HILL REDUCTION THEOREM

Note Title 10/29/2010

Thm $MU^{((G))} \wedge_A S^0 = H\underline{\mathbb{Z}}$

$$MU^{((G))} = R(\infty)$$

Pf By induction on $|G|$.

For $|G| = 1$ stmt is due to Quillen (?)

$$\begin{tikzcd}
EP_+ \wedge R(\infty) \arrow[r] \arrow[d, "L"'] & R(\infty) \arrow[r] \arrow[d] & \tilde{E}P \wedge R(\infty) \arrow[d, "R"] \\
EP_+ \wedge H\underline{\mathbb{Z}} \arrow[r] & H\underline{\mathbb{Z}} \arrow[r] & \tilde{E}P \wedge H\underline{\mathbb{Z}}
\end{tikzcd}$$

P = family of proper subgps

To show R is equiv

(1) Compute $\pi_\star \Phi^{G} R(\infty)$ and $\pi_\star \Phi^{G} H\underline{\mathbb{Z}}$ are abstractly isomorphic MAH

(2) Show $R_\star$ induces the isomorphism MJH

To show L is an equiv

Show $i_H^* R(\infty) \longrightarrow i_H^* H\underline{\mathbb{Z}} = H\underline{\mathbb{Z}}$ MAH

$\quad\quad\quad\; \| $

$\quad\quad\; R_H(\infty)$

$R_{G}(\infty) = MU^{((G))} \wedge_A S^0$

$A = N_{C_2}^{G}\left(S^0[\bar{m}_1, \bar{m}_2, \dots] \right)$

wedge of monomials

$\bigvee_P S^{|P|\rho_2/2}$

The map $A \longrightarrow MU^{((G))}$ is obtained by using ring structure of target

$$\Phi^G R_G(\infty) = \Phi^G\left(MU^{((G))}\right) \wedge_{\Phi^G A} \Phi^G S^0$$

and $\quad \Phi^G MU^{((G))} = MO$

$$\Phi^G(A) = \Phi^G\left(S^0[N\bar{r}_1, N\bar{r}_2 \cdots]\right)$$

$$= S^0[\Phi^G N\bar{r}_1, \Phi^G N\bar{r}_2 \cdots]$$

$$= S^0[h_1, h_2, \cdots] = B$$

$$\Phi^G(S^0) = S^0$$

so $\Phi^{G} R_{G}(\infty) = MO \wedge_{B} S^0$

$= H\mathbb{F}_2[h_1, h_2 \cdots h_i \cdots | i \neq 2^n - 1] \wedge_{B} S^0$

$= H\mathbb{F}_2 \bigwedge_{S^0[b_1, b_3, b_5 \cdots]} S^0$

Can translate this into iterated cofibers, and we get

$$\pi_* \Phi^{G} R(\infty) = \begin{cases} \mathbb{F}_2 & \text{if } * \geq 0 \text{ and } * \text{ even} \\ 0 & \text{else} \end{cases}$$

Can also treat this as a Künneth problem.
There is no multiplicative structure here.

$$\Phi^G H\mathbb{Z} = (S^{\infty\sigma} \wedge H\mathbb{Z})^G$$

$$S^0 \wedge H\mathbb{Z} \longrightarrow S^\sigma \wedge H\mathbb{Z} \longrightarrow S^{2\sigma} \wedge H\mathbb{Z}$$

This leads to a chain cx

$$\mathbb{Z} \xleftarrow{\nabla} \mathbb{Z}[C_2] \xleftarrow{1-\gamma} \mathbb{Z}[C_2] \xleftarrow{1+\gamma} \mathbb{Z}[C_2] \longleftarrow \cdots\cdots$$

Taking fixed pts gives

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \longleftarrow \cdots\cdots$$

$$\pi_i \Phi^G H\mathbb{Z} = \begin{cases} \mathbb{Z}/2 & \text{for } i \geq 0 \text{ and } i \text{ even} \\ 0 & \text{else} \end{cases}$$

Thus $\pi_* \Phi^G H\mathbb{Z}$ and $\pi_* \Phi^G R(\infty)$ are iso

Let $R = i_H^* MU^{((G))}$ $\quad H =$ subgp of index 2

By induction $MU^{((H))}$ has the expected slices

$$R = MU^{((H))} \wedge MU^{((H))}$$

$$= MU^{((H))}[H \cdot \bar{r}_1^G, H \cdot \bar{r}_2^G, \ldots]$$

$$i_H^* R_G(\infty) = i_H^* \left(MU^{((G))} \wedge_A S^0 \right)$$

$$= i_H^* \left(MU^{((G))} \wedge_{MU^{((G))} \wedge A} MU^{((G))} \right)$$

$$= R \wedge_{R \wedge i_H^* A} R$$

1

$$R \wedge i_{\mu}^{*} A = R[G \cdot \bar{m}_1^{G}, \ldots]$$

$$= R[H \cdot \bar{m}_1^{G}, \ldots ; H\gamma \bar{m}_1^{G}, \ldots]$$

$$R_H(\infty) = MU^{((H))} \underset{A_H \wedge MU^{((H))}}{\wedge} MU^{((H))} = R \underset{MU^{((H))}[H \cdot m_1^{H}, \ldots ; H\gamma \bar{m}_1^{G}, \ldots]}{\wedge} MU^{((H))}$$

$$MU^{((H))} \wedge A_H = MU^{((H))}[H \cdot \bar{m}_1^{H}, H \cdot \bar{m}_2^{H}, \ldots]$$

$$= R \underset{R[H \cdot \bar{m}_1^{H}, \ldots ; H \cdot \gamma \bar{m}_1^{G} \ldots]}{\wedge} R$$

2

base change along $MU^{((H))} \to R$

Compare 1 and 2

Thus it suffices to show

$$R[H_0\bar{\mu}_1^H, \ldots ; H\gamma_1\bar{\mu}_1^G \ldots] = R[H_0\bar{\mu}_1^G, \ldots ; H\gamma\bar{\mu}_1^G \ldots]$$

Let I be the ideal gen'd by

$$(H\bar{\mu}_1^G, \ldots ; H\gamma\bar{\mu}_1^G, \ldots)$$

$$\bar{\mu}_1^H = \begin{cases} \bar{\mu}_1^G + \gamma\bar{\mu}_1^G & \text{if } i = 2^n - 1 \\ & \text{or vice versa} \\ \bar{\mu}_1^G & \text{else} \end{cases} \mod I^2$$

This follows from def of $\bar{\mu}_1$

Thus $J := (H\bar{\mu}_1^H, \ldots ; H\gamma\bar{\mu}_1^G, \ldots)$ is I.

Goal: Build a map of associative rings (i.e. A_∞-rings) such that

$$\bar{m}_1^H \longrightarrow \text{polynomial in } \bar{m}_1^G, \gamma\bar{m}_1^G$$

expressing it.

Assume we can:

$$R[H\cdot\bar{m}_1^H, \dots ; H\gamma\bar{m}_1^G \dots] \longrightarrow R[H\cdot\bar{m}_1^G, \dots ; H\gamma\bar{m}_1^G \dots]$$

and its an underlying equivalence

Slices of $MU^{((H))}$ are what we expect

* odd slices vanish
* even slices (isotropic regular cells) $\wedge H\underline{\mathbb{Z}}$

__Lemma__ 1) $R[-]$ also has these properties

2) $f: X \to Y$ with $X + Y$ have these properties and underlying map is an equiv, then f is an equivalence.

This says that under these hypotheses, an ordinary equiv is a G-equiv.

__Pf__ 1) We did this before

2) By considering the slice tower, it suffices to show this for

$X = \left(\begin{matrix}\text{small}\\ \text{regular cell}\end{matrix}\right) \wedge H\underline{\mathbb{Z}}$

$Y = \left(\begin{matrix}\text{small}\\ \text{regular cell}\end{matrix}\right) \wedge H\underline{\mathbb{Z}}$

apply isotropy separation to reduce to case of no induced cells and check it.
EXERCIZE

Def An assoc ring A is weakly comm if it is an A_∞-retract of an E_∞-ring.

$$A \to E \longrightarrow A$$

The rings $R[-]$ above are

weakening comm. Weak commutativity is good enough to construct the maps we need.

$R[-]$ is a retract of $R \circ R$