HILL: THE SLICE TOWER

Note Title 10/26/2010

overview

1. Review of Postnikov towers
2. Equiv. changes
3. Slice theorem: slices of $MU^{((G))}$

I For a spectrum X we have

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & P^n X \arrow[d] & \Sigma^n H\pi_n X \arrow[l] \\
X \arrow[ur] \arrow[r] & P^{n-1} X \arrow[d] & \\
 & \vdots &
\end{tikzcd}$$

$\varprojlim P^n X \simeq X$

$\varinjlim P^n X \simeq *$

We want $\pi_k P^n X = 0$ if $k > n$

$P^n X$ is universal among spaces with

$$\mathrm{Map}(S^{>n}, P^n X) \simeq *$$

If Y is a spectrum that we get from S^{n+1} via cofibers, colimits or extensions, then

$$\mathrm{Map}(Y, P^n X) \simeq *$$

Let $\langle S^{n+1} \rangle$ = full subcat obtained by closing under extensions + colimits

If $Y \in \langle S^{n+1} \rangle$ then $\mathrm{Map}(Y, P^n X) \simeq *$

$\langle S^{n+1} \rangle = \mathcal{J}_{n+1} := \{ n\text{-connected spectra} \}$

P^n is the Dror nullification w.r.t. to $\mathcal{J}_{n+1}$,

Equiv context

* Change meaning of contractibility
  replace by G-contractibility

* Replace S^{n+1} by some S^V

Semi-classical example

① G-equiv $\simeq X$ $\qquad$ $\mathrm{Map}(Y, P^n X) \simeq X$

② Spheres are $\quad G/H_+ \wedge S^{n+1} = G_+ \wedge_H S^{n+1}$

$\mathcal{J}_{n+1} = \langle G_+ \wedge_H S^{n+1}, \dots \rangle$ = n-connected spaces

The resulting tower has fibers $\Sigma^n H \underline{\pi}_n X$

<u>Slice tower</u>

① Consider all H: G-contractibility

② a) $W(k, H) = G_+ \wedge_H S^{k\rho_H}$ ρ_H = reg rep of H

b) $\Sigma^{-1} W(k, H)$

$\dim \Sigma^{\varepsilon} W(k, H)$ = dim of underlying complex where $\varepsilon = 0, -1$

$= k|H| + \varepsilon$

$$\tau_{\geq n} = \langle \Sigma^{\varepsilon} W(k, H) : k|H| + \varepsilon \geq n \rangle$$

Then $\tau_{\geq n+1} \subseteq \tau_n$. We get a tower as before.

It is the slice tower.

For $G = C_2$ this is due to Dugger.
There is a motivic slice story due to Voevodsky and Hopkins–Morel.

Observations Assume for simplicity that $r=0$.

① $G_+ \wedge_H S^n \in \mathcal{T}_{\geq n}$ By induction on $|H|$ it suffices to show $S^n \in \mathcal{T}_{\geq n}$

$S^n \longrightarrow S^{n\rho_G} \longrightarrow$ cofiber built of induced cells in dim $\geq n+1$

$$\underbrace{\Sigma^{-1} S^{n\rho_G}/S^n}_{\text{induced cells of dim } \geq n} \longrightarrow S^n \longrightarrow S^{n\rho} \implies S^n \in \mathcal{T}_{\geq n}.$$

It follows that

$\{(n-1)\text{-conn equiv spectra}\} \subseteq \tau_{\geq n}$

so $P^{n-1}X$ is a quotient of the Postnikov section. Hence $\underline{\pi}_k P^{n-1} X = 0$ for $k > n-1$.

<u>Cor</u> $\varinjlim P^n X = *$

(2) $G_+ \wedge_H S^{m\rho_H}$ is $(m-1)$-connected

$\pi_k (G_+ \wedge_H S^{m\rho_H}) = 0$ if $k < m$

$\Rightarrow$ generators of $\tau_{\geq n}$ are all $\left(\left\lfloor \frac{n}{|G|} \right\rfloor - 1\right)$-conn.

$\Longrightarrow Y \in \tau_n$ is $\left(\left\lfloor \frac{n}{|G|} \right\rfloor\right)$-connected

$$X \longrightarrow P^{n-1}X \quad \text{is} \quad \left(\left\lfloor \frac{n}{|G|} \right\rfloor - 1\right)\text{-connected}$$

Cor $\varprojlim P^n X = X.$

$P_n^n X$ has htpy gps only in a finite range.

<u>Def</u> The <u>slices of X</u> are the layers in its slice tower.

Slice SS has $E_2^{s,t} = \underline{\pi}_{t-s} P_t^t X \Longrightarrow \underline{\pi}_{t-s} X$

Examples (1) $\tau_{\geq 0} = \{\Sigma^{\varepsilon} W(k, H) \mid k|H| + \varepsilon \geq 0\}$

$$\supseteq \{G/H_+\}$$

so $\tau_{\geq 0}$ is the category of (-1)-conn spectra

$$\tau_{\geq -1} \supseteq \{\Sigma^{-1} G/H_+\}$$

so $\tau_{\geq 1}$ is category of (-2)-conn spectra

$$P_{-1}^{-1} X = \Sigma^{-1} H \underline{\pi}_{-1} X$$

Can show $P_0^0 S^0 = H\underline{\mathbb{Z}}$

$$S^{k\rho_G} \wedge \left(G_+ \underset{H}{\wedge} S^{l\rho_H}\right) = G_+ \underset{H}{\wedge} S^{(k|G/H| + l)\rho_H}$$

$\Sigma^{kP_G} : \tilde{\mathcal{J}}_{\geq n} \longrightarrow \tilde{\mathcal{J}}_{\geq n+k|G|}$ is an iso

Cor $\quad P^{n+k|G|} \Sigma^{kP_G} X = \Sigma^{kP_G} P^n X$

Slices for $MU^{((G))} := N_H^G MU_{\mathbb{R}}$ where $G = C_{2^{n+1}}$

$\qquad\qquad = MU^{(2^n)}$

$$\pi_*^{\{e\}} MU^{((G))} = \underline{\pi_* MU^{((G))}}(G/\{e\})$$

$$= \mathbb{Z}[m_1, \ldots \gamma^{2^n-1} m_1, m_2 \ldots] \qquad \gamma^{2^n}(m_i) = (-1)^i m_i$$

Classical Postnikov associated graded

$$H\underline{\mathbb{Z}} \wedge \bigvee_{\substack{P \text{ ranging over} \\ \text{monomials in} \\ \pi_*^{\{e\}} MU^{((G))}}} S^{|P|}$$

(The group acts on the set of monomials)

$$= H\underline{\mathbb{Z}} \wedge \bigvee_{\substack{\text{orbits of} \\ \text{monomials}}} \bigvee_{G/\mathrm{Stab}(P)} S^{|P|}$$

where $\mathrm{Stab}(P)$ means stabilizer up to sign.

There exist C_2-equiv maps

$$S^{i\rho_2} \xrightarrow{\bar{r}_i} MU^{((G))}$$ underlain by r_i.

We replace

$$S^{|P|} \text{ by } S^{\frac{|P|}{H}\rho_H}$$ where $H = Stab(P)$

$$\bigvee_{G/H} S^{|P|} \text{ by } G_+ \wedge_H S^{(|P|/|H|)\rho_H}$$

Slice Theorem: The slice associated graded (A_G) of $MU^{((G))}$ is

$$H\underline{\mathbb{Z}} \wedge \left(\bigvee_{\substack{\text{orbits of} \\ \text{monomials} \\ \text{mod } 2}} G_+ \wedge_H S^{\rho_H |P|/|H|} \right) \quad \text{where } H = \mathrm{Stab}(P)$$

Cor: The slice A_G for $\Sigma^{-k\rho_G} MU^{((G))}$ has the same form.

$$C_2 \subseteq \overset{H}{\overset{\shortparallel}{\mathrm{Stab}(P)}} \subseteq C_{2^{n+1}}$$

$\Rightarrow G_+ \wedge_H S^{-\rho_H}$ is isotropic

The slices of $MU^{((G))}$ or $\Sigma^{-k\rho_G} MU^{((G))}$ are

$H\underline{\mathbb{Z}} \wedge \left(\begin{array}{c}\text{regular} \\ \text{cells}\end{array}\right)$

To compute $\pi_\star^G(-)$ we need

$S^V \wedge$ slice tower of X not

slice tower of $(S^V \wedge X)$.