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Hopf algebras, Witt vectors, and Brown-Gitler spectra

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ABSTRACT. Here we give a very simple construction of Brown-Gitler spectra, using only the existence of certain projective bicommutative Hopf algebras as proved by Schoeller and the analysis of the homology of fibration sequences of infinite loop spaces given by Moore and Smith.

This paper is an outgrowth of [5] but considerably simpler as we are working stably; indeed, Brown-Gitler spectra are a consequence of Brown representability for homology theories, and it is this we wish to explain.

Here is an outline of the method. We will work at a fixed prime p. Let $\mathcal{H}$ be the category of graded, commutative, cocommutative Hopf algebras over the field $\mathbb{F}_p$ of p elements. In $\mathcal{H}$, let $S(n)$ be the free commutative algebra on a single primitive generator of degree n. Thus

$$S(n) = \begin{cases} \mathbb{F}_p[z], & \text{if } p = 2 \text{ or } p > 2 \text{ and } n \text{ even} \\ \Lambda(z), & \text{if } p > 2 \text{ and } n \text{ odd.} \end{cases}$$

and z is of degree n. Here $\Lambda(\cdot)$ denotes the exterior algebra. Then Schoeller proves that there is a projective Hopf algebra $H(n) \in \mathcal{H}$ and a map of Hopf algebras $H(n) \to S(n)$ making $H(n)$ the projective cover of $S(n)$. In fact $H(n) = S(n)$ unless $p = 2$ and $n = 2^m k$, where k is odd and $m > 0$, or $p > 2$ and $n = 2p^m k$ where $(k, p) = 1$ and $m > 0$. In this case,

$$H(n) = \mathbb{F}_p[x_0, x_1, \dots, x_k]$$

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where deg $(x_i) = 2^i k$ or $2p^i k$ for $p = 2$ and $p > 2$ respectively, and $H(n)$ is equipped with the "Witt vector" diagonal. This, and more of Schoeller's work, is explained in Section 1.

Here "projective" means projective in the usual sense: maps out of $H(n)$ in $\mathcal{H}$ lift through surjections. Define an exact functor from $\mathcal{H}$ to abelian groups by

$$D_n K = Hom_{\mathcal{H}}\big(H(n), K\big)$$

for $K \in \mathcal{H}$. Note that if PK denotes the graded vector space of primitives in K, and if $(PK)_n$ denotes the elements of degree n in PK, then

$$(PK)_n \cong Hom_{\mathcal{H}}\big(S(n), K\big).$$

Thus we obtain a natural inclusion of $(PK)_n$ into $D_n K$.

The main result of this paper is the following. Let $H_* = H_*(\ ,\mathbb{F}_p)$.

THEOREM 2.1. *Let $n > 1$ and $n \not\equiv \pm 1 \bmod (2p)$. Then there is a homology theory $B(n)_*$ so that for any CW spectrum X there is a natural isomorphism*

$$B(n)_n X \cong D_n H_* \Omega^\infty X.$$

This is an application of Brown representability, as given in [1]. The wedge and limit axioms hold essentially because $H(n)$ is finitely generated as an algebra. Thus Theorem 2.1 would hold immediately if whenever one had a cofibration sequence

$$X \to Y \to Z$$

of spectra, then one had that

$$H_*\Omega^\infty X \to H_*\Omega^\infty Y \to H_*\Omega^\infty Z \tag{1}$$

was exact as Hopf algebras. Then, because $H(n)$ is projective

$$D_n H_*\Omega^\infty X \to D_n H_*\Omega^\infty Y \to D_n H_*\Omega^\infty Z \tag{2}$$

would be exact. However, the sequence (1) fails to be exact and one must appeal to Moore and Smith to show (2) is exact for $n \not\equiv \pm 1 \bmod (2p)$. This is the content of Section 2.

In Section 3 we identify the representing spectrum $B(n)$ of $B(n)_*$ as a Brown-Gitler spectrum. If $n = 0$, set $B(0)$ to be S^0 completed at p; if $n \equiv \pm 1 \bmod (2p)$, set $B(n) = B(n-1)$. Let A denote the mod p Steenrod algebra.

THEOREM 3.1. *For all $n \geq 0$, $B(n)$ is a p-complete spectrum so that*

(1) $H^*B(n) \cong A/A\{\chi(\beta^\varepsilon P^i) : 2pi + 2\varepsilon > n\}$, *if* $p > 2$, *and*
$H^*B(n) \cong A/A\{\chi(Sq^i) : 2i > n\}$, *if* $p = 2$.

(2) If $\iota : B(n) \to H\mathbb{Z}/(p)$ generates $H^0 B(n)$, then for all CW complexes Z, the reduction

$$\iota_* : B(n)_n Z \to H_n Z$$

is onto.

Part (1) is proved by investigating $D_n H_* K(\mathbb{Z}/(p), k)$ and then part (2) is proved by examining $D_n H_* \Omega^\infty \Sigma^\infty Z$. In (1), $A\{\cdot\}$ denotes the left ideal, χ the canonical anti-automorphism. The results of Theorem 3.1 completely characterize the spectra of Brown and Gitler. See [2] and [3].

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1. Witt vectors and Hopf algebras

Let $\mathcal{H}_0$ be the category of graded commutative, cocommutative connected Hopf algebras over the field $\mathbb{F}_p$ for some fixed prime p. Connected means that $H \in \mathcal{H}_0$ has the property that $H_0 \cong \mathbb{F}_p$. Colette Schoeller [9] notes that $\mathcal{H}_0$ is an abelian and locally noetherian category with a set of projective generators, and that $\mathcal{H}_0$ is equivalent to a category of easily described modules. This section is devoted to an outline of her work.

Let $\mathcal{B}$ be the category of graded, commutative $\mathbb{Z}$ algebras. Fix an integer $k > 0$. Define $C(k) \in \mathcal{B}$ to be the polynomial algebra

$$C(k) = \mathbb{Z}[x_0, x_1, x_2, \ldots]$$

where deg $(x_i) = 2p^i k$. Define elements $w_n \in C(k)$ by

$$w_n = \sum_{i=0}^{n} p^i x_i^{p^{n-i}} = x_0^{p^n} + p x_1^{p^{n-1}} + \cdots + p^n x_n.$$

PROPOSITION 1.1. *There exists a unique Hopf algebra structure on $C(k)$ so that w_n is primitive for $n \geq 0$. With this Hopf algebra structure, $C(k)$ is a cocommutative Hopf algebra.*

It is easy to see that such a structure exists on $\mathbb{Z}[\frac{1}{p}] \otimes C(k)$. For example,

$$\Delta x_0 = \Delta w_0 = x_0 \otimes 1 + 1 \otimes x_0$$

$$\Delta x_1 = x_1 \otimes 1 + 1 \otimes x_1 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x_0^i \otimes x_0^{p-i}.$$

The trick is to show that the diagonal is defined over the integers. For a proof one can appeal to [10, p. 41].

Remark 1.2: Since $C(k)$ is a cocommutative Hopf algebra, the set

$$Hom_{\mathcal{B}}(C(k), R)$$

has a natural structure of an abelian group for every $R \in \mathcal{B}$. If we were to neglect the grading, this group would be the Witt vectors in R.

Now define, for any integer $m \geq 0$, $C(k, m) \subseteq C(k)$ to be the sub-algebra

$$C(k, m) = \mathbb{Z}[x_0, \ldots, x_m]. \tag{1.3}$$

Since $w_n \in C(k,m)$ for $n \leq m$, $C(k,m)$ is a sub-Hopf algebra of $C(k)$.

We can use these Hopf algebras to produce Hopf algebras over $\mathbb{F}_p$. First suppose $p > 2$. Define, for each integer $n > 0$, a Hopf algebra $H(n) \in \mathcal{H}_0$ as follows. If n is odd, let

$$H(n) = \Lambda(x) \tag{1.4.1}$$

be the exterior algebra on a single primitive generator of degree n. If n is even; write $n = 2p^m k$ where $(p,k) = 1$ and set

$$H(n) = \mathbb{F}_p \otimes C(k,m) \tag{1.4.2}$$

with the induced Hopf algebra structure.

For $p = 2$, one must confront the anomaly of signs present in graded $\mathbb{Z}$ algebras, but not in $\mathbb{F}_2$ algebras. To do this, one needs the "doubling" functor Φ. If V is a graded $\mathbb{F}_2$ vector space, define ΦV by

$$(\Phi V)_n = \begin{cases} V_m & n = 2m \\ 0 & n = 2m+1. \end{cases}$$

Now, if n is any integer, write $n = 2^m k$ where $(2,k) = 1$ (i.e., k is odd) and let $H(n)$ be the unique $\mathbb{F}_2$-Hopf algebra so that

$$\Phi H(n) = \mathbb{F}_2 \otimes C(k,m). \tag{1.5}$$

Note that $\mathbb{F}_2 \otimes C(k,m) = \mathbb{F}_2[x_0, \dots, x_m]$ where deg $(x_i) = 2^{i+1}k$; thus

$$H(n) = \mathbb{F}_2[x_0, \dots, x_m]$$

where deg $(x_i) = 2^i k$. In particular, deg $(x_0) = k$ is odd. The following is a result of Schoeller's [9].

PROPOSITION 1.6. *For each n, $H(n) \in \mathcal{H}_0$ is projective; that is, given a diagram*

$$\begin{array}{ccc} & & H \\ & & \downarrow q \\ H(n) & \xrightarrow{f} & K \end{array}$$

in $\mathcal{H}_0$ with q surjective, then a lifting of f exists. The set $\{H(n)\}_{n\geq 1}$ form a set of small projective generators, and $H(n)$ is the projective cover of $S(n)$.

The Hopf algebras $S(n)$ were defined in the introduction.

This is proved by Schoeller [9]. Here "small" means the functor

$$Hom_{\mathcal{H}_0}\big(H(n), \cdot\big)$$

from $\mathcal{H}_0$ to abelian groups commutes with filtered colimits; this is a consequence of the fact that $H(n)$ is a finitely generated algebra. "Set of generators" means that given H in $\mathcal{H}_0$ the H is a quotient of some coproduct of various $H(n)$s.

The set $\{H(n)\}_{n\geq 1}$ can be used to describe $\mathcal{H}_0$ as a category of modules, after Dieudonné. For this one needs some facts about the groups $Hom_{\mathcal{H}_0}\big(H(n), H(m)\big)$.

LEMMA 1.7. *The order of the identity* $H(n) \to H(n)$ *in* $Hom_{\mathcal{H}_0}(H(n), H(n))$ *is*
1.) p *if* n *is odd;*
2.) p^{m+1} *if* $n = 2p^m k$ *with* $(k,p) = 1$ *and* $p > 2$
3.) 2^{m+1} *if* $n = 2^m k$ *with* $(k,2) = 1$ *and* $p = 2$.

This follows immediately from the next result. For $H \in \mathcal{H}_0$ let

$$\xi : H \to H$$

be the Verschiebung. Let, for $p > 2$, $n = 2p^m k$ as in 1.7, or, if $p = 2$, $n = 2^m k$. Then

$$H(n) = \mathbb{F}_p[x_0, \dots, x_m].$$

LEMMA 1.8. *In* $H(n)$, $\xi x_i = x_{i-1}$.

PROOF. Let $[p] : H(n) \to H(n)$ be p times the identity. Since $[p](x) = (\xi x)^p$ and $H(n)$ is a polynomial algebra, it suffices to show that $[p](x_i) = x_{i-1}^p$, with the convention that $x_{-1} = 0$. For that, one examines

$$[p] : C(k,m) \to C(k,m)$$

and in this case $[p](x_i)$, $i \geq 0$, are the unique polynomials so that

$$w_n\big([p](x_0), [p](x_1), \dots\big) = [p]w_n(x_0, x_1, \dots) = pw_n(x_0, x_1, \dots).$$

The second equality uses that w_n is primitive. Induction on i shows $[p](x_i) \equiv x_{i-1}^p \bmod p$.

Now choose an integer n, and assume n is even if $p > 2$. Define maps

$$\varphi : H(n) \longrightarrow H(pn) \tag{1.9.1}$$

$$\psi : H(pn) \longrightarrow H(n) \tag{1.9.2}$$

in $\mathcal{H}_0$ as follows. The map φ is the inclusion of Hopf algebras

$$H(n) = \mathbb{F}_p[x_0, \dots, x_m] \subseteq \mathbb{F}_p[x_0, \dots, x_{m+1}] = H(pn)$$

and ψ is obtained by the composition

$$H(pn) = \mathbb{F}_p[x_0, \dots, x_{m+1}] \to \mathbb{F}_p[x_0, \dots, x_{m+1}]/(x_0) \xrightarrow{f} \mathbb{F}_p[x_0, \dots, x_m]$$

where $f(x_i) = x_{i-1}^p$; that is, ψ is the unique map so that the following diagram commutes

$$\begin{array}{ccc} H(pn) & \xrightarrow{\psi} & H(n) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \varphi} \\ H(pn) & \xrightarrow{[p]} & H(pn) \end{array}$$

where $[p]$ is p times the identity. The following is obvious.

LEMMA 1.10. *The composites* $\varphi\psi : H(pn) \to H(pn)$ *and* $\psi\varphi : H(n) \to H(n)$ *are both p-times the identity.*

Now define a functor $D_*(\cdot)$ on $\mathcal{H}_0$ as follows: D_*H is a graded abelian group with

$$D_nH = Hom_{\mathcal{H}_0}\big(H(n), H\big).$$

There are maps of groups

$$V : D_{pn}H \to D_nH$$
$$F : D_nH \to D_{pn}H$$

(where n is even if $p > 2$) given by $F = Hom_{\mathcal{H}_0}(\psi, 1_H)$ and $V = Hom_{\mathcal{H}_0}(\varphi, 1_H)$. Thus D_* defines a functor (here we apply 1.7 and 1.10)

$$D_* : \mathcal{H}_0 \to \mathcal{D}$$

where $\mathcal{D}$ is the category of graded abelian groups M so that

1.) $M_n = 0$ if $n < 1$, $pM_n = 0$ if n is odd, $p^{m+1}M_n = 0$ if $n = 2p^mk$ with $(p,k) = 1$ and $p > 2$, or $2^{m+1}M_n = 0$ if $n = 2^mk$ with $(2,k) = 1$ and $p = 2$;
2.) there are homomorphisms $V : M_{pn} \to M_n$ and $F : M_n \to M_{pn}$, where n is even if $p > 2$, so that $FV = VF = p$, where "p" is multiplication by p.

Note that $D_*(\cdot)$ is an exact functor and commutes with colimits, by Proposition 1.6.

THEOREM 1.11. *The functor* $D_* : \mathcal{H}_0 \to \mathcal{D}$ *is an equivalence of categories.*

This is one of Schoeller's main theorems: see Section 5 of [9]. D_*H may be called the Dieudonné module of H; this result says that the Dieudonné module determines H.

The functor $D_*(\cdot)$ is computable. In fact, for the proof of Theorem 1.11, Schoeller makes the following calculation. The module $D_*H(n)$ is the "free projective" object in $\mathcal{D}$ characterized by the natural isomorphism

$$Hom_{\mathcal{D}}\left(D_*H(n), M\right) \cong M_n. \tag{1.12}$$

Thus, for example, if $p > 2$ and $n = 2p^mk$, as above, then $D_sH(n) = 0$ unless $s = 2p^tk$ and

$$D_{2p^tk}H(n) \cong \mathbb{Z}/(p^i)$$

where $i = t+1$ if $t \leq m$ and $i = m+1$ if $t \geq m$. If $y_n \in D_nH(n)$ is the generator, then F^jy_n generates $D_{2p^{m+j}k}H(n)$ and V^jy_n generates $D_{2p^{m-j}k}H(n)$. And, obviously,

$$y_n \in D_nH(n) = Hom_{\mathcal{H}_0}\left(H(n), H(n)\right)$$

is the identity $H(n) \to H(n)$.

2. The Dieudonné module yields homology theories

Let $\mathcal{H}$ be the category of graded, commutative, cocommutative Hopf algebras over $\mathbb{F}_p$. Note that the elements of degree 0 form a sub-Hopf algebra $H_0 \subseteq H$ and if $H_+ = \mathbb{F}_p \otimes_{H_0} H$, then there is a natural splitting $H \cong H_0 \otimes H_+$. Note also that $\mathcal{H}_0 \subseteq \mathcal{H}$ is a full sub-category. The functor $D_n(\cdot)$ defined on $\mathcal{H}_0$ in the previous section can be extended to $\mathcal{H}$ by setting, for $H \in \mathcal{H}$,

$$D_n H = Hom_{\mathcal{H}}(H(n), H).$$

Note that since $H(n) \in \mathcal{H}_0$, $D_n H \cong D_n H_+$ and the generalization is illusory.

The main theorem of this section is the following.

THEOREM 2.1. *If $n > 1$ is an integer and $n \not\equiv \pm 1 \bmod (2p)$, then there is a homology theory $B(n)_*$ so that for all spectra X there is a natural isomorphism*

$$B(n)_n X \cong D_n H_* \Omega^\infty X.$$

We will make a remark on $n = 0$ at the end of the section.

For the proof we apply Brown representability [1] to the functor

$$X \to D_n H_* \Omega^\infty X.$$

Since homotopic maps, $f, g : X \to Y$ induce the same map from $D_n H_* \Omega^\infty X$ to $D_n H_* \Omega^\infty Y$, we need to check the following axioms. See Lemmas 2.2 and 2.8.

(B.1) For any wedge of spectra $\vee_\alpha X_\alpha$, the natural map

$$\bigoplus_\alpha D_n H_* \Omega^\infty X_\alpha \to D_n H_* \Omega^\infty (\vee_\alpha X_\alpha)$$

is an isomorphism;

(B.2) If X is a CW spectrum and $\{X_\alpha\}$ is the filtered system of finite CW subspectra, then the natural map

$$\operatorname{colim} D_n H_* \Omega^\infty X_\alpha \to D_n H_* \Omega^\infty X$$

is an isomorphism; and

(B.3) if $X \to Y \to Z$ is a cofibration sequence, then

$$D_n H_* \Omega^\infty X \to D_n H_* \Omega^\infty Y \to D_n H_* \Omega^\infty Z$$

is exact.

LEMMA 2.2. *(B.1) and (B.2) hold for all integers $n \geq 1$.*

PROOF. This is a consequence of the fact that

$$D_n(\cdot) = Hom_{\mathcal{H}}(H(n), \cdot)$$

commutes with all colimits. To see this, note that since $H(n)$ is a finitely generated algebra, $D_n(\cdot)$ commuted with filtered colimits. Then, since $H(n)$ is projective, $D_n(\cdot)$ must commute with all colimits.

Now to be specific,

$$\Omega^\infty(\vee_\alpha X_\alpha) \cong \Pi^\omega_\alpha \Omega^\infty X_\alpha$$

where Π^ω denotes the weak product; only finitely many coordinates are not the base point. Thus

$$H_*\Omega^\infty(\vee_\alpha X_\alpha) = \coprod_\alpha H_*\Omega^\infty X_\alpha$$

where the coproduct on the right is in $\mathcal{H}$. Hence (B.1) holds. For (B.2)

$$\operatorname{colim} \Omega^\infty(X_\alpha) \cong \Omega^\infty \operatorname{colim} X_\alpha \cong \Omega^\infty X$$

and

$$\operatorname{colim} H_*\Omega^\infty X_\alpha = H_* \operatorname{colim} \Omega^\infty X_\alpha$$

since homology commutes with filtered colimits.

Axiom (B.3) requires only a little more work, using the exactness of $D_n(\cdot)$. To begin we need a preliminary result. Let $\mathcal{V}$ be the category of graded $\mathbb{F}_p$ vector spaces W equipped with Verschiebung $\xi : W_{pn} \to W_n$, where n is even if $p > 2$. The indecomposables functor Q on $\mathcal{H}_0$ defines a functor

$$Q : \mathcal{H}_0 \to \mathcal{V}$$

where QH inherits a Verschiebung from H.

LEMMA 2.3. *$Q : \mathcal{H}_0 \to \mathcal{V}$ has a right adjoint $U : \mathcal{V} \to \mathcal{H}_0$.*

PROOF. This is well known, but an easy proof goes as follows. Define a functor $\Psi : \mathcal{D} \to \mathcal{V}$ by

$$\Psi(M) = M/FM$$

and $\xi : \Psi(M) \to \Psi(M)$ is given by $\xi(x + FM) = V(x) + FM$. Then one checks that $QH = \Psi D_*H$. Since Ψ has a right adjoint (namely the inclusion functor $\mathcal{V} \to \mathcal{D}$ that sets $F = 0$ and $V = \xi$ on $W \in \mathcal{V}$) Q must have a right adjoint.

LEMMA 2.4. *Let $M \in \mathcal{V}$, then $D_nU(M) \cong M_n$.*

PROOF. We have

$$\begin{aligned} D_nU(M) &\cong Hom_{\mathcal{H}_0}(H(n), U(M)) \\ &\cong Hom_{\mathcal{D}}(D_*H(n), M) \cong M_n \end{aligned}$$

where M is regarded as an object in $\mathcal{D}$ and we use 1.12.

Now let $K(G, m)$ denote the Eilenberg-MacLane spectrum so that $\pi_*K(G, m) \cong G$ in degree m. Call a sequence of spectra

$$K(G, m) \xrightarrow{i} X \xrightarrow{q} Y \xrightarrow{k} K(G, m+1)$$

a fibration sequence if q is a fibration with fiber $K(G, m)$ classified by k.

LEMMA 2.5. *Given a fibration sequence*

$$K(G, m) \xrightarrow{i} X \xrightarrow{q} Y \xrightarrow{k} K(G, m+1)$$

with $m \geq 1$ and $n \not\equiv \pm 1$ mod $(2p)$ then the sequence of abelian groups

$$D_nH_*\Omega^\infty K(G,m) \xrightarrow{\Omega^\infty i_*} D_nH_*\Omega^\infty X \xrightarrow{\Omega^\infty q_*} D_nH_*\Omega^\infty Y \xrightarrow{\Omega k_*} D_nH_*\Omega^\infty K(G, m+1)$$

is exact.

PROOF. Consider the (classified) principal fibration of spaces

$$\Omega^\infty K(G,m) \xrightarrow{\Omega^\infty i_*} \Omega^\infty X \xrightarrow{\Omega^\infty q_*} \Omega^\infty Y \xrightarrow{\Omega^\infty k_*} \Omega^\infty K(G,m+1).$$

According to Moore and Smith [8]

$$H_*\Omega^\infty X \xrightarrow{\Omega^\infty q_*} H_*\Omega^\infty Y \xrightarrow{\Omega^\infty k_*} H_*\Omega^\infty K(G,m+1)$$

is exact (in $\mathcal{H}$). Thus

$$D_n H_*\Omega^\infty X \to D_n H_*\Omega^\infty Y \to D_n H_*\Omega^\infty K(G,m+1)$$

is exact, since $D_n(\cdot)$ is exact.

Also (same reference) if K is the kernel in $\mathcal{H}$ of

$$H_*\Omega^\infty X \to H_*\Omega^\infty Y$$

there is a factoring

$$\begin{array}{ccc} H_*\Omega^\infty K(G,m) & \xrightarrow{\varphi} & K \\ \Big\downarrow = & & \Big\downarrow \subseteq \\ H_*\Omega^\infty K(G,m) & \xrightarrow{\Omega^\infty i_*} & H_*\Omega^\infty X \end{array}$$

and an exact sequence

(2.5.1) $$H_*\Omega^\infty K(G,m) \xrightarrow{\varphi} K \to U(M) \to \mathbb{F}_p$$

where $M \in \mathcal{V}$ is concentrated in degrees congruent to ± 1 mod $(2p)$. Thus

$$D_n H_*\Omega^\infty K(G,m) \to D_n H_*\Omega^\infty X \to D_n H_*\Omega^\infty Y$$

is exact, by Lemma 2.4.

Remark. The module M can be identified. Let $C \in \mathcal{H}$ be the cokernel in $\mathcal{H}$ of

$$\Omega^\infty k_* : H_*\Omega^\infty Y \to H_*\Omega^\infty K(G,m+1).$$

Since $\Omega^\infty K(G,m+1)$ is an Eilenberg-MacLane space

$$H_*\Omega^\infty K(G,m+1) \cong U(N_1)$$

where $N_1 = QH_*\Omega^\infty K(G,m+1)$. Then $C = U(N_2)$ where $N_1 \xrightarrow{\varphi} N_2$ is a quotient map and $H_*\Omega^\infty K(G,m+1) \to C$ is isomorphic to

$$U\varphi : U(N_1) \to U(N_2).$$

Let N_0 be the kernel of φ. Then the sequence of (2.5.1) is isomorphic to

$$U(\Omega N_1) \to U(\Omega N_2) \to U(\Omega_1 N_0) \to \mathbb{F}_p.$$

Here N_1, N_2, etc. are unstable right modules over the Steenrod algebra and $\Omega(\cdot)$ is the right adjoint of the "suspension" functor of unstable modules. The functor

$\Omega_1(\cdot)$ is the first derived functor, and is well-known to be concentrated in degrees congruent to $\pm 1 \bmod (2p)$. The unstable modules N_1, N_2 etc. become objects in $\mathcal{V}$ by setting ξ to be the top Steenrod operation.

We also need to examine the case $m = 0$.

LEMMA 2.6. *Let $K(G,0) \xrightarrow{i} X \xrightarrow{q} Y \xrightarrow{k} K(G,1)$ be a fibration sequence where $k_* : \pi_1 Y \to G$ onto and suppose $n \not\equiv \pm 1 \bmod (2p)$. Then the sequence of abelian groups*

$$0 \to D_n H_* \Omega^\infty X \xrightarrow{\Omega^\infty q_*} D_n H_* \Omega^\infty Y \xrightarrow{\Omega^\infty k_*} D_n H_* \Omega^\infty K(G,1)$$

is exact.

This follows immediately from the next result, which is a mild extension of the results of Moore and Smith.

LEMMA 2.7. *Let $X \xrightarrow{q} Y \xrightarrow{k} K(G,1)$ be a fibration sequence of spectra with $\pi_1 Y \to G$ onto. Then there is an exact sequence of Hopf algebras*

$$\mathbb{F}_p \to U(M) \to H_* \Omega^\infty X \xrightarrow{\Omega^\infty q_*} H_* \Omega^\infty Y \xrightarrow{\Omega^\infty k_*} H_* \Omega^\infty K(G,1)$$

where M is concentrated in degrees $2p^k - 1$, $k \geq 0$.

PROOF. The Eilenberg-Moore spectral sequence

$$\mathrm{Cotor}^{H_* \Omega^\infty K(G,1)}(H_* \Omega^\infty Y, \mathbb{F}_p) \Rightarrow H_* \Omega^\infty X$$

converges since $H_* \Omega^\infty K(G,1)$ acts trivially on $H_* \Omega^\infty X$. See [4]. But

$$\begin{aligned} &\mathrm{Cotor}^{H_* \Omega^\infty K(G,1)}(H_* \Omega^\infty Y, \mathbb{F}_p) \cong \\ &\qquad H^* \Omega^\infty Y \backslash\backslash \Omega^\infty k_* \otimes \mathrm{Cotor}^{H_* \Omega^\infty K(G,1)//\Omega^\infty k_*}(\mathbb{F}_p, \mathbb{F}_p). \end{aligned}$$

Since $H_* \Omega^\infty K(G,1) \cong U(M_1)$ where M_1 is concentrated in degrees 1 and $2p^k$, $k \geq 0$, and since $\Omega^\infty k_* : \pi_1 \Omega^\infty Y \to G$ is onto, $H_* \Omega^\infty K(G,1)//\Omega^\infty k_* \cong U(M_2)$ where M_2 is in degrees $2p^k$, $k \geq 0$. Finally,

$$\mathrm{Cotor}^{H_* \Omega^\infty K(G,1)//\Omega k_*}(\mathbb{F}_p, \mathbb{F}_p) \cong \Lambda(\Omega M_2)$$

where $\Omega M_2 \cong \mathrm{Cotor}_1$ and $\Lambda(\cdot)$ denotes exterior algebra. This is a spectral sequence of Hopf algebras and it must collapse.

LEMMA 2.8. *Axiom (B.3) holds for all $n \not\equiv \pm 1 \bmod (2p)$.*

PROOF. Consider a cofibration sequence $X \xrightarrow{i} Y \xrightarrow{q} Z$. Since the homotopy fiber of q is weakly equivalent to X, we may assume that this is a fibration sequence, so that

$$\Omega^\infty X \xrightarrow{\Omega^\infty i} \Omega^\infty Y \xrightarrow{\Omega^\infty q} \Omega^\infty Z$$

is a fibration sequence. Next we show that we may assume Y and Z are 0-connected. For if Y_0 and Z_0 are the 0-connected covers, we get a diagram of fiber sequences

$$\begin{array}{ccccc} F & \longrightarrow & Y_0 & \xrightarrow{q_0} & Z_0 \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

where F is the fiber of q_0. Then there is a weak equivalence of 0-connected covers $F_0 \cong X_0$ and a diagram with isomorphisms

$$\begin{array}{ccccc} D_nH_*\Omega^\infty F & \longrightarrow & D_nH_*\Omega^\infty Y_0 & \longrightarrow & D_nH_*\Omega^\infty Z_0 \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ D_nH_*\Omega^\infty X & \longrightarrow & D_nH_*\Omega^\infty Y & \longrightarrow & D_nH_*\Omega^\infty Z. \end{array}$$

This diagram shows that there is no loss of information in passing to the 0-connected covers.

Thus, given Y and Z both 0-connected, let

$$\begin{array}{ccccccccc} \cdots\longrightarrow & Y_{s+1} & \xrightarrow{q_s} & Y_s & \longrightarrow \cdots \longrightarrow & Y_1 & \xrightarrow{q_0} & Z \\ & & & \downarrow k_s & & \downarrow k_1 & & \downarrow k_0 \\ & & & K(G_s, s+1) & & K(G_1, 2) & & K(G_0, 1) \end{array}$$

be the Postnikov tower of Y over Z. If $i_s : X_s \to Y_s$ is defined by the fibration sequence

$$X_s \xrightarrow{i_s} Y_s \to Z$$

and $l_s = k_s i_s$, then there is a tower of fibrations

$$\begin{array}{cccccc} \cdots\longrightarrow & X_{s+1} & \xrightarrow{r_s} & X_s & \longrightarrow \cdots \longrightarrow & X_1 \cong K(G_0, 0) \\ & & & \downarrow l_s & & \downarrow l_1 \\ & & & K(G_s, s+1) & & K(G_1, 2) \end{array}$$

that is the Postnikov tower for X. If $g_{s+1} : K(G_s, s) \to X_{s+1}$ is the inclusion of the fiber of r_s, then $f_{s+1} = i_{s+1}g_{s+1} : K(G_s, s) \to Y_{s+1}$ is the inclusion of the fiber of q_s.

Now if $f : A \to B$ is a map of spectra write $f_\#$ for $D_n\Omega^\infty f_*$.

Suppose $y \in D_nH_*\Omega^\infty Y$ and $q_\#(y) = 0$. We must produce x so that $i_\#(x) = y$. Define $y_s \in D_nH_*\Omega^\infty Y_s$ to be the image of y under projection. Inductively, we will produce $x_s \in D_nH_*\Omega^\infty X_s$ so that $(i_s)_\#(x_s) = y_s$ and $(r_s)_\# x_{s+1} = x_s$. Then, since $D_nH_*K(G_s, s) = 0$ for large s, the result will follow from the diagram

$$\begin{array}{ccc} D_nH_*\Omega^\infty X & \xrightarrow{i_\#} & D_nH_*\Omega^\infty Y \\ \downarrow\cong & & \downarrow\cong \\ \lim D_nH_*\Omega^\infty X_s & \longrightarrow & \lim D_nH_*\Omega^\infty Y_s \end{array}$$

First note that $y_1 = 0$. This follows from 2.6 and the fact that $\pi_0 Y = 0$. Thus we set $x_1 = 0$. Now suppose x_s has been chosen. Since

$$(l_s)_\# x_s = (k_s)_\# (i_s)_\# x_s = (k_s)_\# y_s = 0$$

there exists, by 2.5, a $z \in D_n H_* \Omega^\infty X_{s+1}$ so that $(r_s)_\# z = x_s$. Since

$$(q_s)_\# (y_{s+1} - (i_{s+1})_\# z) = y_s - (i_s)_\# (r_s)_\# z = 0$$

there exists, again by 2.5, a $w \in D_n H_* \Omega^\infty K(G_s, s)$ so that

$$(f_{s+1})_\# w = y_{s+1} - (i_{s+1})_\# z.$$

Set $x_{s+1} = z + (g_{s+1})_\# w$. Then $(r_s)_\# x_{s+1} = (r_s)_\# z = x_s$ and

$$(i_{s+1})_\# x_{s+1} = (i_{s+1})_\# z + (f_{s+1})_\# w = y_{s+1}$$

and the induction step is complete.

Remark 2.9. We could refine the definition of $\mathcal{H}$ as follows. Let $H \in \mathcal{H}$ and

$$\pi_0 H = \{x \in H : \Delta x = x \otimes x\}$$

be the subset of group-like elements. The multiplication of H makes $\pi_0 H$ into an abelian group and there is a natural inclusion of Hopf algebras $\mathbb{F}_p[\pi_0 H] \subseteq H$, where $\mathbb{F}_p[\cdot]$ denotes the group ring. Let $\mathcal{H}_g$ be the full sub-category of H consisting of those $H \in \mathcal{H}$ so that $\mathbb{F}_p[\pi_0 H] \cong H_0$. Then, for example,

$$H_* \Omega^\infty X \in \mathcal{H}_g.$$

Define, for $H \in \mathcal{H}_g$,

$$D_0 H = \pi_0 H = Hom_{\mathcal{H}_g}(\mathbb{F}_p[\mathbb{Z}], H).$$

The Hopf algebra $H(0) = \mathbb{F}_p[\mathbb{Z}]$ is evidently projective in $\mathcal{H}_g$ and Theorem 2.1 holds. Set $B(0) = S^0$. Then

$$B(0)_0 X = \pi_0 X \cong \pi_0 H_* \Omega^\infty X = D_0 H_* \Omega^\infty X.$$

3. The homology theories from Dieudonné modules are represented by Brown-Gitler spectra

The section is devoted to identifying the spectra $B(n)$ of the previous section as Brown-Gitler spectra. It is convenient to define $B(n)$ for all integers n; if $n \not\equiv \pm 1 \bmod (2p)$ then $B(n)$ is defined by Theorem 2.1. If $n \equiv \pm 1 \bmod (2p)$ set $B(n) = B(n-1)$.

THEOREM 3.1. *For* $n \geq 2$ $B(n)$ *is a* p*-complete spectrum so that*

(1) $H^*B(n) \cong A/A\{\chi(\beta^\varepsilon P^i) : 2pi + 2\varepsilon > n\}$, *if* $p > 2$, *and*
$H^*B(n) \cong A/A\{\chi(Sq^i) : 2i > n\}$, *if* $p = 2$.

(2) If $B(n) \to H\mathbb{Z}/(p)$ *is a generator of* $H^*B(n)$, *then*

$$B(n)_n Z \to H_n Z$$

is onto for all CW complexes Z.

These properties uniquely characterize the spectra $B(n)$ as the Brown-Gitler spectra. See [3]. Here χ is the anti-automorphism of the Steenrod algebra.

We begin with the following result.

LEMMA 3.2. $B(n)$ *is bounded below in the sense that* $\pi_k B(n) = 0$ *for* $k < 0$.

PROOF.

$$\begin{aligned}\pi_k B(n) = [S^k, B(n)] &\cong [S^n, B(n) \wedge S^{n-k}] \\ &\cong B(n)_n S^{n-k} \\ &\cong D_n H_* \Omega^\infty S^{n-k}\end{aligned}$$

for $n \not\equiv \pm 1 \bmod (2p)$. If $k < 0$, $n - k > n$ and $D_n H_* \Omega^\infty S^{n-k} = 0$. If $n \equiv \pm 1 \bmod (2p)$, $\pi_k B(n) = \pi_k B(n-1)$.

LEMMA 3.3. *Assume* $n \not\equiv \pm 1 \bmod (2p)$. *There is a natural transformation of homology theories*

$$\iota : B(n)_*(\cdot) \to H_*(\cdot)$$

so that if Z *is a CW complex, then*

$$\iota : B(n)_n Z \to H_n Z$$

is onto.

PROOF. Let X be a spectrum. Then the counit of the adjunction

$$\sigma : \Sigma^\infty \Omega^\infty X \to X$$

defines a map of modules over the Steenrod algebra

$$QH_*\Omega^\infty X \to H_* X$$

and, hence, a map of unstable modules

$$QH_*\Omega^\infty X \to \Omega^\infty H_* X$$

where $\Omega^\infty(\cdot)$ is right adjoint to the inclusion functor of unstable modules into the category of all A-modules. Then let

$$\varphi : H_*\Omega^\infty X \to U(\Omega^\infty H_* X)$$

be the adjoint. This gives a natural map

$$\begin{aligned}D_n H_*\Omega^\infty X &\to D_n U(\Omega^\infty H_* X) \\ &\cong [\Omega^\infty H_* X]_n \subseteq H_n X.\end{aligned}$$

If $n \not\equiv \pm 1 \bmod (2p)$, we obtain the map $B(n)_*(\cdot) \to H_*(\cdot)$. Also if Z is a CW complex $\Omega^\infty H_* \Sigma^\infty Z \cong \widetilde{H}_* Z$ and

$$QH_*\Omega^\infty\Sigma^\infty Z \to \widetilde{H}_* Z$$

is onto. Hence, by exactness,

$$D_n H_* \Omega^\infty \Sigma^\infty Z \to H_n Z$$

is onto.

In order to compute $H^*B(n)$ we discuss Spanier-Whitehead duality for modules over the Steenrod algebra. Let A be the Steenrod algebra and A_* the dual. All A-modules will be right A-modules – for example, H_*X – unless specifically labeled otherwise.

Let M be a finite A-module. Then the Spanier-Whitehead dual of M is a pair (N, μ) consisting of an A-module N and a non-singular pairing

$$\mu : N \otimes M \longrightarrow \mathbb{F}_p$$

that is also a map of A-modules. One often calls N the Spanier-Whitehead dual of M, leaving the map μ implicit.

LEMMA 3.4. *Let M be a finite A-module. Then the Spanier- Whitehead dual of M exists and is unique.*

PROOF. Uniqueness follows by standard methods. To prove existence, define N by the equation

$$N_n = Hom_{\mathbb{F}_p}(M_{-n}, \mathbb{F}_p)$$

and let $\mu : N \otimes M \to \mathbb{F}_p$ be the evaluation homomorphism. Then the A-module structure on N is defined by the equation

$$\mu(x\theta \otimes y) = \mu(x \otimes y\chi(\theta)) \tag{3.4.1}$$

for $\theta \in A$.

Remark 3.5. Note that if N is the Spanier-Whitehead dual of M, then there is an isomorphism

$$\phi : M \longrightarrow N^*,$$

where N^* is the graded dual, given by

$$\phi(y) = \mu(\cdot \otimes y).$$

This reverses degrees: $\phi : M_n \to (N^*)^{-n}$. Also N^* is a left A-module (in the fashion of H^*X) and the equation (3.4.1) implies that

$$\phi(y\theta) = \chi(\theta)\phi(y)$$

for $\theta \in A$.

Example 3.6. Let $G(n) \in \mathcal{U}$ be the free unstable A-module on one generator characterized by the equation

$$Hom_{\mathcal{U}}(G(n), K) \cong K_n \tag{3.6.1}$$

for all unstable A-modules K. Then we know (from [6], for example) that regarded as a cyclic A-module on a generator $\iota_n \in G(n)_n$, there is an isomorphism

$$\begin{aligned} G(n) &\cong \Sigma^n A/\{\beta^\varepsilon P^i \ : \ 2pi + 2\varepsilon > n\}A, \qquad \text{if } p > 2; \\ &\cong \Sigma^n A/\{\mathrm{Sq}^i \ : \ 2i > n\}A, \qquad \text{if } p = 2. \end{aligned}$$

Here $\{\cdot\}A$ denotes the right ideal. It follows from Remark 3.5 that if N is the Spanier-Whitehead dual of $G(n)$, then as left A-modules

$$\begin{aligned} \Sigma^n N^* &\cong A/A\{\chi(\beta^\varepsilon P^i) : 2pi + 2\varepsilon > n\}, \qquad \text{if } p > 2; \\ &\cong A/A\{\chi(\mathrm{Sq}^i) : 2i > n\}, \qquad \text{if } p = 2. \end{aligned}$$

These are exactly the left A-modules of Theorem 3.1.

To characterize the Spanier-Whitehead dual of a module, we have the next result.

LEMMA 3.7. *Let M be a finite A-module. Then the following statements are equivalent:*

1) the A-module N is the Spanier-Whitehead dual of M;

2) there is a natural isomorphism

$$Hom_A(K_1 \otimes M, K_2) \cong Hom_A(K_1, K_2 \otimes N)$$

valid for all A-modules K_1 and K_2; and

3) there is a natural isomorphism

$$Hom_A(M, S) \cong Hom_A(\mathbb{F}_p, S \otimes N)$$

where S runs through the full sub-category of A-modules with objects consisting of A_* and its various suspensions.

PROOF. That parts 1) and 2) are equivalent is standard. Part 3) is a special case of part 2). To prove that part 3) implies part 2), filter K_1 "by skeleta" and take an injective resolution of K_2.

PROOF OF THEOREM 3.1. That $B(n)$ is p-complete follows from the fact that $B(n)$ is connected (Lemma 3.2) and the fact that $H_*\big(B(n), \mathbb{Z}\big)$ is an $\mathbb{F}_p$ vector space. This in turn follows from the case $n \not\equiv \pm 1 \bmod (2p)$, where

$$H_k\big(B(n), \mathbb{Z}\big) \cong B(n)_k H\mathbb{Z} \cong D_n H_* K(\mathbb{Z}, n-k) \cong [QH_* K(\mathbb{Z}, n-k)]_n.$$

We next compute $H^* B(n)$. It is sufficient to examine the case $n \not\equiv \pm 1 \bmod (2p)$. Let $G(n)$ be as in example 3.6. Then the equation 3.6.1 implies that for all A-modules M there is a natural isomorphism

$$Hom_A(G(n), M) \cong [\Omega^\infty M]_n. \tag{3.8.1}$$

Also, if K is a suspension of $H\mathbb{Z}/p$, then there is a natural isomorphism

$$\begin{aligned} B(n)_n K &\cong D_n H_* \Omega^\infty K \\ &\cong [\Omega^\infty H_* K]_n \end{aligned} \tag{3.8.2}$$

Thus, setting $K = \Sigma^{n-k} H\mathbb{Z}/p$, we have natural isomorphisms

$$\begin{aligned} H_k B(n) &\cong B(n)_n \Sigma^{n-k} H\mathbb{Z}/p \\ &\cong Hom_A(G(n), \Sigma^{n-k} A_*) \\ &\cong Hom_{\mathbb{F}_p}(G(n)_{n-k}, \mathbb{F}_p). \end{aligned}$$

Hence, $H_* B(n)$ is finite. Finally, we have

$$B(n)_n K \cong Hom_A(\Sigma^n \mathbb{F}_p, H_* K \otimes H_* B(n)) \tag{3.8.3}$$

because K is a suspension of $H\mathbb{Z}/p$. Combining equation 3.8.1 – 3.8.3, we have a natural isomorphism

$$Hom_A(\Sigma^n \mathbb{F}_p, S \otimes H_* B(n)) \cong Hom_A(G(n), S)$$

where S runs through the full sub-category of all A-modules with objects the suspensions of A_*. Thus, by Lemma 3.7,

$$H_* B(n) \cong \Sigma^n N$$

where N is the Spanier-Whitehead dual of $G(n)$. The calculation of $H^* B(n)$ is then completed by Example 3.6

Part 2 of Theorem 3.1 holds for $n \not\equiv \pm 1 \bmod (2p)$ by Lemma 3.3. Thus the result holds for all n if we can show

$$B(n)_n Z = B(n-1)_n Z \to H_n Z$$

is onto for $n \equiv \pm 1 \bmod (2p)$ and Z a CW complex. We isolate this as a separate lemma.

LEMMA 3.9. *Let $n \equiv \pm 1 \bmod (2p)$ and let Z be a CW complex. Then*

$$\iota : B(n-1)_n Z \to H_n Z$$

is onto.

PROOF. We may assume that Z is connected. Then note that

$$B(n-1)_n Z = B(n-1)_{n-1} \Sigma^{-1} Z$$

and we are confronted with the composition

$$D_{n-1} H_* \Omega^\infty \Sigma^{-1} Z \to [\Omega^\infty H_* \Sigma^{-1} Z]_{n-1} \subseteq H_{n-1} \Sigma^{-1} Z \cong \widetilde{H}_n Z$$

However, this map factors as

$$\begin{aligned} D_{n-1}H_*\Omega^\infty\Sigma^{-1}Z &\longrightarrow [QH_*\Omega^\infty\Sigma^{-1}Z]_{n-1} \\ &\xrightarrow{\sigma} [PH_*\Omega^\infty\Sigma^\infty Z]_n \\ &\xrightarrow{q} [QH_*\Omega^\infty\Sigma^\infty Z]_n \\ &\longrightarrow \widetilde{H}_n Z \end{aligned}$$

where σ is induced from the counit

$$\Sigma\Omega^\infty\Sigma^{-1} \longrightarrow \Omega^\infty\Sigma^\infty Z$$

and the map q is given by the composite

$$PH_*\Omega^\infty\Sigma^\infty Z \xrightarrow{\subseteq} \widetilde{H}_*\Omega^\infty\Sigma^\infty Z \longrightarrow QH_*\Omega^\infty\Sigma^\infty Z.$$

The unlabeled maps are evidentally onto, the map σ is onto because $n \not\equiv 2$ mod $(2p)$ by [8] (see the proof of Theorem 4.3 there) and the map q is onto since $n \not\equiv 0$ mod $(2p)$.

Remark 3.10: After Lemma 1.10 we defined natural transformations (at $p = 2$, say)

$$V : D_{2n}(\cdot) \longrightarrow D_n(\cdot)$$

and

$$F : D_n(\cdot) \longrightarrow D_{2n}(\cdot)$$

so that $VF = FV = 2$. These yield, via Brown representability, maps of spectra

$$f : B(2n) \longrightarrow \Sigma^n B(n)$$

and

$$g : \Sigma^n B(n) \longrightarrow B(2n)$$

so that the composites gf and fg are both twice the identity. The map f is well-known, being the map introduced by Mahowald in [7]. The map g seems less well-known, although it was certainly constructable by previously known methods. Similar remarks hold at other primes.

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