FOUR APPROACHES TO COHOMOLOGY THEORIES WITH REALITY

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Abstract. We give an account of the calculations of the $RO(Q)$-graded coefficient rings of some of the most basic Q-equivariant cohomology theories, where Q is a group of order 2. One purpose is to advertise the effectiveness of the Tate square, showing it has advantages over the slice spectral sequences in algebraically simple cases. A second purpose is to give a single account showing how to translate between the languages of different approaches.

Contents

1. Introduction

Historically, complex orientable cohomology theories (complex K-theory KU, complex cobordism MU, ...) have been the first to be exploited, partly because they tend to be easier to calculate. Real analogues of these (real K-theory KO, real cobordism MO, ...) contain new information but can be hard to approach directly. It is well known that representations over the reals can be viewed as complex representations with an additional conjugate-linear involution: real representations are complex representations with an action of the Galois group $Q = Gal(\mathbb{C}|\mathbb{R})$ of order 2. Atiyah used this technique [2] to define K-theory 'with Reality' (or simply *Real* K-theory with the initial letter capitalized), thereby obtaining a very practical approach to real K-theory by descent. This was then extended to cobordism by Landweber [19], Araki [1] and others. The resulting theories are Q-equivariant cohomology theories represented by Q-spectra.

The same discussion applies if we start with Q-equivariant theories: the complex orientable theories (Atiyah-Segal equivariant K-theory, tom Dieck complex bordism) are much more accessible. Many techniques were developed in the 1990s with applications to equivariant complex orientable theories most clearly in mind.

The point of this article is to describe some well known calculations for Real equivariant theories in the simplest cases (ordinary cohomology with constant integer coefficients and Real connective K-theory), giving several different methods in each case. In particular, despite the fact that the Q-equivariant Real theories are not complex orientable (even though

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the underlying non-equivariant theories are), some of the complex orientable methods are still extremely effective.

1.A. **Notation.** This subsection introduces our notation for some standard machinery. I am grateful to the referee for patiently and objectively pointing out ways in which notational conventions vary. Encouraged by the referee to clarify my conventions, the length of the subsection has increased about fivefold: I hope this will make the article more accessible, and ease communication between different notational tribes. On the other hand, those familiar with the customs of my tribe can skip directly to Subsection 1.C now.

1.A.1. *Grading.* We write σ for the sign representation of Q on $\mathbb{R}$, 1 for the trivial representation, and $\rho = 1 + \sigma$ for the real regular representation. We will be grading our groups over the real representation ring $RO(Q) = \{a + b\sigma \mid a, b \in \mathbb{Z}\}$. We write M_* to indicate grading over $\mathbb{Z}$ and $M_\bigstar$ to denote grading over $RO(Q)$. When we draw pictures, they will be displayed in the plane with the $\mathbb{Z}$ axis horizontal and the σ axis vertical, so that $a + b\sigma$ gets displayed at the point with cartesian coordinates (a, b).

Alternatives: We have followed [15] in using σ for the non-trivial real representation of Q. In the past each author seems to have used a different letter. For example, the present author previously used ξ [9, 12, 3, 4], the letter α is used by Hu and Kriz, and the letter τ is used elsewhere. Perhaps we can unite around σ?

We have used $RO(Q)$ grading both in notation and displays. Atiyah [2], Landweber [19], Araki [1] use $\mathbb{R}^{p,q}$ for $q \oplus p\sigma$ (sic). The Tate twist notation from algebraic geometry used by Dugger [8] is related to $RO(Q)$-grading by $\mathbb{R}^{p+q,q} = p + q\sigma$.

1.A.2. *Mackey functors.* We will use Dress's formulation of Mackey functors for a finite group G [7] as given by a covariant and a contravariant functor on finite G-sets subject to the Mackey condition. A Mackey functor M for a group G is therefore determined by its values on the transitive G-sets G/H together with certain structure maps. If $K \subseteq H$ the projection $\pi^H_K : G/K \longrightarrow G/H$ induces a restriction map $(\pi^H_K)^* = \mathrm{res}^H_K : M(G/H) \longrightarrow M(G/K)$ and an induction map $(\pi^H_K)_* = \mathrm{ind}^H_K : M(G/K) \longrightarrow M(G/H)$, and right multiplication $R_g : G/H \longrightarrow G/H^g$ induces an action of $W_G(H) = N_G(H)/H$ on $M(G/H)$.

We write $\underline{\mathbb{Z}}$ for the constant Mackey functor at $\mathbb{Z}$ (which is to say that the values on all orbits are $\mathbb{Z}$, restriction is the identity and induction is multiplication by the index of the smaller subgroup in the larger one).

1.A.3. *Spectra.* We will be working with cohomology theories $E^*_G(\cdot)$, which are represented by *genuine* G-spectra E (i.e., G-spectra indexed on a complete G-universe) in the sense that $E^*_G(X) = [X, E]^*_G$. Accordingly the integer grading can be extended to an $RO(G)$-grading.

For an orthogonal representation V we write $S(V)$ for the unit sphere and S^V for the one point compactification of V with ∞ as the basepoint. We write $X^G_V := \pi^G_V(X) = [S^V, X]^G$ for the Vth G-equivariant homotopy group of X. As usual $X_V := \pi_V(X) = [S^V, X]$ denotes the Vth non-equivariant homotopy group. The notation extends to virtual representations.

The subcategory of G-spectra of the form G/H_+ is called the *stable orbit category*, and calculation of the maps between them shows that they are generated by the maps $(\pi^H_K)_+$ and their duals, and that a Mackey functor as described above is equivalent to a contravariant functor on the stable orbit category. We write $\underline{\pi}^G_*(X)$ for the Mackey functor $[(\cdot)_+, X]^G_*$ whose value on G/H is $\pi^H_*(X) = [G/H_+, X]^G_*$.

Alternatives: The following is inserted in recognition of the fact that alternative views are held very strongly by some (for example by the referee!).

Some authors (including the present author in the 1980s) prefer to omit the group when writing $RO(Q)$-grading and Mackey functors (writing $\underline{\pi}_*(X)$ where we write $\underline{\pi}^Q_*(X)$ and $\pi_\bigstar(X)$ where we write $\pi^Q_\bigstar(X)$). This is presumably on the grounds of brevity, but it is at the cost of forcing the adoption of a more complicated notation for non-equivariant homotopy. The present choice is based on the ideas (i) that working equivariantly requires explicit acknowledgement whilst working non-equivariantly does not and (ii) that one should be able to substitute a particular grading for $\bigstar$ (so that considering grading zero for instance, $\pi^Q_\bigstar(X)$ becomes $\pi^Q_0(X)$).

1.A.4. *Ordinary cohomology.* A cohomology theory $E^*_G(\cdot)$ satisfying the dimension axiom (i.e., one whose values on cells G/H_+ is entirely in degree 0) is called *ordinary* (or *Bredon*). It is determined by the values on these cells, which define a Mackey functor $M = \underline{\pi}^G_0(E)$. We write $H^*_G(\cdot; M)$ for the cohomology theory and HM for the representing G-spectrum.

1.A.5. *Proto-Euler and Proto-Thom classes.* For any ring spectrum E, the unit map provides, for any representation V a class $\Sigma^V\iota \in E^V_G(S^V)$. A Thom class would be an element $\tau_V \in E^{|V|}_G(S^V)$ which generates a rank one free module. The Euler class (of degree $|V| - V$) would be the pullback of τ_V along the zero section $a_V : S^0 \longrightarrow S^V$.

With $G = Q$ some related classes are important for us. We write $a = a_\sigma : S^0 \longrightarrow S^\sigma$ for the inclusion, and think of it as a homotopy element of degree $-\sigma$.

There are Euler-like classes λ (of degree $1-\sigma$, which does not survive to homotopy), u (of degree $2(1-\sigma)$, important for ordinary cohomology), and U (of degree $4(1-\sigma)$, important for $k\mathbb{R}$).

Alternatives: We follow [15] in using a for the degree $-\sigma$ Euler-like class. The letters e and i are used by some authors.

In [16, Definition 3.4] the class u is called $u_{2\sigma}$, and the class $2u^{-1}$ is called $e_{2\sigma}$. In the more general study of bordism with reality, a class u (of which ours is the image) plays a significant role, and its 2^nth power is an almost-periodicity at the nth chromatic level. This means that $U = u^2$, and that we should think of λ as a square root of u.

1.A.6. *Basepoints.* Our Q spaces will be equipped with a Q-fixed basepoint and cohomology will be reduced.

1.B. **What's the point of this paper?** The paper describes several different methods of calculating the $RO(Q)$-graded homotopy of two well known Real spectra: the integral Eilenberg-MacLane spectrum $H\underline{\mathbb{Z}}$ and connective K-theory with reality $k\mathbb{R}$. The results and the methods are all known. The intention is to describe several different methods in a way that lets one compare them, to provide more details and more pictures than is usually done, and to record the notations used by different authors. The author has found the need to make these comparisons quite often, and that it takes time to pin details down precisely from the existing literature. It is hoped the present summary may be useful to others.

1.C. **Geography of the coefficients.** It is worth highlighting some features of the coefficient rings $X^Q_\bigstar$. The background is that their non-equivariant counterparts are in even degrees. Next the reader should look at the pictures at the the start of Sections 2 and 3) (or at the picture of $tmf_1(3)^Q_\bigstar$ in [13, Subsection 13.C]).

The crudest feature is that the non-zero entries are mostly above the diagonal line of slope -1. Indeed the only exceptions to this are the a-multiple tails in the right half-plane. This is characteristic of theories which are equivariantly connective.

Lemma 1.1. *If X is a connective Q-spectrum then $X^Q_{x+y\sigma}$ is zero below the antidiagonal $x+y=0$ in the half-plane $x<0$, and multiplication by a is an isomorphism below the antidiagonal in the half-plane $x \geq 0$.*

Proof: Non-equivariant connectivity and the cofibre sequence $Q_+ \longrightarrow S^0 \longrightarrow S^\sigma$ shows multiplication by a is surjective from the antidiagonal, and isomorphic below it. The connectivity of X^Q_* shows there are zeros along the negative x axis. $\square$

However, the most striking feature is that the coefficients of the theories described here have 'The Gap':

$$X^Q_{*\rho-i}=0 \text{ for } i=1,2,3$$

so that in the display of the $RO(Q)$-graded coefficients, the three diagonals above the leading diagonal are all zero. It is worth commenting on the implications of this.

Lemma 1.2. *(i) The vanishing of the diagonals $X^Q_{*\rho-i}$ with $i=1,2$ is equivalent to X being* strongly even *in the sense of Meier and Hill* [17] *(i.e., that $X^Q_{*\rho-1}=0$ and the restriction maps $X^Q_{k\rho} \longrightarrow X_{2k}$ are isomorphic for all k).*

(ii) The vanishing of the three diagonals $X^Q_{*\rho-i}$ with $i=1,2,3$ is equivalent to being strongly even and having X_* even.

Proof: Consider the long exact sequence in X-cohomology induced by the sequence $Q_+ \longrightarrow S^0 \longrightarrow S^\sigma$. $\square$

1.D. **The Tate square.** We will make considerable use of the Tate square, so we briefly describe the construction from [10, 12]. This is based on the contractible free Q-space EQ and the join $\tilde{E}Q$ which are related by the cofibre sequence

$$EQ_+ \longrightarrow S^0 \longrightarrow \tilde{E}Q$$

brought to prominence by Carlsson's use of it in his proof of the Segal conjecture [5]. Since Q is a periodic group we use the model $EQ=S(\infty\sigma)$ for a contractible free Q-space and $\tilde{E}Q=S^{\infty\sigma}$ for the join with S^0 to give periodic resolutions as in [10].

We will use Scandinavian notation:

$$X^{hQ}=(X^h)^Q, \text{ where } X^h=F(EQ_+,X)$$
$$X_{hQ}\simeq (X_h)^Q, \text{ where } X_h=EQ_+\wedge X$$
$$X^{tQ}=(X^t)^Q, \text{ where } X^t=F(EQ_+,X)\wedge \tilde{E}Q$$
$$X^{\Phi Q}=(X^\Phi)^Q, \text{ where } X^\Phi=X\wedge \tilde{E}Q.$$

The notation $X^{\Phi Q}$ is justified by the fact that since Q is of prime order, it is the geometric Q-fixed point spectrum.

It is easy to see that there are two linked cofibre sequences

$$\begin{array}{ccccc} X_h & \longrightarrow & X & \longrightarrow & X^{\Phi} \\ \simeq\downarrow & & \downarrow & & \downarrow \\ X_h & \longrightarrow & X^h & \longrightarrow & X^t \end{array}$$

and the indicated equivalence arises since $X \longrightarrow X^h$ is a non-equivariant equivalence. Accordingly the Tate square (the square on the right) is a homotopy pullback, and it is a homotopy pullback of rings if X is a ring.

Typically we proceed as follows

- Calculate $X^{hQ}_{\bigstar}$ using the homotopy fixed point spectral sequence described in Subsection 1.E.
- Infer $X^{tQ}_{\bigstar}$ by inverting a (since $\tilde{E}Q = S^{\infty\sigma}$, and evidently $\pi^Q_{\bigstar}(Y \wedge S^{\infty\sigma}) = \pi^Q_{\bigstar}(Y)[1/a]$).
- Infer $X^{\bigstar}_{hQ}$ from the lower cofibre sequence (which amounts to a simple local cohomology theorem).
- Infer $X^{\Phi Q}_{\bigstar}$ from $X^{tQ}_{\bigstar}$ using Lemma 1.3; since both are a-periodic this follows from the integer graded homotopy groups.
- Infer $X^Q_{\bigstar}$ from the Tate square and the top sequence.

The following is immediate since the fibres of the two verticals in the Tate square are equivalent. The hypothesis on the non-equivariant spectrum shows X_{hQ} is connective and hence that $X^{\Phi Q}$ is connective.

Lemma 1.3. *(*[13, Lemma 11.2]*) Suppose X is a Q-spectrum which is non-equivariantly connective. If $X^Q \longrightarrow X^{hQ}$ is a connective cover then $X^{\Phi Q} \longrightarrow X^{tQ}$ is too.* □

1.E. **The $RO(Q)$-graded homotopy fixed point spectral sequence.** We note that we are trying to calculate the groups

$$X^{hQ}_{a+b\sigma} = [S^{a+b\sigma}, X^h]^Q = [S^{a+b\sigma}, F(EQ_+, X)]^Q = [S^{a+b\sigma} \wedge EQ_+, X]^Q = [S^a, F(S^{b\sigma}, X)^{hQ}].$$

We could consider the separate homotopy fixed point spectral sequences for the various spectra $F(S^{b\sigma}, X)$. To avoid confusion later, we arrange this homologically from the start, so that it is in a (vertically shifted) second quadrant:

$$E^2_{s,t}(b) = H^{-s}(Q; X^{-t}(S^{b\sigma}))) \Rightarrow X_Q^{-(s+t)}(EQ_+ \wedge S^{b\sigma}) = \pi_{s+t}(F(S^{b\sigma}, X)^{hQ})$$

with differentials

$$d^r_{s,t} : E^r_{s,t}(b) \longrightarrow E^r_{s-r,t+r-1}(b).$$

Note that in the E^2-term Q acts on $S^{b\sigma}$ by a degree $(-1)^b$ map, and it is this that makes $X^*(S^{b\sigma})$ a Q-module.

However there is a great advantage to combining these spectral sequences as b varies, especially if X is a ring spectrum, since the pairings

$$S^{b_1\sigma} \wedge S^{b_2\sigma} \longrightarrow S^{(b_1+b_2)\sigma}$$

give the whole construction a multiplicative structure. Although the differentials all lie within the spectral sequence for a single value of b, the spectral sequences all come from the same filtration, so that the Leibniz rule applies across all the spectral sequences.

Definition 1.4. The $RO(Q)$-graded homotopy fixed point spectral sequence for X is the trigraded spectral sequence

$$E^2_{s,t,b} = H^{-s}(Q; X_{t+b\sigma}) \Rightarrow [S^{s+t+b\sigma} \wedge EQ_+, X]^Q =: X^{hQ}_{s+t+b\sigma}$$

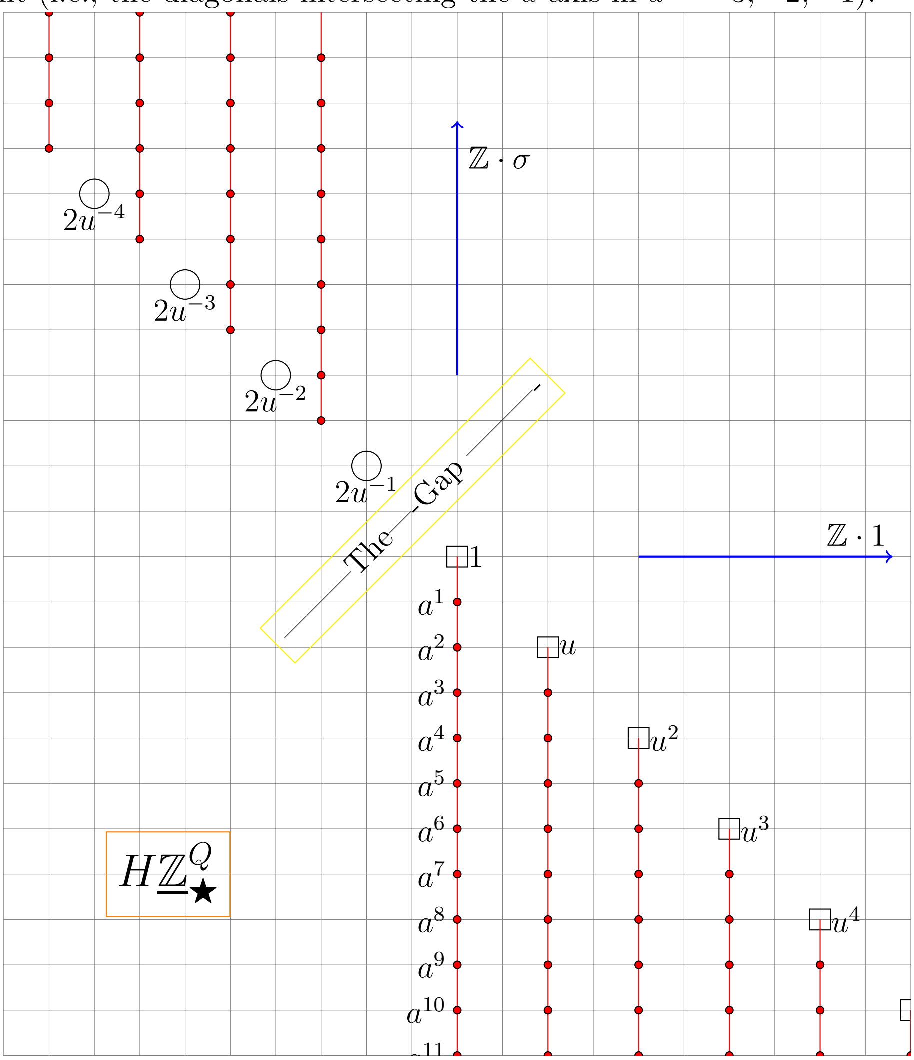
with differentials

$$d^r_{s,t,b} : E^r_{s,t,b} \longrightarrow E^r_{s-r,t+r-1,b}.$$

2. Ordinary cohomology

We may depict the $RO(Q)$-graded coefficients of the Eilenberg-MacLane spectrum, $H\underline{\mathbb{Z}}^Q_\star$ as follows, where squares are copies of $\mathbb{Z}$ and dots are copies of $\mathbb{F}_2$. This was first calculated by Stong [22, 23] and Waner [24] and first published by Caruso [6], but it has appeared in several other places. The Mackey structure is made explicit in [16, Figure 2, Page 405].

One notable feature is 'The Gap', which constitutes the three lines of slope 1 above the unit (i.e., the diagonals intersecting the a axis in $a = -3, -2, -1$).



2.A. **Cell approach.** Note first that

$$H\underline{\mathbb{Z}}^Q_{a+b\sigma} = [S^{a+b\sigma}, H\underline{\mathbb{Z}}] = H^{-a}_Q(S^{b\sigma}; H\underline{\mathbb{Z}}).$$

Since $H\underline{\mathbb{Z}}$ satisfies the dimension axiom, it is natural to calculate this by finding a cell structure for $S^{b\sigma}$.

Indeed, we start from the cofibre sequence

$$Q_+ \longrightarrow S^0 \longrightarrow S^\sigma.$$

Suspending by $n\sigma$ and using the untwisting isomorphism $S^{n\sigma} \wedge Q_+ \simeq S^n \wedge Q_+$, this gives

$$S^n \wedge Q_+ \longrightarrow S^{n\sigma} \longrightarrow S^{(n+1)\sigma}.$$

First we consider a row in the lower half plane, $H\underline{\mathbb{Z}}^Q_{*-n\sigma}$, when $n \geq 0$. We argue as follows. We begin with $H^Q_*(S^0; \underline{\mathbb{Z}}) = \mathbb{Z}$ (by the dimension axiom). Adding an $(n+1)$-cell to deduce $H^Q_*(S^{(n+1)\sigma}; \underline{\mathbb{Z}})$ from $H^Q_*(S^{n\sigma}; \underline{\mathbb{Z}})$ we only ever have to determine the map

$$\mathbb{Z} = H^Q_n(S^n \wedge Q_+; \underline{\mathbb{Z}}) \longrightarrow H^Q_n(S^{n\sigma}; \underline{\mathbb{Z}}) = \mathbb{Z}$$

when n is even. It is always multiplication by 2: for $n = 0$ it is $\pi_* = \mathrm{ind}^Q_1 : \underline{\mathbb{Z}}(Q/1) \longrightarrow \underline{\mathbb{Z}}(Q/Q)$. For larger even n we note inductively that

$$H^Q_n(S^{n\sigma}; \underline{\mathbb{Z}}) \xrightarrow{\cong} H^Q_n(S^{n\sigma}/S^{(n-1)\sigma}; \underline{\mathbb{Z}})$$

is an isomorphism. The composite

$$S^n \wedge Q_+ \longrightarrow S^{n\sigma} \longrightarrow S^{n\sigma}/S^{(n-1)\sigma} \simeq S^n \wedge Q_+$$

is the nth suspension of the case $n = 0$, where it is ind^Q_1.

Next, consider a row in the upper half plane, $H\underline{\mathbb{Z}}^Q_{*+n\sigma}$ when $n \geq 0$. We argue as follows. We begin with $H_Q^*(S^0; \underline{\mathbb{Z}}) = \mathbb{Z}$ (by the dimension axiom). Adding an $(n+1)$-cell to deduce $H_Q^*(S^{(n+1)\sigma}; \underline{\mathbb{Z}})$ from $H_Q^*(S^{n\sigma}; \underline{\mathbb{Z}})$ we only ever have to decide on the map

$$\mathbb{Z} = H_Q^n(S^{n\sigma}; \underline{\mathbb{Z}}) \longrightarrow H_Q^n(S^n \wedge Q_+; \underline{\mathbb{Z}}) = \mathbb{Z}$$

when n is even. For $n = 0$ it is $\pi^* = \mathrm{res}^Q_1 : \mathbb{Z}(Q/Q) \xrightarrow{\cong} \mathbb{Z}(Q/1)$. For larger even n it is always multiplication by 2: we note inductively that

$$H_Q^n(S^{n\sigma}/S^{(n-1)\sigma}) \longrightarrow H_Q^n(S^{n\sigma}; \underline{\mathbb{Z}})$$

is multiplication by 2. The composite

$$S^n \wedge Q_+ \longrightarrow S^{n\sigma} \longrightarrow S^{n\sigma}/S^{(n-1)\sigma} \simeq S^n \wedge Q_+$$

is the nth suspension of the case $n = 0$, where it is the sum $N = 1 + x$ over multiplication by group elements, where $Q = \langle x \rangle$.

2.B. **Quotient approach.** The cohomology of a *space* only depends on the restriction maps in the Mackey coefficients, and if these form a constant coefficient system (i.e., restriction maps are the identity) it follows that it is the nonequivariant cohomology of the quotient: for any Q-space X

$$H_Q^*(X; \underline{\mathbb{Z}}) = H^*(X/Q; \mathbb{Z}).$$

For any $n \geq 0$ we have $S^{(n+1)\sigma} = S^0 * S(n\sigma)$, and hence

$$H_Q^*(S^{(n+1)\sigma}; \underline{\mathbb{Z}}) = H^*(S^0 * \mathbb{R}P^n; \mathbb{Z}).$$

This gives $H\underline{\mathbb{Z}}^Q_{*+n\sigma}$ for $n \geq 0$ from the well known cohomology of projective spaces. This fills in the upper half plane.

For the lower half plane we use the fact that for *free* Q-spectra

$$H_*^Q(X;\underline{\mathbb{Z}}) = H_*(X/Q;\mathbb{Z}).$$

This is not true for arbitrary Q-spectra (or even for Q-spaces). We then use the cofibre sequence

$$S(n\sigma)_+ \longrightarrow S^0 \longrightarrow S^{n\sigma}$$

We have

$$H_*^Q(S(n\sigma)_+;\underline{\mathbb{Z}}) = H_*(\mathbb{R}P^n_+;\mathbb{Z});$$

since S^0 only has homology in degree 0, we need only check that

$$S(n\sigma)_+ \longrightarrow S^0$$

is multiplication by 2 in degree 0. When $n = 0$ this is part of the definition of $\underline{\mathbb{Z}}$, and for higher dimensions, we use the fact that $S(\sigma)_+ \longrightarrow S(n\sigma)_+$ is an isomorphism in degree 0 since only higher cells have been attached. This fills in the lower half plane.

We note in passing that this also shows $H_*^Q(S^{n\sigma};\underline{\mathbb{Z}}) \not\cong H_*^Q(S^{n\sigma}/Q);\mathbb{Z})$ for $n \geq 1$.

2.C. **Tate approach.** The cleanest approach is to use the Tate diagram

$$\begin{array}{ccccc} H\underline{\mathbb{Z}}_h & \longrightarrow & H\underline{\mathbb{Z}} & \longrightarrow & H\underline{\mathbb{Z}}^\Phi \\ \simeq\downarrow & & \downarrow & & \downarrow \\ H\underline{\mathbb{Z}}_h & \longrightarrow & H\underline{\mathbb{Z}}^h & \longrightarrow & H\underline{\mathbb{Z}}^t \end{array}$$

The procedure is to start with homotopy fixed points. From this we can immediately deduce Tate cohomology by inverting a. This then gives homotopy orbits by using the bottom row. These terms only depend on the underlying spectrum (in this case $H\underline{\mathbb{Z}}$) made free. The geometric fixed points depends on the spectrum itself, so needs further input. Then the answer follows from the Tate square.

Lemma 2.1.

$$H\underline{\mathbb{Z}}^{hQ}_\bigstar = BB[u,u^{-1}],$$

where $BB = \mathbb{Z}[a]/(2a)$, $|a| = -\sigma, |u| = 2-2\sigma$ *(a is the Euler class of σ and u is a Thom class).*

Remark 2.2. (i) Notice how this is related to ordinary group cohomology, which occurs along the negative x-axis:

$$H^*(BQ_+;\mathbb{Z}) = H\underline{\mathbb{Z}}^{hQ}_* = \mathbb{Z}[y]/(2y), \text{ where } y = a^2u^{-1} \text{ is of codegree 2.}$$

(ii) The notation BB stands for Basic Block, and the usefulness of the notation will emerge as we consider other examples.

Proof: We describe the $RO(Q)$-graded spectral sequence of Subsection 1.E for $H\underline{\mathbb{Z}}$. It is convenient to describe it from the E_1-term if we choose a good cell structure on EQ_+. Indeed we use the filtration

$$S(\sigma)_+ \subset S(2\sigma)_+ \subset S(3\sigma)_+ \subset \cdots \subset S(\infty\sigma)_+ = EQ_+.$$

There is one free Q-cell in each degree and in cellular homology this gives the standard 2-periodic resolution

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\epsilon} \mathbb{Z}Q \xleftarrow{1-x} \mathbb{Z}Q \xleftarrow{1+x} \mathbb{Z}Q \xleftarrow{1-x} \cdots$$

of $\mathbb{Z}$ over the group ring $\mathbb{Z}Q$.

We may then describe the E^1 term conveniently as

$$E^1_{*,*,*} = \mathbb{Z}[a][\lambda, \lambda^{-1}]$$

where the (s, t, b) gradings are

$$|a| = (-1, 1, -1), |\lambda| = (0, 1, -1).$$

The differential is given by taking $d_1 a = 0$ and

$$d_1 \lambda = 2a.$$

This is immediate from the algebraic resolution since Q acts non-trivially on $H^b(S^{b\sigma})$ for b odd.

It follows that $u = \lambda^2$ is a cycle and that there are no further differentials. $\square$

We now immediately deduce the coefficients of the Tate theory by inverting a, since $S^{\infty\sigma} \simeq S^0[1/a]$, and then the homotopy orbits by taking local cohomology.

Corollary 2.3. *The Tate cohomology is given by*

$$H\underline{\mathbb{Z}}^{tQ}_{\bigstar} = \mathbb{F}_2[a, a^{-1}][u, u^{-1}],$$

and the coefficients of the homotopy orbits by

$$H\underline{\mathbb{Z}}^{\bigstar}_{hQ} = NB[u, u^{-1}]$$

where, $NB = \mathbb{Z} \oplus \Sigma^{-1}\mathbb{F}_2[a]^\vee$. We note also that $NB \simeq R\Gamma_{(a)}(BB)$, i.e., it is the sum of right derived functors of the a-power torsion functor $\Gamma_{(a)}$.

Remark 2.4. The notation NB stands for Negative Block, and as for BB the usefulness of the notation should emerge once we have other examples.

For geometric fixed points we need the following immediate consequence of Lemma 1.3.

Lemma 2.5. *The geometric fixed points are given by $H\underline{\mathbb{Z}}^{\Phi Q}_{\bigstar} = \mathbb{F}_2[a, a^{-1}][u]$.* $\square$

Proof: By definition of the Eilenberg-MacLane spectrum, $(H\underline{\mathbb{Z}})^Q = H\mathbb{Z}$. It is immediate that $(H\underline{\mathbb{Z}})^{hQ}_*$ is zero in positive degrees, and the algebraic resolution shows that $(H\underline{\mathbb{Z}})^{hQ}_0 \xrightarrow{\cong} (H\underline{\mathbb{Z}})^Q_0$ is an isomorphism. The conclusion follows from Lemma 1.3. $\square$

The calculation of $H\underline{\mathbb{Z}}^Q_{\bigstar}$ now concludes by using the Tate square. The only difference between $H\underline{\mathbb{Z}}^Q_{\bigstar}$ and the more regular $H\underline{\mathbb{Z}}^{hQ}_{\bigstar}$ comes from the negative Tate a-columns. The Tate square therefore induces an actual pullback square in gradings with even integer-part. The a-divisible columns in odd integer degrees ≤ -3 come from the cokernel of $(H\underline{\mathbb{Z}})^{hQ}_{\bigstar} \longrightarrow (H\underline{\mathbb{Z}})^{tQ}_{\bigstar}$.

Corollary 2.6.

$$H\underline{\mathbb{Z}}^Q_{\bigstar} = BB[u] \oplus u^{-1} \cdot NB[u^{-1}],$$

where $BB = \mathbb{Z}[a]/(2a)$ as before and

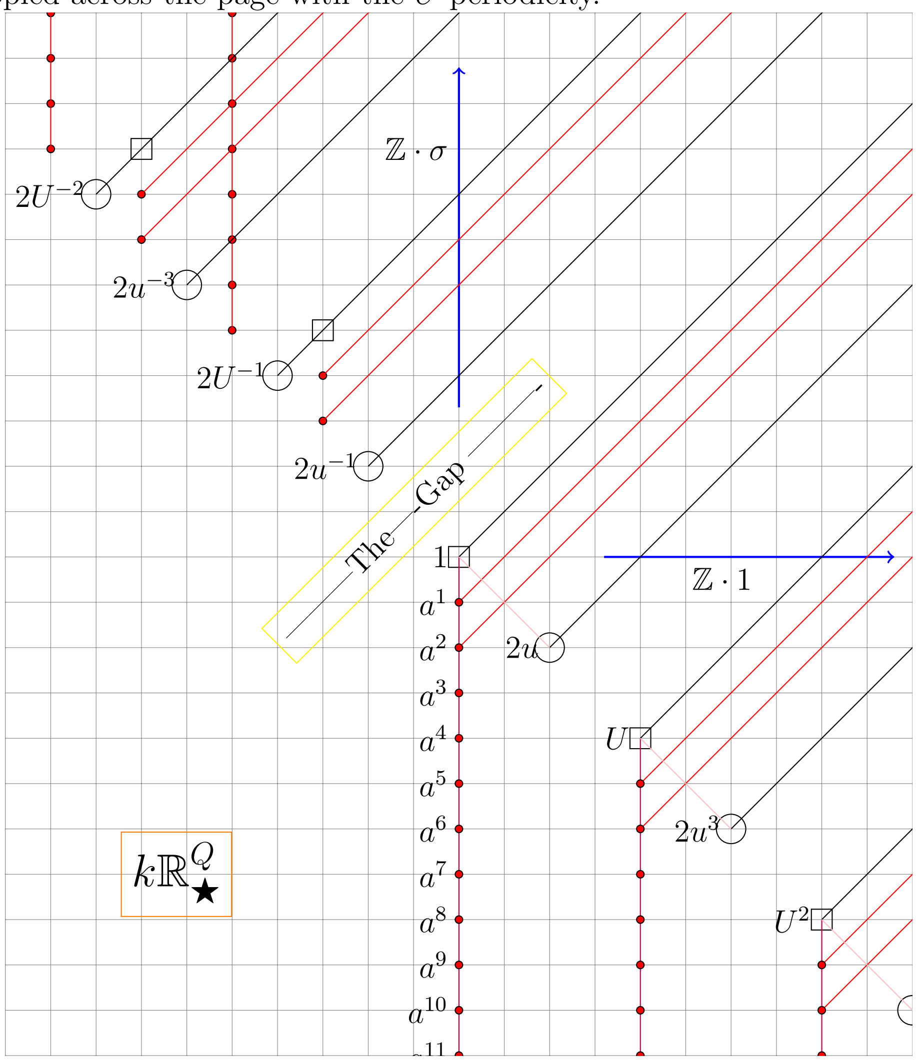
$$NB = \mathbb{Z} \oplus \Sigma^{-\rho}\mathbb{F}_2[a]^\vee.$$

The BB-module structure of NB is as suggested by the notation, and multiplication by u relates factors of NB and BB according to the notation, with the map $u^{-1} \cdot NB \longrightarrow BB$ being the composite $NB \longrightarrow \mathbb{Z} \longrightarrow BB$.

3. K-theory with reality

We now repeat the exercise with the connective version of Atiyah's K-theory with Reality: $k\mathbb{R}$. Again squares are copies of $\mathbb{Z}$ and dots are copies of $\mathbb{F}_2$; circles are also copies of $\mathbb{Z}$, but the generator is twice what one might expect. This calculation first appears in [8], and there are alternative approaches in [4, 13].

One notable feature is 'The Gap', which constitutes the three lines of slope 1 above the unit (i.e., the diagonals intersecting the a axis in $a = -3, -2, -1$.). In the periodic form of the theory (i.e., with the degree ρ class $\overline{v}$ inverted) the a-columns vanish and The Gap is copied across the page with the U-periodicity.



3.A. **Adding cells.** The approach through a cell structure of $S^{b\sigma}$ is a fairly unattractive since the value of the theory on cells is complicated: $k\mathbb{R}^Q_* = ko_*$. On the other hand, it is effectively what was done in [4], by working up the Tate filtration, so we will not repeat it here.

3.B. **Quotient approach.** The good behaviour on quotients is only easy to see on free spectra, so the content can be extracted from the Tate approach by truncating appropriately. Again, it does not seem worth repeating here.

3.C. **Tate approach.** The cleanest approach is to use the Tate diagram

$$\begin{array}{ccccc} k\mathbb{R}_h & \longrightarrow & k\mathbb{R} & \longrightarrow & k\mathbb{R}^\Phi \\ \simeq\downarrow & & \downarrow & & \downarrow \\ k\mathbb{R}_h & \longrightarrow & k\mathbb{R}^h & \longrightarrow & k\mathbb{R}^t \end{array}$$

The only real calculational input is the homotopy fixed points.

Lemma 3.1. *The homotopy fixed point coefficients are*

$$k\mathbb{R}^{hQ}_\bigstar = BB[U, U^{-1}],$$

where $BB = \mathbb{Z}[a, \overline{v}]/(2a, \overline{v}a^3) \oplus 2U \cdot \mathbb{Z}[\overline{v}]$ *is the Basic Block for* $k\mathbb{R}$. *Here* $|a| = -\sigma$ *as before and* $|\overline{v}| = 1 + \sigma$*; the basic block* BB *is copied across the page by the periodicity operator* U *of degree* $|U| = 4 - 4\sigma$.

Remark 3.2. Notice how this is related to ordinary group cohomology, which occurs along the negative x-axis:

$$k\mathbb{R}^{hQ}_{\leq 0} = \mathbb{Z}[Y]/(2Y), \text{ where } Y = a^4U^{-1} \text{ is of codegree 4.}$$

The element Y corresponds to y^2 from $H\underline{\mathbb{Z}}^{hQ}_*$ in Remark 2.2.

Similarly $ko_* = (k\mathbb{R})^{hQ}_{\geq 0}$ occurs along the positive x-axis, with $\eta = a\overline{v}$ generating ko_1, $2u\overline{v}^2$ generating ko_4 and the Bott element $U\overline{v}^4$ generating ko_8.

Proof: We describe the $RO(Q)$-graded spectral sequence of Subsection 1.E for $k\mathbb{R}$. In fact it is convenient to describe it from the E_1-term if we choose a good cell structure on EQ_+. Indeed we use the filtration

$$S(\sigma)_+ \subset S(2\sigma)_+ \subset S(3\sigma)_+ \subset \cdots \subset S(\infty\sigma)_+ = EQ_+.$$

There is one free Q-cell in each degree and in cellular homology this gives the standard 2-periodic resolution

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\epsilon} \mathbb{Z}Q \xleftarrow{1-x} \mathbb{Z}Q \xleftarrow{1+x} \mathbb{Z}Q \xleftarrow{1-x} \cdots$$

of $\mathbb{Z}$ over the group ring $\mathbb{Z}Q$.

We may then describe the E^1 term conveniently as

$$E^1_{*,*,*} = \mathbb{Z}[a, \overline{v}][\lambda, \lambda^{-1}]$$

where the (s, t, b) gradings are

$$|a| = (-1, 1, -1), |\lambda| = (0, 1, -1), |\overline{v}| = (0, 1, 1).$$

The differential is given by making $a, \overline{v}$ into d_1 cycles and

$$d_1\lambda = 2a.$$

This is clear from the algebraic resolution since we know Q acts to negate odd powers of v and is of degree -1 on $S^{b\sigma}$ for b odd.

This means that $u = \lambda^2$ is a d_1-cycle and we may calculate

$$E^2_{*,*,*} = \{\mathbb{Z}[a]/(2a)\}\,[\overline{v}][u, u^{-1}].$$

The next differential is the one piece of topological input. It is forced by the fact that $\eta^4 = 0$ (from the stable homotopy of S^0). The differential is given by making $a, \overline{v}$ into cycles again and

$$d_2 u = \overline{v}a^3.$$

This makes $U = u^2$ into a d_2-cycle and

$$E^3 = \left\{\mathbb{Z}[a, \overline{v}]/((2a, \overline{v}a^3) \oplus (2u) \cdot \mathbb{Z}[\overline{v}]\right\}[U, U^{-1}],$$

where the relations $(2u) \cdot a = 0$ and $(2u)^2 = 4u^2 = 4U$ are left implicit. Now we find that all subsequent differentials on $a, \overline{v}$ or U have target in a zero group, so the spectral sequence has collapsed. □

Because $S^{\infty\sigma} \simeq S^0[1/a]$ it is then easy to calculate Tate cohomology by inverting a and homotopy orbits as local cohomology.

Corollary 3.3.

$$k\mathbb{R}^{tQ}_{\bigstar} = \mathbb{F}_2[a, a^{-1}][U, U^{-1}],$$

$$k\mathbb{R}_{hQ}^{\bigstar} = NB[U, U^{-1}],$$

where $NB = (2, \overline{v}) \oplus \Sigma^{-1}\mathbb{F}_2[a]^\vee$ is the Negative Block for $k\mathbb{R}$, more conceptually described as the right derived functors of the a-power torsion functor $\Gamma_{(a)}$:

$$NB = R\Gamma_{(a)}(BB).$$

For geometric fixed points we apply Lemma 1.3.

Lemma 3.4. *The geometric fixed points are given by $k\mathbb{R}^{\Phi Q}_{\bigstar} = \mathbb{F}_2[a, a^{-1}][U]$*

Proof: By construction $KO = (K\mathbb{R})^Q$. Since $(K\mathbb{R})^{tQ} \simeq *$ (by nilpotence of a in $(K\mathbb{R})^{hQ}_{\bigstar}$) we see $(K\mathbb{R})^Q \simeq (K\mathbb{R})^{hQ}$. Now the behaviour of the homotopy fixed point spectral sequence for $K\mathbb{R}$ shows that of $k\mathbb{R}$ and hence that $(k\mathbb{R})^Q \longrightarrow (k\mathbb{R})^{hQ}$ is the connective cover. We may therefore apply Lemma 1.3 to obtain the conclusion. □

Finally, we return to the Tate square, and read off the conclusion: the only difference between $k\mathbb{R}^Q_{\bigstar}$ and the more regular $k\mathbb{R}^{hQ}_{\bigstar}$ comes from the negative Tate a-columns.

Corollary 3.5. *The $RO(Q)$-graded homotopy of $k\mathbb{R}$ is as follows*

$$k\mathbb{R}^Q_{\bigstar} = BB[U] \oplus U^{-1} \cdot NB[U^{-1}],$$

where the action of U across the boundary

$$U^{-1} \cdot NB \longrightarrow BB$$

factors out the $H^1_{(a)}$ towers and includes the a-power torsion in BB.

3.D. **The Bockstein spectral sequence.** Dugger [8] has shown that there is a cofibre sequence

$$\Sigma^\rho k\mathbb{R} \xrightarrow{\overline{v}} k\mathbb{R} \longrightarrow H\underline{\mathbb{Z}}$$

where $\rho = 1+\sigma$. (This can be proved by noting the homotopy fixed point spectral sequence gives an element $\overline{v}$, and slice filtration methods easily show the mapping cone is $H\underline{\mathbb{Z}}$). We use the Dugger sequence to calculate $k\mathbb{R}^Q_\bigstar$ from $H\underline{\mathbb{Z}}^Q_\bigstar$ by the Bockstein spectral sequence. This spectral sequence happens to coincide with the slice spectral sequence for $k\mathbb{R}$, as we make explicit in Subsection 3.E, but we will not make use of this.

Accordingly (using the conventions of [4, Section 4.1]) we consider the spectral sequence obtained by taking $RO(Q)$-graded homotopy of the diagram

$$\begin{array}{ccccccccc}
k\mathbb{R} & \longleftarrow & \Sigma^\rho k\mathbb{R} & \longleftarrow & \Sigma^{2\rho} k\mathbb{R} & \longleftarrow & \Sigma^{3\rho} k\mathbb{R} & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z} & & \Sigma^\rho H\underline{\mathbb{Z}} & & \Sigma^{2\rho} H\underline{\mathbb{Z}} & & \Sigma^{3\rho} H\underline{\mathbb{Z}} & &
\end{array}$$

to obtain a spectral sequence

$$E^1 = H\underline{\mathbb{Z}}^Q_\bigstar[\hat{\overline{v}}] \Rightarrow k\mathbb{R}^Q_\bigstar$$

where $\hat{\overline{v}}$ is a formal variable of bidegree $(1,\rho)$. Since we have already calculated $H\underline{\mathbb{Z}}^Q_\bigstar$ this is straightforward once we have found a reasonable way to display the information. The standard thing to do in the $\mathbb{Z}$-graded case (for $\Sigma ko \longrightarrow ko \longrightarrow ku$ for example) is that for each row of constant s we write ku_* horizontally, shifted to the right by s, so that the differential d_r takes on the Adams form of going up r rows and one column to the left (i.e., subtracting 1).

In our present context, for each fixed s we have a whole $RO(Q)$-plane. Accordingly we will refer to this counterpart of the sth row as the sth *floor*. Once again we arrange that d^r goes up from the sth floor to the $(s+r)$th floor. If we place $\pi^Q_\bigstar(\Sigma^{s\rho}H\underline{\mathbb{Z}})$ on the sth floor, d^r again subtracts 1: symbolically, for any $\alpha \in RO(Q)$,

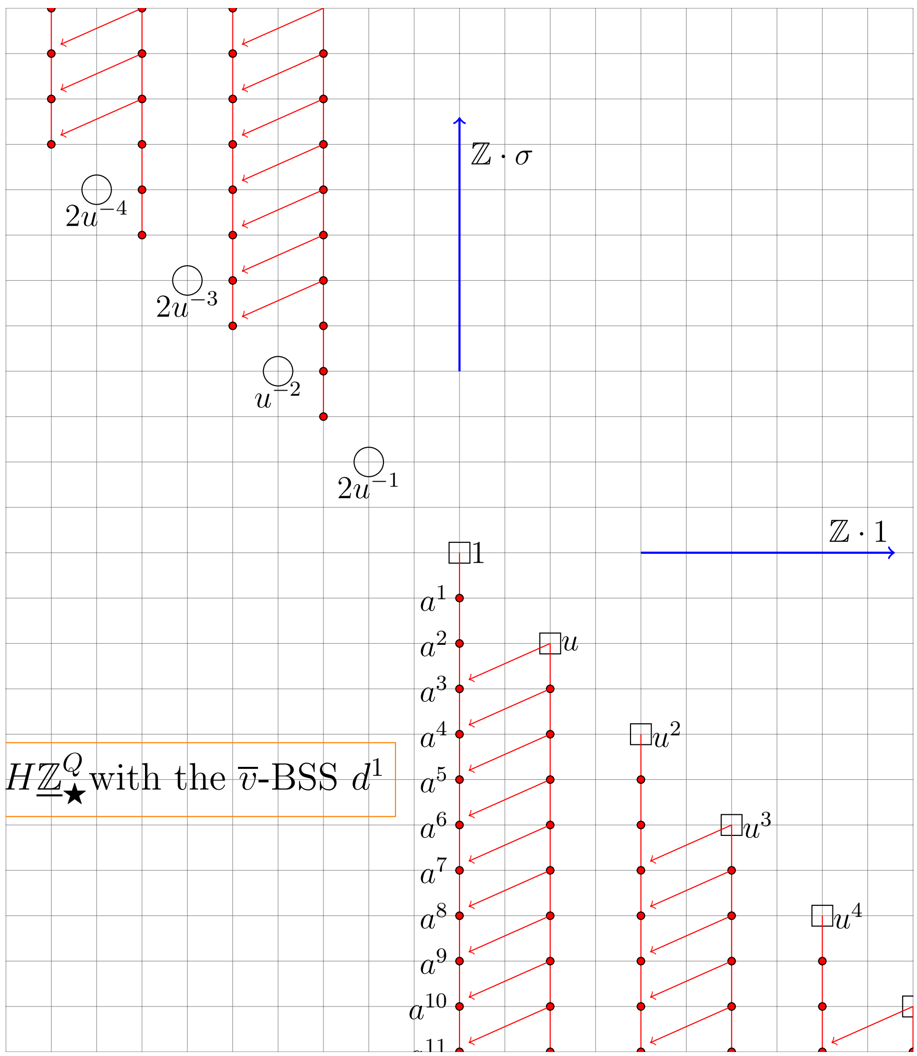
$$d^r : H\underline{\mathbb{Z}}^Q_\alpha \longrightarrow (\Sigma^{r\rho}H\underline{\mathbb{Z}})^Q_{\alpha-1} = H\underline{\mathbb{Z}}^Q_{\alpha-r\rho-1} = H\underline{\mathbb{Z}}^Q_{\alpha-r-1-r\sigma}$$

The value on x is defined by finding the image of x in $k\mathbb{R}^Q_{\alpha-1}$, dividing by $\overline{v}^{r-1}$ and looking at the image in $H\underline{\mathbb{Z}}^Q_\bigstar$ again. We need only determine it in a few cases and then propogate this using the multiplicative structure.

For those with old-fashioned habits, it is natural to arrange the calculation conventionally in the upper half-plane with the box at (n,s) containing $H\underline{\mathbb{Z}}^Q_{n-s+*\sigma}$, then d^r relates (n,s) to $(n-1,s+r)$ and reduces the σ-grading by r. We then find the value of $ku^Q_{n+*\sigma}$ by adding up the terms for fixed n

On the other hand this places a strain on visualization, and an alternative is to pile copies of $H\underline{\mathbb{Z}}^Q_\bigstar$ vertically above each other and look from above. Each $H\underline{\mathbb{Z}}^Q_{a+b\sigma}$ that you see then represents a whole pile, $H\underline{\mathbb{Z}}^Q_{a+b\sigma}[\hat{\overline{v}}]$, of copies. Then the differential d^r raises the floor in the pile by r and as indicated above it reduces $a+b\sigma$ by $(r+1)+r\sigma$. Finding the answer then involves adding up entries from different stacks with total degree constant.

We illustrate this for d^1, which has degree $-2-\sigma$:



To justify the displayed d^1, note that it is induced by a map $\theta : H\underline{\mathbb{Z}} \longrightarrow \Sigma^{2+\sigma}H\underline{\mathbb{Z}}$, which is the analogue of the integral lift of Sq^3 in the non-equivariant setting. The stable operations of $H\underline{\mathbb{Z}}$ are relatively complicated, but in our case it suffices to observe that θ induces a map $\overline{\theta} : b \longrightarrow \Sigma^3 b$ where $b = F(EQ_+, H\mathbb{Z}/2)$ represents mod 2 Borel cohomology. The stable operations in this case are easy to determine: we immediately see $b^*_Q b = H^*(BQ_+;\mathbb{Z}/2)\otimes\mathcal{A}^*$ (the operations are studied further in [11]). In any case, we see that for representations V

$$\theta : H^n_Q(S^V;\underline{\mathbb{Z}}) = H\underline{\mathbb{Z}}^Q_{V-n} \longrightarrow H\underline{\mathbb{Z}}^Q_{V-n-2-\sigma} = H^{n+2}_Q(S^{V-\sigma})$$

is

$$\theta : H^{n-1}_Q(S(V);\underline{\mathbb{Z}}) \longrightarrow H^{n+1}_Q(S(V-\sigma);\underline{\mathbb{Z}})$$

for $n\geq 2$ and provided V contains σ. When $n-1$ is even, reduction mod 2 is an isomorphism and this agrees with

$$\overline{\theta} : b^{n-1}_Q(S(V)) \longrightarrow b^{n+1}_Q(S(V-\sigma)) = b^{n+2}_Q(S(V)).$$

Such an operation is a linear combination $\lambda Sq^3 + \mu Sq^2 Sq^1$. We need only argue that $\lambda \neq 0$, since this forces $\mu = 0$ (so as to get $(d^1)^2 = 0$). There are several approaches to understanding this operation. The first is to argue from the fact that $\eta^4 = 0$ (this involves looking at the whole BSS as displayed in Figure 1 below). The second is to study operations sufficiently to see the behaviour of d^1 is forced by the non-equivariant situation. Perhaps the most elementary is to begin by understanding the situation in geometric fixed points (i.e., after inverting a). After the fact, we know that the cofibre sequence

$$\Sigma k\mathbb{R}^{\Phi Q} = (\Sigma^\rho k\mathbb{R})^{\Phi Q} \longrightarrow k\mathbb{R}^{\Phi Q} \longrightarrow H\underline{\mathbb{Z}}^{\Phi Q}$$

has $H\underline{\mathbb{Z}}_*^{\Phi Q} = \mathbb{F}_2[x]$ where x is of degree 2 and $k\mathbb{R}_*^{\Phi Q} = \mathbb{F}_2[x^2]$. However, we do not need to use this full analysis of the cofibre sequence to see that the operation is non-trivial from the 2 column to the 0 column: we only need to know that $k\mathbb{R}_1^{\Phi Q} = 0$, which is easily checked from the Dugger sequence and known values of $H\underline{\mathbb{Z}}_\bigstar^Q$. This then determines the operation, which gives all the values. Using any one of these methods shows that d^1 is as displayed.

We could now jump straight to displaying $E^2 = E^\infty$. One simply notes that the $s = 0$ line consists of kernels of d^1 and each of the higher rows consists of the homology (with the sth row shifted s steps to the right).

Alternatively, instead of dealing with d^1 in piles, we could have used the traditional display, and we will describe this next. Each box for constant (n, s) contains a group graded on multiples of σ. Because of the almost-periodicity of $H\underline{\mathbb{Z}}_\bigstar^Q$ of $2(1-\sigma)$ and for typographical reasons we have arranged that $n(1-\sigma)$ is the midpoint of the (n, s)-box. In other words, the element x of (n, s)-degree $(2, 0)$ is represented by horizontal translation and corresponds to u of degree $2(1-\sigma)$.

We will now display the E^1-term in Figure 1 and then the $E^2 = E^\infty$ term in Figure 2. The squares and circles denote copies of $\mathbb{Z}$ and the dots denote copies of $\mathbb{F}_2$. The larger (pale blue) blobs pick out the slice spectral sequence as detailed in the next section.

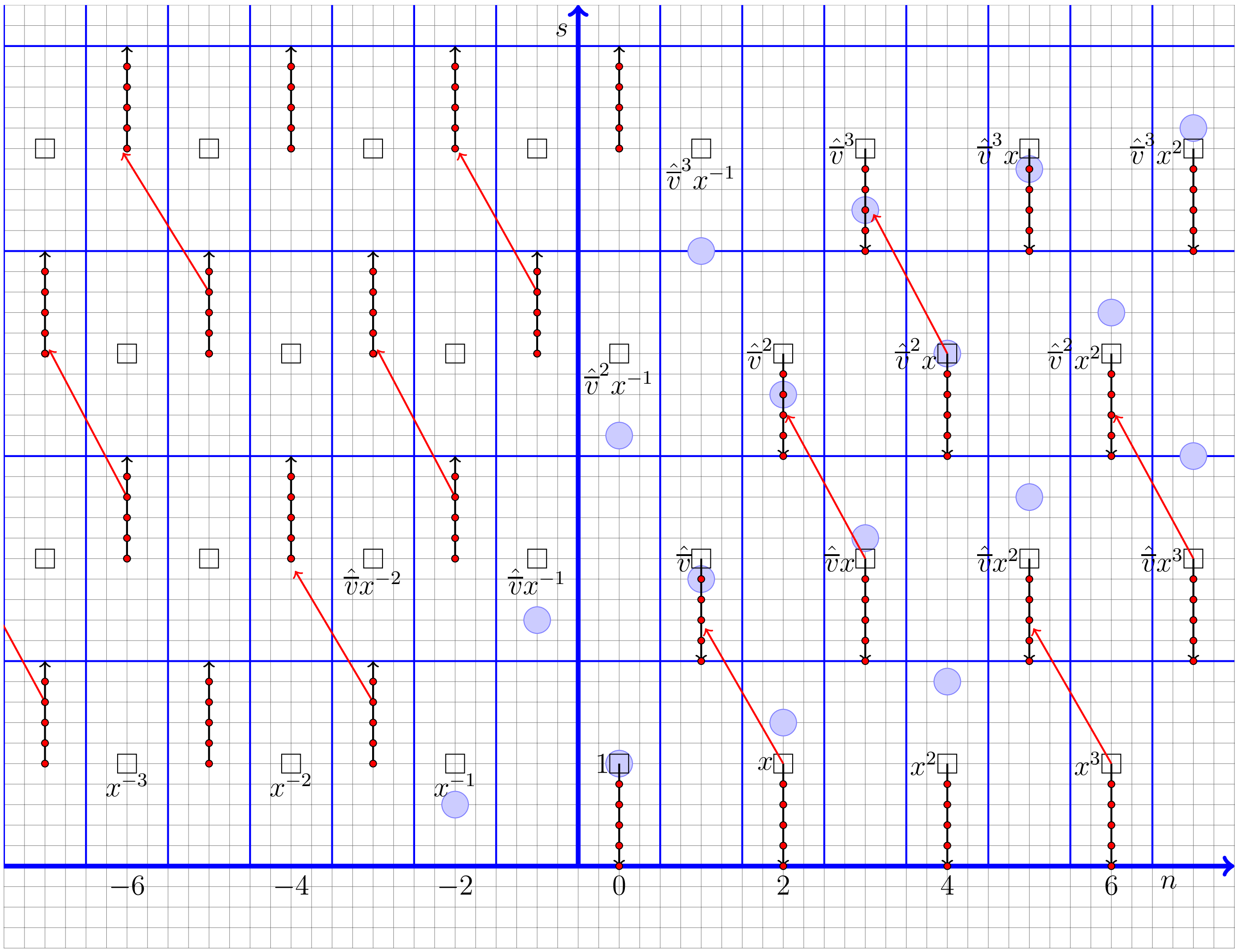


FIGURE 1. The $\overline{v}$-Bockstein spectral sequence: $E_1 = H\underline{\mathbb{Z}}^Q_\bigstar[\hat{\overline{v}}] \Rightarrow k\mathbb{R}^Q_\bigstar$. Squares are copies of $\mathbb{Z}$, small (red) dots are copies of $\mathbb{F}_2$. The d^1 differentials are indicated by displaying the top or bottom degree component between the relevant boxes. The larger (blue) blobs select out the degrees occurring in one slice spectral sequence.

3.E. **The slice spectral sequence (SSS).** The SSS is usually presented as a mechanism for calculating the integer graded part $\pi^Q_*(X)$, with $RO(Q)$-graded part obtained by looking at suspensions of X [14]. Since suspension by ρ just shifts the slices, we may combine the suspensions and view it as a method for calculating the $RO(Q)$-graded homotopy. We will not give a detailed account of the SSS here since in this special case it is contained in the BSS. This short section is designed so that those already familiar with the SSS can pick it out from the BSS as described in the previous subsection.

For $k\mathbb{R}$, the slice filtration is precisely the same as the $\overline{v}$-Bockstein filtration, so that the $RO(Q)$-graded BSS and SSS are the same. The one piece of added value in the SSS is

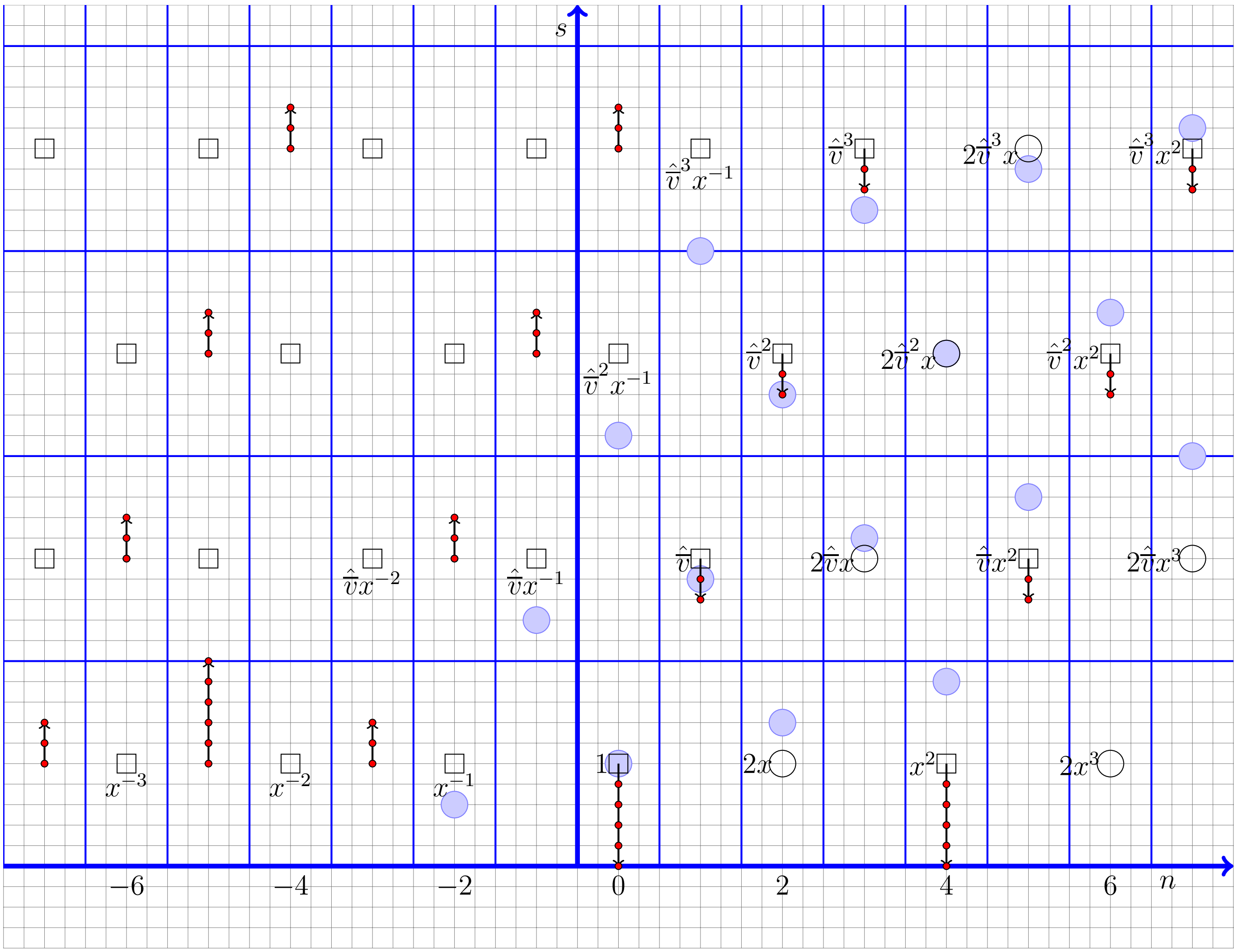


FIGURE 2. $E_2 = E_\infty$ of the $\bar{v}$-Bockstein spectral sequence: $E_1 = H\underline{\mathbb{Z}}^Q_\star[\hat{\bar{v}}] \Rightarrow k\mathbb{R}^Q_\star$. Squares are copies of $\mathbb{Z}$. Large circles are copies of $\mathbb{Z}$ of index 2 in E_1. Small (red) dots are copies of $\mathbb{F}_2$. The larger (blue) blobs select out the degrees occurring in one slice spectral sequence. The Gap occurs in this picture most prominently as the three zeroes below the diagonal squares emanating from $(n, s) = (-2, 0)$, but it also requires three zeroes along the diagonal emanating from $(n, 0)$ for any $n \leq -1$.

the way that the usual ($\mathbb{Z}$-graded) display highlights vanishing lines and 'The Gap'. This information is present in the BSS displays above, but it is not so easy to spot.

Just to be completely explicit, for $k\mathbb{R}$ the negative slices are all zero and the odd slices are all zero. The positive even slices are suspensions of $H\underline{\mathbb{Z}}$: in the notation of [15] $P^{2k}_{2k}k\mathbb{R} = \Sigma^{k\rho}H\underline{\mathbb{Z}}$ for $k \geq 0$. In Adams grading $(n, s) = (2k - s, s)$ the usual SSS has

$$\pi^Q_{2k-s}(P^{2k}_{2k}k\mathbb{R}) = H^Q_{2k-s}(S^{k\rho}; \underline{\mathbb{Z}}) = H^Q_{k-s}(S^{k\sigma}; \underline{\mathbb{Z}}) = H\underline{\mathbb{Z}}^Q_{k-s-k\sigma},$$

so that the homotopy groups of one particular slice occur along a line of slope -1. The appropriate chart is displayed clearly as Figure 7 in [16, Page 414].

Note that the homotopy group above, at coordinates $(x_S, y_S) = (2k-s, s)$ in the SSS, appears at coordinates $(x_B, y_B) = (2k-s, k)$ in the BSS. Thus $(x_B, y_B) = (x_S, (x_S+y_S)/2)$ and $(x_S, y_S) = (x_B, 2y_B - x_B)$. The BSS differential d_B^1 corresponds to the SSS differential d_S^3. The slice spectral sequence simply selects a subspectral sequence given by one particular σ-grading in each box.

The larger (pale blue) blobs in the two $\overline{v}$-**B**SS charts are the potentially nonzero entries in the **S**SS for $\pi_*^Q(k\mathbb{R})$ (i.e., the integer graded part). We have displayed five diagonal rows of blobs (in adding $(1,1)$ to (n,s), σ is subtracted). The diagonal emanating from $(n,s) = (-2,0)$ does not contain any non-zero SSS entries. The diagonal emanating from $(n,s) = (0,0)$ contains 1, and $a^k\hat{\overline{v}}^k$ (corresponding to η^k) for $k \geq 1$. Of course $1, a\hat{\overline{v}}$ and $a^2\hat{\overline{v}}^2$ survive to E^∞ to give η and η^2 and higher powers are the images of differentials. The first non-zero entry on the diagonal emanating from $(n,s) = (2,0)$ is on the $s=2$ line, where it is $\hat{\overline{v}}^2 x$, and after that $a^k\hat{\overline{v}}^{2+k}x$ for $k \geq 1$. These all support nonzero differentials, so only $2\hat{\overline{v}}^2 x$ survives (to give the generator of $\pi_4^Q(k\mathbb{R}) = \pi_4(ko)$). In our display, only one blob is visible on diagonal emanating from $(n,s) = (4,0)$, and it is zero.

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