GLIMPSES OF BOTT PERIODICITY

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Abstract. Bott periodicity sits at the intersection of different fields of mathematics and physics: Morse Theory, Clifford Algebras, K-Theory, and others. This paper discusses the the weak form of *Bott Periodicity* in both the real and complex case through the viewpoint of the original proof done via Morse Theory. The paper is organized in two main parts: the first part looks at the complex case and the second part looks at the real case.

Contents

1. Background

For the sake of brevity, this paper will give minimal background material and dive into the proof(s) of Bott Periodicity. The two parts of this paper study the original formulation of Bott Periodicity. The original proof of complex Bott periodicity uses Morse theory to show that $\Omega^2 U \cong U$ where $U = colim_n U(n)$ is the infinite unitary group.

2. Morse Theory

Morse originally invented Morse theory to study the energy functional on the infinite dimensional space of loops in a manifold. While it was intended to understand geodesic motion, Bott employed this theory to show that the homotopy groups of the unitary group U, the orthogonal group O, and the symplectic group Sp are periodic. A consequence of Bott periodicity is that the infinite unitary space is an infinite loop space (and, hence, an E_∞-space). The power of Morse theory should not be understated; for example, Smale proved the h-cobordism and generalized Poincare conjectures via Morse theory and surgery cobordisms.

The proof of Bott periodicity given here focuses on the main lemmas while the details can be found in Milnor's *Morse theory*[1]. Morse theory essentially connects the structure of differentiable manifolds with the behavior of certain real-valued functions called *Morse functions*. Most of the normal functions people deal with are Morse or can be perturbed to become Morse. The advantage of Morse Theory is that the homotopy type of M can be established via the behavior of f at the

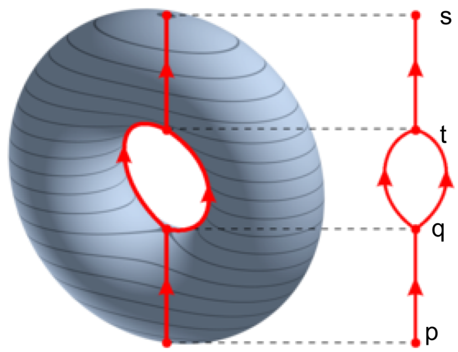

Figure 1: The height function on a torus from Wikipedia.

non-degenerate critical points where the Hessian of f is nonsingular. Moreover, each critical point can be given an index λ that is determined by the Hessian of f at that point. This shows that M has the homotopy type of a CW-complex with a cell-dimension λ for each critical point.

Example 2.1. Height function on the torus

The classic example of a Morse function is the height function on a torus standing on one end (Figure 1). A height function $h : T \to \mathbb{R}$ can be assigned to the image that indicates the minimal distance from each point of the torus to the plane. The points p, q, t, and s are the critical points of h, and an index can be assigned to each of them which counts the number of independent directions along which one can move such that h decreases. We see that $\text{index}(p) = 0,\ \text{index}(q) = 1,\ \text{index}(t) = 1$, and $\text{index}(s) = 2$. As it turns out, information about the index corresponds to information about a CW-complex. Without fleshing the details out, giving T the corresponding homotopy type of a CW-complex with one 0-cell, two 1-cells, and one 2-cell allows us to deduce topological information from geometric information.

Analogously, through the lens of Riemannian Geometry, we can study the path space of geodesics on a Riemannian manifold N. In particular, consider the non-minimal geodesics connecting points p and q in N. Keeping in mind the Index Theorem (which will be mentioned later), each non-minimal geodesic can be assigned an index λ. This is done by counting the conjugate points along the geodesic. This gives us the **Fundamental Theorem of Morse Theory** which states that the path space between p and q in N has the homotopy type of a CW-complex which contains a cell of dimension λ for each geodesic of index λ. Note that this is analogous to assigning an index to each non-degenerate critical point.

Moreover, information is gained when we consider the **Morse lemma**: the lemma allows us to directly analyze the neighborhoods of non-degenerate critical points on a manifold. Let p_0 be a non-degenerate critical point of a smooth function $f : M \to \mathbb{R}$ where M is an n-manifold. We can choose a local coordinate system $(x_1, \cdots, x_n)$ centered at p_0 such that

$$f = -x_1^2 - x_2^2 - \ldots - x_\lambda^2 + x_\lambda^2 + \ldots + x_n^2 + c$$

where $c = f(p_0)$ is some constant and λ is the index of f at the critical point p_0. The Morse Lemma will be an important tool in Bott Periodicity (and, moreover, tends to be vital in calculations involving Morse theory).

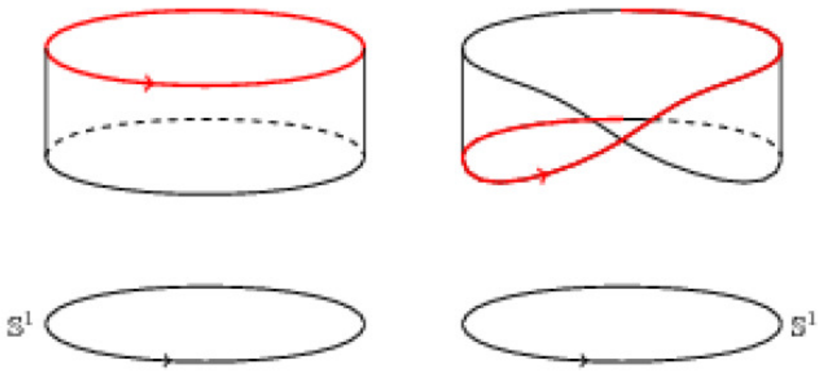

Figure 2: The Mobius band (right) is a nontrivial bundle compared to the cylinder (left), a trivial bundle, from [HEP].

2.1. **Unitary Case.** For the unitary group, the fact that $\pi_{i-1}(U) \cong \pi_{i+1}(U)$ for $i \geq 1$ shows us that the homotopy groups are two-periodic. Before diving into the proof, remarks on linear algebra, Jacobi fields, and fiber bundles will be mentioned.

First, consider $c = (c_1, \ldots, c_n)$ and $d = (d_1, \ldots, d_n) \in \mathbb{C}^n$, and note that the *Hermitian inner product* is $\langle c, d\rangle = \sum_1^n c_i\overline{d_i}$. Furthermore, a matrix A is *unitary* if it satisfies $\langle Ac, Ad\rangle = \langle c, d\rangle$. Therefore $A \in \mathbb{C}^{n\times n}$ is unitary if and only if $AA^* = I_n$. Moreover, the unitary group $U(n)$ is the group of n by n unitary matrices in $\mathbb{C}^{n\times n}$ under matrix multiplication. Moreover, A is unitary if and only if its columns are orthonormal.

Next, suppose $c : [0, a] \to M$ is a geodesic. We say that a vector field J along a geodesic c is a *Jacobi field* if it satisfies the equation $\frac{D^2J}{dt^2} + R(\frac{dc}{dt}, J)\frac{dc}{dt} = 0$, where R is the *curvature*. Note that these are the solutions to an ODE, and so they are uniquely determined by their initial conditions J_0 and $\frac{DJ}{dt}(0)$. The geometric significance of Jacobi fields comes from considering a parametrized surface $s : (-\epsilon, \epsilon) \times [0, 1] \to M$ where $s(u, t)$ is a geodesic for each $u \in (-\epsilon, \epsilon)$. Now, we have that the *variation vector field* $V_t : \frac{dt}{du}(0, t)$ is a Jacobi field along $s(0, t)$ and that conversely every Jacobi field along a geodesic $c : [0, 1] \to M$ can be obtained from a similar variation of c through geodesics.

Definition 2.2. A **fiber bundle** over B with fiber F is defined as a surjective map $p : E \to B$ where every $p \in B$ has a neighborhood U such that $p^{-1}(U)$ is homeomorphic to $U \times F$ via $\gamma : p^{-1}(U) \to U \times F$ and $p = \mathrm{proj}_U(U \times F) \circ \gamma$.

Example 2.3. The Cartesian product $E = B \times F$ is known as the **trivial fiber bundle.**

Example 2.4. The (open) **Mobius strip** is a non trivial bundle whose base space is the circle and whose fiber is the real line. Intuitively, you make it by twisting a strip of paper. Formally, you can consider the trivial bundle $([0, 1] \times \mathbb{R})$ and identify $(0, x)(0, x)$ with $(1, -x)(1, -x)$.

Example 2.5. A **vector bundle** is a special class of the fiber bundle where the fiber is a vector space.

A fiber bundle is written as $F \to E \to B$. We can think of E as being somehow put together from B and F. Intuitively, we can think of E as "almost" – or at least, locally – a product of F and B. Thus, the fiber bundle $F \to E \to B$ tells us that locally E looks like the product of $B \times F$ even if the two globally differ.

In particular, the following types of fiber bundles will be important.

Example 2.6. The inclusion $i : U(n) \to U(n+1)$ defined by

$$A \to \left(\begin{array}{cc} 1 & 0 \\ 0 & A \end{array} \right)$$

and the map $j : U(n+1) \to S^{2n+1} \subset \mathbb{C}^{n+1}$ that takes a matrix to its first column give the following fiber bundle:

$$U(n) \to U(n+1) \to S^{2n+1}$$

This induces the following long exact sequence of homotopy groups:

$$\ldots \to \pi_{i+1}(S^{2n+1}) \to \pi_i(U(n)) \to \pi_i(U(n+1)) \to \pi_i(S^{2n+1}) \to \ldots$$

Note that for $2n+1 > i$ we have that $\pi_i(S^{2n+1}) = 0$. Hence, we have that $\pi_i(U(n)) \cong \pi_i(U(n+1)) \cong \pi_1(U(n+2)) \cong \ldots$ for $2n+1 > i$, and that $\pi_i(U(n))$ is independent of n for big enough n. Thus, we have that $\pi_i(U) \cong \pi_i(U(n))$ for sufficiently large n.

Example 2.7. The **complex Steifel manifold**, $V_k(\mathbb{C}^n)$, is the coset space $U(2k)/U(k)$. It is the space of k-tuples of orthonormal vectors in $\mathbb{C}$. We have that $U(n)$ acts transitively on $V_k(\mathbb{C}^n)$ via matrix multiplication. Moreover, by noting that an n-frame of $\mathbb{C}^n$ is fixed by a member of $U(n)$ which acts non-trivially only on the k frame of the orthogonal complement, we obtain the following fiber bundle:

$$U(n-k) \to U(n) \to V_k(\mathbb{C}^n)$$

So, from looking at the associated long exact homotopy sequence, we have that $\pi_i(V_k(\mathbb{C}^n))$ is trivial for all $i \leq 2(k-n)$.

Example 2.8. The **complex Grassmann manifold**, $G_m(\mathbb{C}^n)$, can be identified with the coset space $U(2m)/U(m) \times U(m)$. It is the collection of m dimensional subspaces of $\mathbb{C}^n$ for some $m \leq n$. Take the mapping $V_m(\mathbb{C})^n) \to G_m(\mathbb{C}^n)$ that sends an m-frame in $\mathbb{C}^n$ to the m-dimensional subspace it spans and note that the fibers here are the collections of m-frames spanning the same m-dimensional subspace. So, we have the following fiber bundle:

$$U(m) \to V_m(\mathbb{C}^n) \to G_m(\mathbb{C}^n)$$

In addition, we have the long exact sequence of homotopy groups:

$$\ldots \to \pi_i(V_m(\mathbb{C}^n)) \to \pi_i(G_m(\mathbb{C}^n)) \to \pi_{i-1}(U(m)) \to \pi_{i-1}(V_m(\mathbb{C}^n)) \to \ldots$$

Thus, $\pi_{i-1}(U(m)) \cong \pi_i(G_m(\mathbb{C}^n)), i < 2(n-m)$.

Definition 2.9. The **special unitary group** is the subgroup $SU(m) \subset U(M)$ of matrices with determinant 1.

Note that the determinant map $U(m) \to S^1 \subset \mathbb{C}$ gives us a fiber bundle $SU(m) \to U(m) \to S^1$. We then have $\pi_i(SU(m)) \cong \pi_i(U(m))$ and $\pi_i(SU(m)) \cong \pi_i(U)$ for $i > 1$ from the long exact sequence of homotopy groups:

$$\ldots \to \pi_{i+1}(S^1) \to \pi_i(SU(m)) \to \pi_i(U(m)) \to \pi_i(S^1) \to \ldots$$

Thus, the last two examples give us

$$\pi_{i-1}(U(m)) \cong \pi_i(G_m(\mathbb{C}^{2m})) \text{ and } \pi_{i+1}(SU(2m)) \cong \pi_{i+1}(U(m))$$

for sufficiently large m and $1 \leq i \leq 2m$. Returning to Bott periodicity, we now want to prove that $\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i+1} U(2m)$ for large m (Theorem 23.3 in [1]). This is difficult to do directly; hence we will look at the space Ω^α of minimal geodesics in $SU(2m)$ from I to $-I$ and use the tools of Morse Theory mentioned before.

Lemma 2.10. $\pi_i(G_m(\mathbb{C}^{2m})) \cong \pi_i(\Omega^\alpha)$

Proof. This uses calculus of variations applications to Morse Theory. Check lemma 23.1 in [1].

Consider the Lie group $SU(2m)$. We identify the tangent space $T_I SU(2m)$ with the space of $2m$ by $2m$ skew-Hermitian matrices with trace 0. The exponential map

$$\exp(A) = 1 + A + \frac{1}{2!}A^2$$

sends $T_I SU(2m)$ into $SU(2m)$.

Choose some $A \in T_I SU(2m)$ such that $\exp(A) = -I$ (which we do in order to obtain the minimal geodesic) and define $\gamma(t) = \exp(tA)$ for $t \in [0, 1]$. Since A is skew-Hermitian, it may be diagonalized $D = TAT^{-1}$. Then

$$\exp(D) = T \exp(A) T^{-1} = -I,$$

so we may assume that A is diagonal without loss of generality. Since A is skew-Hermitian, its (diagonal) entries a_j are purely imaginary and sum to zero. In fact, we may write $a_j = ik_j\pi$ where each k_j is odd and $\sum_j k_j = 0$.

Skipping a few small steps, we have that a geodesic $\exp(tA)$ has length $\|A\| = \sqrt{\langle A, A\rangle} = \pi\sqrt{k_1^2 + \ldots + k_{2m}^2}$. It is also the case that all minimal geodesics from $-I$ to I in $SU(2m)$ have the form $\exp(tA)$, where A is a diagonal matrix with m entries equal to $i\pi$ and m entries equal to $-i\pi$. Hence, A is uniquely determined by its eigenspace for $i\pi$, an m-dimensional subspace of $\mathbb{C}^{2m}$. Hence, we see that the minimal geodesic from I to $-I$ is equivalent to a matrix in $SU(2m)$ with two eigenspaces of m-dimensions corresponding to $i\pi$ and $-i\pi$.

The second step is to show that the matrix $SU(2m)$ with two eigenspaces of m-dimensions, corresponding to $i\pi$ and $-i\pi$, is equivalent to the eigenspace corresponding to $i\pi$ as a point in $G_m(2m)$. This is an algebraic map where it suffices to show one of the directions is continuous (since the Grassmannian is compact and the space of matrices is Hausdorff). Without writing out the explicit computation, the Grassmannian can be considered by the local coordinates given by $m \times 2m$ matrices. In turn, we use the local coordinates to explicitly construct the map to a matrix in $SU(2m)$ with eigenvalues $i\pi$ and $-i\pi$. Since all these matrices are conjugate to each other, determining one of the eigenspaces give you the explicit conjugating matrix. From here, we get our desired $\pi_i(G_m(\mathbb{C}^{2m})) \cong \pi_i(\Omega^\alpha)$.

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Lemma 2.11. $\pi_i(\Omega^\alpha) \cong \pi_{i+1}(SU(2m))$, *for* $i \leq 2m$

This proof hinges on conjugate points and different parts of Morse Theory. Before getting into the proof, we'll define the following:

Definition 2.12. Along a curve γ, $\gamma(a)$ and $\gamma(b)$ are conjugate if there is a non-zero Jacobi field J that vanishes at both of them.

Recall that a Jacobi field satisfies $\dfrac{D^2J}{dt^2} + R(\dfrac{d\gamma}{dt}, J)\dfrac{d\gamma}{dt} = 0$ where $\dfrac{D}{dt}$ is the covariant derivative and R is the curvature.

The next few theorems will be used but not proven. The purpose of these theorems is to show that the index of a geodesic may be obtained by counting conjugate points along it. We may think of geodesics as "straight lines." A path $c : [a, b] \to M$ is a geodesic if the acceleration $\dfrac{D}{dt}\dfrac{dc}{dt}$ is 0.

Theorem 2.13. *Index Theorem: The (finite) index λ of a geodesic γ from $\gamma(0) = I$ to $\gamma(1) = -I$ is equal to the number of points such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ (counted with multiplicity).*

Theorem 2.14. *Consider a differentiable manifold M. For $p, q \in M$, the space of minimal geodesics from p to q is a topological manifold Ω, and if every non-minimal geodesic from p to q has index $\geq \lambda_0$ then $\pi_i(\Omega) \cong \pi_{i+1}(M)$ for $i = 0, \ldots, \lambda_0 - 2$.*

Proof and discussion of the above two theorems can be found at the end of Chapter 23 in [1].

We now compute a lower bound on the indices of *non-minimal* geodesics in order to establish the above equivalence of homotopy groups.

Theorem 2.15. *Every non-minimal geodesic γ from I to -I in $SU(2m)$ has index $\geq 2m + 2$*

Consider a non-minimal geodesic γ of $SU(2m)$ from I to $-I$, and note that it can be written in the form $\gamma(t) = \exp(tA)$ where $\gamma(0) = I$ and $\gamma(1) = -I$. Without loss of generality, A is diagonal, with entries $k_1\pi i, \ldots k_{2m}\pi i$ where k_j odd, $\sum k_j = 0$, and $k_1 \leq k_2 \leq \ldots k_{2m}$.

To find conjugate points of $\gamma(t) = \exp(tA), 0 \leq t \leq 1$, consider the following lemma:

Lemma 2.16. *Conjugate pairs occur along the above geodesic whenever $t = \frac{\pi k}{\sqrt{e_i}} \in [0, 1]$, where k is a non-zero integer and e_i is a positive eigenvalue of the map K_V: $T_I SU(2m) \to T_I SU(2m)$ defined by $K_V(W) = R(V, W)V$ where $V = \dfrac{d\gamma}{dt}(0)$ and R is the curvature tensor.*

Proof. By noting that K_V is self-adjoint (which follows from the symmetry condition), we have that

$$\langle K_V(W), W'\rangle = \langle R(V, W)V, W'\rangle = \langle R(V, W')V, W\rangle = \langle W, K_V(W')\rangle$$

and so there is an orthonormal eigenbasis $U_1, \ldots U_{2m}$ of $T_I SU(2m)$ such that $K_V(U_i) = e_iU_i$. Since K_V is self adjoint, we can extend $V, U_1, \ldots, U_{2m}$ to a parallel vector field $\tilde{V}, \tilde{U_1}, \ldots \tilde{U}_{2m}$ along γ such that $\tilde{V}(0) = V, \tilde{U}_i(0) = U_i$ for $i = 1, \ldots, 2m$. Since $SU(2m)$ is a locally symmetric, compact, connected Lie group, we have that it is globally symmetric. Hence, we have that $K_{\tilde{V}}(\tilde{U_1}) = R(\tilde{V}, \tilde{U_1})\tilde{V}$ is a parallel vector field along γ that extends along $K_{\tilde{V}}(\tilde{U_1}) = e_i\tilde{U_1}$. For the last step, take $W_i = \sin(\sqrt{e_i})U_i$ for $e_i > 0$, and note

$$\frac{D^2W_i}{dt^2} + K_V(W_i) = \frac{d^2}{dt^2}(\sin(\sqrt{e_i}))U_i + \sin(\sqrt{e_i})e_iU_i = 0$$

Hence W_i is a Jacobi field that vanishes at multiples of $\pi/\sqrt{e_i}$, giving us all the conjugate points along the geodesic for $t \in [0,1]$ with needed multiplicities.

The positive eigenvalues of $K_V : T_ISU(2m) \to T_ISU(2m)$ are $\frac{\pi^2}{4}(k_j - k_l)^2, k_j \neq k_l$. The calculation for this relies on the fact that $R(A,W)A = \frac{1}{4}[[A,W],A]$, where $[X,Y] = XY - YX$, and so

$$K_A(W) = \frac{1}{4}[[A,W],A]$$

Direct computation shows $K_A(W) = \frac{\pi^2}{4}((k_j - k_l)^2 w_{j,l})$. Therefore $K_A(W)$ has corresponding conjugate points $t = \frac{2}{k_j - k_l}, \frac{4}{k_j - k_l}, \frac{6}{k_j - k_l}, \frac{8}{k_j - k_l}, \cdots$. The Index theorem tells us that the index is equal to:

$$\sum_{k_j > k_l} 2(\frac{k_j - k_l}{2} - 1) = \sum_{k_j > k_l} (k_j - k_l - 2)$$

Since the geodesic is non-minimal, we have that $\sum_{i=1}^{2} mk_i = 0$ where not all $k_i = \pm 1$. Hence, there are three cases: m of the k_i are positive and m are negative, at least $m+1$ of the k_i are positive, and at least $m+1$ of the k_i are negative. One-line index calculations for each case show that the index is greater than or equal to $2m+2$.

At this point, we can fully approach Bott Periodicity. Recall we have that $\pi_0(U) \cong \pi_0(U(1))$ and $\pi_1(U) \cong \pi_1(U(1))$. Note that $U(1)$ can be identified with the unit circle in $\mathbb{C}$. Thus, computing the homotopy groups, we have that:

$$\pi_n(U) \cong \begin{cases} 0 & n = 0 \mod 2 \\ \mathbb{Z} & n = 1 \mod 2 \end{cases} \tag{2.17}$$

This completes the section of the (weak form) of the Unitary Case.

2.2. **Orthogonal Case.** The *orthogonal group*, $O(n)$, is the group of real $n \times n$ matrices A such that $A^TA = 1$.

Theorem 2.18. *Bott Periodicity for Orthogonal Group: For $i \geq 0, \pi_i(O) \cong \pi_{i+8}(O)$*

For the complex case, we introduced the special unitary group and worked with the space of minimal geodesics. Here, we introduce the space of complex structures $\Omega_1(n)$ and look at the space of geodesics on it.

Definition 2.19. J is a *complex structure* on $\mathbb{R}^n$ if $J \in O(n)$ and $J^2 = -I_n$.

Define $\Omega_k(n)$ to be the space of complex structures that anti-commute with fixed $J_1, \dots J_{k-1}$. By the choice of indexing, Ω_1 is the set of all complex structures. It is natural for us to define $\Omega_0(n) = O(n)$ since $\Omega_k(n) \subset \Omega_{k-1} \subset \dots \subset \Omega_1(n) \subset O(n)$. Recall that we have a natural inclusion $O(n) \to O(n+1)$, so we may take the direct limit, denoted O. Similarly, we have inclusions $\phi : \Omega_k(n) \to \Omega_k(n+n')$ for any n' (see page 138 in Milnor's *Morse Theory*); thus, we may take a direct limit of those as well.

We now present a few lemmas about the structure of $\Omega_k(n)$.

Lemma 2.20. *Each $\Omega_k(n)$ is a smooth, totally geodesic submanifold of $O(n)$.*

Proof. We may find a neighborhood of I_n in O_n where all points can be uniquely expressed as $\exp(A)$ for some small, skew-symmetric A; similarly, for each $J \in \Omega_k(n)$, we may find some neighborhood where every point in it can be written as $J\exp(A)$ for a small, skew-symmetric matrix A. Note that $J\exp(A) \in \Omega_k(n)$ if and only if J is a complex structure that anti-commutes with $k-1$ fixed complex structures $J_1, \dots, J_{k-1}$. The first property tells us that $J^{-1}AJ + A = 0$. We see this because

$$-\exp(J^{-1}AJ)\exp(A) = J^2\mathrm{exp}(J^{-1}AJ)\mathrm{exp}(A) = (J\mathrm{exp}(A))^2 = -I$$

The second property tells us that $AJ_i = J_iA$, which we may see by a similar argument. Thus, locally $\Omega_k(n)$ takes the values $J\exp(A)$ as A varies over some linear subspace of the tangent space at the identity. Thus, $\Omega_k(n)$ is a smooth, totally geodesic submanifold of $O(n)$.

□

Lemma 2.21. *For fixed J_l, the space of minimal geodesics from J_l to $-J_l \in \Omega_l(n)$ is homeomorphic to $\Omega_{l+1}(n)$ for $0 \leq l < k$.*

Proof. The key observation is that the fixed anti-commuting complex structures $J_1, \cdots, J_{l-1}$ determine $\Omega_l(n)$; therefore, if we can fix some $J_l \in \Omega_l(n)$, then we determine $\Omega_{l+1}(n)$.

We now construct a minimal geodesic in $\Omega_l(n)$ from J_l to $-J_l$. First, choose some $J \in \Omega_{l+1}(n)$ and define $A = J_lJ$. We have that $A^2 = -JJ_lJ_lJ = -I$, which implies that A is a complex structure. For $i < l$, $AJ_i = -J_lJ_iJ = J_iA$ and $AJ_l = J = -J_lA$. Now, $\gamma(t) = J_l\exp(\pi tA)$ is the geodesic we seek.

Now, note that A is skew-symmetric (since $A^2 = -I$ and $AA^T = I$) and that $\exp(\pi tA)$ defines a geodesic in $O(n)$. Since A is skew-symmetric, there exists an element $T \in O(n)$ so that TAT^{-1} we get a matrix of the form

$$\mathrm{C} = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_k \end{pmatrix}$$

where each $A_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}$ with a_i positive. Furthermore, we get that

$$\exp(\pi tA_i) = \sum_{k\geq 0} \frac{1}{k!}A_i^k = \begin{pmatrix} \cos(\pi a_i) & \sin(\pi a_i) \\ -\sin(\pi a_i) & \cos(\pi a_i) \end{pmatrix}$$

Now, we see that $\exp(\pi C) = \exp(\pi TAT^{-1}) = T\exp(\pi A)T^{-1} = -I$ if and only if the a_i are odd integers, so the same holds true for $\exp(\pi A)$. Since this is a geodesic, we will have that the a_i are all equal to 1.

Now, given a minimal geodesic γ from J_l to $-J_l$, we wish to associate it with an element of $\Omega_{l+1}(n)$. To do so, we write $\gamma(t) = J_l \exp(\pi t A)$ for some skew-symmetric A. We claim that $J_l A \in \Omega_{l+1}(n)$. To see this, change basis in order to obtain a block-diagonal matrix B whose elements all have absolute-value 1 (by minimality). Then we have that

$$B^2 = -I \text{ and } A^2 = -I$$

and A is a complex structure. Recall that we had $AJ_l = -J_l A$ and so

$$(J_l A)^2 = -AJ_lJ_lA = -A(J_lA)J_l = -J_l(J_lA)$$

Recall that for $i < 1$ we had that $AJ_i = J_iA$ so we have that

$$(J_lA)J_i = J_lJ_iA = -J_i(J_lA)$$

Thus, $J_lA \in \Omega_{l+1}(n)$.

□

Lemma 2.22. *For each $k \geq 0$, there is a real valued function g_k such that any non-minimal geodesic from J to $-J$ in $\Omega_k(n)$ has index at least $g_k(n)$ where $g_k(n)$ tends to infinity as $n \to \infty$*

Proof. This proof will have a few cases to go through. For $k = 0$, choose a non-minimal geodesic γ and take the matrix A such that $\gamma(t) = \exp(\pi t A)$. We have that A is skew-symmetric and $\exp(\pi A) = -I$. We may choose some orthonormal basis and obtain a matrix of the form

$$\begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_k \end{pmatrix}$$

where

$$A_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}$$

for odd integers $0 < a_1 \leq a_2 \ldots \leq a_m$.

Next, we find the index of $\exp(t\pi A)$ by counting the number of conjugate points. As we have discussed earlier, the conjugate points are determined by the eigenvalues. Recall that

$$K_{\pi A}W = \frac{\pi^2}{4}[[A, W], A]$$

Then, the eigenvalues of $K_{\pi A}$ are as follows:

$$\frac{\pi^2(a_i + a_j)^2}{4}, i \neq j$$

Thus, we have conjugate points at $\frac{2}{a_i + a_j}$ and $\frac{4}{a_i + a_j}$. There are $\frac{a_i + a_j - 1}{2}$ such values between 0 and 1, and thus we have at least

$$\sum_{a_i \neq a_j} \frac{a_i + a_j - 1}{2} = \sum_{a_i < a_j} a_i + a_j - 1$$

conjugate points via the index theorem. Note that we must have $a_m \geq 3$ by non-minimality of the geodesic. Thus, the index is at least $\sum_{i=1}^{m-1}(1+3-1) = 2m-2 = n-2$ and that $g_0(n) = n-2$.

The case for $k > 0$ is similar, but more involved. A detailed discussion may be found on pages 143-148 of [1].

The combination of the three lemmas shows that $\pi_i(\Omega_k(n)) \cong \pi_{i-1}(\Omega_{k+1}(n))$ for sufficiently large n. Moreover, passing to the direct limit as $n \to \infty$ gives us that $\pi_h(\Omega_0) \cong \pi_{h-i}(\Omega_i)$ for $0 \leq i \leq h$. To show 8-periodicity, one can describe each Ω_k in terms of Ω_{k-1} for $l = 0, 1, 2, \ldots, 8$. Pages 138-141 in do this in detail. Furthermore, we have the space $\Omega_8(16n)$ is diffeomorphic to orthogonal group $O(n)$. Passing to the limit, we see that $\Omega_8 \cong O$. Be a previous theorem, we have that $\pi_h(\Omega_8) \cong \pi_{h+1}(\Omega_7) \cong \ldots \cong \pi_{h+8}(\Omega_0)$.

Thus, $\pi_h(O) \cong \pi_{h+8}(O)$. $\square$

Because of this, it suffices to calculate $\pi_i(O)$ just for $0 \leq i \leq 7$. We outline how to calculate π_0 and π_1. Before proceeding, it is worth noting that $\pi_k(O(n))$ stabilizes to $\pi_k(O)$, so only the calculations of $\pi_k(O(n))$ for large n are needed.

- $\pi_0(O)$: The key fact needed here is that $O(n)$ has two path connected components. We see $\pi_0(O(n))$ stabilizes to $\pi_0 \cong \mathbb{Z}/2\mathbb{Z}$.
- $\pi_1(O)$: Note that for large n, $\pi_1(O(n)) \cong \pi_0(\Omega_1(n))$. We now count the path connected components of $\Omega_1(n)$. First consider the case where $n = 2$. Note that for any $A \in \Omega_1(2)$, there exists some $T \in O(2)$ such that

$$TAT^{-1} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

  Solving for θ as $\dfrac{\pi}{2} + n\pi, n \in \mathbb{Z}$ (since $A^2 = -I$), we get TAT^{-1} is either

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  or

$$B' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

  Hence $A = SBS^{-1}$ for some $S \in O(2)$. Now, for any even n, A is similar to a matrix of the following form:

$$\begin{pmatrix} B & 0 & \ldots & 0 & \\ 0 & B & \ldots & 0 & \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & B & \end{pmatrix}$$

  The change of basis matrix is orthogonal, and thus has determinant ± 1. Thus, there are exactly two path components, and we have that $\pi_1(O) \cong \pi_0(\Omega_1) \cong \mathbb{Z}/2\mathbb{Z}$.
- $\pi_2(O) \cong 0$ calculations are similar to the previous case and rely on the quaternions.

We end up with:

$$\pi_n(O) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n = 0, 1 \mod 8 \\ \mathbb{Z} & n = 3, 7 \mod 8 \\ 0 & n = 2, 4, 5, 6 \mod 8 \end{cases} \tag{2.23}$$

This completes the section on the (weak form of the) orthogonal case.

Acknowledgments. It is a pleasure to thank my mentor Professor Peter May for suggesting the subject. A very big thank you to the always helpful Dylan Wilson and my partner in math, Cameron Krulewski. Amin Idelhaj, Matt Scalamndre, Maeve Coates-Welsh, Eric Chen, Ping Chung, and many others – you are wonderful, and I am grateful to all of you.

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