^{i_0},\dots,v_{n-1}^{i_{n-1}}))$.

Since $M(p^{i_0},\dots,v_{n-1}^{i_{n-1}})$ is $L_{n-1}$ acyclic,

$$L_n(X \wedge M(p^{i_0},\dots,v_{n-1}^{i_{n-1}})) = L_{K(n)}(X \wedge M(p^{i_0},\dots,v_{n-1}^{i_{n-1}})).$$

We need to verify that $L_{K(n)}$ commutes with this inverse limit. Clearly the inverse limit is local, so we need only check that the map

$$X \wedge K(n) \to [\varprojlim_I L_n(X \wedge M(p^{i_0},\dots,v_{n-1}^{i_{n-1}}))] \wedge K(n)$$

induced by the inclusion of the bottom cells in the generalized Moore spectra is an equivalence. To do this, let $D$ be some fixed generalize Moore spectrum of



type $n$. We consider the diagram

$$\begin{array}{ccc} X \wedge K(n) & \longrightarrow & [\varprojlim_I L_n(X \wedge M(p^{i_0},\dots,v_{n-1}^{i_{n-1}}))] \wedge K(n) \\ \downarrow & & \downarrow \\ X \wedge D \wedge K(n) & \longrightarrow & [\varprojlim_I L_n(X \wedge M(p^{i_0},\dots,v_{n-1}^{i_{n-1}}) \wedge D)] \wedge K(n). \end{array}$$

(Since $D$ is finite, it doesn't matter where it is placed in the parentheses in the lower right item.) $D \wedge K(n)$ is a wedge of $K(n)$'s indexed over the cells of $D$, so the vertical maps are inclusions of a summand in a wedge.

On the other hand, once $\alpha$ is sufficiently large, $M(\alpha) \wedge D$ is a wedge of copies of $D$ indexed by the cells of $M(\alpha)$ and the map

$$M(\alpha') \wedge D \to M(\alpha) \wedge D$$

is the identity on the $D$ indexed by the bottom cells and trivial on the others if $\alpha$ is sufficiently large and $\alpha'$ is sufficiently larger than $\alpha$. It follows that the bottom map is an equivalence, and hence that the top one is also. □

The theories $\mathcal{K}_{n,*}(-)$ are of interest to us because of Theorem 1.3. We owe the reader a proof of (ii) $\Leftrightarrow$ (iii) which we supply at this point.

COMPLETION OF PROOF OF THEOREM 1.3. The implication (iii) implies (ii) is easy; $K(n)_*(Z)$ is just the reduction of $\mathcal{K}_{n,*}(Z)$ modulo $(p, v_1, \dots, v_{n-1})$.

To show the other implication, assume $K(n)_*(Z) = K(n)_*$.

We claim that

$$[E(n)/(p^{i_0},\dots,v_{n-1}^{i_{n-1}})]_*(Z) = [E(n)/(p^{i_0},\dots,v_{n-1}^{i_{n-1}})]_*. \tag{15}$$

The proof is by induction on $N = i_0 + \cdots + i_{n-1}$. The base case is $N = n$, in which case $E(n)/(p^{i_0},\dots,v_{n-1}^{i_{n-1}}) = K(n)$ so (15) holds.

First we observe (by induction) that $[E(n)/(p^{i_0},\dots,v_{n-1}^{i_{n-1}})]_*(Z)$ is a monogenic $E(n)_*$-module, with generator in even degree. This is true by hypotheses if $N = n$. If $N > n$, let $j$ be such that $i_j > 1$.

The cofibration

$$\begin{aligned} &\Sigma^{2p^j-2} E(n)/(p^{i_0},\dots,v_j^{i_j-1},\dots,v_{n-1}^{i_{n-1}}) \\ &\quad \to E(n)/(p^{i_0},\dots,v_{n-1}^{i_{n-1}}) \to E(n)/(p^{i_0},\dots,v_j^1,\dots,v_{n-1}^{i_{n-1}}) \end{aligned} \tag{16}$$

shows that $[E(n)/(p^{i_0},\dots,v_{n-1}^{i_{n-1}})]_*(Z)$ is concentrated in even degrees (since this is true for the homology theories represented by the end terms of cofibration(16)).

We also have a cofibration

$$\begin{aligned} &\Sigma^{2p^j-1} E(n)/(p^{i_0},\dots,v_{n-1}^{i_{n-1}}) \xrightarrow{v_j} E(n)/(p^{i_0},\dots,v_{n-1}^{i_{n-1}}) \to \\ &E(n)/(p^{i_0},\dots,v_j^1,\dots,v_{n-1}^{i_{n-1}}) \vee \Sigma^{(i_j-1)(2p^j-2)+1} E(n)/(p^{i_0},\dots,v_j^1,\dots,v_{n-1}^{i_{n-1}}) \end{aligned} \tag{17}$$

It follows that if $[E(n)/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})]_*(Z)$ has more than one generator, then so does the quotient of multiplication by $v_j$. But by cofibration (17), this is just $[E(n)/(p^{i_0}, \ldots, v_j^1, \ldots, v_{n-1}^{i_{n-1}})]_*(Z)$, which is monogenic by inductive hypothesis.

Once we know $[E(n)/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})]_*(Z)$ is monogenic, it follows that the map from $E(n)/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$ to $E(n)/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}) \wedge Z$ given by a generator is a surjection, and hence by counting (only finitely many elements in each dimension) an injection.

So by induction, (15) holds for all $N$. This implies that

$$\varprojlim_{(i_0,\ldots,i_{n-1})} E(n)/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})_* = \varprojlim_{(i_0,\ldots,i_{n-1})} E(n)/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})_*(Z)$$

which shows that $\mathcal{K}_{n,*}(Z) = \mathcal{K}_{n,*}(S^0) = E_n$. □

To describe our algebraic approximation, suppose $X \in \mathrm{Pic}_n$. Then $\mathcal{K}_{n,*}(X)$ is isomorphic to some suspension of $E_n$. So we have a map

(18) $\qquad \mathrm{Pic}_n \to \{\text{isomorphism classes of rank 1 } E_n\text{-}S_n\text{-modules}\}.$

By an $E_n$-$S_n$-module, we mean an $E_n$-module $M$ with an $S_n$-action on $M$ commuting with the $E_n$-action in the following way:

$$\begin{array}{ccc} E_n \otimes M & \xrightarrow{g \otimes g} & E_n \otimes M \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & M. \end{array}$$

for any $g \in S_n$.

We would like to understand the set on the right side of (18) a bit better. Let $K$ be some rank one $E_n$-$S_n$-module. Because the $S_n$ action is compatible with the $E_n$-module structure on $K$, to describe $K$ it suffices to describe the $S_n$ action on $K_0 \simeq \mathbf{W}_{\mathbf{F}_{p^n}}[[u_1, \ldots, u_{n-1}]]$. So

(19) $\qquad \{\text{isomorphism classes of rank 1 } E_n\text{-}S_n\text{-modules}\} =$
$\qquad\qquad \{\text{isomorphism classes of rank 1 } \mathbf{W}_{\mathbf{F}_{p^n}}[[u_1, \ldots, u_{n-1}]]\text{-}S_n\text{-modules}\}.$

Now both of these are actually groups (the group operation is tensor product), and by Proposition 8.4, the right hand side of (19) is isomorphic to

$$H^1(S_n; \mathbf{W}_{\mathbf{F}_{p^n}}[[u_1, \ldots, u_{n-1}]]^\times).$$

PROPOSITION 7.5. *The map* $\alpha : \mathrm{Pic}_n \to H^1(S_n; \mathbf{W}_{\mathbf{F}_{p^n}}[[u_1, \ldots, u_{n-1}]]^\times)$ *is an injection if* $n^2 \le q$ *and* $p > 2$.

PROOF. If $n^2 \le q = 2p-2$ and $p > 2$ the spectral sequence of Proposition 7.4 collapses. This is because the sparseness of the Adams-Novikov spectral sequence [**18**, 4.4.2] tells us the first possible differential is $d_{q+1}$, and [**18**, 6.2.10] says that differential is too long to exist in this spectral sequence.



Now suppose $X$ is in the kernel of $\alpha$. Then the $E_2$ term of the Adams-Novikov spectral sequence for $X$ is isomorphic to that for $L_{K(n)}S^0$, so there is a class in the $E_2$ term corresponding to $S^0 \to L_{K(n)}S^0$. If we use this class to map $S^0$ to $X$ (using the collapsing of the spectral sequence), and extend to $L_{K(n)}S^0 \to X$, by naturality we see that it is an isomorphism on $E_2$ terms (we are mapping the $E_2$ term for the sphere (a ring) to the other $E_2$ term (a copy of the same ring) by sending 1 to 1). □

At this point it is convenient to prove the following.

PROPOSITION 7.6. *$\mathrm{Pic}_n$ is a set.*

PROOF. We note that Proposition 7.5 implies that $\mathrm{Pic}_n$ is a set if $p$ is sufficiently large. For general $p$ we show that $\mathrm{Pic}_n$ has a finite filtration so that each sub-quotient is a set.

To define the filtration, we consider the spectral sequence of Proposition 7.4. This spectral sequence collapses at some finite $E_{m+1}$ (depending only on $p$ and $n$) since the same is true for the localized Adams-Novikov spectral sequence in Proposition 7.1 and this spectral sequence is obtained from an inverse limit of those.

Let $F_0 = \mathrm{Pic}_n$. Let $X \in \mathrm{Pic}_n$ be in $F_1$ if the image of $X$ is 0 under the map in (18). (This is equivalent to $\mathcal{K}_{n,*}(X) = \mathcal{K}_{n,*}(S^0)$ as a module over $S_n$.) If $X \in F_1$, let $\iota_X$ be the fundamental class in the $E_2$ term of the spectral sequence for $X$ ($\iota_X = 1$ in the $E_2$ term of that spectral sequence). $X \in F_i$ if $d_2(\iota_X) = \cdots = d_i(\iota_X) = 0$. Note then that $F_m = F_{m+1} = \cdots$ since the spectral sequence collapses at $E_{m+1}$, and by the proof of Proposition 7.5, $F_m = \{L_{K(n)}S^0\}$.

$F_0/F_1$ is clearly isomorphic to the image of the map (18), and hence is a set. We claim that $F_i/F_{i+1}$ is in one-to-one correspondence with a subset of the possible $d_{i+1}(\iota)$'s.

Suppose $X$ and $X'$ are in $F_i$ and $d_{i+1}(\iota_X) = d_{i+1}(\iota_{X'})$. Let $Y \in \mathrm{Pic}_n$ be such that $L_{K(n)}(X \wedge Y) \simeq L_{K(n)}S^0$, so $\iota_{X \wedge Y}$ is a permanent cycle. It follows (from the multiplicative property of Adams spectral sequences [**18**, Theorem 2.3.3]) that $d_{i+1}(\iota_X) = -d_{i+1}(\iota_Y)$, and hence that $L_{K(n)}(X' \wedge Y) \in F_{i+1}$.

In other words, if $X$ and $X'$ are any two elements of $F_i$ such that $d_{i+1}(\iota_X) = d_{i+1}(\iota_{X'})$, $X$ and $X'$ "differ" by an element of $F_{i+1}$.

Since this is a finite filtration on $\mathrm{Pic}_n$, and the associated graded is a set, so is $\mathrm{Pic}_n$. □

## 8. Digression on Picard groups of $G$-modules

The material in this section is standard, but we reproduce it here in the form in which we use it.

Suppose $G$ is a finite group, $R$ is a commutative ring, and $G$ acts on $R$ by ring automorphisms.

Let $C$ be the category of $R$ modules with "compatible" $G$ actions. $M \in C$ means $G$ acts on $M$ by group homomorphisms and the compatibility axiom is the commutativity of

$$\begin{array}{ccc} R \otimes M & \xrightarrow{g \otimes g} & R \otimes M \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & M. \end{array}$$

Notice that $C$ is a symmetric monoidal category, where the operation on underlying $R$ modules, is $\otimes_R$ and the $G$ action is the diagonal one. We will denote the operation by $\wedge$.

Definition 8.1. *$Pic_C$ is the abelian group of isomorphism classes of invertible objects in $C$, that is the set of $M \in C$ such that there is some $N \in C$ with $M \wedge N \simeq R$.*

Lemma 8.2. *If $R$ is local, $M, N$ are $R$ modules such that $M \otimes_R N \simeq R$ and $M$ is finitely generated, then $M \simeq R$. If $R$ is noetherian we can drop the finitely generated hypothesis on $M$.*

Proof. There is a proof of the first claim in [4, 2.5.4.3]. The proof is by considering the isomorphism $M \otimes_R N \simeq R$ modulo the maximum ideal $\mathfrak{m}$ of $R$ to see that $M/\mathfrak{m}M$ is isomorphic to the residue field of $R$. Then Nakayama's lemma implies $M$ is isomorphic to $R$.

If $R$ is noetherian, then since $\mathfrak{m}$ is finitely generated, the fact that $M/\mathfrak{m}M$ is the residue field implies $M$ is finitely generated automatically. □

Corollary 8.3. *If $R$ is local and noetherian then $M \in Pic_C$ if and only if the underlying $R$-module of $M$ is free of rank one.*

Proof. The "only if" follows from above. To see the "if," take $M^{-1}$ the $R$-module $\mathrm{Hom}_R(M, R)$ with the $G$-action

$$(gf)(m) = g(f(g^{-1}m)).$$

It is easy to check that the evaluation map

$$M \otimes_R \mathrm{Hom}_R(M, R) \to R$$

is a $G$-map. □

From now on, we assume $R$ is a ring such that $M \otimes_R N \simeq R$ implies $M \simeq R$ (note that $\mathbf{W}_{\mathbf{F}_{p^n}}[[u_1, \ldots, u_{n-1}]]$ is local and noetherian). Then $\mathrm{Pic}_C$ is the group of isomorphism classes of free rank one $G$-$R$-modules. We want to prove the following proposition.

Proposition 8.4. *$Pic_C = H^1(G, R^\times)$.*



The proof proceeds in two steps. First we identify $\mathrm{Pic}_C$ with the group of crossed homomorphism modulo principal crossed homomorphisms, and then we identify that group with the cohomology group. The second step can be found in [10, X.3].

Definition 8.5. *A crossed homomorphism from a group $G$ to a $G$-module $V$ is a set map $f : G \to V$ such that $f(gh) = {}^g f(h) + f(g)$. The set of crossed homomorphism from $G$ to $V$ forms a group by using the group operation in $V$.*

Suppose $M$ is in $\mathrm{Pic}_C$, and $e \in M$ is an $R$-module generator. This defines a crossed homomorphism

$$G \xrightarrow{f_{M,e}} R^\times$$

as follows: let $g \in G$, then $ge$ is another $R$-module generator of $M$ since $g$ is an automorphism of $R$, so ${}^g e = u_g e$ where $u_g \in R^\times$. Let $f_{M,e}(g) = u_g$. Since $u_{gh} e = {}^{gh} e = {}^g(u_h e) = {}^g u_h {}^g e = {}^g u_h u_g e$ it follows that $f_{M,e}$ is a crossed homomorphism. (we are using multiplicative notation for the group $R^\times$).

It is easy to check that if $(M, e)$ and $(N, e')$ are elements of $\mathrm{Pic}_C$ together with specific choices of $R$-module generators, then $f_{M \wedge N, e \otimes e'} = f_{M,e} \cdot f_{N,e'}$.

We define the group $\widetilde{\mathrm{Pic}}_C$ to be the set of elements of $\mathrm{Pic}_C$ together with specific choices of $R$-module generators. There is an obvious quotient homomorphism (forget the generator)

$$\Pi : \widetilde{\mathrm{Pic}}_C \to \mathrm{Pic}_C,$$

and the correspondence $(M, e) \to f_{M,e}$ is a homomorphism from $\widetilde{\mathrm{Pic}}_C$ to the group of crossed homomorphisms.

This homomorphism is actually an isomorphism: if $f$ is a crossed homomorphism $G \to R^\times$, define a $G$ action on $M = R, e = 1$ by ${}^g e = f(g)e$, ${}^g m = {}^g m f(g) e$. This gives a compatible group action, and $f = f_{M,e}$.

Using this isomorphism to identify $\widetilde{\mathrm{Pic}}_C$ with the group of crossed homomorphisms, the kernel of $\Pi$ is the set of crossed homomorphisms of the form $g \to u_g \in R^\times$ when $u_g r = {}^g r$ for some fixed $r \in R^\times$. These are called *principal* crossed homomorphisms.

We've just proven the first isomorphism in

$$\mathrm{Pic}_C \cong \frac{\{\text{crossed homomorphisms: } G \to R^\times\}}{\{\text{principal crossed homomorphisms}\}} \cong H^1(G; R^\times).$$

We sketch a verification of the second equality replacing $R^\times$ with any $G$-module $V$. Recall the bar complex: $B_n = \mathbb{Z}[G]^{\times(n+1)}$ and $H^*(G; V)$ is the cohomology of the cocomplex $\mathrm{Hom}_G(B_*, V)$.

Now $\mathrm{Hom}_{\mathrm{sets}}(G^n, V) = \mathrm{Hom}_G(B_n, V)$ by the correspondence

$$\{(g_1, \ldots, g_n) \to f(g_1, \ldots, g_n)\} \leftrightarrow \{g[g_1 | \cdots | g_n] \to {}^g f(g_1, \ldots, g_n)\}.$$

Using this identification, it is straightforward to verify that $f \in Z^1 \Leftrightarrow f$ is a crossed homomorphism to $V$, and that $f \in B^1 \Leftrightarrow f$ is a principal crossed homomorphism to $V$.

## 9. The $p$-adic integers embed in $\mathrm{Pic}_n$

In this section we will construct a homomorphism $\mathbf{Z}_p \hookrightarrow \mathrm{Pic}_n$, that extends the obvious homomorphism $\mathbf{Z} \hookrightarrow \mathrm{Pic}_n$ obtained by $m \mapsto L_{K(n)}S^{m|v_n|}$.

This construction will generalize the construction summarized in (3) in section 1.

We denote by $M^0(p^{i_0}, \dots, v_{n-2}^{i_{n-2}})$ any finite spectrum with top cell in dimension 0, satisfying

$$BP_*(M^0(p^{i_0}, \dots, v_{n-2}^{i_{n-2}})) = BP_*/(p^{i_0}, \dots, v_{n-2}^{i_{n-2}})$$

up to suspension. (The restriction on the dimension of the top cell puts the bottom cell in dimension $1-n-i_1|v_1|-\cdots-i_{n-2}|v_{n-2}|$, which gives the dimension of the omitted suspension above.) We abbreviate the ideal $(p^{i_0}, \dots, v_{n-2}^{i_{n-2}})$ by $(V^I)$.

We will use the direct system of spectra

$$M^0(p^{i_0}, \dots, v_{n-2}^{i_{n-2}}) \xrightarrow{\phi} M^0(p^{j_0}, \dots, v_{n-2}^{j_{n-2}}) \to \cdots$$

(with $j_m \geq i_m$ for all $m$), such that $\phi$ induces

$$1 \in BP_*/(V^I) \mapsto p^{j_0 - i_0} v_1^{j_1 - i_1} \cdots v_{n-2}^{j_{n-2} - i_{n-2}} \in BP_*/(V^J).$$

and commutes with the pinch maps to the top cells.

The nilpotence theorem ensures that given a multi-index $I'$, there is a multi-index $I$ with $i_m \geq i'_m$ for all $m$ such that $M(V^I)$ exists. It also guarantees that for any multi-index $I$ such that $M(V^I)$ exists, there is a multi-index $J$ with $j_m > i_m$ such that a map $\phi$ as above exists. By another application of the nilpotence theorem, we can fix a cofinal sequence of spectra in this direct system.

Now fix a multi-index $I$ so that $M^0(V^I)$ is in our sequence, and fix a $\lambda \in \mathbf{Z}_p$. We will construct a spectrum we call $M(\lambda)(V^I)$. This is one of two main steps in our construction. We will then use the collection of $M(\lambda)(V^I)$'s as $I$ varies to construct a spectrum $S_\lambda$ such that $L_{K(n)}S_\lambda \in \mathrm{Pic}_n$ and $S_\lambda \wedge M^0(V^I) \simeq M(\lambda)(V^I)$.

Let $t$ be such that $M^0(V^I)$ has a $v_{n-1}^t$ map, and define

$$M_r^0(V^I) = M^0(p^{i_0}, \dots, v_{n-2}^{i_{n-2}}, v_{n-1}^{tr}) = \mathrm{cofiber}([v_{n-1}^t]^r).$$

Since $M_r^0(V^I)$ is defined using iterates of the $v_{n-1}^t$ map, these spectra come with maps

$$M_r^0(V^I) \xrightarrow{\iota_r} M_{r+1}^0(V^I)$$

inducing the obvious map on $BP_*$ (multiplication by $v_{n-1}^t$).



Before going on, we briefly describe how we are about to construct $M(\lambda)(V^I)$. If $\lambda$ happens to be an ordinary integer, we can make the spectrum

$$\Sigma^{\lambda|v_n|}M^0(p^{i_0}, \dots, v_{n-2}^{i_{n-2}}, v_{n-1}^{\infty}) = \varinjlim_{\iota_r} \Sigma^{\lambda|v_n|}M^0(v^{i_0}, \dots, v_{n-2}^{i_{n-2}}, v_{n-1}^{tr}) = \varinjlim_{\iota_r} \Sigma^{\lambda|v_n|}M_r^0(V^I).$$

If $\lambda$ is some $p$-adic integer instead, we observe that $\Sigma^m L_n M_r^0(V^I)$ depends only on the residue of $m$ modulo the degree of a $v_n$ self map of $M_r^0(V^I)$. So for each $r$, one can define $\Sigma^{\lambda|v_n|}L_n M_r^0(V^I)$ to be the $k$th suspension of $L_n M_r^0(V^I)$ where $k = \lambda|v_n|$ modulo the degree of a $v_n$ self map of $M_r^0(V^I)$. What we want to do is fit these together in a direct system and take the limit.

Now we continue with our construction. Each $M_r^0(V^I)$ admits a $v_n^{k_r}$ map for some $k_r$. Without loss of generality, assume that $k_{r+1}$ is a power of $p$ times $k_r$, and that these maps commute with $\iota_r$. We are allowed to make these assumptions by Theorem 9 of [9].

Let $x_r \in \mathbf{Z}$ be a decreasing sequence with $x_r \equiv \lambda$ modulo $k_r$. Then $x_{r-1}-x_r = u_{r-1}k_{r-1}$. To construct $M(\lambda)(V^I)$ we take the direct limit of

$$\Sigma^{x_1|v_n|}M_1^0(V^I) \xrightarrow{[v_n^{k_1}]^{u_1}} \Sigma^{x_2|v_n|}M_1^0(V^I) \xrightarrow{\Sigma^{x_2}\iota_1} \Sigma^{x_2|v_n|}M_2^0(V^I) \xrightarrow{[v_n^{k_2}]^{u_2}}$$
$$\to \Sigma^{x_3|v_n|}M_2^0(V^I) \xrightarrow{\Sigma^{x_3}\iota_2} \Sigma^{x_3|v_n|}M_3^0(V^I) \to \cdots.$$

Because the maps $[v_n^{k_r}]^{u_r}$ are $K(n)_*$ isomorophisms,

$$K(n)_*(M(\lambda)(V^I)) = K(n)_*(\varinjlim_{\iota_r} M_r^0(V^I)) = K(n)_*(M^0(V^I)).$$

Note that $M(\lambda)(V^I)$ is not very well defined. Firstly, it obviously depends on the original choice of $M^0(V^I)$, and not necessarily just $I$. Besides that, it may depend on the choices of the $x_r$ and hence of the $k_r$. Nevertheless we have the following proposition.

PROPOSITION 9.1. *$L_n M(\lambda)(V^I)$ is well defined up to the choice of $M^0(V^I)$.*

PROOF. We begin by assuming we have one system of $v_n$ maps which we will call $v_n^{k_r}$ and another system we call $v_n^{k'_r}$.

Now make choices of $x_r$ and $x'_r$ as in the construction for $M(\lambda)(V^I)$. The choices of $x_r$ ensure that $L_n\Sigma^{x_r}M_r^0(V^I)$ and $L_n\Sigma^{x_r}M_{r-1}^0(V^I)$ are independent of the choice. Also, $x_r - x'_r$ is divisible by the smaller of $k_r$ or $k'_r$, so there is a $v_n$ map between any stage of the two direct systems in one direction or the other. This map commutes with $\iota_r$ by assumption, and after applying $L_n$ is invertible. This gives an equivalence between the two direct systems after applying $L_n$.

We also have to check independence of the choice of $t$ and $v_{n-1}^t$ map on $M^0(V^I)$. By the nilpotence theorem, it suffices to compare the construction for $v_{n-1}^t$ to that for $t' = st$ and $v_{n-1}^{t'} = [v_{n-1}^t]^s$. In this case, a choice of $v_n^{k_r}$ and $x_r$

for $v_{n-1}^{t}$ gives us $k_r' = k_{sr}$ and $v_n^{k_r'} = v_n^{k_{sr}}$, and the system for $t'$ is obviously a cofinal subsystem of the system for $t$. □

PROPOSITION 9.2. *Let*

$$\phi : M^0(V^I) \to M^0(V^J)$$

*be a map in our sequence of spectra (so $J \geq I$).*

(i) *There are choices of $t$ and $v_{n-1}^t$ (one choice of $t$ for both $M^0(V^I)$ and $M^0(V^J)$) so that $\phi$ induces maps*

$$\phi_r : M_r^0(V^I) \to M_r^0(V^J)$$

*commuting with the pinch maps to the top cells and taking*

$$1 \in BP_*/(V^I, v_{n-1}^{tr}) \mapsto p^{j_0 - i_0} v_1^{j_1 - i_1} \cdots v_{n-2}^{j_{n-2} - i_{n-2}} \in BP_*/(V^J, v_{n-1}^{tr}).$$

(ii) *The spectra $M(\lambda)(V^I)$ and $M(\lambda)(V^J)$ can be defined so that the $\phi_r$ induce a map*

$$\bar{\phi} : M(\lambda)(V^I) \to M(\lambda)(V^J).$$

PROOF. To see that the first condition can be satisfied, just take $t$ large enough so that $v_{n-1}^t$ maps exist on both spectra, and then invoke the centrality result of the nilpotence theorem (to take $t$ even larger if necessary) in order that

$$\begin{array}{ccc} M^0(V^I) & \xrightarrow{\phi} & M^0(V^J) \\ {\scriptstyle v_n^t}\downarrow & & \downarrow{\scriptstyle v_n^t} \\ \Sigma^{-t|v_n|} M^0(V^I) & \xrightarrow{\phi} & \Sigma^{-t|v_n|} M^0(V^J) \end{array}$$

commutes. Define $\phi_1$ on the cofiber to make the ladder of cofibrations obtained by extending downward commute.

Then inductively define $\phi_r$ so that the ladder of cofiber sequences

$$\begin{array}{ccccc} M_1^0(V^I) & \longrightarrow & M_r^0(V^I) & \longrightarrow & M_{r+1}^0(V^I) \\ {\scriptstyle \phi_1}\downarrow & & {\scriptstyle \phi_r}\downarrow & & \downarrow{\scriptstyle \phi_{r+1}} \\ M_1^0(V^J) & \longrightarrow & M_r^0(V^J) & \longrightarrow & M_{r+1}^0(V^J) \end{array}$$

commutes.

To satisfy (ii), we note that by the nilpotence theorem if we take the $k_r$'s sufficiently large, both $M_r^0(V^I)$ and $M_r^0(V^J)$ have $v_n^{k_r}$ maps commuting with $\phi_r$. □

We can now complete our construction; take

$$S(\lambda) = \varinjlim_I L_n M(\lambda)(V^I).$$



By construction,

$$\begin{aligned} K(n)_*(S(\lambda)) &= \varinjlim_I K(n)_* M(\lambda)(V^I) \\ &\doteq \varinjlim_I K(n)_* \Sigma^{x_1} M^0(p^{i_0}, \dots, v_{n-2}^{i_{n-2}}, v_{n-1}^{\infty}) \\ &= K(n)_* \Sigma^{x_1} M^0(p^{\infty}, \dots, v_{n-1}^{\infty}) = \Sigma^{x_1} K(n)_*, \end{aligned}$$

so $L_{K(n)} S(\lambda) \in \mathrm{Pic}_n$.

PROPOSITION 9.3. (i) *The map $\mathbf{Z}_p \to Pic_n$ by*

$$(20) \qquad \lambda \mapsto L_{K(n)} S(\lambda)$$

*is a homomorphism.*

(ii) *The map (20) is an injection.*

PROOF. To show our map is a homomorphism, we first note that once we choose a map

$$M^0(V^I) \wedge M^0(V^I) \to M^0(V^I)$$

(by mapping out to the top cell of, say, the leftmost factor of the spectrum on the left) the diagonal of the smash product of the direct systems defining $M(\lambda)(V^I)$ and $M(\lambda')(V^I)$ maps to the direct system defining $M(\lambda + \lambda')(V^I)$.

These maps commute as $I$ increases, giving a map $S(\lambda) \wedge S(\lambda') \to S(\lambda + \lambda')$ that is a $K(n)_*$-isomorphism.

Finally, to show this homomorphism is injective, suppose

$$L_{K(n)} S(\lambda) = L_{K(n)} S^0.$$

We observe that

$$S(\lambda) \wedge M^0(V^I) \simeq L_n M(\lambda)(V^I)$$

for any $\lambda$.

By our hypothesis then,

$$\begin{aligned} &L_{K(n)} M(\lambda)(V^I) \simeq L_{K(n)}[S(\lambda) \wedge M^0(V^I)] = \\ &\quad L_{K(n)}[L_{K(n)} S(\lambda) \wedge M^0(V^I)] = L_{K(n)}[L_{K(n)} S^0 \wedge M^0(V^I)] = L_{K(n)} M^0(V^I). \end{aligned}$$

Hence for each $r$, we have a map $L_{K(n)} M_r^0(V^I) \to L_{K(n)} M(\lambda)(V^I)$ that is non-zero on the $K(n)$ homology coming from the top cells. But this implies $\Sigma^{x_r|v_n|} L_{K(n)} M_r^0(V^I) \simeq L_{K(n)} M_r^0(V^I)$ for all $r$, and hence $\lambda = 0$. □

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# ALL COMPLEX THOM SPECTRA ARE HARMONIC

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ABSTRACT. In this paper, we prove that all complex Thom spectra are harmonic and all real Thom spectra are harmonic at $p>2$.

## 1. Introduction.

A spectrum is called <u>harmonic</u> at a prime $p$ if its $p$-localization is local in the sense of Bousfield [2] with respect to the homology theory associated with the spectrum

$$K(0)\vee K(1)\vee K(2)\vee\ldots$$

where $K(0) = K(\mathbb{Q},0)$ and $K(n)$ for $n>0$ are the periodic Morava K-theories (see [9]). A spectrum is <u>harmonic</u> if it is harmonic at all primes. Harmonic spectra were extensively studied by Ravenel [9], who showed that they have very strong and interesting properties. For example, an elementary observation shows that if two harmonic connective spectra $X,Y$ have the same $n$-connected cover (for any chosen $n$), then they differ only by a wedge of rational Eilenberg-MacLane spectra.

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