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Algebraic K-Theory

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Abstract. Algebraic K-theory has seen further progress during the last three years. One important aspect of this recent progress has been a better conceptual understanding of motivic filtrations on K-theory and the systematic use of localizing invariants and related concepts. Progress on motivic cohomology has also played an important role concerning foundations as well as applications.

Mathematics Subject Classification (2020): 14xx, 19xx.

Introduction by the Organizers

The workshop *Algebraic K-theory* was well attended by over fifty participants from various backgrounds. Most participants attended in person, but a smaller group attended online. The workshop covered a wide range of topics in algebraic K-theory and its applications. Two major lines of investigation were emphasized during the workshop. One was the definition and application of motivic filtrations to algebraic K-theory of schemes, and, more generally, to localizing invariants applied to stable ∞-categories of geometric origin. Another line of results was concerned with excision and motivic homotopy theory.

Results related to motivic filtrations

At the workshop, Antieau, Elmanto, Iwasa, Krause, Morin, and Morrow reported on results concerning the definition and application of motivic filtrations of K-theory and related theories.

Krause reported on ongoing work with Antieau and Nikolaus, which extends the definition of Bhatt–Scholze prismatic cohomology to allow for the base to be a general δ-ring, as opposed to a prism. This extended theory facilitates faithfully flat descent for prismatic cohomology, and Krause used these methods to produce an algorithm which determines the structure of the algebraic K-groups of $\mathbb{Z}/p^{n+1}$, and more generally, of $\mathcal{O}_K/\mathfrak{m}^{n+1}$ for $[K : \mathbb{Q}_p] < \infty$. These results demonstrate the spectacular progress made possible by methods based on motivic filtrations.

Morin gave a new conjectural formula for special values of zeta functions of arithmetic schemes in terms of the determinants of Weil-étale cohomology and of derived de Rham cohomology relative to the ring $\mathbb{S}$ of animated integers. The novelty of this work was Morin's realization that the latter precisely combines the derived de Rham cohomology relative to the ring of integers $\mathbb{Z}$ and an archimedean correction factor defined in earlier work of Flach–Morin. So the animated integers somehow encode archimedean information! Morin defined derived de Rham cohomology relative to $\mathbb{S}$ as the graded pieces of a refined motivic filtration of topological cyclic homology that recovers the Bhatt–Morrow–Scholze filtration of completed topological cyclic homology.

Elmanto and Morrow both reported on work that extends the motivic filtration of algebraic K-theory to schemes quasi-compact and quasi-separated (qcqs) over a field by showing that the square formed by the cyclotomic trace map and its sheafification for the cdh-topology is cartesian and that the latter map carries Voevodsky's slice filtration on the domain to the Bhatt–Morrow–Scholze filtration on the target. As an application, these results show that the negative K-groups of a (not necessarily noetherian) scheme qcqs over a field vanish below minus the valuative dimension of the scheme.

Iwasa reported on work with Annala that characterizes the algebraic K-theory of qcqs schemes as the initial sheaf of spectra for the Zariski topology which admits a Chern class of line bundles and satisfies the projective bundle formula. He also explained that, by replacing the Zariski topology by the étale topology, one obtains an analogous characterization of Clausen's Selmer K-theory.

Antieau explained forthcoming work by Hahn–Raksit–Wilson, which defines a new filtration called the "even filtration" of every commutative algebra in spectra. It turns out that, in many cases, this very general filtration recovers "motivic" filtrations, the earlier definition of which relied on geometric input. These examples include the Bhatt–Morrow–Scholze filtration of p-adically completed topological cyclic homology, and Morin's refinement thereof to uncompleted topological cyclic homology presented at the workshop. It remains an unresolved question as to whether the even filtration of algebraic K-theory of a scheme agrees with the motivic filtration thereof in the cases where the latter is defined.

Results related to excision and motivic homotopy theory

Hoyois and Tamme reported on results concerning Milnor excision, and Asok and Bachmann reported on results concerning motivic homotopy theory.

Tamme explained his work with Land on a general method for understanding the structure of the K-theory of a pushout of associative algebras in spectra. He applied these methods to determine the total fiber of the square of K-theory spectra obtained from the conductor square of the group ring $\mathbb{Z}[C_p]$, a problem which first appears in Milnor's book on algebraic K-theory. The result, which has also been obtained independently by Krause–Nikolaus, is that the total fiber in question is equivalent to the K-theory of a free associative $\mathbb{F}_p$-algebra in spectra on a generator of degree 2, the structure of which was previously determined by Bayindir–Moulinos. This is the first time that the difference between the K-theory of a $\mathbb{Z}$-order in a semisimple $\mathbb{Q}$-algebra and the K-theory of a maximal $\mathbb{Z}$-order that contains it has been determined.

Asok reported on work with Bachmann and Hopkins which shows that for a scheme affine and smooth of relative dimension $d+1$ over an algebraically closed field of characteristic 0, a rank d vector bundle splits off a trivial rank 1 summand if and only if its dth Chern class vanishes. The result employs a version of the Freudenthal suspension theorem in unstable motivic homotopy theory.

Ravi reported on work with Khan which extends the definition of motivic stable homotopy theory to (scalloped algebraic) stacks, generalizing a definition by Hoyois of equivariant stable motivic homotopy theory for group scheme actions.

Bachmann reported on his result that the ∞-category of p-local motivic spectra over $\mathbb{F}_p$ is $\mathbb{Z}$-linear. In fact, more is true in that the commutative ring in spectra given by the endomorphisms tensor unit is an animated ring. He explained that this result supports a conjecture of Hopkins–Morel that $\mathbb{F}_p$ is obtained from the motivic cobordism spectrum MGL by annihilating the ideal defined by the additive formal group law.

Hoyois reported on work with Elmanto, Iwasa, and Kelly which shows that the ∞-category of motivic spectra satisfies Milnor excision. In particular, for any scheme S and any motivic spectrum E over S, the cohomology theory $E(-)$ on S-schemes takes Milnor squares of S-schemes to cartesian squares of spectra.

Results related to motives and number theory

In addition to the work of Morin already mentioned, Binda, Jansen, and Scholbach reported on work related to motives and number theory.

Jansen reported on work with Clausen showing that the exit-path ∞-category of the reductive Borel–Serre compactification of the classifying space of a reductive group G over $\mathbb{Q}$ is equivalent to an explicit 1-category, whose objects are the parabolic subgroups of G, and whose morphisms encode parabolic induction. This leads to a definition of a new unstable algebraic K-theory associated with a finite projective module over a ring, which, as opposed to the Quillen plus-construction of the classifying space of the automorphism group of the module, is well-behaved for all fields, and, conjecturally, for all rings.

Binda reported on work with Kato and Vezzani affirming a p-adic version due to Fontaine–Jannsen of Deligne's monodromy-weight conjecture in the case of a smooth complete intersection in a projective toric variety over a complete discrete valuation field of a mixed characteristic with perfect residue field. The proof follows Scholze's proof of Deligne's ℓ-adic monodromy-weight conjecture in the analogous situation, replacing Scholze's almost purity theorem and tilting equivalence by motivic versions based on earlier work of Ayoub and Ayoub–Gallauer–Vezzani.

Scholbach explained work with Richarz, which aims at a motivic version of the Satake equivalence for split reductive groups over global fields.

Miscellaneous results

The workshop also featured a number of results outside the three groupings above, notably, the resolution of the "redshift conjecture" reported by Land.

Land explained work with Mathew, Meier, and Tamme on the interaction of algebraic K-theory and the chromatic filtration of the ∞-category of commutative algebras in spectra. Together with work of Burklund–Schlank–Yuan, this shows that the hth graded piece for the chromatic filtration of a commutative algebra in spectra is nonzero if and only if the $(h+1)$th graded piece for the chromatic filtration its algebraic K-theory is so, in agreement with the "redshift" philosophy. More informally, this shows that algebraic K-theory displays Bott periodicity at all chromatic levels.

Finally, Pirutka reported on work with Cadoret on birational geometry and Milnor K-theory; Bräunling explained work with Groechenig, which produces non-torsion classes in the higher K-groups of varieties over a field, as defined by Zakharevich; and Hornbostel reported on analogues of the Quillen formal group for Sp-oriented motivic cohomology theories.

Workshop: Algebraic K-Theory

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Abstracts

Purity in chromatic localizations of algebraic K-theory

Markus Land

(joint work with Akhil Mathew, Lennart Meier and Georg Tamme)

The purpose of this talk was to give some perspective and new results on higher chromatic analogs of the (proven) Lichtenbaum–Quillen conjecture on the algebraic K-theory of schemes. To put this into context, I recalled the following: First, one considers the following localizing invariant of stable ∞-categories

$$K^{\mathrm{Sel}} = \mathrm{TC} \times_{L_{KU}\mathrm{TC}} L_{KU}K$$

first introduced by Clausen in his work on Artin reciprocity laws. Here, L_{KU} refers to the Bousfield localization functor on spectra at the cohomology theory KU. Selmer K-theory satisfies étale descent so one obtains a canonical factorization

$$K \to K^{\mathrm{et}} \to K^{\mathrm{Sel}}$$

of the tautological map $K \to K^{\mathrm{Sel}}$ through the étale sheafification K^{et} of K-theory.

First, I recalled how Selmer K-theory of schemes can be described (sometimes) in terms of more familiar chromatic localizations. In order to do so, I recalled the telescopic spectra $T(n) = \mathbb{S}/(p, \dots, v_{n-1})[v_n^{-1}]$, and their corresponding Bousfield localizations $L_{T(n)}$, as well as the localizations $L_n^f = L_{T(0)\oplus\cdots\oplus T(n)}$ which, as I briefly touched upon, can be thought of in terms of the moduli stack of formal groups, more specifically the open substack of formal groups of height at most n (in the latter case) and its closed substack of formal groups of height exactly n (in the former case).

Example 1. *Let X be a scheme over* $\mathrm{spec}(\mathbb{Z}[\frac{1}{p}])$*. Then $K^{\mathrm{Sel}}(X)_{(p)} = L_1^f K(X)_{(p)}$, and $K^{\mathrm{Sel}}(X)^\wedge_p = L_{T(1)}K(X)$. We have that $L_{T(1)}K(X)/p = K(X)/p[v_1^{-1}]$, where we think of v_1 as a Bott element in algebraic K-theory, so that $L_{T(1)}K(X)/p$ is* Bott periodic*. In the opposite case, when X is p-complete, one has $K^{\mathrm{Sel}}(X)^\wedge_p = \mathrm{TC}(X)^\wedge_p$.*

In [3], Clausen–Mathew show, making use of deep earlier work of several authors, that the map $K^{\mathrm{et}} \to K^{\mathrm{Sel}}$ induces an isomorphism on π_* for $* \geq -1$ on qcqs spectral algebraic spaces, and that the map $K \to K^{\mathrm{Sel}}$, and hence also the map $K \to K^{\mathrm{et}}$, induces an isomorphism on p-local π_* for $*$ bigger than certain invariants of the residue fields k like the mod p (virtual) Galois cohomological dimension (if $\mathrm{char}(k) \neq p$) or the dimension of Kähler differentials of k over k^p (if $\mathrm{char}(k) = p$).

I then discussed the following table of implications which one might wish to understand for $R \in \mathrm{CAlg}(\mathrm{Sp})$.

$$R \text{ is } (T(0) = L_0^f)\text{−local} \xRightarrow{\text{LQ}} K(R)_{(p)} \text{ is asympt. } L_1^f\text{−local}$$

$$\begin{array}{ccc} R \text{ is asympt. } L_{n-1}^f\text{−local} & \xRightarrow[\text{chrom. redshift}]{\text{chrLQ}} & K(R)_{(p)} \text{ is asympt. } L_n^f\text{−local} \\ \Downarrow & & \Downarrow \\ R \text{ is } T(m) - \text{acyclic} & \xLeftrightarrow[\text{chrom. heightshift}]{} & K(R) \text{ is } T(m+1) - \text{acyclic} \end{array}$$

where $m \geq n$. The first line is, by the above explanations, the conclusion of the Lichtenbaum–Quillen conjecture. Here the word *asymptotically* comes in two flavours, the *qualitative* result that the fibre of the map $K(R)_{(p)} \to L_1^f K(R)_{(p)}$ is bounded above, and the quantitative refinement specifying a bound and perhaps also describing $L_{T(1)}K(R)$ in some way. The second line is a direct higher chromatic analog, hence called *chromatic Lichtenbaum–Quillen* and coined *chromatic redshift* by Ausoni and Rognes – with the idea that the application of K-theory shifts chromatic periodicity. In remarkable works, the (quantitative) chromatic Lichtenbaum–Quillen conjectures have been verified in [2, 1] for $BP\langle 1\rangle$ and $BP\langle 2\rangle$, and qualitatively in [5] for $BP\langle n\rangle$ for general n.

The lower vertical implications are trivial, and one may wonder instead how algebraic K-theory shifts *chromatic height*. In particular, for the higher chromatic LQ conjecture to hold, one should better be able to show that the algebraic K-theory of an L_{n-1}^f-local ring is $T(m+1)$-acyclic for all $m \geq n$. Our results (together with work of further authors as I explain below) give the stronger lower equivalence and establishes chromatic height shift in full generality.

I observed that, for an algebra in spectra R, the following two numbers deserve the name *height*:

$$\inf\{n \geq -1 \mid L_{T(n+1)}R = 0\} \quad \text{and} \quad \sup\{n \geq -1 \mid L_{T(n)}R \neq 0\}.$$

It is a theorem of Hahn that (if finite) these numbers agree for *commutative* algebras. Chromatic height shift then asks for the equality

$$\text{height}(K(R)) = \text{height}(R) + 1.$$

In the rest of the talk, I sketched how our main result, together with work of other authors, establishes this equality for commutative algebras. The first theorem I addressed is the main result of [6].

Theorem 1. *Let $A \in \mathrm{Alg}(\mathrm{Sp})$.*

(1) *For $n \geq 1$, we have $L_{T(n)}K(A) \xrightarrow{\simeq} L_{T(n)}K(L_n^f A)$.*
(2) *For $n \geq 2$, we have $L_{T(n)}K(A) \xrightarrow{\simeq} L_{T(n)}K(L_{T(1)\oplus\cdots\oplus T(n)}A)$.*

Though I have not discussed a proof in my talk, I sketch here the main steps in the proof of (1) above.

Proof.

(1) First, we show that there exists an N such that for $f\colon A \to B$ an N-connective L_n^f-equivalence of connective ring spectra, the induced map $L_{T(n)}K(A) \to L_{T(n)}K(B)$ is an equivalence. This uses the plus construction model of $\Omega_0^\infty K(A)$, and a result of Bousfield which gives an N (depending only on n) such that the suspension spectrum functor takes N-connective L_n^f-equivalences of spaces (defined via v_n-periodic homotopy groups) to L_n^f-equivalences of spectra, together with the consequence of the existence of the Bousfield–Kuhn functor that a spectrum X is, $T(n)$-locally, naturally a retract of $\Sigma^\infty\Omega_0^\infty X$. As a consequence, we obtain that for any L_n^f-acyclic ring spectrum A, the map $A \to \tau_{\leq N}A$ induces an equivalence on $L_{T(n)}K(-)$.

(2) We then use (1) and the main theorem of [7] together with an induction over the Postnikov-tower of $\tau_{\leq N}A$ and show that $L_{T(n)}K(-)$ is *truncating* in the sense of [7] on L_n^f-acyclic ring spectra (for $n \geq 1$, this implies that $L_{T(n)}K(A) = 0$ for an L_n^f-acyclic connective ring spectrum A).

(3) For a stable category $\mathcal{C}$ such that for all objects $X \in \mathcal{C}$ the endomorphism spectrum $\mathrm{End}_{\mathcal{C}}(X)$ is L_n^f-acyclic, we show that $L_{T(n)}K(\mathcal{C}) = 0$. This proceeds by writing $K(\mathcal{C})$ as the geometric realisation $|[n] \mapsto K^{\mathrm{add}}(S_n\mathcal{C})|$ and $K^{\mathrm{add}}(S_n\mathrm{C})$ as a filtered colimit of $K(A_i)$ for $A_i = \mathrm{End}_{S_n\mathcal{C}}(X_i)$. The assumptions are such that A_i is a connective L_n^f-acyclic ring spectrum, so (2) applies.

(4) Finally, we note that the fibre of the map $K(A) \to K(L_n^f A)$ is given by the K-theory of the full stable subcategory of Perf(A) consisting of the L_n^f-acyclic objects. Thus (3) applies to show the theorem.

□

Corollary 1. *Let A be a ring spectrum and $n \geq 2$. Then the maps $L_{T(n)}K(A) \leftarrow L_{T(n)}K(\tau_{\geq 0}A) \to L_{T(n)}\mathrm{TC}(\tau_{\geq 0}A)$ are equivalences.*

Using this, an extension of Hahn's result on $T(n)$-acyclicity of commutative algebras in spectra, equivariant algebraic K-theory, and a clever inductive argument, in [4] the authors show the following complementary vanishing result:

Theorem 2. *For $n \geq 2$, we find that $L_{T(n)}K(L_{n-2}^f\mathbb{S})$ vanishes.*

Together with Theorem 1 above, we obtain the following *purity theorem*:

Theorem 3. *For $n \geq 1$, we have $L_{T(n)}K(A) \xrightarrow{\simeq} L_{T(n)}K(L_{T(n)\oplus T(n-1)}A)$.*

As an immediate consequence we obtain.

Corollary 2. *Let $A \in \mathrm{CAlg}(\mathrm{Sp})$. Then* $\mathrm{height}(K(A)) \leq \mathrm{height}(A) + 1$.

The converse then follows by combining work of Yuan [8] and forthcoming work of Burklund–Schlank–Yuan. Their theorems can be phrased as follows, (1) is due to Yuan and makes use of Corollary 1 above, while (2) is due to Burklund–Schlank–Yuan and has nothing to do with K-theory.

Theorem 4. *Let $n \geq 1$.*

(1) *Let E_n be a height n Lubin–Tate theory. Then* $\mathrm{height}(K(E_n)) = n+1$.
(2) *Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ with $L_{T(n)}A \neq 0$. Then there exists a Lubin–Tate theory E_n and a map of commutative algebras $A \to E_n$.*

In particular, $\mathrm{height}(K(A)) \geq \mathrm{height}(K(E_n)) = n+1$.

Together, these results establish that for $A \in \mathrm{CAlg}(\mathrm{Sp})$ with $\mathrm{height}(A) \geq 1$, we have $\mathrm{height}(K(A)) = \mathrm{height}(A) + 1$, as promised.

References

[1] G. Angelini-Knoll, C. Ausoni, D. Culver, E. Höning, and J. Rognes, *Algebraic K-theory of elliptic cohomology* arXiv:2204.05890.
[2] C. Ausoni and J. Rognes, *Algebraic K-theory of topological K-theory* Acta Math. **188**, 1–39, 2002.
[3] D. Clausen and A. Mathew, *Hyperdescent and étale K-theory* Invent. Math. **225** (2021), no. 3, 981–1076.
[4] D. Clausen, A. Mathew, N. Naumann, and J. Noel, *Descent and vanishing in chromatic algebraic K-theory via group actions*, arXiv:2011.08233.
[5] J. Hahn and D. Wilson, *Redshift and multiplication for truncated Brown-Peterson spectra*, to appear in Ann. of Math. (2022).
[6] M. Land, A. Mathew, L. Meier, and G. Tamme, *Purity in chromatic localizations of algebraic K-theory* arXiv:2001.10425 (2020).
[7] M. Land and G. Tamme, *On the K-theory of pullbacks*, Ann. of Math. (2) **190** (2019), no. 3, 877–930.
[8] A. Yuan, *Examples of chromatic redshift in algebraic K-theory*, arXiv:2111.10837.

On the K-theory of $\mathbb{Z}/p^n$

Achim Krause
(joint work with Ben Antieau, Thomas Nikolaus)

In recent work, we develop new methods to study the K-theory of rings such as $\mathbb{Z}/p^n$. Specifically, let

- K be a p-adic number field, i.e. a finite extension of $\mathbb{Q}_p$, of ramification index e and residue field degree f,
- $\varpi \in \mathcal{O}_K$ a choice of uniformizer,
- $n \geq 1$ some number.

We generally study K-theory of rings of the form $\mathcal{O}_K/\varpi^n$. In fact, it is easy to recover the integral K-theory $K_*(R)$ from the p-completed K-theory $K_*(R;\mathbb{Z}_p)$ by Gabber rigidity and Quillen's computation of $K_*(\mathbb{F}_q)$, so we focus on p-completed K-theory. We highlight two prior important results on K-theory of rings of the form $\mathcal{O}_K/\varpi^n$:

Theorem 1 (Hesselholt-Madsen[1]). *For $R = \mathbb{F}_q[z]/z^n$, we have*

$$K_{2i-2}(R;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & \text{for } i = 1 \\ 0 & \text{for } i > 1 \end{cases}, \qquad |K_{2i-1}(R;\mathbb{Z}_p)| = q^{i\cdot(n-1)} \text{ for all } i$$

Theorem 2 (Angeltveit[2]). *For $R = \mathbb{Z}/p^n$, we have*

$$K_{2i-2}(R;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & \text{for } i = 1 \\ 0 & \text{for } 1 < i < p \\ \mathbb{Z}/p & \text{for } i = p \end{cases}, \quad |K_{2i-1}(R;\mathbb{Z}_p)| = p^{i\cdot(n-1)} \text{ for } i \leq p-1$$

In both cases, the actual group structures of the odd-degree groups are identified, we omit them for simplicity. Notably, the orders of the K-groups look similar through a range, but start to differ right at the edge of the known range for $\mathbb{Z}/p^n$. It is also notable that the even groups tend to vanish, although this phenomenon ends right at the edge of the known range for $\mathbb{Z}/p^n$.

We extend on these results by providing and implementing an algorithm to compute arbitrary K-groups of general rings of the form $\mathcal{O}_K/\varpi^n$. Specifically, we show:

Theorem 3 (AKN). *For fixed $\mathcal{O}_K$, fixed $n \geq 1$ and fixed $i \geq 1$, there is an explicit three-term cochain complex*

$$\mathbb{Z}_p^{f\cdot(ni-1)} \to \mathbb{Z}_p^{2f\cdot(ni-1)} \to \mathbb{Z}_p^{f(ni-1)},$$

with $H^0 \cong 0$, $H^1 \cong K_{2i-1}(\mathcal{O}_K/\varpi^n)$, and

$$H^2 \cong \begin{cases} K_{2i-2}(\mathcal{O}_K/\varpi^n) \text{ if } i > 1 \\ 0 \text{ if } i = 1 \end{cases}$$

The maps in the chain complex can be determined algorithmically, and computed in practice with a computer. We obtain tables such as Figure 1.

At $p = 2$, only the first two rows were previously known through Angeltveit's result. Notably, it appears that the even K-groups start to vanish again in high degrees. Based on the description from Theorem 3, we are able to prove this:

Theorem 4 (AKN, "Even vanishing theorem"). *For $R = \mathcal{O}_K/\varpi^n$ and*

$$i > \left(\frac{p}{p-1}\right)^2 \cdot (p^{\lfloor \frac{n}{e} \rfloor} - 1),$$

we have

$$K_{2i-2}(R;\mathbb{Z}_p) = 0, \qquad |K_{2i-2}(R;\mathbb{Z}_p)| = p^{fi\cdot(n-1)}.$$

Concretely, this means that the *orders* of $K_*(\mathcal{O}_K/\varpi^n)$ and $K_*(\mathbb{F}_{p^f}[z]/z^n)$ agree in large degrees. The actual group structure seems to differ in arbitrarily large degrees though.

The description in Theorem 3 is based on trace methods and prismatic cohomology. By a theorem of Dundas-Goodwillie-McCarthy, the cyclotomic trace map

$$K_*(R;\mathbb{Z}_p) \to \mathrm{TC}(R)$$

is an isomorphism in nonnegative degrees for $R = \mathcal{O}_K/\varpi^n$, since these are nilpotent extensions of finite fields. It thus suffices to study topological cyclic homology

FIGURE 1. The 2-torsion part of $K_*(R)$ for some rings of the form $R = \mathbb{Z}/2^k$. Shown are the orders of the cyclic summands (written as powers of 2).

$R:$	$\mathbb{Z}/4$	$\mathbb{Z}/8$	$\mathbb{Z}/16$	$\mathbb{Z}/32$
K_1	2^1	$2^1, 2^1$	$2^1, 2^2$	$2^1, 2^3$
K_2	2^1	2^1	2^1	2^1
K_3	2^3	$2^3, 2^2$	$2^3, 2^4$	$2^3, 2^6$
K_4	0	2^1	2^2	2^3
K_5	2^3	$2^1, 2^6$	$2^1, 2^1, 2^9$	$2^1, 2^2, 2^{12}$
K_6	0	0	2^1	2^1
K_7	$2^1, 2^3$	$2^4, 2^4$	$2^1, 2^4, 2^8$	$2^1, 2^1, 2^4, 2^{11}$
K_8	0	0	2^1	2^2
K_9	$2^1, 2^1, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^4$	$2^1, 2^1, 2^2, 2^{12}$	$2^1, 2^1, 2^1, 2^2, 2^2, 2^{17}$
K_{10}	0	0	0	2^1
K_{11}	$2^1, 2^5$	$2^1, 2^1, 2^1, 2^2, 2^2, 2^5$	$2^3, 2^3, 2^{12}$	$2^1, 2^3, 2^5, 2^{16}$
K_{12}	0	0	0	2^1
K_{13}	$2^1, 2^2, 2^4$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^1, 2^3, 2^{15}$	$2^1, 2^1, 2^1, 2^1, 2^3, 2^{22}$
K_{14}	0	0	0	2^1
K_{15}	$2^1, 2^1, 2^1, 2^5$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^2, 2^5, 2^{15}$	$2^1, 2^1, 2^2, 2^3, 2^5, 2^{21}$
K_{16}	0	0	0	2^1
K_{17}	$2^1, 2^1, 2^1, 2^3, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^3, 2^9$	$2^1, 2^2, 2^2, 2^2, 2^3, 2^{17}$	$2^1, 2^1, 2^2, 2^2, 2^2, 2^3, 2^{26}$
K_{18}	0	0	0	0
K_{19}	$2^2, 2^3, 2^5$	$2^1, 2^3, 2^4, 2^{12}$	$2^3, 2^3, 2^4, 2^{20}$	$2^3, 2^3, 2^3, 2^4, 2^{27}$
K_{20}	0	0	0	0

$\mathrm{TC}(R)$. The traditional approach to doing so goes through a sequence of spectral sequences, which leads to technical difficulties in absence of a good algebraic description of the result. Prismatic cohomology, originally discovered by Bhatt-Morrow-Scholze while studying topological periodic homology TP (and the closely related negative topological cyclic homology TC^-), provides a shortcut. They construct a filtration $\mathrm{F}^{\geq *}_{\mathrm{BMS}}$ on all terms in the defining fiber sequence

$$\mathrm{TC}(R) \to \mathrm{TC}^-(R) \to \mathrm{TP}(R),$$

whose i-th associated graded is given by a fiber sequence

$$\mathbb{Z}_p(i)(R)[2i] \to \mathrm{N}^{\geq i}\widehat{\Delta}_R\{i\}[2i] \to \widehat{\Delta}_R\{i\}[2i].$$

Here the two right-hand terms are a certain twist of absolute (Nygaard-completed) prismatic cohomology of R. The left-hand term is the i-th *syntomic complex* of R, essentially defined as the fiber of the right-hand map. Crucially, all terms are essentially algebraic. In our case $R = \mathcal{O}_K/\varpi^n$, it turns out that the two right-hand terms are both concentrated in degree $2i-1$, and so the fiber is concentrated in degrees $2i-1$ and $2i-2$. So the spectral sequence recovering $\mathrm{TC}_*(R)$ from the homotopy groups of the syntomic complexes $\mathbb{Z}_p(i)(R)[2i]$ degenerates, and it suffices to compute those.

Absolute prismatic cohomology is difficult to compute, in contrast to relative prismatic cohomology, which often admits generators-and-relations descriptions via prismatic envelopes. In what follows, we recover absolute prismatic cohomology from certain relative prismatic cohomologies by a kind of descent. This is inspired by observations in THH, specifically the following two:

(1) In [3], we observe the fiber sequence

$$\mathrm{THH}(R) \to \mathrm{THH}(R/\mathbb{S}[z]) \to \Sigma^2\,\mathrm{THH}(R/\mathbb{S}[z])$$

relating absolute THH and THH relative a given element of R.

(2) In [4], Liu and Wang construct a limit diagram

$$\mathrm{TP}(R) \longrightarrow \mathrm{TP}(R/\mathbb{S}[z]) \rightrightarrows \mathrm{TP}(R/\mathbb{S}[z_0, z_1]) \rightrightarrows\!\!\!\!\rightarrow \dots$$

together with analogues for TC^- and TC.

A priori, an analogous diagram in terms of prismatic cohomologies relative $\mathbb{Z}_p[[z_0, \dots, z_n]]$ doesn't make sense, since there is no choice of prism structure on these δ-rings preserved by all the cosimplicial maps. In this case, however, one can check by hand that the prismatic envelopes describing Δ_{R/A^n} for different choices of prism structure are all canonically isomorphic. This motivates the following surprising observation:

Theorem 5. *Prismatic cohomology $\Delta_{R/A}$ extends canonically to a functor on a category of pairs $A \to R$ where A is a δ-ring and R any A-algebra.*

If the kernel of $A \to R$ contains a distinguished nonzerodivisor, this agrees with the usual relative prismatic cohomology, but implies functoriality in arbitrary δ-ring maps. In particular, it is now possible to construct the following diagram:

Proposition 6. *There is a limit diagram*

$$\Delta_R \longrightarrow \Delta_{R/\mathbb{Z}_p[[z]]} \rightrightarrows \Delta_{R/\mathbb{Z}_p[[z_0,z_1]]} \rightrightarrows\!\!\!\!\rightarrow \dots$$

(and twisted versions) which for $R = \mathcal{O}_K/\varpi^n$, made into a $\mathbb{Z}_p[[z]]$ algebra via $z \mapsto \varpi$, degenerates to a cochain complex of cohomological dimension 1.

Based on this limit description of absolute prismatic cohomology, we obtain a description of the syntomic complex $\mathbb{Z}_p(i)(\mathcal{O}_K/\varpi^n)$ as total fiber of a square of the form

$$\begin{tikzcd} \mathrm{N}^{\geq i}\Delta^{(1)}_{(\mathcal{O}_K/\varpi^n)/\mathbb{Z}_p[[z]]}\{i\} \arrow[r] \arrow[d] & \ker \arrow[d] \\ \Delta^{(1)}_{(\mathcal{O}_K/\varpi^n)/\mathbb{Z}_p[[z]]}\{i\} \arrow[r] & \ker \end{tikzcd}$$

where the terms labeled ker arise as kernel of the second differential in the cochain complexes arising from Proposition 6. All four terms admit descriptions in terms of prismatic envelopes, which lead to a filtration induced by the z-adic filtration. This has the property that $\mathrm{F}^{\geq ni} \subseteq \mathrm{N}^{\geq i}$, from which we prove that the total fiber of the square does not change when we truncate to $\mathrm{F}^{[1,ni-1]}$. This makes all four terms free $\mathbb{Z}_p$-modules of rank $f \cdot (ni-1)$, leading to the explicit description of Theorem 3.

References

[1] Hesselholt, Lars, and Ib Madsen. "Cyclic polytopes and the K-theory of truncated polynomial algebras." Inventiones mathematicae 130.1 (1997): 73-98.
[2] Angeltveit, Vigleik. "On the algebraic K-theory of Witt vectors of finite length." arXiv preprint arXiv:1101.1866 (2011).
[3] Krause, Achim, and Thomas Nikolaus. "Bökstedt periodicity and quotients of DVRs." arXiv preprint arXiv:1907.03477 (2019).
[4] Liu, Ruochuan, and Guozhen Wang. "Topological cyclic homology of local fields." arXiv preprint arXiv:2012.15014 (2020).

The reductive Borel–Serre compactification and unstable algebraic K-theory

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(joint work with Dustin Clausen)

For a fixed ring A and finitely generated projective A-module M, there is a map $BGL(M) \to K(A)$. An *unstable algebraic K-theory* is an intermediary anima thorough which this map factorises, which is built entirely from linear algebra internal to M as is $BGL(M)$, but which is ideally closer in nature to the K-theory space $K(A)$. Classical models for unstable algebraic K-theory include Quillen's plus-construction ([7]) and Volodin K-theory ([9]).

We propose a new model for unstable algebraic K-theory ([1]), which is given as the realisation of a category $RBS(M)$, the *reductive Borel–Serre category*, defined as follows

- objects are splittable flags in M: $\mathcal{F} = (0 \subset M_1 \subset \cdots \subset M_{d-1} \subset M)$.
- morphisms $\mathcal{F} \to \mathcal{F}'$ are given by the set $\{g \in GL(M) \mid g\mathcal{F} \leq \mathcal{F}'\}/U_{\mathcal{F}}$, where the poset relation is given by refinement of flags, i.e. $\mathcal{F} \leq \mathcal{F}'$ if $\mathcal{F}$ refines $\mathcal{F}'$, and $U_{\mathcal{F}}$ is the group of elements preserving $\mathcal{F}$ and inducing the identity on the associated graded of $\mathcal{F}$.
- composition is induced by multiplication in $GL(M)$.

The empty flag $(0 \subset M)$ has automorphism group $GL(M)$, so we get a map $BGL(M) \to |RBS(M)|$. The map $|RBS(M)| \to K(A)$ is arises from a different description of $RBS(M)$ where objects are interpreted as associated gradeds of an unspecified flag. This relies on the observation that giving a flag $\mathcal{F}$ in some sense overspecifies the object, since we are only interested in what morphisms do on the associated graded. This alternative description allows us to make a more refined statement, showing that our unstable models "stabilise" to recover the actual the K-theory space:

Theorem 1. *Let $\mathcal{M}$ be a set of representatives of finitely generated projective A-modules. Then the category $\coprod_{M\in\mathcal{M}} RBS(M)$ has a natural structure of a monoidal category and the K-theory space $K(A)$ identifies with the group completion of its realisation: $K(A) \simeq \Omega B|\coprod_{M\in\mathcal{M}} RBS(M)|$*

The proof of this requires a wealth of simplicial manipulations, but ultimately relies on identifying $B|\coprod RBS(M)|$ with Quillen's Q-construction. If we replace $RBS(M)$ by $BGL(M)$, this is essentially Segal's definition of algebraic K-theory ([8]). An important technical difference, however, is that whereas $\coprod BGL(M)$ is symmetric monoidal, $\coprod RBS(M)$ is *only* monoidal.

An important part of this story is that the categories $RBS(M)$ have important geometric origins. The reductive Borel–Serre compactification is one of the classical compactifications of locally symmetric spaces. It is an important and interesting geometric object which comes equipped with a natural stratification. Its stratified homotopy type (or exit path ∞-category) is given by a 1-category which can be described purely algebraically in terms of parabolic subgroups, unipotent radicals and group actions ([3]). The categories $RBS(M)$ are in fact direct generalisations of the this stratified homotopy type, using the correspondence between flags and parabolic subgroups.

From now on, we make the technical assumption that M be split Noetherian, i.e. any descreasing chain of splittable submodules stabilises.

The following identification of what happens to the map $BGL(M) \to |RBS(M)|$ of π_1 already gives some indication of the nature of $|RBS(M)|$. Let $E(M) \subset GL(M)$ be the subgroup generated by the elements which induce the identity on the associated graded of *some* flag. This is a variant of the usual group $E_n(A)$ generated by elementary matrices, and for $n \geq 2 + sr(A)$, $E_n(A) = E(A^n)$ ([10]).

Theorem 2 ([1])**.** *The map $GL(M) = \pi_1 BGL(M) \to \pi_1|RBS(M)|$ surjective with kernel $E(M)$.*

For a large class of rings, this calculation completely captures the difference between $BGL(M)$ and $|RBS(M)|$:

Theorem 3 ([1])**.** *For a ring A with many units in the sense of* [6] *and a fintely projective module M such that any summand of M is free, then the map $BGL(M) \to |RBS(M)|$ exhibits $|RBS(M)|$ as the plus-construction of $BGL(M)$ with respect to $E(M)$.*

An example of this setting, is when A local commutative with infinite residue field, and then M can be any finitely generated projective module. The proof of this relies heavily on work of Nesterenko–Suslin on rings with many units: these are a class of rings for which the maps $P_{\mathcal{F}} \to P_{\mathcal{F}}/U_{\mathcal{F}}$ are homology-isomorphisms.

For finite fields, it is a very different story.

Theorem 4 ([1])**.** *Let k be a finite field of characteristic p and V a finitely generated k-vector space. Then*

- *the map $BGL(V) \to |RBS(V)|$ is a $\mathbb{Z}[1/p]$-homology isomorphism,*
- *the map $|RBS(V)| \to *$ is a $\mathbb{F}_p$-homology isomorphism.*

The $\mathbb{F}_p$-homology of the general linear groups over k is non-trivial, rather complicated and still largely unknown ([5, 4]). But it vanished upon stabilisation, so it does not contribute to $K(k)$ ([7]). Away from the characteristic on the other hand, the homology groups are well-understood ([7]). So these models exactly remove complicated part which anyway dies upon stabilisation. This result should properly be attributed to Grodal ([2]) but we provide a different proof inspired by the geometric origins of $RBS(M)$. To prove that the $\mathbb{F}_p$-homology vanishes, we reduce to showing that the $\mathbb{F}_p$-cohomology of the Steinberg representation of $GL_n(k)$ vanishes.

We shall finish of with some of the many questions yet to be answered about these models for unstable algebraic K-theory:

1. We have maps $BGL_n(A) \to |RBS(A^n)|$ for all n and taking colimits we obtain a stable version $BGL_\infty(A) \to |RBS(A^\infty)|$. In what generality do these maps exhibit $|RBS(A^\infty)|$ as the plus-construction of $GL_\infty(A)$? For all fields and a large class of rings with many units this is true by our calculations.
2. What kind of homological stability properties do the reductive Borel–Serre categories satisfy?
3. The realisation $|\coprod RBS(M)|$ is a priori just an $\mathbb{E}_1$-space. Does it satisfy higher homotopy commutativity?
4. A more vague question is that of exploiting the geometric origins. One could try to identify explicit constructible sheaves on the reductive Borel–Serre compactification as representations of its stratified homotopy type and investigate if these can be generalised to the algebraic situation as representations of the categories $RBS(M)$.

References

[1] D. Clausen, M. Ø Jansen, *The reductive Borel–Serre compactification as a model for unstable algebraic K-theory*, Preprint: arXiv:2108.01924 (2021).
[2] J. Grodal, *Endotrivial modules for finite groups via homotopy theory*, Preprint: arXiv:1608.00499 (2018).
[3] M. Ø Jansen, *The stratified homotopy type of the reductive Borel–Serre compactification*, Preprint:arXiv:9810017 (2020).
[4] A. Lahtinen, D. Sprehn, *Modular characteristic classes for representations over finite fields*, Advances in Mathematics **323** (2018), 1–37.
[5] R. Milgram, S. Priddy, *Invariant theory and* $H^*(\mathrm{GL}_n(\mathbf{F_p}); \mathbf{F_p})$, Journal of Pure and Applied Algebra **44** (1987), 291–302.
[6] Y. Nesterenko, A. Suslin, *Homology of the full linear group over a local ring, and Milnor's K-theory*, Mathematics of the USSR-Izvestiya **34** (1990), 121.
[7] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Annals of Mathematics (1972), 552–586.
[8] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.
[9] A. Suslin, *Stability in algebraic K-theory*, Algebraic K-Theory (1982), 304–333.
[10] L. Vaserstein, *On the stabilization of the general linear group over a ring*, Math. USSR Sbornik **8** (1969), 383-400.

On the motivic cohomology of schemes

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(joint work with Matthew Morrow)

Let k be a field and X a smooth k-scheme. The work of various authors [3, 5, 7, 10, 11, 12] constructs an Atiyah-Hirzebruch style spectral sequence (also commonly known as the *motivic spectral sequence*)

$$\mathrm{H}^{i-j}_{\mathrm{mot}}(X;\mathbb{Z}(-j)) \Rightarrow \mathrm{K}_{-i-j}(X),$$

which diffracts the algebraic K-theory of X into its motivic cohomology. We explain an extension of this spectral sequence to the case when X is a qcqs scheme over $\mathbb{F}_p$, using recent advances in the subject. Here, we discuss the story in characteristic $p > 0$; the characteristic zero story will appear in Morrow's article in this volume.

Theorem 1. *Let* $\mathrm{Sch}_{\mathbb{F}_p}$ *denote the category of qcqs* $\mathbb{F}_p$*-schemes. There exists, for each* $j \geq 0$*, presheaves*

$$\mathbb{Z}(j)^{\mathrm{mot}} : \mathrm{Sch}^{\mathrm{op}}_{\mathbb{F}_p} \to \mathcal{D}(\mathbb{Z}),$$

and a filtration on (Thomason-Trobaugh) K*-theory*

$$\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{K} \to \mathrm{K} : \mathrm{Sch}^{\mathrm{op}}_{\mathbb{F}_p} \to \mathcal{DF}(\mathbb{S})_+{}^{1},$$

which enjoy the following properties:

(1) *the filtration is multplicative and exhaustive. It is complete whenever* X *has finite valuative dimension.*
(2) *Each* $\mathrm{Fil}^{\geq j}_{\mathrm{mot}}\mathrm{K}$ *and* $\mathbb{Z}(j)^{\mathrm{mot}}$ *is a finitary, Nisnevich sheaf.*
(3) *There is a canonical identification of graded pieces*

$$\mathrm{gr}^j_{\mathrm{mot}}\mathrm{K} \simeq \mathbb{Z}(j)^{\mathrm{mot}}[2j];$$

(4) *if* ℓ *is coprime to* p *then we have a canonical equivalence:*

$$\mathbb{Z}(j)^{\mathrm{mot}}/\ell \simeq L_{\mathrm{cdh}}\tau^{\leq j}R\Gamma_{\acute{e}t}(-;\mu_\ell^{\otimes j});$$

(5) *at the prime* p*, we have a cartesian square*

$$\begin{array}{ccc} \mathbb{Z}(j)^{\mathrm{mot}}/p & \longrightarrow & \mathbb{Z}/p(j)^{\mathrm{syn}} \\ \downarrow & & \downarrow \\ R\Gamma_{\mathrm{cdh}}(-;\Omega^j_{\log})[-j] & \longrightarrow & R\Gamma_{\mathrm{eh}}(-;\Omega^j_{\log})[-j]. \end{array}$$

(6) *For any* $r \geq 1$*, there is a first chern class*

$$c_1(\mathcal{O}(1)) \in H^2(\mathbb{Z}(1)^{\mathrm{mot}}(\mathbb{P}^r_X));$$

such that the map

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \oplus \cdots \oplus \mathbb{Z}(j-r)^{\mathrm{mot}}(X)[-2r] \xrightarrow{\pi^*\oplus\cdots\oplus\pi^*(-)\cup c_1(\mathcal{O}(1))^r} \mathbb{Z}(j)^{\mathrm{mot}}(\mathbb{P}^r_X)$$

is an equivalence, i.e., it satisfies the $\mathbb{P}^r$-bundle formula.

[1] This denotes the ∞-category of filtered spectra equipped with an augmentation.

(7) *If X is an (essentially-)smooth $\mathbb{F}_p$-scheme, then we have a canonical identification:*

$$\mathbb{Z}(j)^{\mathrm{mot}}(X) \simeq z^j(X, \bullet)[-2j],$$

where $z^j(X, \bullet)$ is Bloch's cycle complex.

(8) *We have a canonical equivalence:*

$$L_{\mathrm{cdh}}\mathbb{Z}(j)^{\mathrm{mot}} \simeq L_{\mathbb{A}^1}\mathbb{Z}(j)^{\mathrm{mot}}.$$

The presheaf $\mathbb{Z}(j)^{\mathrm{mot}}$ is constructed by modifying a cdh-local version of the theory (which is related to "conventional motivic homotopy theory") with *syntomic cohomology*, built from prismatic cohomology [2]. The former is a presheaf of complexes, $\mathbb{Z}(j)^{\mathrm{cdh}}$, constructed by cdh-sheafifying the left Kan extension of Voevodsky's complexes from smooth k-schemes to all qcqs k-schemes; the details of this construction will appear in joint work with Tom Bachmann and Matthew Morrow [1]. An important property of this construction is its value after p-completion:

$$\mathbb{Z}_p(j)^{\mathrm{cdh}}(X) \simeq R\Gamma_{\mathrm{cdh}}(X; W\Omega^j_{\log})[-j],$$

a cdh-extension of the Geisser-Levine theorem [6]. On the other hand, we have the p-adic syntomic complexes $\mathbb{Z}_p(j)^{\mathrm{syn}}$ whose cdh-sheafification we can compute as:

$$L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X) \simeq R\Gamma_{\mathrm{eh}}(X; W\Omega^j_{\log})[-j];$$

whence we have a map $\mathbb{Z}_p(j)^{\mathrm{cdh}} \to L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}(X)$. The presheaf $\mathbb{Z}(j)^{\mathrm{mot}}$ is then defined as the following pullback:

$$\begin{array}{ccc} \mathbb{Z}(j)^{\mathrm{mot}} & \longrightarrow & \mathbb{Z}_p(j)^{\mathrm{syn}} \\ \downarrow & & \downarrow \\ \mathbb{Z}(j)^{\mathrm{cdh}} & \longrightarrow & L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}. \end{array}$$

Similar arguments also produce a map $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{KH} \to L_{\mathrm{cdh}}\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{TC}$, where $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{TC}$ is the motivic filtration coming from [2]. The motivic filtration on K-theory is then defined by an analogous pullback diagram. The construction of our motivic filtration is a filtered refinement of the following cartesian square:

$$\begin{array}{ccc} \mathrm{K} & \xrightarrow{\mathrm{tr}} & \mathrm{TC} \\ \downarrow & & \downarrow \\ \mathrm{KH} & \longrightarrow & L_{\mathrm{cdh}}\mathrm{TC}. \end{array} \tag{1}$$

This existence of this cartesian square follows from the fact 1) that fiber of the cyclotomic trace satisfies cdh-descent [9] and that 2) KH identifies with $L_{\mathrm{cdh}}K$ [8].

The key to comparing our construction with Voevodsky's original complexes is based on the following result, joint with Bachmann and Morrow:

Theorem 2. *Let X be a qcqs $\mathbb{F}_p$-scheme, then*

(1) $\mathbb{Z}(j)^{\mathrm{cdh}}(\mathbb{A}^1_X) \simeq \mathbb{Z}(j)^{\mathrm{cdh}}(X)$ *and,*
(2) *if X is essentially smooth over $\mathbb{F}_p$, then $\mathbb{Z}(j)^{\mathrm{cdh}}(X)$ recovers Voevodsky's motivic cohomology.*
(3) $L_{\mathrm{cdh}}\mathbb{Z}_p(j)^{\mathrm{syn}}$ *satisfies the $\mathbb{P}^1$-bundle formula.*

These are motivic refinements of facts about localizing invariants: 1) that $L_{\mathrm{cdh}}\mathrm{K}$ is $\mathbb{A}^1$-invariant, 2) that $\mathrm{K} \simeq \mathrm{KH}$ on regular noetherian schemes and 3) $L_{\mathrm{cdh}}\mathrm{TC}$ satisfies the $\mathbb{P}^1$-bundle formula (which is a consequence of the bicartesian square (1)).

Theorem 2.(1) implies Theorem 2.(2) using the formalism of Gersten resolutions [4]. Theorem 2.2 then, in turn, produces a non-obvious map from $\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{EssSm}_{\mathbb{F}_p}}$ to Voevodsky's motivic cohomology which we proved to be a retraction. It then suffices to prove that a summand of $\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{EssSm}_{\mathbb{F}_p}}$ is zero. We reduce to checking this on all characteristic $p > 0$ fields after verifying that $\mathbb{Z}(j)^{\mathrm{mot}}|_{\mathrm{EssSm}_{\mathbb{F}_p}}$ satisfies a form of Gersten injectivity:

Theorem 3. *Let A be a regular local $\mathbb{F}_p$-algebra with fraction field F, then for all $i, j \geq 0$ the map*

$$\mathrm{H}^i(\mathbb{Z}(j)^{\mathrm{mot}}(A)) \to \mathrm{H}^i(\mathbb{Z}(j)^{\mathrm{mot}}(F)),$$

is injective.

This last result follows the observation of [4] that a $\mathbb{P}^1$-bundle formula can be used in lieu of $\mathbb{A}^1$-invariance to verify Gersten-type statements.

References

[1] T. Bachmann, E. Elmanto, M. Morrow, *The cdh motivic cohomology of schemes in mixed characteristics*, Forthcoming.
[2] B. Bhatt, M. Morrow, P. Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **129** (2019), 199–310.
[3] S. Bloch, *A spectral sequence for motivic cohomology*, Preprint (1995).
[4] J.L. Colliot-Thélène, R. Hoobler, B. Kahn, *The Bloch-Ogus-Gabber theorem*, Fields Inst. Commun. **16** (1997), 31–94.
[5] E. Friedlander, A. Suslin, *The spectral sequence relating algebraic K-theory to motivic cohomology*, Ann. Sci. École Norm. Sup. **35** (2002), 773–875.
[6] T. Geisser, M. Levine, *The K-theory of fields in characteristic p*, Invent. Math. **139** (2000), 459–493.
[7] D. Grayson, *Weight filtrations via commuting automorphisms*, K-Theory **9** (1995) , 139–172.
[8] M. Kerz, F. Strunk, G. Tamme, *Algebraic K-theory and descent for blow-ups*, Invent. Math. **2** (2018), 523–577.
[9] M. Land, G. Tamme, *On the K-theory of pullbacks*, Ann. of Math. (2) **3** (2019), 877–930.
[10] M. Levine, *Techniques of localization in the theory of algebraic cycles*, J. Algebraic Geom. (2001) **10**, 299–363.
[11] M. Levine, *The homotopy coniveau tower*, J. Topol. (2008) **1**, 217–267.
[12] V. Voevodsky, *A possible new approach to the motivic spectral sequence for algebraic K-theory*, Contemp. Math. (2002) **293**, 371–379.

On the p-adic weight-monodromy conjecture for complete intersections in toric varieties

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(joint work with Hiroki Kato, Alberto Vezzani)

Let K be a non-archimedean local field of mixed characteristic $(0,p)$ with ring of integers $\mathcal{O}_K$ and residue field k, and let $K_0 = W(k)[1/p]$ be the maximal unramified sub-extension of K over $\mathbb{Q}_p$. We let φ denote the lift of the arithmetic Frobenius in $\mathrm{Gal}(K_0/\mathbb{Q}_p)$. Let Y be a smooth and proper scheme over K with a proper flat model $\mathcal{Y}$ over $\mathcal{O}_K$. We let $\mathcal{Y}_s$ denote its special fibre.

When $\mathcal{Y}$ is smooth over $\mathcal{O}_K$, the crystalline cohomology groups

$$H^i_{\mathrm{crys}}(\mathcal{Y}_s/W(k))[1/p]$$

come equipped naturally with a φ-semilinear endomorphism Φ, called the *Frobenius endormorphism*. According to the Weil conjectures for the crystalline cohomology, the action of Φ on $H^i_{\mathrm{crys}}(\mathcal{Y}_s/W(k))[1/p]$ is pure of weight i that is, the generalised eigenvalues of (the linearised version of) Φ have norm $|k|^{i/2}$ via any embedding $K \subset \mathbb{C}$.

When $\mathcal{Y}$ is *semistable* over $\mathcal{O}_K$, a replacement for crystalline cohomology is given by the Hyodo-Kato [HK94] log crystalline cohomology

$$H^i_{\mathrm{HK}}(Y) = H^i_{\mathrm{HK}}(\mathcal{Y}_s/W(k)^0)[1/p],$$

where $\mathcal{Y}_s$ here refers to the special fibre of $\mathcal{Y}$ equipped with its natural log structure, and $W(k)^0$ is the log structure on $W(k)$ associated to the morphism of monoids $\mathbb{N} \to W(k), 1 \mapsto 0$. Again, $H^i_{\mathrm{HK}}(Y)$ naturally comes with a semilinear endomorphism Φ, but this time it is also equipped with a nilpotent endomorphism N, called the *monodromy operator*, which satisfy the equality $N\Phi = p\Phi N$. Together, they give rise to two different filtrations on the groups $V = H^i_{\mathrm{HK}}(Y)$. The first one, called the *weight filtration*, is characterised by the property that the graded pieces $\mathrm{gr}^W_n V$ are Frobenius-pure of weight n. The second one, called the *monodromy filtration*, is characterised by the fact that $N^i \colon \mathrm{gr}^M_i V \xrightarrow{\simeq} \mathrm{gr}^M_{-i} V$ is an isomorphism on the graded quotients for all i.

This is similar to the ℓ-adic situation: in that case, a Frobenius and a monodromy operator arise naturally on $H^i_\ell(Y_{\overline{K}}) = H^i_{\acute{e}t}(Y_{\overline{K}}, \mathbb{Q}_\ell)$ thanks to Grothendieck's quasi-unipotency theorem. As above, they give rise to two different filtrations on the cohomology groups, and the *weight-monodromy conjecture*, due to Deligne [Del71], predicts that they agree, up to a shift, i.e. that the j-th graded quotient $\mathrm{gr}^M_j H^i_\ell(Y_{\bar{K}})$ of the monodromy filtration is Frobenius-pure of weight $i+j$.

The p-adic analogue, as formulated by Jannsen, can be stated as follows

Conjecture 1 ([Jan89, Page 347])**.** *The i-th Hyodo-Kato cohomology group $H^i_{\mathrm{HK}}(Y_{\bar{K}})$ is* quasi-*pure of weight i, i.e. the j-th graded quotient $\mathrm{gr}^M_j H^i_{\mathrm{HK}}(Y_{\bar{K}})$ of the monodromy filtration is Frobenius-pure of weight $i+j$.*

As Beilinson observed [Bei13], a Hyodo-Kato cohomology $H^i_{\mathrm{HK}}(Y_{\overline{K}})$ can be actually defined for an arbitrary proper and smooth variety Y, regardless of the singularities on the special fiber. In fact, the complex $R\Gamma_{\mathrm{HK}}(Y_{\overline{K}}) \in \mathcal{D}_{(\varphi,N)}(\mathbb{Q}_p^{\mathrm{nr}})$ of (φ, N)-modules computing the Hyodo-Kato cohomology can be seen as a *refinement* of the de Rham cohomology $R\Gamma_{\mathrm{dR}}(Y_{\overline{K}})$, equipping the latter with an extra $\mathbb{Q}_p^{\mathrm{nr}}$-structure, as well as a Frobenius and monodromy operator. In particular, the p-adic weight-monodromy conjecture 1 makes sense for any smooth and proper K-variety Y.

To our knowledge, the known cases of the p-adic weight-monodromy conjecture are essentially of the following kind: when the dimension of Y is ≤ 2 as shown by Mokrane [Mok93, Théorème 5.3, Corollaire 6.2.3] and when Y is a p-adically uniformized variety, as shown independently by de Shalit [dS05] and Ito [Ito05a]. The ℓ-adic weight-monodromy conjecture is also known in the above cases (see [RZ82, Ito05a]), and it also known in the case where Y is a smooth complete intersection in a smooth and projective toric variety. This was shown by Scholze in his seminal paper [Sch12] by using his theory of perfectoid spaces to reduce the problem to the case of an equi-characteristic local field K, which had already been proved by Deligne in [Del80].

Considering that an equi-characteristic version of the p-adic weight monodromy conjecture is also known (it has been proved by Lazda and Pal [LP16] by reducing it to some essential computations made by Crew, following Deligne), it is natural to ask if Scholze's strategy can be adapted to reduce the statement (for nice complete intersections) to the equi-characteristic case. We answer positively to this expectation, by proving the following Theorem.

Theorem 2. *Let K be a finite extension of $\mathbb{Q}_p$ and Y be a smooth scheme-theoretic complete intersection inside a projective smooth toric variety over K. Then the p-adic weight-monodromy conjecture holds for Y.*

In order to explain the outline of the proof, let us briefly recall Scholze's strategy in the special case where $Y = V(f)$ is a smooth hypersurface in a projective space $\mathbb{P}^N_K$, given by the zero locus of a homogeneous polynomial $f \in H^0(\mathbb{P}^N_K, \mathcal{O}(d))$. First, without altering the statement, we can replace K with a perfectoid field $K_\infty = \widehat{K(p^{1/p^\infty})}$. In a nutshell, Scholze's proof is based on the following ingredients:

(A) There exists, thanks to a result of Huber [Hub98a], a small (analytic) open neighbourhood $Y(\epsilon) = \{x \in (\mathbb{P}^N_K)^{\mathrm{an}} \mid |f(x)| \leq \epsilon\}$ of (the analytification of) Y inside $(\mathbb{P}^N_K)^{\mathrm{an}}$ such that $H^n_\ell(Y^{\mathrm{an}}) = H^n_\ell(Y(\epsilon))$,

(B) We may find [Sch12, Proposition 8.7] a hypersurface Z' inside $\pi^{-1}(Y(\epsilon))$ defined over some finite extension of $k(\!(T)\!)$. Here, π denotes the (continuous) map of topological spaces (or even of étale topoi)

$$|\mathbb{P}^N_{K^\flat_\infty}| \cong |(\mathbb{P}^N_{K^\flat_\infty})^{\mathrm{perf}}| \cong |(\mathbb{P}^N_{K_\infty})^{\mathrm{perf}}| = \varprojlim_{\varphi} |\mathbb{P}^N_{K_\infty}| \to |\mathbb{P}^N_{K_\infty}|$$

where the second isomorphism is given by the tilting equivalence.

(C) Using the almost purity theorem [Sch12, Theorem 7.12] and the invariance of ℓ-adic cohomology under analytification [Hub96, Theorem 3.8.1] we can define a canonical Galois-equivariant morphism

$$H^n_\ell(Y) = H^n_\ell(Y(\epsilon)) \to H^n(\pi^{-1}(Y(\epsilon))) \to H^n_\ell(Z), \tag{1}$$

where Z is a smooth alteration of Z'.

Having the map (1) at disposal, it is easy to show that the cohomology $H^n_\ell(Y)$ can be realised as a direct summand of the cohomology of Z by a Poincaré duality trick.

While ingredient (B) is purely geometric (and as such it is not specific to the ℓ-adic setting), Steps (A) and (C) use quite essentially the fact that one is working with ℓ-adic cohomology, and it is unclear a priori how to adapt them to the p-adic situation. For example, in Step (C) the morphism between the étale sites doesn't directly induce a map on the Hyodo-Kato cohomology groups. Similarly, the existence of a tubular neighbourhood that does not alter the Hyodo-Kato cohomology was not known.

Our approach is to re-interpret the whole problem using a suitable motivic framework, that allow us to overcome these difficulties.

The first key ingredient is the so-called motivic tilting equivalence $\mathbf{RigDA}(\mathbb{C}_p) \cong \mathbf{RigDA}(\mathbb{C}_p^\flat)$ established in [Vez19a]. It is stated in terms of the theory of *motives of rigid analytic varieties* $\mathbf{RigDA}(K)$ introduced by Ayoub [Ayo15]. This actually helps in our situation, since the Hyodo-Kato cohomology can be extended to rigid analytic varieties thanks to the work of Colmez-Nizioł [CN19], and it can be shown to be motivic (in the sense that satisfies rigid-étale descent and $\mathbb{B}^1$-invariance). The motivic tilting equivalence implies that any Weil cohomology theory defined for rigid analytic varieties over K can be tilted to a Weil cohomology theory defined for rigid analytic varieties over $K^\flat$ (and viceversa). In particular, it allows us to "tilt" the Hyodo-Kato cohomology $R\Gamma_{\mathrm{HK}}$ on the category of smooth rigid analytic varieties over $\mathbb{C}_p$ to obtain a cohomology theory $R\Gamma^\flat_{\mathrm{HK}}$ on the category of smooth rigid analytic varieties over $\mathbb{C}_p^\flat$, which is equipped with a functorial (φ, N)-structure. We then show that the new cohomology theory $R\Gamma^\flat_{\mathrm{HK}}$ compares to the classical Hyodo-Kato cohomology and hence, by Lazda-Pal, it satisfies the weight monodromy conjecture:

Theorem 3. *Let Z be a smooth proper variety over a finite extension of $\mathbb{F}_p(\!(p^\flat)\!)$ with residue field k, having a semistable formal model $\mathcal{Z}$ with log special fiber Z_0. Then we have a canonical quasi-isomorphism*

$$R\Gamma^\flat_{\mathrm{HK}}(Z^{\mathrm{an}}_{\mathbb{C}_p^\flat}) \cong R\Gamma_{\mathrm{HK}}(Z_0/W(k)^0) \otimes_{W(k)[1/p]} \mathbb{Q}_p$$

of complexes of (φ, N)-modules.

In order to prove this theorem we show a comparison result between *log motives* over the special fibre and *formal log motives*.

Theorem 4. *Let $\mathfrak{S}$ be a quasi-coherent integral log formal scheme of finite Krull topological dimension. The special fibre functor induces an equivalence*

$$\mathbf{logFDA}(\mathfrak{S}) \cong \mathbf{logDA}(\mathfrak{S}_s)$$

This is compatible, in a suitable sense, with the motivic tilting equivalence. Note that the non-log case has been known for a long time, where it is a consequence (or rather a special case) of the Morel-Voevodsky localisation property.

The last crucial ingredient is a motivic version of Huber's result on the existence of an analytic tubular neighbourhood that doesn't alter the ℓ-adic cohomology. Our version can be stated as follows.

Theorem 5. *Let $X \to S$ be a qcqs smooth morphism of smooth rigid analytic varieties over K. Let s be a K-rational point of S and let X_s be the fiber of X over it. Then, for any sufficiently small open neighborhood U of s isomorphic to $\mathbb{B}^N_K$, the natural morphism from the motive of X_s to the one of $X \times_S U$ in* $\mathbf{RigDA}(K)$ *is invertible.*

This result is actually a quite immediate consequence of the "spreading out" property of rigid analytic motives shown in [AGV20, Theorem 2.8.14]. Note that, in particular, we can take a tubular neighborhood that does not change the ℓ-adic cohomology independently on ℓ.

Having all these results at disposal, we can finally complete the proof following Scholze's blueprint.

References

[AGV20] Joseph Ayoub, Martin Gallauer, and Alberto Vezzani. The six-functor formalism for rigid analytic motives. Preprint arXiv:2010.15004 [math.AG], 2020.

[Ayo15] Joseph Ayoub. Motifs des variétés analytiques rigides. *Mém. Soc. Math. Fr. (N.S.)*, (140-141):vi+386, 2015.

[Bei13] Alexander Beilinson. On the crystalline period map. *Cambridge Journal of Mathematics*, 1(1):1—51, 2013.

[CN19] Pierre Colmez and Wiesława Nizioł. On p-adic comparison theorems for rigid analytic varieties, i. *arXiv preprint arXiv:1905.04721*, 2019.

[Del71] Pierre Deligne. Théorie de hodge. I. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 425–430. Gauthier-Villars, Paris, 1971.

[Del80] Pierre Deligne. La conjecture de Weil II. *Publ. Math. IHES*, 52:137–252, 1980.

[dS05] Ehud de Shalit. The p-adic monodromy-weight conjecture for p-adically uniformized varieties. *Compos. Math.*, 141(1):101–120, 2005.

[Fal02] Gerd Faltings. Almost étale extensions. *Astérisque*, 279:185–270, 2002.

[HK94] Osamu Hyodo and Kazuya Kato. Semi-stable reduction and crystalline cohomology with logarithmic poles. *Astérisque*, 223:221–268, 1994.

[Hub96] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces.* Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Hub98a] R. Huber. A finiteness result for direct image sheaves on the étale site of rigid analytic varieties. *J. Algebraic Geom.*, 7(2):359–403, 1998.

[Ito05a] Tetsushi Ito. Weight-monodromy conjecture for p-adically uniformized varieties. *Invent. Math.*, 159(3):607–656, 2005.

[Jan89] Uwe Jannsen. On the l-adic cohomology of varieties over number fields and its Galois cohomology. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 315–360. Springer, New York, 1989.

[LP16] Christopher Lazda and Ambrus Pál. *Rigid cohomology over Laurent series fields*, volume 21 of *Algebra and Applications*. Springer, [Cham], 2016.

[Mok93] A. Mokrane. La suite spectrale des poids en cohomologie de Hyodo-Kato. *Duke Math. J.*, 72(2):301–337, 1993.

[RZ82] M. Rapoport and Th. Zink. Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik. *Invent. Math.*, 68(1):21–101, 1982.

[Sch12] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.

[Vez19a] Alberto Vezzani. A motivic version of the theorem of Fontaine and Wintenberger. *Compos. Math.*, 155(1):38–88, 2019.

Topological Hochschild Homology and Zeta-values

Baptiste Morin

(joint work with Matthias Flach)

Let $\mathcal{X}/\mathbb{Z}$ be a proper regular arithmetic scheme and let $n \in \mathbb{Z}$. Under some assumptions, we have a perfect complex of abelian groups $R\Gamma_{W,c}(\mathcal{X};\mathbb{Z}(n))$. Using deep work of Antieau and Bhatt–Morrow–Scholze, we define a perfect complex $R\Gamma(\mathcal{X};L\Omega^{<n}_{\mathcal{X}/\mathbb{S}})$ which we think of as derived de Rham relatively to the sphere spectrum modulo the n-step of the Hodge filtration. When Beilinson's regulator behaves as expected, there is a canonical isomorphism

$$\lambda_{\mathcal{X};n} \colon \mathbb{R} \xrightarrow{\sim} \left(\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X};\mathbb{Z}(n)) \underset{\mathbb{Z}}{\otimes} \det_{\mathbb{Z}} R\Gamma(\mathcal{X};L\Omega^{<n}_{\mathcal{X}/\mathbb{S}}) \right) \underset{\mathbb{Z}}{\otimes} \mathbb{R}$$

and we conjecture that

$$\underline{C(\mathcal{X};n)}\colon \qquad \det(\lambda_{\mathcal{X};n})) = \pm\, \zeta^*(\mathcal{X};n)$$

where $\zeta^*(\mathcal{X};n)$ denotes the leading coefficient in the Taylor development of $\zeta(\mathcal{X};s)$ near $s=n$. We also denote by $\zeta(\mathcal{X}_\infty;s)$ the archimedean Euler factor and by $A(\mathcal{X}) \in \mathbb{Q}_{>0}$ the Bloch conductor. The conjecture $C(\mathcal{X};n)$ and $C(\mathcal{X};d-n)$ together with the expected functional equation implies that the number

$$\zeta^*(\mathcal{X}_\infty;d-n)\cdot\zeta^*(\mathcal{X}_\infty;n)^{-1}\cdot A(\mathcal{X})^{\frac{d}{2}-n}$$

has the following description. Consider the cohomology of the archimedean fiber $R\Gamma_W(\mathcal{X}_\infty;\mathbb{Z}(n))$, which is a perfect complex of abelian groups. Duality for Deligne cohomology gives a canonical isomorphism

$$\begin{aligned}\varepsilon_{\mathcal{X};n} \colon \mathbb{R} \xrightarrow{\sim} \Big(&\det{}^{-1}_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty;\mathbb{Z}(n)) \underset{\mathbb{Z}}{\otimes} \det_{\mathbb{Z}} R\Gamma(\mathcal{X};L\Omega^{<n}_{\mathcal{X}/\mathbb{S}}) \\ &\underset{\mathbb{Z}}{\otimes} \det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty;\mathbb{Z}(d-n)) \underset{\mathbb{Z}}{\otimes} \det_{\mathbb{Z}} R\Gamma(\mathcal{X};L\Omega^{<d-n}_{\mathcal{X}/\mathbb{S}}) \Big) \underset{\mathbb{Z}}{\otimes} \mathbb{R}\end{aligned}$$

Theorem. $\det(\varepsilon_{\mathcal{X};n}) = \pm\, z\eta^*(\mathcal{X}_\infty;d-n)\cdot\zeta^*(\mathcal{X}_\infty;n)^{-1}\cdot A(\mathcal{X})^{\frac{d}{2}-n}$.

Birational geometry and Milnor K-theory

Alena Pirutka
(joint work with Anna Cadoret)

The main question of this talk is to understand how much information about the field is encoded in its Milnor K-theory: *does the Milnor K-ring $K_*^M(F)$ determine the isomorphism class of the field F?*

In general the answer is negative (one could take two solvably closed subfields of $\overline{\mathbb{Q}}$), the more suitable class of fields to study is given by function fields over algebraically closed (or finite) fields. In this case, the above question could be thought of as a discrete analogue of the birational anabelian program [3, 4, 5], where one is interested in reconstructing a field k from its absolute Galois group G_k, its pro-ℓ-completion $\mathcal{G}_k$, the abelianization $\mathcal{G}_k^a = \mathcal{G}_k/[\mathcal{G}_k, \mathcal{G}_k]$, and $\mathcal{G}_k^c = \mathcal{G}_k/[\mathcal{G}_k, [\mathcal{G}_k, \mathcal{G}_k]]$ the canonical central extension. For example, $K_1^M(k) = k^*$ and $\varprojlim H^1(G_k, \mu_{\ell^n}) = \varprojlim k^*/k^{*,\ell^n} = \hat{k}^*$ by Kummer theory, so that the Milnor K-theory already sees $k^* \subset \hat{k}^*$.

The main theorem of this talk states that if k is a finite or algebraically closed field, if F_1, F_2 are finitely generated fields over k of transcendence degree ≥ 2, such that one has an isomorphism $\phi : K_*^M(F_1) \xrightarrow{\sim} K_*^M(F_2)$, then $F_1 \simeq F_2$. A more functorial version indentifies isomorphisms of such fields with isomorphisms of the Milnor K-theory, upto inversion $\phi \mapsto \phi^{-1}$. When the characteristic of k is zero, it is a theorem of Bogomolov-Tschinkel [2].

Since $K_*^M(F)$ determines the multiplicative structure (as $K_1^M(F) = F^*$), it remains to reconstruct the additive structure. This is done in three steps:

- A result from classical projective geometry says that the multiplicative structure and lines determine the field.
- Using powerful and difficult results on Milnor K-theory (Bloch-Kato conjecture, or Bass-Tate conjecture if the base field is finite) one can prove that $K_*^M(F)$ gives algebraic dependence over k.
- The technical and conceptual heart of the proof is to reconstruct lines in F^*/k^* from algebraic dependence.

In order to achieve the third step we notice that an element $x_1 - cx_2$ of a line through x_1 and x_2 belongs to the intersection $\overline{k(\frac{x_1}{x_2})}^F \cdot x_2 \cap \overline{k(\frac{y_1}{y_2})}^F \cdot y_2$, where we write $\overline{k(x)}^F$ for the algebraic closure of $k(x)^*$ in F^*, $c = \frac{c_1}{c_2}$, and $y_i = x_i - c_i$. Note that the intersection $\overline{k(\frac{x_1}{x_2})}^F \cdot x_2 \cap \overline{k(\frac{y_1}{y_2})}^F \cdot y_2$ is defined using the multiplicative structure and algebraic dependence. The main lemma gives a converse statement modulo p-powers (where p is the characteristic of k): for an appropriate choice of x_1 and x_2, the latter intersection could only contain elements of the form $x_1 - cx_2$, for $y_i = x_i - c_i$. Although the quotient F^*/F^{*p} is small, an additional analysis allows to fully reconstruct lines, as desired.

References

[1] A. Cadoret, A. Pirutka, *Reconstructing function fields from Milnor K-theory*, Algebra Number Theory **15** (2021), no. 9, 2261–2288.

[2] F. Bogomolov, Y. Tschinkel, *Milnor K_2 and field homomorphisms*, pp. 223–244 in Geometry, analysis, and algebraic geometry, edited by H.-D. Cao and S.-T. Yau, Surv. Differ. Geom. **13**, International, Somerville, MA, 2009.

[3] F. Bogomolov, Y. Tschinkel, *Introduction to birational anabelian geometry*, pp. 17–63 in Current developments in algebraic geometry, edited by L. Caporaso et al., Math. Sci. Res. Inst. Publ. **59**, Cambridge Univ. Press, 2012.

[4] F. Pop, *On the birational anabelian program initiated by Bogomolov, I*, Invent. Math. **187**:3 (2012), 511–533. MR Zbl

[5] F. Pop, *Recovering function fields from their decomposition graphs*, pp. 519–594 in Number theory, analysis and geometry, edited by D. Goldfeld et al., Springer, 2012.

A universality of K-theory: Algebraic Snaith's theorem

Ryomei Iwasa

(joint work with Toni Annala)

Our main result in [AI22b] is an algebraic analogue of Snaith's theorem in [Sna79] and a non-$\mathbb{A}^1$-localized refinement of [GS09, SØ09]. To fix the notation, let S be a qcqs scheme and let St_S denote the ∞-topos of Zariski sheaves on smooth S-schemes. We consider $B\mathbb{G}_m$ as an $\mathbb{E}_\infty$-monoid in St_S. Then its stabilization $\mathbb{S}[B\mathbb{G}_m]$ is an $\mathbb{E}_\infty$-algebra in $\mathrm{Sp}(\mathrm{St}_S)$. We say that an $\mathbb{S}[B\mathbb{G}_m]$-module E in $\mathrm{Sp}(\mathrm{St}_S)$ *satisfies projective bundle formula* if, for every $n \geq 1$ and every smooth S-scheme X, the map

$$\sum_{i=0}^{n} \beta^i \colon \bigoplus_{i=0}^{n} E(X) \to E(\mathbb{P}^n_X)$$

is an equivalence, where β is the Bott element $1 - [\mathcal{O}(-1)]$. By abstract reason, there exists a localization

$$L_{\mathrm{pbf}} \colon \mathrm{Mod}_{\mathbb{S}[B\mathbb{G}_m]}(\mathrm{Sp}(\mathrm{St}_S)) \to \mathrm{Mod}_{\mathbb{S}[B\mathbb{G}_m]}(\mathrm{Sp}(\mathrm{St}_S))$$

whose essential image is spanned by $\mathbb{S}[B\mathbb{G}_m]$-modules which satisfy projective bundle formula. Let K denote the non-connective K-theory, which we regard as a sheaf of spectra on Sm_S. Then the main theorem is stated as follows.

Theorem. *There is a canonical equivalence of sheaves of spectra on* Sm_S

$$L_{\mathrm{pbf}}\mathbb{S}[B\mathbb{G}_m] \simeq K.$$

In other words, the non-connective K-theory is a universal $\mathbb{S}[B\mathbb{G}_m]$-module which satisfies projective bundle formula (and Zariski descent). Similarly, we show that the Selmer K-theory, in the sense of [Cla17], is a universal $\mathbb{S}[B\mathbb{G}_m]$-module which satisfies projective bundle formula and étale descent.

The basic idea of the proof is to regard the projective bundle formula as $\mathbb{P}^1$-periodicity and work in a category where $\mathbb{P}^1$ is formally inverted. This leads to our formulation of (non-$\mathbb{A}^1$-local) motivic spectra.

Definition. We define the ∞-*category* $\mathrm{Sp}_{\mathbb{P}^1}(S)$ *of motivic spectra over* S to be the formal inversion of the pointed projective space $\mathbb{P}^1$ in St_{S*}.

We develop a theory of Chern classes in this generality and calculate the cohomology of $B\mathrm{GL}_n$ by adopting the argument in [AI22a]. Then, as an application, we prove the universality of the (Selmer) K-theory.

References

[AI22a] T. Annala, R. Iwasa, *Cohomology of the moduli stack of algebraic vector bundles.* 2022, arXiv:2104.09559.

[AI22b] T. Annala, R. Iwasa, *Motivic spectra and universality of K-theory.* 2022, arXiv:2204.03434.

[Cla17] D. Clausen, *A K-theoretic approach to Artin maps.* 2017, arXiv:1703.07842.

[GS09] D. Gepner, V. Snaith, *On the motivic spectra representing algebraic cobordism and algebraic K-theory.* Doc. Math. 14 (2009), 359–396.

[Sna79] V. Snaith, *Algebraic cobordism and K-theory.* Mem. Amer. Math. Soc. 21 (1979), no. 221, vii+152 pp.

[SØ09] M. Spitzweck, P. A. Østvær, *The Bott inverted infinite projective space is homotopy algebraic K-theory.* Bull. Lond. Math. Soc. 41 (2009), no. 2, 281–292.

On the Freudenthal suspension theorem in unstable motivic homotopy theory

Aravind Asok

(joint work with Tom Bachmann, Michael J. Hopkins)

Theorem 1. *Suppose* k *is an algebraically closed field having characteristic* 0 *and* $d \geq 1$ *is an integer. If* X *is a smooth affine* k*-variety of dimension* $d+1$, *and* E *is a rank* d *vector bundle on* X, *then* E *splits off a trivial rank* 1 *summand if and only if* $0 = c_d(E) \in CH^d(X)$.

Remark 2. *Tbe result above answers in the affirmative "Murthy's conjecture" in characteristic* 0. *The statement is immediate if* $d = 1$ *in which case it holds without restriction on the characteristic of* k. *The statement also holds when* $d = 2, 3$ *if* k *has characteristic not equal to* 2 *by previous work of J. Fasel and the first author* [AF14, AF15].

Suppose k is a field having characteristic 0. Write Sm_k for the category of smooth k-schemes. We write Spc_k for a suitable category of spaces (e.g., simplicial presheaves on Sm_k) and $\mathrm{H}(k)$ for the Morel–Voevodsky unstable motivic homotopy category of k [MoVo99]. Traditionally, $\mathrm{H}(k)$ is obtained by a two-step Bousfield localization of Spc_k: one first inverts Nisnevich local weak equivalences and then $\mathbb{A}^1$-weak equivalences.

Theorem 1 is established using techniques of motivic homotopy theory. Combining the $\mathbb{A}^1$-homotopy classification of vector bundles and the existence of $\mathbb{A}^1$-fiber sequences of the form

$$\mathbb{A}^n \setminus 0 \longrightarrow BGL_{n-1} \longrightarrow BGL_n,$$

techniques of obstruction theory reduce the verification of Theorem 1 to understanding the $\mathbb{A}^1$-homotopy theory of $\mathbb{A}^n \setminus 0$.

Write S^1 for the space $\mathbb{A}^1/\{0,1\}$ and $\mathbb{G}_m$ for $\mathbb{A}^1 \setminus 0$ pointed by 1. We set $S^p := (S^1)^{\wedge p}$ and $S^{p,q} = S^p \wedge \mathbb{G}_m^{\wedge q}$. For any pointed space (X,x), one defines homotopy (Nisnevich) sheaves $\pi_p^{\mathbb{A}^1}(X,x)$. These sheaves detect $\mathbb{A}^1$-weak equivalences and we may define $\mathbb{A}^1$-n-connectedness by imposing vanishing conditions on homotopy sheaves. One can define, more generally, $\pi_{p,q}^{\mathbb{A}^1}(X,x)$, and these sheaves may be identified with $\pi_p^{\mathbb{A}^1}(\Omega^q_{\mathbb{G}_m} X)$, where $\Omega^q_{\mathbb{G}_m} X$ is the q-fold $\mathbb{G}_m$-loop space of X.

One knows that $\mathbb{A}^n \setminus 0 \sim S^{n-1,n}$ and $\mathbb{P}^1 \cong S^{1,1}$. F. Morel's foundational unstable connectivity theorem asserts that $\mathbb{A}^1$-localization preserves connectivity. One deduces immediately that $\mathbb{A}^n \setminus 0$ is $\mathbb{A}^1$-$(n-2)$-connected. Morel also computed the first non-vanishing homotopy sheaf of $S^{p,q}$ in various situations. For example, if $p \geq 2$ and $q \geq 1$, then $\pi_{p,i}^{\mathbb{A}^1}(S^{p,q}) \cong \mathbf{K}^{MW}_{q-i}$. where $\mathbf{K}^{MW}_r$ is the so-called Milnor–Witt K-theory sheaf (see [Mor12] for all these results).

Granted Morel's computations, the required information to establish Theorem 1 is contained in the next non-vanishing $\mathbb{A}^1$-homotopy sheaf $\pi_n(\mathbb{A}^n \setminus 0)$. Previous work of the first author and J. Fasel (exposed at a previous Oberwolfach meeting) gave a regular form for this result, which could be deduced from a suitable version of the Freudenthal suspension theorem for $\mathbb{P}^1$-suspension.

While Morel established a Freudenthal suspension theorem for S^1-suspension that looks formally identical to the classical case, simple computations show that Freudenthal suspension for $\mathbb{P}^1$-suspension requires further hypotheses. For example, it is easy to see that $\pi_p^{\mathbb{A}^1}(S^p) \cong \mathbb{Z}$ for all $p \geq 1$, and therefore, for $p \geq 2$, the map $S^p \to \Omega_{\mathbb{P}^1} S^{p+1,1}$ is not an isomorphism on homotopy sheaves in degree p.

Intuitively speaking, S^p is not sufficiently "$\mathbb{G}_m$-connected". To make this precise, one proceeds as follows, roughly mimicking one construction of the classical Postnikov tower using Bousfield localization. Consider the left Bousfield localization of $\mathrm{H}(k)$ generated by the maps $\mathbb{G}_m^{\wedge n+1} \times X \to X, X \in \mathrm{Sm}_k$; write L_n for the resulting localization functor.

Definition 3. *We will say that a space X is $\mathbb{G}_m$-n-connected if $X \sim \mathrm{L}_n X$ and $\mathbb{G}_m$-n-truncated if $\mathrm{L}_n X \cong *$.*

By construction, any pointed space that is of the form $\mathbb{G}_m^{\wedge n+1} \wedge X$ is $\mathbb{G}_m$-n-connected. One says that a space X is (p,q)-connected if it is p-connected and $\mathbb{G}_m$-q-connected. In particular, $\mathbb{A}^n \setminus 0$ is $(n-2, n-1)$-connected. From the definitions, it is not hard to show that a pointed, connected space X is $\mathbb{G}_m$-n-truncated if and only if the $\mathbb{G}_m$-loop space $\Omega^{n+1}_{\mathbb{G}_m} X$ is contractible. In particular, by a result of Morel it follows that such an X is $\mathbb{G}_m$-n-truncated if and only if $\pi_i^{\mathbb{A}^1}(X,x)_{-n-1} = 0$ for all i. It follows that the motivic Eilenberg–Mac Lane space $K(\mathbb{Z}(n), 2n)$ is $\mathbb{G}_m$-n-truncated for any $n \geq 0$.

This notion of $\mathbb{G}_m$-connectivity is well-behaved in that $\mathbb{G}_m$-connectedness and truncatedness is preserved by taking suitable fibers and cofibers. To see this, one appeals to a comparison between unstable and S^1-stable homotopy theory. In the S^1-stable context one uses results about existence of $\mathbb{G}_m$-deloopings based on the work of the second author and M. Yakerson [BY20, Bac21]. One nice consequence of these results is that one can establish a Whitehead theorem using motives, at least over fields k having finite étale 2-cohomological dimension: a map f of $(1,1)$-connected spaces such that $H\mathbb{Z}\wedge f$ is an isomorphism is an $\mathbb{A}^1$-weak equivalence.

Theorem 4 ($\mathbb{P}^1$-Freudenthal suspension theorem)**.** *Assume $1 \leq p \leq q$ are integers. If (X,x) is a pointed (p,q)-connected space, then the unit map*

$$X \longrightarrow \Omega_{\mathbb{P}^1}\Sigma_{\mathbb{P}^1}X$$

has $\mathbb{A}^1$-homotopy fiber that is at least $(2p, 2q+1)$-connected.

Theorem 4 can be reduced to the case of motivic Eilenberg–Mac Lane spaces. The assembly maps

$$a_n : \mathbb{P}^1\wedge K(\mathbb{Z}(n), 2n) \longrightarrow K(\mathbb{Z}(n+1), 2n+2)$$

defining the motivic Eilenberg-Mac Lane spectrum can be used to factor the identity map on $K(\mathbb{Z}(n), 2n)$ through the unit of the loop suspension adjunction:

$$K(\mathbb{Z}(n), 2n) \xrightarrow{u} \Omega_{\mathbb{P}^1}\Sigma_{\mathbb{P}^1}K(\mathbb{Z}(n), 2n) \xrightarrow{\Omega_{\mathbb{P}^1}a_n} \Omega_{\mathbb{P}^1}K(\mathbb{Z}(n+1), 2n+2) \cong K(\mathbb{Z}(n), 2n).$$

In that case, there is a fiber sequence $\mathrm{fib}(u) \longrightarrow * \longrightarrow \mathrm{fib}(\Omega_{\mathbb{P}^1}a_n)$, i.e., $\mathrm{fib}(u) \cong \Omega\,\mathrm{fib}(\Omega_{\mathbb{P}^1}a_n)$. To establish the result, it then suffices to establish a suitable connectivity bound on $\mathrm{fib}(a_n)$. By Blakers–Massey style results, one then reduces to establishing the following bound on the connectivity of $\mathrm{cof}(a_n)$.

Lemma 5. *The space $\mathrm{cof}(a_n)$ is $(2n+1, 2n)$-connected;*

Sketch of proof. The proof of Lemma 5 proceeds by analyzing the geometry of symmetric powers. Following Voevodsky, write Quot_{Σ_r} for the unique colimit preserving extension of the functor on quasi-projective Σ_r-schemes to motivic spaces with Σ_r-action given by taking the quotient. One defines $\mathrm{Sym}^r(\mathbb{P}^{1\wedge n}) \cong \mathrm{Quot}_{\Sigma_r}(\mathbb{P}^{1\wedge n\times r})$. To analyze the connectivity of a_n, one uses Voevodsky's motivic Dold-Thom theorem: $K(\mathbb{Z}(n), 2n) \cong \mathrm{Sym}^\infty(\mathbb{P}^{1\wedge n})$ and the fact that the latter space is a colimit of spaces of the form $\mathrm{Sym}^r(\mathbb{P}^{1\wedge n})$.

Let us write ρ for the standard r-dimensional representation of Σ_r, which decomposes as $\bar{\rho}\oplus\mathbf{1}$, where $\mathbf{1}$ is the trivial representation. To make explicit the Σ_r-action we identify $\mathrm{Sym}^r(\mathbb{P}^{1\wedge n})$ with $\mathrm{Quot}_{\Sigma_r}(Th(\mathbb{A}(\rho^{\oplus n})))$, viewing $\mathbb{A}(\rho^{\oplus n})$ as a trivial vector bundle over a point. The identification $\rho^{\oplus n} \cong \mathbf{1}^{\oplus n}\oplus\bar{\rho}^{\oplus n}$ and the usual properties of Thom spaces imply $\mathrm{Quot}_{\Sigma_r}(Th(\mathbb{A}(\rho^{\oplus n}))) \cong \Sigma^n_{\mathbb{P}^1}\mathrm{Quot}_{\Sigma_r}(Th(\mathbb{A}(\bar{\rho}^{\oplus n})))$.

The assembly map a_n arises from the sequence of inclusions $\mathbf{1}\oplus n\rho \subset \rho\oplus n\rho \cong (n+1)\rho$ by applying $\mathrm{Quot}_{\Sigma_r}(Th(\mathbb{A}(-)))$, i.e., it is a map

$$\Sigma^{r+1}_{\mathbb{P}^1}\mathrm{Quot}_{\Sigma_r}Th(\mathbb{A}(\bar{\rho}^{\oplus n})) \longrightarrow \Sigma^{r+1}_{\mathbb{P}^1}\mathrm{Quot}_{\Sigma_r}Th(\mathbb{A}(\bar{\rho})^{\oplus n+1}).$$

We may then identify $\mathrm{cof}(a_n)$ with a colimit of spaces of the form

$$\Sigma\mathrm{Quot}_{\Sigma_r}Th(\mathbb{A}_{\mathbb{A}(\bar{\rho})\setminus 0}(\bar{\rho}^{\oplus n}\oplus 1)),$$

viewing $\mathbb{A}(\bar{\rho}^{\oplus n}\oplus 1)$ as a trivial vector bundle over $\mathbb{A}(\bar{\rho})\setminus 0$.

Following Nakaoka and Voevodsky (see [AD01] and [Voe04, §4]), we analyze the Thom space above by stratifying $\mathbb{A}(\bar{\rho}) \setminus 0$ by stabilizer type. The connectivity of the Thom space in question can be bounded below by the Thom spaces of normal bundles to each of the stabilizers. One analyzes these normal bundles using a bit of representation theory of the symmetric group.

Since $\bar{\rho}$ is an irreducible representation of Σ_r, the only fixed point in $\mathbb{A}(\bar{\rho})$ is the origin, i.e., every point of $\mathbb{A}(\bar{\rho}) \setminus 0$ has a non-trivial stabilizer. Each stabilizer in Σ_r is a partition subgroup and the the stabilizers appearing in $\mathbb{A}(\bar{\rho}) \setminus 0$ are proper partition subgroups. A proper partition subgroup of Σ_r is of the form $\Sigma_{r_1} \times \cdots \times \Sigma_{r_s}$ where $\sum_i r_i = r$; in particular, a non-trivial such subgroup necessarily has more than 1 factor. If H is a partition subgroup of Σ_r, then $\mathrm{Res}^{\Sigma_r}_H(\rho)$ decomposes as a direct sum of the standard r_i-dimensional representation of Σ_{r_i}, each of which splits off a trivial summand. By induction, this observation yields the required connectvity estimate. □

Remark 6. *The restriction on the characteristic of the base field arises because of our need to analyze symmetric powers, which may be singular. More generally, a version of the suspension theorem also holds after inverting the exponential characteristic of the base field.*

References

[AD01] G. Z. Arone and W. G. Dwyer Partition complexes, Tits buildings and symmetric products *Proc. London Math. Soc. (3)*, 82(1): 229–256, 2001.

[AF14] A. Asok and J. Fasel. A cohomological classification of vector bundles on smooth affine threefolds. *Duke Math. J.*, 163(14):2561–2601, 2014.

[AF15] A. Asok and J. Fasel. Splitting vector bundles outside the stable range and $\mathbb{A}^1$-homotopy sheaves of punctured affine spaces. *J. Amer. Math. Soc.*, 28(4):1031–1062, 2015.

[Bac21] T. Bachmann. The zeroth $\mathbb{P}^1$-stable homotopy sheaf of a motivic space. *Accepted for publication in Journal of the Institute of Mathematics of Jussieu*, 2021. https://arxiv.org/abs/2003.12021.

[BY20] T. Bachmann and M. Yakerson. Towards conservativity of $\mathbb{G}_m$-stabilization. *Geometry & Topology*, 24(4):1969–2034, 2020.

[Mor12] F. Morel. *$\mathbb{A}^1$-Algebraic Topology over a Field*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2012.

[MoVo99] F. Morel and V. Voevodsky., *$\mathbb{A}^1$-homotopy theory of schemes. Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[Voe04] Voevodsky, V. On the zero slice of the sphere spectrum. *Tr. Mat. Inst. Steklova* 246:106-115, 2004. Algebr. Geom. Metody, Svyazi i Prilozh.; reprinted in *Proc. Steklov Inst. Math.* 2004, 246(3): 93–102, 2004.

Algebraic K-Theory and Zakharevich K-Theory

Oliver Braunling
(joint work with Michael Groechenig)

In the talk I have presented the proof of the following result.

Theorem 1 (B.–Groechenig [BG21]). *For each odd $n \geq 3$, we have*

$$\dim_{\mathbb{Q}}(K_n(\mathsf{Var}_{\mathbb{C}}) \otimes \mathbb{Q}) = \infty.$$

Here $K(\mathsf{Var}_{\mathbb{C}})$ refers to the K-theory spectrum of varieties. Let us recall what that is. The theory begins with the following definition.

$$K_0(\mathsf{Var}_k) = \frac{\mathbb{Z}\{[X] \text{ for } X \text{ a } k\text{-variety}\}}{[X] = [Z] + [U] \text{ for every } Z \hookrightarrow X \overset{\circ}{\hookleftarrow} U}, \tag{1}$$

where $Z \hookrightarrow X \overset{\circ}{\hookleftarrow} U$ refers to the decomposition of the variety X into a closed subvariety Z and its open complement U. Defining

$$[X] \cdot [Y] := [X \times_k Y],$$

the abelian group $K_0(\mathsf{Var}_k)$ becomes a commutative ring. This definition originates from a letter from Grothendieck to Serre in 1964 (which is reprinted in [CS04]), at the time when Grothendieck was experimenting with the first ideas to set up a theory of motives. Despite being interesting in its own right, the ring $K(\mathsf{Var}_{\mathbb{C}})$ received a lot more attention after the work of Kontsevich on motivic integration.

Theorem 2 (Kontsevich, 1995). *Let X_1, X_2 be K-equivalent[1] varieties over the complex numbers, this means that there exists a diagram*

$$\begin{tikzcd} & Y \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ X_1 & & X_2, \end{tikzcd}$$

where X_1, X_2 and Y are proper smooth over $\mathbb{C}$, the maps h_i are proper birational, and the pullbacks of the respective canonical bundles to Y are isomorphic, i.e. $h_1^* K_{X_1} \cong h_2^* K_{X_2}$. Then the varieties have the same Hodge numbers, i.e. $h^{p,q}(X_1) = h^{p,q}(X_2)$.

Kontsevich proved this as an application of his theory of motivic integration, extending an earlier result of Batyrev which concerned the Betti numbers and was proved using p-adic integration. Really, Kontsevich's theory shows much more: There is a suitable completion of $K_0(\mathsf{Var}_k)$, let us call it $\widehat{K_0(\mathsf{Var}_k)}$, and any invariant which factors over this completion satisfies the conclusion of the above theorem. For example, over finite fields, point counting is such an invariant,

$$K_0(\mathsf{Var}_{\mathbb{F}_q}) \to \mathbb{Z}, \qquad [X] \mapsto \#X(\mathbb{F}_q),$$

as any decomposition $Z \hookrightarrow X \overset{\circ}{\hookleftarrow} U$ is a disjoint decomposition of the set of $\mathbb{F}_q$-points.

[1]The letter K is unrelated to K-theory in this context.

Similarly, the above theorem on Hodge numbers (Theorem 2) comes from such a map to mixed Hodge structures,

$$K_0(\mathsf{Var}) \longrightarrow K_0(\mathsf{MHS}) \qquad (\mathsf{MHS}{:=}\text{mixed Hodge structures})$$

using the mixed Hodge structure on the cohomology $H_c^*(X)$ with compact supports. In Motivic Integration such maps are known as *motivic measures*, whereas people with a background in motives would perhaps rather call it a *realization.* Since the definition of K_0 in Equation 1 strongly resembles the definition of K_0 for an abelian category, one could hope that many developments in Algebraic K-Theory have counterparts for $K_0(\mathsf{Var}_k)$, and at the very least one could hope that there are higher K-groups or a full K-theory spectrum having $K_0(\mathsf{Var}_k)$ as its π_0. Inna Zakharevich was the first to construct such a theory [Zak17b]. Later, Jonathan Campbell found a different definition resembling the Waldhausen $S_\bullet$-construction [Cam19]. At first, both definitions look completely different, but by a very fundamental recent advance in the foundations of the whole theory due to Campbell and Zakharevich, we know now that both definitions give the same invariant [CZ21]. As a result, we now have an E_∞-ring spectrum $K(\mathsf{Var}_k)$ whose π_0 recovers Equation 1.

This raises the question whether this theory has anything interesting to say. Does it? This has already been answered positively by Zakharevich, who used it to give a new perspective on the multiplication by the affine line[2] $\mathbb{A}^1$ on $K(\mathsf{Var}_k)$, see [Zak17a]. However, this work only really uses $K_1(\mathsf{Var}_k)$. Hence, the role of K-theory classes in higher degrees remains somewhat unclear. What are they good for? And, do they exist at all?

Problem 3. *Are there any classes in $K_n(\mathsf{Var}_k)$ for $n \geq 1$?*

The first construction of non-trivial classes in higher degrees is due to Campbell, Wolfson and Zakharevich [CWZ19]. It is based on maps

$$\mathbb{S} \longrightarrow K(\mathsf{Var}_k)$$

from the sphere spectrum. A tricky part is to ensure that the classes in the image of this map (i.e. on the level of homotopy groups) are actually non-zero. This is also solved in [CWZ19] by developing an ℓ-adic realization for $K(\mathsf{Var}_k)$.

However, this construction by design can only ever produce torsion classes. Zakharevich has later found a different construction, but also only exhibiting torsion classes. Hence, it had remained open whether non-torsion classes exist at all. Our Theorem 1 gives an affirmative answer.

The proof has three steps: First, if A is an abelian variety over a perfect field k with $\mathcal{O} := \mathrm{End}(A)$, there is an incoming map

$$K(\mathcal{O}) \longrightarrow K(\mathsf{Var}_k)[A^{-1}]^\times,$$

where A^{-1} refers to inverting the class in π_0 corresponding to $[A]$. We only construct this map after taking a connective cover and on the level of spaces.

[2]in Motivic Integration this is called $\mathbb{L}$ instead, $\mathbb{L} = \mathbb{A}^1$, inspired by L in Hodge theory, or more broadly referring to Lefschetz hyperplane arguments

Second, there are outgoing maps

$$K(\mathsf{Var}_k) \longrightarrow K(\mathcal{A})$$

to all sorts of abelian categories $\mathcal{A}$, which come from the following general machine:

Theorem 4 (B.–Groechenig–Nanavaty, [BGN21]). *Let k be a perfect field. Suppose $\mathcal{M}$ is a cohomology theory of k-varieties satisfying $\mathbb{A}^1$-invariance and h-descent and having values in an abelian category $\mathcal{A}$ (in positive characteristic, demand that $\mathcal{A}$ is $\mathbb{Z}[\frac{1}{p}]$-linear). Then there is a map of spectra*

$$K(\mathsf{Var}_k) \longrightarrow K(\mathcal{A}),$$

which, roughly speaking, on K_0 sends X to $\sum_n (-1)^n H^n_{\mathcal{M}}(X)$.

A similar result was proven independently and around the same time by Joshua Lieber [Lie21], as part of his PhD thesis supervised by Matilde Marcolli. The results are different. Lieber only deals with $\mathbb{Q}$-coefficients, but he treats far more general bases S than just the perfect fields of the above theorem.

The proof of the above work is based on the viewpoint of Vologodsky [Vol12], mixed with ideas of Kelly in positive characteristic [Kel17].

Theorem 4 should be general enough to generalize all classical motivic measures from K_0 to all of $K(\mathsf{Var}_k)$, if they resemble a cohomology theory. However, the theorem is clearly not general enough to encompass invariants as in Röndig's vision [Roe16].

The last step in the proof of Theorem 1 studies the composition of both maps. The key idea is that the cohomology of abelian varieties looks like an exterior algebra on H^1. Therefore, restricting to weight -1, one can (when done properly) construct something like a section (rationally, and after connective cover) to the incoming map $K(\mathcal{O}) \to K(\mathsf{Var}_k)[A^{-1}]^{\times}$. Choosing abelian varieties with complex multiplication for bigger and bigger CM fields then leads to the result.

A version of the result where mixed Hodge structures are replaced by 1-motives, yields a similar result in positive characteristic. This uses the derived Albanese of Barbieri-Viale and Kahn [BVK16] (also with Ayoub [ABV09]).

References

[ABV09] J. Ayoub and L. Barbieri-Viale, *1-motivic sheaves and the Albanese functor*, J. Pure Appl. Algebra **213** (2009), no. 5, 809–839. MR 2494373

[BG21] O. Braunling and M. Groechenig, *Classes in Zakharevich K-groups constructed from Quillen K-theory*, `https://arxiv.org/abs/2109.01136` (2021).

[BGN21] O. Braunling, M. Groechenig, and A. Nanavaty, *The standard realizations for the K-theory of varieties*, `https://arxiv.org/abs/2107.01168` (2021).

[BVK16] L. Barbieri-Viale and B. Kahn, *On the derived category of 1-motives*, Astérisque (2016), no. 381, xi+254. MR 3545132

[Cam19] J. A. Campbell, *The K-theory spectrum of varieties*, Trans. Amer. Math. Soc. **371** (2019), no. 11, 7845–7884. MR 3955537

[CS04] P. Colmez and J.-P. Serre (eds.), *Grothendieck-Serre correspondence, bilingual edition*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], American Mathematical Society and Société Mathématique de France, Paris, 2004.

[CWZ19] J. Campbell, J. Wolfson, and I. Zakharevich, *Derived ℓ-adic zeta functions*, Adv. Math. **354** (2019), 106760, 53. MR 3993931

[CZ21] J. A. Campbell and I. Zakharevich, *Devissage and Localization for the Grothendieck Spectrum of Varieties*, `https://arxiv.org/abs/1811.08014` (2021).

[Kel17] S. Kelly, *Voevodsky motives and ldh-descent*, Astérisque (2017), no. 391, 125. MR 3673293

[Lie21] J. Lieber, *From Six Functors Formalisms to Derived Motivic Measures*, `https://arxiv.org/abs/2109.07065` (2021).

[Roe16] O. Roendigs, *The Grothendieck ring of varieties and algebraic K-theory of spaces*, `https://arxiv.org/abs/1611.09327` (2016).

[Vol12] V. Vologodsky, *Hodge realizations of 1-motives and the derived Albanese*, J. K-Theory **10** (2012), no. 2, 371–412. MR 3004172

[Zak17a] I. Zakharevich, *The annihilator of the Lefschetz motive*, Duke Math. J. **166** (2017), no. 11, 1989–2022. MR 3694563

[Zak17b] ———, *The K-theory of assemblers*, Adv. Math. **304** (2017), 1176–1218. MR 3558230

Generalized cohomology of algebraic stacks

Charanya Ravi

(joint work with Adeel Khan)

The goal of this talk was to present the main constructions from [KhRa]. Motivic homotopy theory provides a framework for the study of generalized or extraordinary cohomology theories in algebraic geometry, such as motivic cohomology, algebraic K-theory, and algebraic cobordism. To develop cohomology theory of stacks, we introduce an extension of the motivic stable homotopy category to a large class of algebraic stacks. This gives a systematic approach to studying the existing cohomology theories for algebraic stacks, along with introducing some new cohomology theories.

Cohomology theories of quotient stacks correspond to equivariant cohomology of schemes. Traditionally, equivariant cohomology theories in algebraic geometry are defined via algebraic versions of the Borel construction. For example, equivariant Chow groups [EG] and equivariant algebraic cobordism [Kri1, HML] are all defined this way. The primary example of a non-Borel-type cohomology theory in algebraic geometry is algebraic K-theory, which is "genuine" by nature. Our goal is to construct a "genuine" and a "lisse-extended" motivic homotopy category for algebraic stacks along with the formalism of six operations. Such that in the case of quotient stacks, objects in the respective categories represent genuine and Borel-type equivariant cohomology theories.

The construction of genuine **SH** is given for a class of algebraic stacks, called *scalloped* stacks, which includes for instance tame Deligne–Mumford or Artin stacks with separated diagonal as well as quotients of qcqs algebraic spaces by nice linear algebraic groups.

Theorem 1. *The assignment $\mathcal{X} \mapsto \mathbf{SH}(\mathcal{X})$, together with the formalism of six operations, extends from quasi-compact and quasi-separated algebraic spaces to scalloped algebraic stacks. More precisely, we have the following operations:*

(1) *For every scalloped stack $\mathcal{X}$, a pair of adjoint bifunctors $(\otimes, \underline{Hom})$.*

(2) *For every morphism of scalloped stacks $f : \mathcal{X} \to \mathcal{Y}$, an adjoint pair*

$$f^* : \mathbf{SH}(\mathcal{Y}) \to \mathbf{SH}(\mathcal{X}), \quad f_* : \mathbf{SH}(\mathcal{X}) \to \mathbf{SH}(\mathcal{Y}).$$

(3) *For every representable morphism of finite type $f : \mathcal{X} \to \mathcal{Y}$ between scalloped stacks $\mathcal{X}$ and $\mathcal{Y}$, an adjoint pair*

$$f_! : \mathbf{SH}(\mathcal{X}) \to \mathbf{SH}(\mathcal{Y}), \quad f^! : \mathbf{SH}(\mathcal{Y}) \to \mathbf{SH}(\mathcal{X}).$$

Moreover, these satisfy various identities including the base change and projection formulas, homotopy invariance, purity isomorphism, and localization triangle.

Our constuction of the genuine $\mathbf{SH}$ is mainly inspired by Hoyois's construction of the genuine equivariant motivic homotopy theory for quasi-projective G-schemes [Ho]. For the quotient of a G-quasi-projective scheme X by a nice group G, $\mathbf{SH}([X/G])$ recovers Hoyois's equivariant stable motivic homotopy category $\mathbf{SH}^G(X)$. Thus in this case, our work in particular removes the quasi-projectivity hypotheses in [Ho].

The construction of the lisse-extended $\mathbf{SH}$ is done via the following formal procedure. Given any qcqs algebraic stack $\mathcal{X}$ (not necessarily scalloped), denote by $\mathrm{Lis}_{\mathcal{X}}$ the ∞-category of pairs $(U, u : U \to \mathcal{X})$, where U is a qcqs algebraic space and $u : U \to \mathcal{X}$ is a smooth morphism. The *lisse extension* $\mathbf{SH}_{\triangleleft}(\mathcal{X})$ is the homotopy limit of ∞-categories

$$\mathbf{SH}_{\triangleleft}(\mathcal{X}) = \lim_{(U,u)\in \mathrm{Lis}_{\mathcal{X}}} \mathbf{SH}(U),$$

over the $*$-inverse image functors. The six functor formalism also extends to this setting.

Let $\mathrm{C}^\bullet$, $\mathrm{C}^{\mathrm{BM}}_\bullet$ (resp. $\mathrm{C}^\bullet_\triangleleft$, $\mathrm{C}^{\mathrm{BM}}_{\triangleleft,\bullet}$) denote the cohomology and Borel–Moore homology spectrum defined by taking coefficients in $\mathbf{SH}$ (resp. $\mathbf{SH}_\triangleleft$). Using the formalism of six operations, we get various functorialities, products, euler classes and properties like localization, Mayer–Vietoris, homotopy invariance and derived invariance for these cohomology and Borel-Moore homology theories. In particular, our proof of the representability of homotopy K-theory in the genuine $\mathbf{SH}$, as a result establishes derived invariance and cdh descent for homotopy K-theory.

The motivic cohomology, algebraic K-theory, and algebraic cobordism spectra give rise to lisse-extended motivic spectra $\mathbb{Z}^{\triangleleft}_{\mathcal{X}}$, $\mathrm{KGL}^{\triangleleft}_{\mathcal{X}}$, $\mathrm{MGL}^{\triangleleft}_{\mathcal{X}} \in \mathbf{SH}_\triangleleft(\mathcal{X})$ over $\mathcal{X}$, simply by virtue of stability under $*$-inverse image. If $\mathcal{X}$ is scalloped, then these are moreover the images of their genuine counterparts by a canonical functor (which commutes with colimits and $*$-inverse image)

$$\mathbf{SH}(\mathcal{X}) \to \mathbf{SH}_\triangleleft(\mathcal{X}).$$

However, this functor is far from being an equivalence, so that the corresponding cohomology theories are very different.

Example 2. *In the case of* $\mathrm{KGL}^{\triangleleft}_{[X/G]}$, *the canonical map*

$$\pi_0\mathrm{K}([X/G]) \to \pi_0\mathrm{C}^\bullet_\triangleleft([X/G], \mathrm{KGL}),$$

induced by the functor $\mathbf{SH}([X/G]) \to \mathbf{SH}_{\triangleleft}([X/G])$ *(for G nice), is not bijective if G is nontrivial. In fact, it exhibits the target as a completion of the source (see* [Kri2]*).*

We show that for quotient stacks, cohomology with coefficients in any of the lisse-extended cohomology theories above can be computed via Totaro and Morel–Voevodsky's algebraic approximation of the Borel construction. For example, in the case of motivic cohomology we have:

Theorem 3. *Let G be a linear algebraic group over a field k of characteristic zero. Let X be a smooth G-quasi-projective k-scheme. Then for every $n, s \in \mathbb{Z}$ there are canonical isomorphisms*

$$\pi_s \mathrm{C}^{\bullet}_{\triangleleft}([X/G], \mathbb{Z})\langle n\rangle \simeq CH^n_G(X, s)$$

where on the right-hand side are the Edidin–Graham equivariant higher Chow groups [EG]*. The result also holds for characteristic $p > 0$, up to inverting p.*

Remark 4. *For general coefficients $\mathcal{F}$, we have isomorphisms*

$$\mathrm{C}^{\bullet}_{\triangleleft}([X/G], \mathcal{F})\langle n\rangle \simeq \lim_i \mathrm{C}^{\bullet}([X/G] \times_{BG} U_i, \mathcal{F})\langle n\rangle \tag{1}$$

where $(U_i)_i$ is a sequence of algebraic approximations to the Borel construction. This is deduced from a stronger comparison of motivic stable homotopy types. On π_0 we have isomorphisms $H^n_{\triangleleft}([X/G], \mathcal{F}) \simeq \lim_i H^n([X/G] \times_{BG} U_i, \mathcal{F})$ by [Kh].

In particular, although the right-hand side has been considered in the case of 1 algebraic cobordism in [HML, Kri1], it was not known to satisfy the right-exact localization sequence. The localization sequences for lisse-extended cobordism, which in fact even extend to long-exact sequences using the higher group, in particular establishes the above mentioned right exact localization sequence in view of [Kh]. In general, we regard the lisse extension as a good way to define "Borel-type" extensions of arbitrary generalized cohomology theories.

We expect the homotopical properties of cohomology theories for stacks developed above to have several useful applications. As an example, we have the following Atiyah-Bott localization theorem for oriented cohomology theories of algebraic stacks which we prove in an ongoing joint work with Dhyan Aranha, Adeel Khan, Alexei Latyntsev and Hyeonjun Park. The above results make it possible to obtain such a result for oriented theories in $\mathbf{SH}_{\triangleleft}$, for e.g. motivic cohomology and algebraic cobordism.

Theorem 5. *Let G be an fppf affine group scheme over a connected noetherian ring k. Let $i : Z \to X$ a G-equivariant closed immersion of qcqs algebraic stacks over k with affine stabilizers. Let $\Sigma \subset \pi_0 \mathrm{C}^{\bullet}(BG)\langle * \rangle$ be a subset such that for every geometric point x of $X \setminus Z$ there exists a rank one G-representation $\mathcal{L} \in Pic(BG)$ with $c_1(\mathcal{L}) \in \Sigma$ and $c_1(\mathcal{L}|_{BSt^G_x}) = 0$, where St^G_x denotes the G-stabilizer at x. Then the direct image map*

$$i_* : \mathrm{C}^{BM}_{G,\bullet}(Z)\langle * \rangle[\Sigma^{-1}] \to \mathrm{C}^{BM}_{G,\bullet}(X)\langle * \rangle[\Sigma^{-1}]$$

is invertible.

References

[EG] D. Edidin, W. Graham, *Equivariant intersection theory*. Invent. Math. **131** (1998), no. 3, 595–644.

[HML] J. Heller, J. Malagón-López, *Equivariant algebraic cobordism*. J. Reine Angew. Math. **684**, 87–112 (2013).

[Ho] M. Hoyois, *The six operations in equivariant motivic homotopy theory*, Adv. Math. **305**, 197–279 (2017); corrigendum ibid. **333**, 1293–1296 (2018).

[Kri1] A. Krishna, *Equivariant cobordism of schemes*. Doc. Math. **17** (2012), 95–134.

[Kri2] A. Krishna, *The completion problem for equivariant K-theory*. J. Reine Angew. Math. **740** (2018), 275–317.

[KhRa] A. A. Khan, C. Ravi, *Generalized cohomology theories for algebraic stacks*. arXiv: 2106.15001 (2021).

[Kh] A. A. Khan, *Homology of Artin stacks*, In preparation.

The motivic Satake equivalence and a question about the Drinfeld lemma

Jakob Scholbach

(joint work with Timo Richarz)

1. The motivic Satake equivalence

The motivic Satake equivalence is a unification of several geometric Satake equivalences in the literature. To state it, recall that for a reductive group G over a field k, such as $G = \mathrm{GL}_n$, the loop group LG and the positive loop group L^+G are given by

$$LG := G(k((t))) \supset L^+G := G(k[[t]]).$$

The quotient $\mathrm{Gr}_G := LG/L^+G$ is called the affine Grassmannian. The classical Satake isomorphism relates the *spherical Hecke algebra*, i.e., the space of compactly supported functions on the double cosets $L^+G \setminus LG/L^+G$ to the Grothendieck group of representations of the Langlands dual group $\widehat{G}$. Examples of dual groups include $\widehat{GL_n} = GL_n$, $\widehat{PGL_n} = SL_n$. The geometric Satake equivalences due, in several similar versions, to many authors including Ginzburg, Mirkovic–Vilonen, Richarz, Zhu, Fargues–Scholze, enhance this isomorphism in a geometric way. For example, there is an equivalence

$$\mathrm{Rep}(\widehat{G}_{/\mathbf{Q}_\ell}) = \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \mathbf{Q}_\ell)$$

between algebraic representations of the dual group defined over $\mathbf{Q}_\ell$, and the category of L^+G-equivariant perverse ℓ-adic sheaves on the affine Grassmannian over an algebraically closed field k. Under this equivalence the highest weight representations V^μ correspond to intersection sheaves $\mathrm{IC}_{\mathrm{Gr}_G^{\le\mu}}$. For $k = \mathbf{C}$, there is similar statement for perverse sheaves on the complex (infinite-dimensional) analytic space associated to Gr_G. Theorem 1 below confirms the idea that Satake's idea can be stated without reference to any Weil cohomology theory.

We introduce a category $\mathrm{DTM}(\mathrm{Gr}_G)$ of *stratified Tate motives* on Gr_G, i.e., those motives whose restriction to the $\mathrm{Gr}_G^{\le\mu}$ are Tate motives. The notion of stratified Tate motives is due to Soergel–Wendt in the case of flag varieties. All our motives are with rational coefficients. For G being defined over a scheme satisfying the Beilinson–Soulé conjecture, the cellular nature of the stratification of Gr_G by L^+G-orbits leads to the existence of a t-structure on $\mathrm{DTM}(\mathrm{Gr}_G)$. The heart of this t-structure, denoted $\mathrm{MTM}(\mathrm{Gr}_G)$, consists of precisely those motives whose ℓ-adic realization are perverse sheaves, as encountered in the ℓ-adic version of Satake above. In fact, objects in this category are naturally equivariant with respect to the L^+G-action on Gr_G, so we denote this category more symmetrically by $\mathrm{MTM}(L^+G \setminus LG/L^+G)$. This abelian category is generated under extensions by the intersection motives $\mathrm{IC}_{\mathrm{Gr}_G^{\leq\mu}}$. The subcategory consisting only of direct sums of such motives (but no extensions) is denoted by $\mathrm{MTM}(L^+G \setminus LG/L^+G)^{ss}$.

Theorem 1 ([RS21])**.** *Let k be a finite field, a number field, a function field $\mathbf{F}_q(t)$ and G/k a split reductive group. Then there is an equivalence of Tannakian categories*

$$\mathrm{Rep}(\widehat{G} \rtimes \mathbf{G}_m) \cong \mathrm{MTM}(L^+G \setminus LG/L^+G)^{ss}.$$

In comparison to the proofs of the existing Satake equivalences, the work going into the proof of Theorem 1 is chiefly to ensure that all relevant functors, notably the convolution product, preserve (stratified) Tate motives. This is achieved by inspecting the cellular structure of Gr_G and related objects such as the affine flag variety and convolution affine Grassmannians.

2. A question related to the Drinfeld lemma

The motivic Satake equivalence in Theorem 1 is a step of a program aiming to apply motivic methods to the Langlands parametrization due to V. Lafforgue [Laf18]. To put it into perspective recall that V. Lafforgue, following Drinfeld and L. Lafforgue, uses the Satake equivalence in order to construct the intersection complexes on the moduli stack of shtukas. Up to issues related to constructibility and lisseness of sheaves, which are resolved by Lafforgue and Xue's later work, and which we ignore here, a main point in Lafforgue's work is a functor (and similarly with more factors)

$$\mathrm{Rep}(\widehat{G}^2_{/\mathbf{Q}_\ell}) \to \mathrm{LocSys}((X^{\mathrm{Weil}}) \times_{\overline{\mathbf{F}}_p} (X^{\mathrm{Weil}})).$$

Here X is a smooth proper curve over $\mathbf{F}_p$ and the right hand category consists of tuples (F, α_1, α_2) consisting of an ℓ-adic local system F on $\overline{X}^2 := X^2 \times_{\mathbf{F}_p} \overline{\mathbf{F}}_p$, and two (appropriately commuting) isomorphisms $\alpha_k : \mathrm{Frob}_k^* F \stackrel{\cong}{\to} F$, with $\mathrm{Frob}_k : \overline{X}^n \to \overline{X}^n$ being the Frobenius on the k-th copy of X, and the identity on all other factors, including $\overline{\mathbf{F}}_p$. A lemma of Drinfeld is then used in order to decompose such local systems with partial Frobenius "descent" data into two local systems on the individual factors $\overline{X}$.

A first obstacle in doing this motivically is the fact that the above functor uses a t-structure truncation, whose existence in the context of motives is among the deepest motivic conjectures. This leads to the question how to phrase the Drinfeld lemma for the *derived* category of constructible sheaves.

Theorem 2 ([HRS20]). *For algebraic varieties $X_1, X_2/\mathbf{F}_p$ the exterior product yields an equivalence*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_1^{\mathrm{Weil}}) \otimes_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\overline{\mathbf{F}}_p)} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_2^{\mathrm{Weil}}) \stackrel{\cong}{\to} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_1^{\mathrm{Weil}} \times_{\overline{\mathbf{F}}_p} X_2^{\mathrm{Weil}}).$$

The categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_k^{\mathrm{Weil}})$ consist of pairs (F, α_k) with $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\overline{X}_k)$ (bounded complex of constructible ℓ-adic sheaves), and $\alpha_k : \mathrm{Frob}^*_{X_k} F \stackrel{\cong}{\to} F$ similarly as above. This theorem might be called a *derived Drinfeld lemma* or, from a different perspective, a *categorical Künneth formula.* It tells us precisely which sheaves on $\overline{X}_1 \times_{\overline{\mathbf{F}}_p} \overline{X}_2$ can be decomposed into ones on the factors. Incidentally, it also exhibits an interesting kinship of such Weil sheaves and (derived categories) of quasi-coherent sheaves or D-modules, where similar tensor product formulae hold.

Given such a satisfactory situation in the ℓ-adic context, it is a natural, if lofty question to ask for a Drinfeld lemma for motives: can break up motives on $X_1^{\mathrm{Weil}} \times_{\overline{\mathbf{F}}_p} X_2^{\mathrm{Weil}}$ into Weil motives on the individual factors? This also seems to be the major road-block towards a motivic refinement of V. Lafforgue's work on the Langlands parametrization.

As a plausibility check for such a far-fetched goal, we are investigating, again in joint ongoing work with Timo Richarz, the following seemingly sensible question:

Question 3. *Let X_1, X_2 be smooth projective curves over $\mathbf{F}_p$. Is there an equivalence*

$$\mathrm{DTM}(X_1^{\mathrm{Weil}}) \otimes_{\mathrm{DTM}(\overline{\mathbf{F}}_p)} \mathrm{DTM}(X_2^{\mathrm{Weil}}) \stackrel{\cong}{\to} \mathrm{DTM}(X_1^{\mathrm{Weil}} \times_{\overline{\mathbf{F}}_p} X_2^{\mathrm{Weil}})?$$

Here, $\mathrm{DTM}(X_k^{\mathrm{Weil}})$ consists of *Tate* motives (as opposed to arbitrary motives) on $\overline{X}_k$, again with a Weil datum as in the case of ℓ-adic sheaves as above etc. In the context of Theorem 2, the usual Künneth formula implies that the exterior product is already fully faithful for sheaves on $\overline{X}_k$ etc., as opposed to X_k^{Weil}. The Künneth formula for motivic cohomology fails: for weight one motivic cohomology (and geometrically connected X_k) the situation is such that the quotient of the subgroup

$$\mathrm{H}^1(\overline{X}_1, \mathbf{Q}(1)) \oplus \mathrm{H}^1(\overline{X}_2, \mathbf{Q}(1)) \subset \mathrm{H}^1(\overline{X}_1 \times_{\overline{\mathbf{F}}_p} \overline{X}_2, \mathbf{Q}(1))$$

is isomorphic to $\mathrm{Hom}_{\mathrm{Ab/Isogeny}}(\mathrm{Jac}_{\overline{X}_1}, \mathrm{Jac}_{\overline{X}_2})$. However, the map $\mathrm{id} - \mathrm{Frob}^*_{X_k}$ acts as an isogeny on the Jacobians $\mathrm{Jac}_{\overline{X}_k}$, causing the cokernel to vanish after passing to homotopy fixed points with respect to the two Frob^*_k. This shows that the presence of "Weil" in Question 3 is crucial, already for the full faithfulness of the functor.

References

[HRS20] Tamir Hemo, Timo Richarz, and Jakob Scholbach. Constructible sheaves on schemes and a categorical künneth formula, 2020. `arXiv:arXiv:2012.02853`.

[Laf18] Vincent Lafforgue. Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale. *J. Amer. Math. Soc.*, 31(3):719–891, 2018.

[RS21] Timo Richarz and Jakob Scholbach. The motivic Satake equivalence. *Math. Ann.*, 380(3-4):1595–1653, 2021. URL: `https://doi.org/10.1007/s00208-021-02176-9`, `doi:10.1007/s00208-021-02176-9`.

A Residual Intersection Formula in Hermitian K-theory

Kirsten Wickelgren
(joint work with Tom Bachmann)

If X is a scheme and $Z \supset Z'$ are closed subschemes, we denote by $Z : Z' = Z(\mathrm{Ann}(I(Z')/I(Z)))$ the scheme associated to the ideal quotient. In some cases $Z : Z'$ has underlying set $\overline{Z \setminus Z'}$, and in general we think of $Z : Z'$ as an algebro-geometric enhancement of the closure of the complement of Z' in Z. Suppose that Z is locally cut out by s equations but has codimension $< s$. If $Z : Z'$ does have codimension s, then we say $Z : Z'$ is a *residual intersection*. This is a classical notion.

For local complete intersection morphisms, duality for coherent sheaves is related to the determinant of the cotangent complex. Eisenbud and Ulrich have recently given coherent duality results in the setting of certain residual intersections [4]. In [1], we twist their description of the dualizing object, globalizing their result. (See loc. cit. Theorem 1.2.) We use this to describe an exceptional pushforward in Hermitian K-theory as follows.

For $f : Y \to X$ a proper map of schemes, there is an adjunction between derived categories D_c^+ of bounded below complexes of Zariski sheaves with coherent cohomology

$$f_* : D_c^+(Y) \to D_c^+(X) \qquad D_c^+(Y) \leftarrow D_c^+(X) : f^!,$$

and Grothendieck–Serre duality produces a canonical isomorphism in $D_c^+(X)$

$$f_* \operatorname{Hom}_Y(F, f^! G) \xrightarrow{\sim} \operatorname{Hom}_X(f_* F, G)$$

For $\pi_X : X \to \operatorname{Spec} k$ a Cohen-Macaulay k-scheme, $\omega_X := \pi_X^! \mathcal{O}_k$ is a dualizing object concentrated in a single degree. The resulting duality gives $D_c^+(X)$ the structure of a Poincaré ∞-category in the sense of [2]. The work of [2] builds an associated symmetric Hermitian K-theory spectrum, $\mathrm{GW}(X, \omega_X)$. Hermitian K-theory spectra are also constructed with other inputs and methods in [5] [6] and build on ideas of Karoubi, Quillen, and others. A symmetric bilinear form $G \times G \to \omega_X$ such that the associated map $G \to \operatorname{Hom}(G, \omega_X)$ is an equivalence determines a point in $\mathrm{GW}(X, \omega_X)$. For $f : Y \to X$ with both X and Y Cohen-Macaulay k-schemes, f_* determines a Poincaré functor $(D_c^+(Y), \omega_Y) \to (D_c^+(X), \omega_X)$ by Grothendieck–Serre duality. The machinery of [2] then produces a pushforward

$$\mathrm{GW}(Y, \omega_Y) \to \mathrm{GW}(X, \omega_X)$$

Combining with the results of [4] and [1, Theorem 1.2], we obtain the following pushforward.

Theorem 1. *Let X be a Gorenstein scheme and let $Z \supset Z'$ be closed subschemes such that Z is locally cut out by s equations, Z' is codimension $g < s$, and $Y := Z : Z'$ is a residual intersection satisfying the strong hypothesis of Eisenbud–Ulrich. Let J denote the ideal associated to Z, I denote the ideal associated to Z', K denote the ideal associated to the residual intersection Y, and $t = s - g$.*

Then the closed immersion $i : Y \hookrightarrow X$ has an associated pushforward in Hermitian K-theory

$$i_* : \mathrm{GW}(Y, I^{t+1}/JI^t) \to \mathrm{GW}_Y(X, \det J/JK[s])$$

The analogous pushforward exists in Witt groups, also called L-groups in [2].

Excess and residual intersections arise frequently in enumerative geometry. For example, Chasles' Theorem that there are 3264 (and not $6^5 = 7776$:) smooth conics tangent to 5 general conics in $\mathbb{P}^2_{\mathbb{C}}$ is a residual intersection computation in Chow groups [3]. Duality for coherent sheaves gives a quadratic enrichment to certain results in enumerative geometry over $\mathbb{C}$, providing information about the field of definition of solutions over a potentially non-algebraically closed field. This enrichment can also be naturally understood in the context of $\mathbb{A}^1$-homotopy theory and oriented Chow groups.

In this setting, it is not immediate that a residual intersection computation has an enrichment. An orientation condition may fail. Even if the orientation data is present, "invariance of number" may fail in the enriched sense, even when it is present over $\mathbb{C}$. We show that there is an enrichment of Chasles' theorem in this context: Over an arbitrary field of characteristic not 2, there are

$$\frac{3264}{2}(\langle 1\rangle + \langle -1\rangle)$$

smooth conics tangent to 5 general conics when counted with a certain weight.

We use the exceptional pushforward of Theorem 1 to give a residual intersection formula enriched in quadratic forms: In the setting of Theorem 1, suppose that X is regular of dimension s, Z arises as the vanishing locus of a section σ of a vector bundle V of rank s, $g = 1$,and Z' is the largest closed subscheme of Z such that none of its associated points is closed. In other words, the vanishing locus of σ is dimension 1 instead of dimension 0 and we are removing Z', where Z' is the locus of pure dimension 1. We can compute the Euler class $[e(V, \sigma)]$ in the twisted Witt group (or L-group of [2]) $W^V_Z(X)$ from the residual intersection $Y = Z : Z'$.

Theorem 2. *In Witt cohomology $W^V_Z(X)$, the Euler class is computed*

$$[e(V, \sigma)] = i_*[I/J]$$

where I/J denotes the class of the multiplication map $I/J \times I/J \to I^2/IJ$ in $W(Y, I^2/IJ)$.

The equality in Theorem 2 not hold in $\mathrm{GW}^V_Z(X)$. One might expect this failure, because the Euler class "should" have a contribution from Z'. From this point of view, the Theorem 2 is saying the something of the form "the contribution from Z' is hyperbolic and the contribution from Y is $[I/J]$."

References

[1] T. Bachmann and K. Wickelgren, *On quadratically enriched excess and residual intersections*, ArXiv 2112.05960.

[2] B. Calmès, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, and W.Steimle. *Hermitian k-theory for stable ∞-categories i, ii, iii.* ArXiv 2009.07223, 2009.07224, 2009.07225, 2020.

[3] D. Eisenbud and J. Harris. *3264 and all that – a second course in algebraic geometry.* Cambridge University Press, Cambridge, 2016.
[4] D. Eisenbud and B. Ulrich. *Duality and socle generators for residual intersections*, Journal für die reine und angewandte Mathematik, 2019 (756):183–226, 2019.
[5] J. Hornbostel *A1-representability of Hermitian K-theory and Witt groups* Topology 44 (2005), no. 3, 661–687
[6] M. Schlichting *Hermitian K-theory of exact categories* J. K-Theory 5 (2010), no. 1, 105–165.

Thoughts on Weibel's conjecture

Matthew Morrow
(joint work with Elden Elmanto)

Weibel's highly influential conjecture on negative K-groups (proved in 2016 by Kerz–Strunk–Tamme [7]) states the following: for X a Noetherian scheme of finite Krull dimension d,

(1) $K_{-n}(X) = 0$ for $n > d$,
(2) $K_{-d}(X) \cong H^d_{\mathrm{cdh}}(X, \mathbb{Z})$, and
(3) $K_{-n}(X) \overset{\simeq}{\rightarrow} K_{-n}(\mathbb{A}^r_X)$ for $n \geq d$ and all $r \geq 0$.

The most important of these is (3), from which one obtains $K_{-n}(X) \overset{\simeq}{\rightarrow} KH_{-n}(X)$ for all $n \geq d$; parts (1) and (2) then follow formally from the equivalence $KH \simeq L_{\mathrm{cdh}} K_{\geq 0}$ (i.e., homotopy invariant K-theory is the cdh sheafification of connective K-theory [7]) and resulting cdh descent spectral sequence.

We believe that negative K-groups should continue to be studied to stimulate the development of algebraic K-theory and (topological) cyclic homology. In the talk we considered the following questions:

(1) What happens in Weibel's conjecture if X is not assumed to be Noetherian?
(2) Do there exist geometric/cohomological descriptions of negative K-groups other than the formula for K_{-d} in (2) above?

Example 1. This example is essentially due to Weibel [8]. Let X be a 2 dimensional Noetherian scheme and consider the Nisnevich descent spectral sequence $E_2^{ij} = H^i_{\mathrm{Nis}}(X, K_{-j}) \Rightarrow K_{i-j}(X)$. Using Drinfeld's lemma that K_{-1} vanishes Nisnevich locally [4], one easily obtains a short exact sequence

$$0 \longrightarrow H^2_{\mathrm{Nis}}(X, \mathbb{G}_m) \longrightarrow K_{-1}(X) \longrightarrow H^1_{\mathrm{Nis}}(X, \mathbb{Z}) \longrightarrow 0$$

describing $K_{-1}(X)$ in terms of reasonable geometric invariants.

Before the following non-Noetherian example, recall Jaffard's notion of the *valuative dimension* $\dim_v A$ of a ring A [6]: it is defined to be the maximum of rank V, as V varies over all valuation rings occurring as subextensions $A/\mathfrak{p} \subseteq V \subseteq k(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subseteq A$. Jaffard proved that $\dim_v A$ is always $\geq$ the Krull dimension of A, which we will denote by $\dim_K A$; equality holds if A is Noetherian. The importance of valuative dimension to algebraic K-theory and motivic cohomology is the following theorem:

Theorem 2 ([5]). *Let A be a ring of finite valuative dimension d. Then the homotopy dimension of the topos of cdh sheaves on Sch_S^{fp} is bounded above by d. In particular, for any cdh sheaf of abelian groups $\mathcal{F}$ on Sch_S^{fp}, the cohomology groups $H^i_{cdh}(A, \mathcal{F})$ vanish for $i > d$.*

Example 3. Let A be a perfect $\mathbb{F}_p$-algebra of finite valuative dimension d (which would coincide with the Krull dimension if A were the perfection of a Noetherian $\mathbb{F}_p$-algebra). Then $K(A) \xrightarrow{\sim} KH(A)$ by Weibel [9, 10] and Antieau–Mathew–Morrow [1]. Examining the cdh descent spectral sequence for $KH \simeq L_{cdh}K_{\geq 0}$ we see that

(1) $K_{-n}(A) = 0$ for $n > d$,
(2) $K_{-d}(A) \cong H^d_{\mathrm{cdh}}(X, \mathbb{Z})$, and there is an exact sequence

$$K_{-d-2}(A) \to H^{d-2}_{\mathrm{cdh}}(A, \mathbb{Z}) \xrightarrow{\delta} H^d_{\mathrm{cdh}}(X, \mathbb{G}_m) \to K_{-d+1}(X) \to H^{d-1}_{\mathrm{cdh}}(X, \mathbb{Z}) \to 0.$$

The action of the Frobenius shows that the image of δ is killed by $p-1$.

The examples suggest that Weibel's conjecture might remain true for non-Noetherian schemes, as long as we work with valuative dimension, and that geometric descriptions of other negative K-groups may indeed exist. The goal of the talk was to present further results in the case of schemes of characteristic zero:

Theorem 4 (Elmanto–M.). *Let X be a quasi-compact, quasi-separated $\mathbb{Q}$-scheme of finite valuative dimension d. Then*

(1) *$K_{-n}(X) = 0$ for $n > d$,*
(2) *$K_{-d}(X) \cong H^d_{cdh}(X, \mathbb{Z})$, and*
(3) *$K_{-n}(X) \xrightarrow{\sim} KH_{-n}(X)$ for $n \geq \dim_K X$.*

A tool is the non-$\mathbb{A}^1$-invariant motivic cohomology theory for equicharacteristic schemes which we are developing; we restrict the statement here to characteristic zero, as the case of characteristic p was the focus of Elmanto's talk at the workshop:

Theorem 5 (Elmanto-M.). *Voevodsky's motivic cohomology $\mathbb{Z}(j)^{mot}(-)$ for smooth schemes over characteristic zero fields admits an extension to all quasi-compact, quasi-separated $\mathbb{Q}$-schemes X, such that the following properties hold:*

(1) *there is an Atiyah–Hirzebruch spectral sequence $E_2^{ij} = H^{i-j}_{mot}(X, \mathbb{Z}(-j)) \Rightarrow K_{-i-j}(X)$, and*
(2) *there is a homotopy cartesian square*

$$\begin{array}{ccc} \mathbb{Z}(j)^{mot}(X) & \longrightarrow & R\Gamma_{Nis}(X, \widehat{L\Omega}^{\geq j}_{X/\mathbb{Q}}) \\ \downarrow & & \downarrow \\ \mathbb{Z}(j)^{cdh}(X) & \longrightarrow & R\Gamma_{cdh}(X, \widehat{L\Omega}^{\geq j}_{X/\mathbb{Q}}) \end{array}$$

where $\mathbb{Z}(j)^{cdh}(X)$ is the Friedlander–Suslin–Voevodsky-style, $\mathbb{A}^1$-invariant, cdh-local motivic cohomology of X, which is being developed for arbitrary qcqs schemes in a separate project with Bachmann and Elmanto.

Now assume that X is a qcqs $\mathbb{Q}$-scheme of finite valuative dimension d. It follows from the construction of $\mathbb{Z}(j)^{\mathrm{cdh}}$ that it is cdh locally supported in degrees $\leq j$, whence Theorem 2 implies that $\mathbb{Z}(j)^{\mathrm{cdh}}(X)$ is supported in degrees $\leq j+d$. To show that the same vanishing bound holds for $\mathbb{Z}(j)^{\mathrm{mot}}(X)$, it is equivalent (thanks to the homotopy cartesian square in Theorem 5 and cdh descent for Hodge-completed derived de Rham cohomology) to show that the natural map $H^n_{\mathrm{Nis}}(X, \Omega^j) \to H^n_{\mathrm{cdh}}(X, \Omega^j)$ is surjective for all $n \geq \dim_K X$. This surjectivity is due to Cortiñas–Haesemeyer–Schlichting–Weibel for smooth varieties over characteristic zero fields [2, 3]; to extend their result to our general qcqs X we pass through localising invariants and results of Kerz–Strunk–Tamme [7].

From the vanishing bound for $\mathbb{Z}(j)^{\mathrm{mot}}(X)$ and the Atiyah–Hirzebruch spectral sequence, Theorem 4 follows. But one also obtains a partial cohomological description of $K_{-d+1}(X)$ similar to the examples presented above; more precisely, from the Atiyah–Hirzebruch spectral sequence one reads off an exact sequence

$$K_{-d+2}(X) \to H^{d-2}_{\mathrm{cdh}}(X, \mathbb{Z}) \xrightarrow{\delta} H^{1+d}_{\mathrm{mot}}(X, \mathbb{Z}(1)) \to K_{-d+1}(X) \to H^{d-1}_{\mathrm{cdh}}(X, \mathbb{Z}) \to 0.$$

Bruno Kahn pointed out to us after the talk that if Adams operators work as usual in our context then the differential δ should be zero.

References

[1] Antieau, B., Mathew, A., and Morrow, M. K-theory of perfectoid rings. `arXiv:2203.06472v1` (2022).

[2] Cortiñas, G., Haesemeyer, C., Schlichting, M., and Weibel, C. Cyclic homology, cdh-cohomology and negative K-theory. *Ann. of Math. (2) 167*, 2 (2008), 549–573.

[3] Cortiñas, G., Haesemeyer, C., and Weibel, C. K-regularity, *cdh*-fibrant Hochschild homology, and a conjecture of Vorst. *J. Amer. Math. Soc. 21*, 2 (2008), 547–561.

[4] Drinfeld, V. Infinite-dimensional vector bundles in algebraic geometry: an introduction. In *The unity of mathematics*, vol. 244 of *Progr. Math.* Birkhäuser Boston, Boston, MA, 2006, pp. 263–304.

[5] Elmanto, E., Hoyois, M., Iwasa, R., and Kelly, S. Cdh descent, cdarc descent, and Milnor excision. *Math. Ann. 379*, 3-4 (2021), 1011–1045.

[6] Jaffard, P. *Théorie de la dimension dans les anneaux de polynomes.* Mémor. Sci. Math., Fasc. 146. Gauthier-Villars, Paris, 1960.

[7] Kerz, M., Strunk, F., and Tamme, G. Algebraic K-theory and descent for blow-ups. *Invent. Math. 211*, 2 (2018), 523–577.

[8] Weibel, C. The negative K-theory of normal surfaces. *Duke Math. J. 108*, 1 (2001), 1–35.

[9] Weibel, C. A. Mayer-Vietoris sequences and mod p K-theory. In *Algebraic K-theory, Part I (Oberwolfach, 1980)*, vol. 966 of *Lecture Notes in Math.* Springer, Berlin, 1982, pp. 390–407.

[10] Weibel, C. A. Homotopy algebraic K-theory. In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, vol. 83 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1989, pp. 461–488.

K-theory of pushouts of associative rings

Georg Tamme
(joint work with Markus Land)

In previous joint work [4] we associated to any pullback square of ring spectra

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd} \tag{1}$$

a commutative diagram of ring spectra

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ A' \arrow[r] \arrow[drr, bend right] & A' \odot_A^{B'} B \arrow[dr] & \\ & & B' \end{tikzcd} \tag{2}$$

such that any localizing invariant sends the inner square in (2) to a pullback square. While we know $A' \odot_A^{B'} B$ as a spectrum – it is simply the (derived) tensor product $A' \otimes_A B$, we do not understand its ring structure in general. For the applications in [4] and some further structural properties of localizing invariants the concrete ring structure is not important. However, for computational advances a better understanding of the ring structure is desirable.

In the talk, I first described the following generalization of the $\odot$-construction: Assume given ring spectra A' and B, and a pointed (B, A')-bimodule M. The basepoint of M induces maps $A' \to M$ and $B \to M$, and we can form the pullback $A = A' \times_M B$. It turns out that A is a ring spectrum in a natural way, and the maps to A' and B are maps of ring spectra. The bimodule M determines a functor $(-) \otimes_B M \colon \mathrm{Perf}(B) \to \mathrm{Mod}(A')$. Using this, one can mimic the construction of [4] to obtain a ring spectrum $A' \odot_A^M B$, and a square of ring spectra

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & A' \odot_A^M B \end{tikzcd}$$

which becomes cartesian upon applying any localizing invariant.

In particular, assume given maps of ring spectra $C \to A'$ and $C \to B$. Then we form the pointed bimodule $M = B \otimes_C A'$ and define A accordingly. In this situation, we can describe $A' \odot_A^M B$ as a ring spectrum.

Theorem. *In the above situation, the canonical map*

$$A' \amalg_C B \to A' \odot_A^M B$$

is an equivalence. Here the pushout is formed in the category of ring spectra.

I discussed two examples:

(1) Let k be any commutative ring, and consider the rings $C = k[x, y]$, $A' = k[x]$ and $B = k[y]$ with the canonical maps $C \to A'$ and $C \to B$. In this situation, the bimodule M is simply k, and the pullback $A' \times_M B$ is the ring of the coordinate axes in the plane, $A = k[x, y]/(xy)$. In this situation, I explained how to compute the pushout $A' \amalg_C B$ to be $k[t]$, the free k-algebra on a generator in degree 2. Thus we obtain a pullback square of K-theory spectra

$$\begin{array}{ccc} K(k[x,y]/(xy)) & \longrightarrow & K(k[y]) \\ \downarrow & & \downarrow \\ K(k[x]) & \longrightarrow & K(k[t]). \end{array}$$

For instance, if k is a perfect field of positive characteristic, the relative K-groups $K_*(k[t], k)$ have been computed by Bayindir and Moulinos [1]. By our result, this computation is equivalent to the computation of the K-theory of the coordinate axes relative to k. Note that the latter has previously been computed by Geller–Reid–Weibel [2] for k a regular noetherian $\mathbb{Q}$-algebra, and by Hesselholt [3] for k a regular noetherian $\mathbb{F}_p$-algebra.

(2) Let $C = \mathbb{Z}[x]$, $A' = \mathbb{Z}$, $B = \mathbb{Z}[\zeta_p]$ for a prime p. Consider the maps $\mathbb{Z}[x] \to \mathbb{Z}$ sending x to 1 and $\mathbb{Z}[x] \to \mathbb{Z}[\zeta_p]$ sending x to ζ_p, respectively. The bimodule M in this case identifies with $\mathbb{F}_p$, and the resulting pullback square is the Rim square for $\mathbb{Z}[C_p]$. In the talk, I indicated how one could prove that $A' \amalg_C B \simeq \tau_{\geq 0}\mathbb{Z}^{tC_p}$, the connective cover of the C_p-Tate construction on $\mathbb{Z}$. It is known that the ring spectrum $\tau_{\geq 0}\mathbb{Z}^{tC_p}$ is equivalent to $\mathbb{F}_p[t]$, the free $\mathbb{F}_p$-algebra on a generator of degree 2. So we have a pullback of K-theory spectra

$$\begin{array}{ccc} K(\mathbb{Z}[C_p]) & \longrightarrow & K(\mathbb{Z}[\zeta_p]) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}) & \longrightarrow & K(\mathbb{F}_p[t]). \end{array}$$

In particular, we see that the total fibre of the K-theory of the Rim square is the same as that of the K-theory of the standard Milnor square of $\mathbb{F}_p[x, y]/(xy)$ which is known by (1) above.

I point out that the same pullback square has also been established by different means by Krause and Nikolaus (unpublished).

References

[1] H. Bayindir and T. Moulinos, *Algebraic K-theory of* THH($\mathbb{F}_p$), Trans. Amer. Math. Soc. **375** (2022), 4177–4207

[2] S. Geller, L. Reid, and C. Weibel, *The cyclic homology and K-theory of curves*, J. Reine Angew. Math. **393** (1989), 39–90.

[3] L. Hesselholt, *On the K-theory of the coordinate axes in the plane*, Nagoya Math. J. **185** (2007) 93–109.

[4] M. Land and G. Tamme, *On the K-theory of pullbacks*, Ann. of Math. **190** no. 3 (2019), 877–930.

4-valued ternary laws and 2-valued formal group laws

Jens Hornbostel

(joint work with David Coulette, Frédéric Déglise, Jean Fasel)

The classical work of Quillen and others on one-dimensional commutative formal group laws (FGL for short) shows how complex universal cohomology theories $A^*(-)$ on topological spaces lead to chern classes $c_1^A, c_2^A, ..., c_n^n \in A^*(BGl_n)$, which by evaluating c_1 on products of line bundles then yield FGLs $F_A(x,y)$. The universal oriented cohomology theory is complex cobordism MU^* whose coefficient ring $MU^*(pt)$ is a free polynomial ring, namely the *Lazard ring*, which also is the coefficient ring of the universal FGL. This picture has been translated to algebraic geometry by Levine and Morel [LM], who in particular construct algebraic cobordism $\Omega^*(-)$ as the universal oriented cohomology on smooth varieties over a given base field k of characteristic zero. Similar to toplogy, the associated FGLs to algebraic K-theory and to Chow groups are the multiplicative (i.e. $x+y-\beta xy$) and the additive one.

In a series of articles including [PW], Panin and Walter have introduced and studied *symplectic oriented cohomology theories* in algebraic geometry, which come with a good theory of Borel classes $b_1, b_2, ...$ (sometimes also called Pontryagin classes) for symplectic vector bundles. They construct MSL^{**} as the universal example, and also discuss the example of hermitian K-theory (aka higher Grothendieck Witt groups) in detail. Chow-Witt groups are yet another example [HW], and all oriented cohomology theories are also symplectically oriented. Unlike for MU^* or Ω^*, the coefficient ring MSp^* is only known rationally even in topology, but it's 2-primary torsion is known only in low degrees, see [Ko, section 8] for computations of $MSp_{\leq 100}$. (I thank Achim Krause for pointing out this reference after my talk.)

Refining FGLs, one would like to assign an algebraic structure to symplectic oriented cohomology theories, but there are two obstructions: The product $(E,\phi)\otimes(F,\psi)$ of two symplectic plane bundles has rank 4 rather than 2, and its fibers no longer carry a symplectic=anti-symmetric form but a symmetric one. The idea, which first appears in unpublished notes of Walter and then was studied more seriously in [DF], is to take as input the triple product of such rank 2 bundles, as output the four non-zero classes $b_1, ..., b_4$, and then to study the corresponding algebraic structure. Working over the base ring $GW(\mathbb{Z}) = \mathbb{Z}_\epsilon = \mathbb{Z}[\epsilon]/(\epsilon^2-1)$, this leads to the following definition:

Definition 1. Let R be a commutative $\mathbb{Z}_\epsilon$-algebra. A 4-valued formal ternary law, FTL for short, with coefficients in R is a $(4,3)$ power series with coefficients in R

$$F_t(x,y,z) = 1 + F_1(x,y,z)t + F_2(x,y,z)t^2 + F_3(x,y,z)t^3 + F_4(x,y,z)t^4$$

satisfying the following properties:

(1) *Neutral element.* The $(4,1)$-series $F_t(x,0,0)$ is split with roots x and $-\epsilon.x$ each with multiplicity 2, *i.e.* $F_t(x,0,0) = (1+xt)^2(1-\epsilon xt)^2$.
(2) *Symmetry/Commutativity.* The element $F_t(x,y,z)$ of $R[[x,y,z]][t]$ is a fixed point under permutation of the variables x,y,z.
(3) *Associativity.* Given formal variables (x,y,z,u,v), one has the following equality
$$F_t\big(F_t(x,y,z),u,v\big) = F_t\big(x,F_t(y,z,u),v\big)$$
where we have used a suitable substitution operation.
(4) ϵ*-Linearity.* We have
$$F_t(-\epsilon x,y,z) = F_{-\epsilon t}(x,y,z).$$
(5) *Geometricity.* We have
$$F_4(x,x,0) = 0.$$

Explicitly, we may write
$$F_t(x,y,z) = 1 + \sum_{i,j,k\geq 0, 1\leq l\leq 4} a^l_{ijk} x^i y^j z^k t^l$$
where we put the element $a^l_{i,j,k}$ in degree $i+j+k-l$.

Arguing similarly as in the case of FGLs, one can show that the free $\mathbb{Z}_\epsilon$-algebra generated by all $a^l_{i,j,k}$ and relations according to the above constraints yield the universal FTL over a certain universal ring $\mathcal{W}$.

Spelling out the above associativity condition leads to 16 equations. In practice, one often thinks of FTLs in terms of its four t-linear roots, applying the usual algeraic splitting principle. Also note that setting $\epsilon = -1$ means to look at the "oriented" or "complex" part.

Defining morphisms by power series, we obtain the categories of 4-valued FTLs.

With the help of a computer, we made several computations on the coefficient ring in the universal case in degrees ≤ 2. In the much easier case where we invert 2, this reduces to the following in non-positive degrees:

Theorem 2. *Let $\mathcal{W}^{\leq 0}[\frac{1}{2}]$ be the sub-ring of $\mathcal{W}[\frac{1}{2}]$ generated by variables of non-positive degree. Then, one has*
$$\mathcal{W}^{\leq 0}[\frac{1}{2}] \simeq \mathbb{Z}_\epsilon[\frac{1}{2}][a^3_{111}]/\langle (a^3_{111}-40)(1-\epsilon), (a^3_{111}-8)(a^3_{111}+8)(1+\epsilon)\rangle$$
with coefficients
$$\begin{array}{llll} a^1_{100} = 2(1-\epsilon) & & & \\ a^2_{200} = 2(1-2\epsilon) & a^2_{110} = 2(1-\epsilon) & & \\ a^3_{300} = 2(1-\epsilon) & a^3_{210} = -2(1-\epsilon) & a^3_{111} & \\ a^4_{400} = 1 & a^4_{310} = -2(1-\epsilon) & a^4_{220} = 2(1-2\epsilon) & a^4_{211} = 2(1-\epsilon). \end{array}$$

Some time ago, Buchstaber (see e.g. [Bu]) studied another refinement of FGLs, namely 2-valued FGLs. These may be essentially classified in those of type I

– having applications in topology – and type II. We establish explicit functors between these three categories as follows

$$\begin{tikzcd} FGL \arrow[rr, "W"] \arrow[dr, "B"'] & & FTL \\ & (2-FGL)_I \arrow[ur, "C"'] & \end{tikzcd}$$

and study their properties. For example, the functor N is neither full nor faithful. We also show that the FTL of Chow-Witt groups is not in the image of the functor C, even when refining everything over $\mathbb{Z}_\epsilon$. Further details may be found in the recent preprint [CDFH].

References

[Bu] V. M. Buchtaber, *Two-valued formal groups. Algebraic theory and applications to cobordism. I*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 1044–1064.

[CDFH] D. Coulette, F. Déglise, J. Fasel and J. Hornbostel *Formal ternary laws and Buchstaber's 2-groups*, Preprint 2021, arXiv:2112.03646.

[DF] F. Déglise and J. Fasel, *The Borel character*, to appear in the Journal of the IMJ.

[FH] J. Fasel and O. Haution, *The stable Adams operations on Hermitian K-theory*, Preprint (revised) 2022, arXiv:2005.08871.

[HW] J. Hornbostel and M. Wendt Chow-Witt rings of classifying spaces for symplectic and special linear groups, J. Topol. **12** (2019), 916–966.

[Ko] S. O. Kochman *Symplectic cobordism and the computation of stable stems*, Mem. Amer. Math. Soc. **104** (1993).

[LM] M. Levine and F. Morel,textitAlgebraic cobordism, Springer Monographs in Mathematics, 2007.

[PW] I. Panin and C. Walter, *Quaternionic Grassmannians and Borel classes in algebraic geometry*, Algebra i Analiz **33** (2021), 136–193.

Motivic stable homotopy theory is $\mathbb{Z}$-linear at the characteristic

Tom Bachmann

1. Statement of result

Recall that if $\mathcal{C}$ is a symmetric monoidal, semiadditive ∞-category, then the endomorphism space of the unit $\mathrm{Map}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ upgrades to an $\mathcal{E}_\infty$-ring space.

Theorem. *There is a (canonical) morphism of $\mathcal{E}_\infty$-ring spaces*

$$\mathbb{Z} \to \mathrm{Map}_{\mathcal{SH}(\mathbb{F}_p)_{(p)}}(\mathbb{1}, \mathbb{1}).$$

Here by $\mathbb{Z}$ I denote the usual (discrete, commutative) ring and $\mathcal{SH}(\mathbb{F}_p)_{(p)}$ denotes the p-local motivic stable homotopy category of the field with p elements (see e.g. [BH21, §4.1] for a quick definition).

An immediate amplification is the following: if X is an $\mathbb{F}_p$-scheme and $E, F \in \mathcal{SH}(X)_{(p)}$, then $\mathrm{Map}(E, F)$ is a $\mathbb{Z}$-module.

Example. We have $K_*(\mathbb{F}_p)_{(p)} \simeq \mathbb{Z}_{(p)}$, by famous work of Quillen. In particular $K(\mathbb{F}_p) \simeq \mathrm{Map}_{\mathcal{SH}(\mathbb{F}_p)_{(p)}}(\mathbb{1}, \mathrm{KGL}_{(p)})$ is a $\mathbb{Z}$-module, as needed.

The theorem says that, in some sense, motivic homotopy theory at the characteristic "has no homotopy theory left", that is, essentially belongs to the realm of homological algebra. Unfortunately I know of no concretely useful application of this fact.

Remark. The following stronger result is true (and may be proved by similar methods): if $E \in \mathcal{SH}(X)_{(p)}$ (where X is an $\mathbb{F}_p$-scheme) is a *normed spectrum* in the sense of [BH21], then $\mathrm{Map}(\mathbb{1}, E)$ is an animated commutative ring.

2. Backstory

The so-called *Hopkins–Morel conjecture* states that over any scheme S, the canonical map

$$\mathrm{MGL}/(x_1, x_2, \dots) \to \mathrm{H}\mathbb{Z} \in \mathcal{SH}(S)$$

is an equivalence. Here MGL denotes Voevodsky's algebraic cobordism spectrum, $\mathrm{H}\mathbb{Z}$ denotes Spitzweck's motivic cohomology spectrum, and $x_1, x_2, \dots$ are polynomial generators of the Lazard ring. Work of Spitzweck [Spi18] shows that it suffices to prove this over prime fields, and Hoyois has shown [Hoy15] that over fields, the result holds after inverting the exponential characteristic. Thus the only remaining case is to show that

$$\mathrm{MGL}/(p, x_1, x_2, \dots) \xrightarrow{\simeq} \mathrm{H}\mathbb{Z}/p \in \mathcal{SH}(\mathbb{F}_p).$$

Let us explore for a moment some consequences of this (particular case of the) conjecture. Further work of Spitzweck [Spi10] determines all the slices of MGL/p. Using the theorem of Geisser–Levine [GL00], one deduces that the slice and postnikov filtrations coincide on MGL/p, and consequently

$$\pi_{**}(\mathrm{MGL}/p) \simeq L_* \otimes \Omega^*_{log}$$

(placed in some appropriate degrees that I shall not elaborate on here). Using the motivic Adams–Novikov spectral sequence one deduces further that

$$\pi_{**}(\mathbb{1}_{(p)})(\mathbb{F}_p) \simeq \mathrm{Ext}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*)_{(p)}.$$

Making the degrees on the right hand side explicit, one finds in particular that $\pi_{*,0}(\mathbb{1}_{(p)})(\mathbb{F}_p) \simeq \mathbb{Z}_{(p)}$, and so $\mathrm{Map}_{\mathcal{SH}(\mathbb{F}_p)_{(p)}}(\mathbb{1}, \mathbb{1}) \simeq \mathbb{Z}_{(p)}$. Thus the main theorem follow from the Hopkins–Morel conjecture.

3. Proof ideas

(More details can be found in [Bac22].)

3.1. Given an ∞-category $\mathcal{C}$ with pullbacks, write $\mathrm{Span}(\mathcal{C})$ for the span category. That is, objects in $\mathrm{Span}(\mathcal{C})$ are the same as objects in $\mathcal{C}$, but morphisms are given by spans

$$\mathrm{Map}_{\mathrm{Span}(\mathcal{C})}(X, Y) = \{X \leftarrow T \rightarrow Y\};$$

composition is by pullback. The category $\mathrm{Span}(\mathrm{Fin})$ encodes the theory of commutative monoids, in the sense that (1) its nonabelian derived category $\mathcal{P}_\Sigma(\mathrm{Fin})$ is equivalent to the category of $\mathcal{E}_\infty$-monoids, and (2) given any presentably symmetric monoidal, semiadditive ∞-category $\mathcal{C}$, there is a unique left adjoint symmetric monoidal functor

$$c : \mathcal{P}_\Sigma(\mathrm{Fin}) \rightarrow \mathcal{C}.$$

The right adjoint c^* of c is lax symmetric monoidal (c being symmetric monoidal) and hence we obtain

$$c^*(\mathbb{1}) \in \mathrm{CAlg}(\mathcal{P}_\Sigma(\mathrm{Fin})).$$

This is nothing but $\mathrm{Map}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$, viewed as an $\mathcal{E}_\infty$-ring.

3.2. Note that even though Fin is a 1-category, $\mathrm{Span}(\mathrm{Fin})$ is a $(2, 1)$-category (the mapping spaces are groupoids). For example $\mathrm{Map}_{\mathrm{Span}(\mathrm{Fin})}(*, *)$ is just the groupoid of finite sets and isomorphisms. On the other hand, there is a natural equivalence between the *homotopy category* of $\mathrm{Span}(\mathrm{Fin})$ and the category of finitely generated, free (discrete) abelian monoids. From this one deduces easily that the main theorem is equivalent to factoring the functor

$$c : \mathcal{P}_\Sigma(\mathrm{Fin}) \rightarrow \mathcal{SH}(\mathbb{F}_p)_{(p)}$$

through the homotopy category of the source.

3.3. In order to do this, we use ∞-categories enriched in $\mathcal{P}(\mathrm{Sm}_{\mathbb{F}_p})$. We can view $\mathcal{SH}(\mathbb{F}_p)_{(p)}$ as enriched by declaring that

$$\underline{\mathrm{Map}}(E, F)(X) = \mathrm{Map}_{\mathcal{SH}(X)_{(p)}}(E_X, F_X).$$

Denote the resulting enriched ∞-category by $\underline{\mathcal{SH}(\mathbb{F}_p)_{(p)}}$. Write FEt_S for the category of finite étale schemes over S. By a similar construction as for $\mathcal{SH}(\mathbb{F}_p)$, we obtain an enriched ∞-category $\underline{\mathrm{Span}(\mathrm{FEt}_{\mathbb{F}_p})}$. Note that for example

$$\underline{\mathrm{Map}}_{\underline{\mathrm{Span}(\mathrm{FEt}_{\mathbb{F}_p})}}(*, *) \simeq L_\Sigma \coprod_{n \geq 0} B_{et}\Sigma_n. \tag{1}$$

The category $\underline{\mathrm{Span}(\mathrm{FEt}_{\mathbb{F}_p})}$ is an enriched avatar of $\mathrm{Span}(\mathrm{Fin})$. It will thus suffice to construct a (sufficiently canonical) functor

$$\underline{c} : \underline{\mathrm{Span}(\mathrm{FEt}_{\mathbb{F}_p})} \rightarrow \underline{\mathcal{SH}(\mathbb{F}_p)_{(p)}}$$

and the factor it through the homotopy category of the source. The construction of the functor uses the so-called *ambidexterity property* of $\mathcal{SH}(-)$, and I shall not elaborate on it here further. The produce the factorization, observe that all mapping presheaves in $\underline{\mathcal{SH}(\mathbb{F}_p)_{(p)}}$ are p-local motivic infinite loop spaces. Thus if

$$L : \mathcal{P}(\mathrm{Sm}_{\mathbb{F}_p}) \to \mathcal{P}(\mathrm{Sm}_{\mathbb{F}_p})$$

denotes the Bousfield localization at maps inverted by the p-local infinite suspension spectrum functor, then all mapping spaces in $\underline{\mathcal{SH}(\mathbb{F}_p)_{(p)}}$ are automatically L-local. Since the functor L preserves finite products, it is possible to apply it to all the mapping spaces in an ∞-category enriched in $\mathcal{P}(\mathrm{Sm}_S)$ and obtain a new enriched category. By what we just said, doing this to $\overline{\mathcal{SH}(\mathbb{F}_p)_{(p)}}$ has no effect. Thus it will be enough to prove that when applying this to $\overline{\underline{\mathrm{Span}(\mathrm{FEt}_{\mathbb{F}_p})}}$, we obtain a discrete category. By a slight generalization of Equation $\overline{(1)}$, for this it suffices to show the following. If G is a finite constant group, then

$$LB_{et}G = * \in \mathcal{P}(\mathrm{Sm}_{\mathbb{F}_p}).$$

The case where G is a p-groups is a folklore result of Morel–Voevodsky (in this case $B_{et}G$ is even $\mathbb{A}^1$-contractible); the passage to all finite groups is via a transfer argument (and this is why in L we built in p-local stabilization).

References

[Bac22] Tom Bachmann. Motivic stable homotopy theory is strictly commutative at the characteristic. arXiv:2111.02320.

[BH21] Tom Bachmann and Marc Hoyois. Norms in motivic homotopy theory. *Astérisque*, 425, 2021. arXiv:1711.03061.

[GL00] Thomas Geisser and Marc Levine. The k-theory of fields in characteristic p. *Inventiones mathematicae*, 139(3):459–493, 2000.

[Hoy15] Marc Hoyois. From algebraic cobordism to motivic cohomology. *Journal für die reine und angewandte Mathematik*, 2015(702):173–226, 2015.

[Spi10] Markus Spitzweck. Relations between slices and quotients of the algebraic cobordism spectrum. *Homology, Homotopy and Applications*, 12(2):335–351, 2010.

[Spi18] Markus Spitzweck. A commutative $\mathbb{P}^1$-spectrum representing motivic cohomology over Dedekind domains. *Memoires Soc. Math. France*, 157, 2018.

Milnor excision for motivic spectra

Marc Hoyois

(joint work with Elden Elmanto, Ryomei Iwasa, Shane Kelly)

The title of the talk refers to the following theorem, where $\mathrm{SH}(A)$ denotes the Morel–Voevodsky stable ∞-category of motivic spectra over A:

Main Theorem ([4]). *Let $A \to B$ be a morphism of commutative rings mapping an ideal $I \subset A$ isomorphically onto an ideal $J \subset B$. Then the following square of ∞-categories is cartesian:*

$$\begin{array}{ccc} \mathrm{SH}(A) & \longrightarrow & \mathrm{SH}(B) \\ \downarrow & & \downarrow \\ \mathrm{SH}(A/I) & \longrightarrow & \mathrm{SH}(B/J). \end{array}$$

In particular, every motivic spectrum $E \in \mathrm{SH}(A)$ gives rise to a cartesian square of spectra

$$\begin{array}{ccc} R\Gamma(\operatorname{Spec} A, E) & \longrightarrow & R\Gamma(\operatorname{Spec} B, E) \\ \downarrow & & \downarrow \\ R\Gamma(\operatorname{Spec} A/I, E) & \longrightarrow & R\Gamma(\operatorname{Spec} B/J, E). \end{array}$$

Examples of such morphisms $(A, I) \to (B, J)$ include the coordinate axes

$$(k[x, y]/(xy), (x)) \to (k[x], (x)),$$

the "Rim square"

$$(\mathbb{Z}[x]/(x^p - 1), (x - 1)) \to (\mathbb{Z}[\zeta_p], (\zeta_p - 1)),$$

and the desingularization of an affine curve. Another key example, first considered by Huber and Kelly [5], is the localization map $(V, \mathfrak{p}) \to (V_{\mathfrak{p}}, \mathfrak{p}V_{\mathfrak{p}})$ for $\mathfrak{p}$ a prime ideal in a valuation ring V.

Examples of cohomology theories $R\Gamma(\,\cdot\,, E)$ defined on all schemes include:

- $\mathbb{A}^1$-invariant algebraic K-theory $\mathrm{KH} = L_{\mathbb{A}^1}\mathbb{K}$, where $\mathbb{K}$ denotes localizing (i.e., nonconnective) K-theory (Weibel).
- $\mathbb{A}^1$-invariant motivic cohomology (Spitzweck), which is known to agree with the cdh motivic cohomology of Elmanto-Morrow in equicharacteristic (and is expected to in general).
- $\mathbb{A}^1$-invariant symmetric Grothendick-Witt theory $L_{\mathbb{A}^1}\mathbb{GW}^{\mathrm{s}}$, where $\mathbb{GW}^{\mathrm{s}}$ denotes the localizing Grotendieck-Witt theory of homotopy-symmetric forms (Calmès-Harpaz-Nardin).
- Étale cohomology with coefficients in $\mathcal{F}$, where $\mathcal{F}$ is a torsion étale sheaf of abelian groups over a scheme S, whose torsion is coprime to the residual characteristics of S.

The main theorem follows from Theorems A and B below using ideas of Bhatt and Mathew [1].

Theorem A ([4])**.** *Let V be a valuation ring and $\mathfrak{p} \subset V$ a prime ideal. Then the following square of ∞-categories is cartesian:*

$$\begin{array}{ccc} \mathrm{SH}(V) & \longrightarrow & \mathrm{SH}(V_{\mathfrak{p}}) \\ \downarrow & & \downarrow \\ \mathrm{SH}(V/\mathfrak{p}) & \longrightarrow & \mathrm{SH}(\kappa(\mathfrak{p})). \end{array}$$

Using the localization property and the finitary nature of SH, Theorem A is reduced to the following statement: if V has finite rank and

$$\operatorname{Spec} V_{\mathfrak{p}} - \{\mathfrak{p}\} \xrightarrow{h} \operatorname{Spec} V_{\mathfrak{p}} \xrightarrow{j} \operatorname{Spec} V$$

are the canonical open immersions, then the natural transformation

$$j_! h_! \to j_* h_! \colon \mathrm{SH}(\operatorname{Spec} V_{\mathfrak{p}} - \{\mathfrak{p}\}) \to \mathrm{SH}(\operatorname{Spec} V)$$

is an isomorphism. Using the cdh descent of motivic spectra proved by Cisinski, this is in turn implied by the same statement for $\mathrm{SH}_{\mathrm{cdh}}$, which is the analogue of

SH defined using the cdh site instead of the smooth Nisnevich site. The point is that pushforwards along quasi-compact open immersions are compatible with the cdh topology, allowing us to further reduce the statement to the level of presheaves. There it becomes an easy direct computation in light of the following fact, which uses that V is a valuation ring in an essential way: if X is a connected scheme, the image of any map $X \to \operatorname{Spec} V$ is an interval in the specialization poset.

Theorem B ([3])**.** *Let X be a qcqs scheme. Then the homotopy dimension of the cdh ∞-topos of X is at most the valuative dimension of X.*

Recall that an ∞-topos has *homotopy dimension* $\leq d$ if every d-connective sheaf admits a global section. This implies in particular that abelian cohomology vanishes in degrees $> d$ (since cohomology classes of degree $n+1$ classify n-gerbes, which are n-connective). The *valuative dimension* of a scheme is a variation of the Krull dimension introduced by Jaffard, which is surprisingly well-behaved for non-noetherian schemes. For an integral scheme X with fraction field K, $\mathrm{vdim}(X)$ is the supremum of the lengths of chains of valuation subrings of K centered on X, and one can extend this notion to arbitrary schemes by taking the supremum over all irreducible components. The valuative dimension agrees with the Krull dimension for locally noetherian schemes as well as for valuation rings, but it has the advantage of being a birational invariant in general.

The proof of Theorem B uses the Riemann-Zariski space $\mathrm{RZ}(X)$ of an integral scheme X, which is limit of all blow-ups of X (one can first reduce to the integral case using the fact that the space of generic points of a qcqs scheme is totally separated). The canonical map $p\colon \mathrm{RZ}(X) \to X$ induces a geometric morphism of ∞-topoi

$$p^*\colon \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}_X^{\mathrm{fp}}) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{RZ}(X)).$$

The valuative dimension of X turns out to be the Krull dimension of $\mathrm{RZ}(X)$, which by a theorem of Clausen and Mathew is an upper bound for the homotopy dimension of $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{RZ}(X))$ [2]. Hence any $\mathrm{vdim}(X)$-connective cdh sheaf on X has a section over some blow-up of X, which can be descended to a section over X using cdh descent and the induction hypothesis.

References

[1] B. Bhatt, A. Mathew, *The arc topology*, Duke Math. J. **170** (2021), no. 9, 1899–1988

[2] D. Clausen, A. Mathew, *Hyperdescent and étale K-theory*, Invent. Math. **225** (2021), 981–1076

[3] E. Elmanto, M. Hoyois, R. Iwasa, S. Kelly, *Cdh descent, cdarc descent, and Milnor excision*, Math. Ann. **379** (2021), 1011–1045

[4] E. Elmanto, M. Hoyois, R. Iwasa, S. Kelly, *Milnor excision for motivic spectra*, to appear in J. reine angew. Math. **779** (2021), 223–235

[5] A. Huber, S. Kelly, *Differential forms in positive characteristic, II: cdh-descent via functorial Riemann-Zariski spaces*, Algebra Number Theory **12** (2018), no. 3, 649–692

The even filtration after J. Hahn, A. Raksit, and D. Wilson

Benjamin Antieau

In forthcoming work [5], Jeremy Hahn, Arpon Raksit, and Dylan Wilson introduce a filtration on any $\mathbf{E}_\infty$-ring R. Their filtration recovers many filtrations in the literature, including the double-speed décalage of the Adams–Novikov filtration [4], the Hochschild–Kostant–Rosenberg filtration on Hochschild homology for quasi-lci rings [1], the Bhatt–Morrow–Scholze filtration on p-adic THH for p-quasisyntomic rings [3], and the filtration of Morin on integral THH of quasi-lci rings with bounded torsion [6].

1. The even filtration, variants

An $\mathbf{E}_\infty$-ring R is **even** if its odd-degree homotopy groups vanish. Examples of even $\mathbf{E}_\infty$-rings include the complex cobordism spectrum MU, the connective complex K-theory spectrum ku, or the Eilenberg–Mac Lane spectrum associated to any discrete commutative ring. Non-examples include the sphere spectrum $\mathbf{S}$ and the connective real K-theory spectrum ko.

The **even filtration** $\mathrm{F}^\star_{\mathrm{ev}}R$ on an $\mathbf{E}_\infty$-ring R is defined to be the limit over all maps $R \to A$, where A is an even $\mathbf{E}_\infty$-ring, of $\tau_{\geq 2\star}A$, the double-speed Postnikov, or Whitehead, filtration of A. The result is a multiplicative decreasing filtered $\mathbf{E}_\infty$-ring spectrum, i.e., an $\mathbf{E}_\infty$-algebra object in filtered spectra.

Alternatively, the even filtration $\mathrm{F}^\star_{\mathrm{ev}}R$ is obtained by **right Kan extension** of the natural inclusion functor from even $\mathbf{E}_\infty$-rings into all $\mathbf{E}_\infty$-rings along the map $\tau_{\geq 2\star}$ from even $\mathbf{E}_\infty$-rings to filtered $\mathbf{E}_\infty$-rings. This definition possibly produces an object in a larger set-theoretic universe, although in all cases considered in this abstract, the limit is small if R is.

The even filtration is always complete: $\mathrm{F}^\infty_{\mathrm{ev}}R = \lim_{n\mapsto\infty} \mathrm{F}^n_{\mathrm{ev}}R \simeq 0$. It is not currently known if it is always exhaustive; instead, there is a canonical map $R \to \mathrm{F}^{-\infty}_{\mathrm{ev}}R$. In the main cases of interest in this abstract, this map is an equivalence.

If R is connective, then the even filtration on R is concentrated in non-negative weights in the sense that $\mathrm{gr}^i_{\mathrm{ev}}R \simeq 0$ for $i < 0$.

There are variants where one works in p-complete $\mathbf{E}_\infty$-rings, or in $\mathbf{E}_\infty$-rings with S^1-action, or in cyclotomic $\mathbf{E}_\infty$-rings, or with various combinations of these. An important example is the even filtration on the homotopy fixed points of an $\mathbf{E}_\infty$-ring R with S^1-action. This filtration $\mathrm{F}^\star_{\mathrm{ev},S^1}R$ on R^{hS^1} is defined as the limit over $R \to A$ of $\tau_{\geq 2\star}(A^{hS^1})$, where A is an even $\mathbf{E}_\infty$-ring with S^1-action and the map $R \to A$ is S^1-equivariant. Note that this construction is not typically the same as applying homotopy fixed points to the S^1-equivariant even filtration on R itself. Similarly, one has $\mathrm{F}^\star_{\mathrm{ev},tS^1}R$, a filtration on R^{tS^1}. If R is an $\mathbf{E}_\infty$-algebra in p-complete cyclotomic spectra, there is a filtration $\mathrm{F}^\star_{\mathrm{ev},p,\mathrm{cyc}}R$ on $\mathrm{TC}(R)$ obtained by taking a limit over maps to even $\mathbf{E}_\infty$-algebras A in p-complete cyclotomic spectra of $\mathrm{fib}\left(\tau_{\geq 2\star}(A^{hS^1}) \xrightarrow{\mathrm{can}-\varphi} \tau_{\geq 2\star}(A^{tS^1})\right)$. Note that while A^{hS^1} and A^{tS^1} are even if A is, this is not necessarily the case for $\mathrm{TC}(A)$. For example, $\mathrm{THH}(\mathbf{F}_p)$ is even, but $\pi_{-1}\mathrm{TC}(\mathbf{F}_p)$ is non-zero.

Hahn, Raksit, and Wilson introduce motivic filtrations on various homological invariants as follows:

- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{HH}(R/k) = \mathrm{F}^\star_{\mathrm{ev}}\mathrm{HH}(R/k)$,
- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{HC}^-(R/k) = \mathrm{F}^\star_{\mathrm{ev},hS^1}\mathrm{HH}(R/k)$,
- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{HP}(R/k) = \mathrm{F}^\star_{\mathrm{ev},tS^1}\mathrm{HH}(R/k)$,
- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{THH}(R) = \mathrm{F}^\star_{\mathrm{ev}}\mathrm{THH}(R)$,
- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{TC}^-(R) = \mathrm{F}^\star_{\mathrm{ev},hS^1}\mathrm{THH}(R)$, and
- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{TP}(R) = \mathrm{F}^\star_{\mathrm{ev},tS^1}\mathrm{THH}(R)$

with p-complete versions

- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{HH}(R/k;\mathbf{Z}_p) = \mathrm{F}^\star_{\mathrm{ev},p}\mathrm{HH}(R/k;\mathbf{Z}_p)$,
- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{HC}^-(R/k;\mathbf{Z}_p) = \mathrm{F}^\star_{\mathrm{ev},p,hS^1}\mathrm{HH}(R/k;\mathbf{Z}_p)$,
- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{HP}(R/k;\mathbf{Z}_p) = \mathrm{F}^\star_{\mathrm{ev},p,tS^1}\mathrm{HH}(R/k;\mathbf{Z}_p)$,
- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{THH}(R;\mathbf{Z}_p) = \mathrm{F}^\star_{\mathrm{ev},p}\mathrm{THH}(R;\mathbf{Z}_p)$,
- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{TC}^-(R;\mathbf{Z}_p) = \mathrm{F}^\star_{\mathrm{ev},p,hS^1}\mathrm{THH}(R;\mathbf{Z}_p)$,
- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{TP}(R;\mathbf{Z}_p) = \mathrm{F}^\star_{\mathrm{ev},p,tS^1}\mathrm{THH}(R;\mathbf{Z}_p)$, and
- $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{TC}(R;\mathbf{Z}_p) = \mathrm{F}^\star_{\mathrm{ev},p,\mathrm{cyc}}\mathrm{THH}(R;\mathbf{Z}_p)$.

2. Even faithfully flat maps

A map $R \to S$ of $\mathbf{E}_\infty$-rings is **evenly faithfully flat**, or **eff**, if for every map $R \to A$ of $\mathbf{E}_\infty$-rings, where A is even, the pushout $A \otimes_R S$ is even and $\pi_*(A \otimes_R S)$ is faithfully flat as an ungraded $\pi_* A$-module.

A warning: the notion of faithful flatness used here is *not* the usual notion of faithful flatness in derived algebraic geometry. The latter asks for $\pi_0 R \to \pi_0 S$ to be faithfully flat as a map of ordinary commutative rings and for the natural maps $\pi_i R \otimes_{\pi_0 R} \pi_0 S \to \pi_i S$ to be isomorphisms for all $i \in \mathbf{Z}$.

There are obvious p-complete variants, which are left to the reader to consider.

The main technical theorem on even faithful flatness, which applies as well to the variants, is that the even filtration descends along eff maps

Theorem 2.1 (HRW). *If $R \to S$ is eff and $S^\bullet$ denotes the Čech/descent/Amitsur complex of the map, then $\mathrm{F}^\star_{\mathrm{ev}}R \simeq \mathrm{Tot}(\mathrm{F}^\star_{\mathrm{ev}}S^\bullet)$.*

Proof. Let $\mathrm{Aff}^{\mathrm{ev}}_{\mathrm{eff}}$ be the site consisting of the opposite of the category $\mathrm{CAlg}^{\mathrm{ev}}$ of even $\mathbf{E}_\infty$-rings, equipped with the even faithfully flat topology: covers consist of finite collections $\{A \to B_i\}$ of maps of even $\mathbf{E}_\infty$-rings such that $A \to \prod_i B_i$ is even faithfully flat. One proves, using ordinary faithfully flat descent, that the even filtration defines a sheaf $\mathrm{F}^\star_{\mathrm{ev}}\mathcal{O}$ of filtered $\mathbf{E}_\infty$-rings on $\mathrm{Aff}^{\mathrm{ev}}$. For any $\mathbf{E}_\infty$-ring R, there is a sheaf of spaces $\mathrm{Spev}(R)$ on $\mathrm{Aff}^{\mathrm{ev}}$ given by $(\mathrm{Spev}(R))(A) \simeq \mathrm{Map}_{\mathbf{E}_\infty}(R, A)$. The even filtration on R is obtained by evaluating $\mathrm{F}^\star_{\mathrm{ev}}\mathcal{O}(\mathrm{Spev}(R))$. The theorem follows from the fact that if $R \to S$ is even faithfully flat, then $\mathrm{Spev}(S) \to \mathrm{Spev}(R)$ is an epimorphism (in the sense of ∞-topoi), so $\mathrm{F}^\star_{\mathrm{ev}}(\mathrm{Spev}(R))$ can be computed using the Čech cover of the map of sheaves, which is checked to be Spev of $S^\bullet$. □

3. Comparison theorems

Hahn, Raksit, and Wilson prove several comparison theorems which relate their even filtration to filtrations in the literature.

3.1. **ANSS.** Let $\mathbf{S} \to \mathrm{MU}^\bullet$ be the augmented cosimplicial $\mathbf{E}_\infty$-ring obtained by taking the descent complex of $\mathbf{S} \to \mathrm{MU}$.

Theorem 3.1 ("Novikov descent", HRW). *For any $\mathbf{E}_\infty$-ring R, the natural map*

$$\mathrm{F}^\star_{\mathrm{ev}}R \simeq \mathrm{Tot}(\mathrm{F}^\star_{\mathrm{ev}}(R \otimes_{\mathbf{S}} \mathrm{MU}^\bullet)).$$

Proof. It is enough to check for $R = \mathbf{S}$, in which case the question comes down to showing that $\mathbf{S} \to \mathrm{MU}$ is even faithfully flat. But, if A is an even $\mathbf{E}_\infty$-ring, then, $\pi_*(A \otimes_{\mathbf{S}} \mathrm{MU}) \cong \pi_* A[b_1, b_2, \ldots]$ where the b_i have even degree. □

There are variants for rings with S^1 action obtained by giving MU the trivial S^1 action.

Corollary 3.2 (HRW). *There is an equivalence $\mathrm{F}^\star_{\mathrm{ev}}\mathbf{S} \simeq \mathrm{Tot}(\tau_{\geq 2\star}\mathrm{MU}^\bullet)$.*

Proof. In this case, each term of $\mathrm{MU}^\bullet$ is itself even, so the even filtration is the double-speed Postnikov filtration. □

Corollary 3.3 (HRW). *If $R \otimes_{\mathbf{S}} \mathrm{MU}$ is even, then $\mathrm{F}^\star_{\mathrm{ev}}R$ is the (double-speed) décalage of the Adams–Novikov filtration.*

For any R, $\mathrm{F}^\star_{\mathrm{ev}}R$ is an $\mathrm{F}^\star_{\mathrm{ev}}\mathbf{S}$-algebra. The stable ∞-category of p-complete $\mathrm{F}^\star_{\mathrm{ev}}\mathbf{S}$-module has been studied before in [4, 7] under the name **synthetic spectra**. It is shown in these sources that p-complete $\mathrm{F}^\star_{\mathrm{ev}}\mathbf{S}$-modules are equivalent to p-complete cellular motivic spectra over $\mathbf{C}$. It follows that $\mathrm{F}^\star_{\mathrm{ev}}R$ gives a natural synthetic (motivic) analogue of R for any $\mathbf{E}_\infty$-ring R.

3.2. **HKR.** The even filtration also recovers various "HKR" filtrations. A map $k \to R$ of commutative rings is quasi-lci if the algebraic cotangent complex $\mathrm{L}_{R/k}$ has Tor-amplitude contained in $[0, 1]$. Examples include smooth maps or local complete intersection morphisms. Besides the HKR filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R/k)$ on $\mathrm{HH}(R/k)$, there are so-called Beilinson filtrations $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k)$ and $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(R/k)$. These filtrations have been studied for many years in the rational setting. They were defined by [3] in the p-complete case and by [1] in the integral case.

Theorem 3.4 (HRW). *If $k \to R$ is a quasi-lci, then*

1. $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{HH}(R/k) \simeq \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R/k)$,
2. $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{HC}^-(R/k) \simeq \mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k)$, *and*
3. $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{HP}(R/k) \simeq \mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(R/k)$.

In particular,

(1) $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{HH}(R/k) \simeq \mathrm{L}\Omega^i_{R/k}[i]$,
(2) $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{HC}^-(R/k) \simeq \widehat{\mathrm{L}\Omega}^{\geq i}_{R/k}[2i]$, *and*
(3) $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{HP}(R/k) \simeq \widehat{\mathrm{L}\Omega}_{R/k}[2i]$,

where $\widehat{\mathrm{L}\Omega}_{R/k}$ *denotes Hodge-complete derived de Rham cohomology.*

Proof. The idea is to reduce to showing that the natural map $\mathrm{HH}(\mathbf{Z}[x]/\mathbf{Z}) \to \mathbf{Z}[x]$ is even faithfully flat when x is a degree 0 generator. Then, one checks in this case directly that one obtains the HKR filtration. The reduction goes by (1) showing that if $S \to R$ is a quotient of S where $\mathrm{L}_{R/S}$ has Tor-amplitude in $[1,1]$, then $\mathrm{HH}(R/S)$ is even and (2) showing that if $k \to S \to R$ is a factorization of $k \to R$ into a smooth map followed by a quotient as above, then $\mathrm{HH}(R/k) \to \mathrm{HH}(R/S)$ is eff. □

An analogous result holds in the p-complete case for p-quasisyntomic k-algebras R.

By shearing down and remembering the sheared down of the circle action, one obtains crystalline cohomology from the HKR filtration. In particular, there is a close connection between the site of even animated k-algebras A with S^1-action and non-S^1-equivariant maps $R \to A$ and divided power thickenings of A.

3.3. **BMS2.** The even filtration recovers various motivic filtrations on THH and p-complete THH of reasonable commutative rings studied in [3] by Bhatt, Morrow, and Scholze. Write BMS to denote these filtrations.

Theorem 3.5 (HRW). *If R is p-quasisyntomic, then*

(1) $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{THH}(R;\mathbf{Z}_p) \simeq \mathrm{F}^\star_{\mathrm{BMS}}\mathrm{THH}(R;\mathbf{Z}_p)$,
(2) $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{TC}^-(R;\mathbf{Z}_p) \simeq \mathrm{F}^\star_{\mathrm{BMS}}\mathrm{TC}^-(R;\mathbf{Z}_p)$,
(3) $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{TP}(R;\mathbf{Z}_p) \simeq \mathrm{F}^\star_{\mathrm{BMS}}\mathrm{TP}(R;\mathbf{Z}_p)$, *and*
(4) $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{TC}(R;\mathbf{Z}_p) \simeq \mathrm{F}^\star_{\mathrm{BMS}}\mathrm{TC}(R;\mathbf{Z}_p)$,

In particular,

(1) $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{THH}(R;\mathbf{Z}_p) \simeq \mathrm{gr}^i_{\mathrm{N}}\Delta_R[2i]$,
(2) $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{TC}^-(R;\mathbf{Z}_p) \simeq \mathrm{N}^{\geq i}\widehat{\Delta}_R\{i\}[2i]$,
(3) $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{TP}(R;\mathbf{Z}_p) \simeq \widehat{\Delta}_R\{i\}[2i]$, *and*
(4) $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{TC}(R;\mathbf{Z}_p) \simeq \mathbf{Z}_p(i)(R)$,

where $\widehat{\Delta}_R$ *is absolute prismatic cohomology completed with respect to the Nygaard filtration* $\mathrm{N}^{\geq\star}\Delta_R$ *and* $\mathbf{Z}_p(i)(R)$ *is the* i*th syntomic complex of* R.

The even filtration also recovers the motivic filtration on integral variants studied by Morin in [6] and discovered independently by Bhatt–Lurie [2]. Write MBL for these filtrations.

Theorem 3.6 (HRW). *If R is quasi-lci over $\mathbf{Z}$ with bounded p-torsion for all prime numbers p, then*

(1) $\mathrm{F}^{\star}_{\mathrm{mot}}\mathrm{THH}(R) \simeq \mathrm{F}^{\star}_{\mathrm{MBL}}\mathrm{THH}(R)$,
(2) $\mathrm{F}^{\star}_{\mathrm{mot}}\mathrm{TC}^{-}(R) \simeq \mathrm{F}^{\star}_{\mathrm{MBL}}\mathrm{TC}^{-}(R)$, *and*
(3) $\mathrm{F}^{\star}_{\mathrm{mot}}\mathrm{TP}(R) \simeq \mathrm{F}^{\star}_{\mathrm{MBL}}\mathrm{TP}(R)$.

4. Chromatically quasisyntomic $\mathbf{E}_\infty$-rings

Hahn, Raksit, and Wilson also introduce a new class of $\mathbf{E}_\infty$-rings for which the even filtration on TC is particularly well-behaved. Time did not permit an extended discussion of this point, but I did state their main theorem.

Definition 4.1 (HRW)**.** A map $k \to R$ of $\mathbf{E}_\infty$-rings is **quasi-lci** if the algebraic cotangent complex $\mathrm{L}_{\pi_* R/\pi_* k}$ has Tor-amplitude in $[0,1]$. A map $k \to R$ of $\mathbf{E}_\infty$-rings is **chromatically quasi-lci** if $k \otimes_{\mathbf{S}} \mathrm{MU} \to R \otimes_{\mathbf{S}} \mathrm{MU}$ is a quasi-lci map of *even* $\mathbf{E}_\infty$-rings. An $\mathbf{E}_\infty$-ring R is chromatically quasisyntomic if $R \otimes_{\mathbf{S}} \mathrm{MU}$ is even, $\mathbf{Z} \to \pi_*(R \otimes_{\mathbf{S}} \mathrm{MU})$ is quasi-lci, and $\pi_*(R \otimes_{\mathbf{S}} \mathrm{MU})$ has bounded p-power torsion for all prime numbers p. The ∞-category of chromatically quasisyntomic $\mathbf{E}_\infty$-rings is written as $\mathrm{XQSyn} \subseteq \mathrm{CAlg}$.

For example, ku is chromatically quasisyntomic.

Theorem 4.2 (HRW)**.** *If* $R \in \mathrm{XQsyn}$*, then the natural fiber sequence*

$$\mathrm{TC}(R;\mathbf{Z}_p) \to \mathrm{TC}^{-}(R;\mathbf{Z}_p) \to \mathrm{TP}(R;\mathbf{Z}_p)$$

upgrades canonically to a filtered fiber sequence

$$\mathrm{F}^{\star}_{\mathrm{mot}}\mathrm{TC}(R;\mathbf{Z}_p) \to \mathrm{F}^{\star}_{\mathrm{mot}}\mathrm{TC}^{-}(R;\mathbf{Z}_p) \to \mathrm{F}^{\star}_{\mathrm{mot}}\mathrm{TP}(R;\mathbf{Z}_p).$$

The non-trivial point of theorem is that the motivic filtrations on $\mathrm{TC}^{-}(R;\mathbf{Z}_p)$ and $\mathrm{TP}(R;\mathbf{Z}_p)$ are defined only using the circle action on $\mathrm{THH}(R;\mathbf{Z}_p)$. It is something of a miracle, already exploited by [3], that the corresponding motivic filtrations are preserved by the $\mathrm{can} - \varphi$ map used to define $\mathrm{TC}(R;\mathbf{Z}_p)$. It also follows that the graded pieces of the motivic filtrations on $\mathrm{TC}(R;\mathbf{Z}_p)$ form some kind of analogue of syntomic cohomology for ring spectra.

References

[1] B. Antieau, *Periodic cyclic homology and derived de Rham cohomology*, Ann. K-Thy. **4** (2019), no. 3, 505–519.
[2] B. Bhatt and J. Lurie, *Absolute prismatic cohomology*, arXiv preprint arXiv:2201.06120.
[3] B. Bhatt, M. Morrow, and P. Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, Pub. Math. IHES **129** (2019), 199–310.
[4] B. Gheorghe, D. Isaksen, A. Krause, and N. Ricka, *C-motivic modular forms*, arXiv preprint arXiv:1810.11050.
[5] J. Hahn, A. Raksit, and D. Wilson, *The even filtration*, forthcoming.
[6] B. Morin, *Topological Hochschild homology and zeta-values*, arXiv preprint arXiv:2011.11549.
[7] P. Pstrągowski, *Synthetic spectra and the cellular motivic category*, arXiv preprint arXiv:1803.01804.

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