

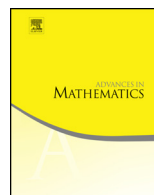


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An Additivity Theorem for the interchange of E_n structures

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ABSTRACT

Let \mathcal{A} and \mathcal{B} be operads and let X be an object with an \mathcal{A} -algebra and a \mathcal{B} -algebra structure. These structures are said to interchange if each operation $\alpha : X^n \rightarrow X$ of the \mathcal{A} -structure is a homomorphism with respect to the \mathcal{B} -structure and vice versa. In this case the combined structure is codified by the tensor product $\mathcal{A} \otimes \mathcal{B}$ of the two operads. There is not much known about $\mathcal{A} \otimes \mathcal{B}$ in general, because the analysis of the tensor product requires the solution of a tricky word problem.

Intuitively one might expect that the tensor product of an E_k -operad with an E_l -operad (which encode the multiplicative structures of k -fold, respectively l -fold loop spaces) ought to be an E_{k+l} -operad. However, there are easy counterexamples to this naive conjecture. In this paper we essentially solve the word problem for the nullary, unary, and binary operations of the tensor product of arbitrary topological operads and show that the tensor product of a cofibrant E_k -operad with a cofibrant E_l -operad is an E_{k+l} -operad. It follows that if \mathcal{A}_i are E_{k_i} operads for $i = 1, 2, \dots, n$, then $\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$ is at least an $E_{k_1 + \dots + k_n}$ operad, i.e. there is an $E_{k_1 + \dots + k_n}$ -operad \mathcal{C} and a map of operads $\mathcal{C} \rightarrow \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$.

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1. Introduction

Two algebraic structures are said to interchange if the structure maps of one structure are homomorphisms with respect to the second structure and vice versa. A precise definition will be given below.

Interchange features are abundant in algebra, category theory, algebraic topology and related fields. A well-known exercise in introductory algebra is to show that two interchanging group structures coincide and are abelian group structures. In iterated loop space theory interchanging loop structures provide rich algebraic structures. In the theory of n -categories the interchange of the various category structures is of central interest, and one of the main problems in the search for the “best” notion of a weak n -category is the determination of the “right” notion of interchange.

In the present paper we address the interchange of E_n structures extending a program originally suggested by J.M. Boardman [5] in the context of a recognition principle for n -fold loop spaces. This program has experienced a revival of interest for various reasons: In connection with the research on weak n -categories, A_∞ -categories, and Segal categories the question of the uniqueness of n -fold delooping machines for $1 \leq n \leq \infty$ has become of interest again. The solution offered in [15] has gaps. The analysis of the delooping problem brings up the question of interchanging E_1 structures, usually called A_∞ structures.

Kontsevich’s generalization of Deligne’s Hochschild cohomology conjecture to algebras over the little n -cubes operad and the dual problem about the multiplicative structure of (topological) Hochschild homology lead directly into interchange problems of E_n structures and A_∞ structures.

A space X has two interchanging structures encoded by operads \mathcal{B} and \mathcal{C} iff it has a structure encoded by the tensor product $\mathcal{B} \otimes \mathcal{C}$. Our main result is

Additivity Theorem. *If \mathcal{B} is an E_k operad and \mathcal{C} is an E_l operad and both are cofibrant, then $\mathcal{B} \otimes \mathcal{C}$ is an E_{k+l} operad.*

Our choice of weak equivalences are maps of operads $f : \mathcal{B} \rightarrow \mathcal{C}$ such that each map $f(n) : \mathcal{B}(n) \rightarrow \mathcal{C}(n)$ is a Σ_n -equivariant homotopy equivalence, i.e. we work with the Strøm model structure on the underlying collections. It is not known that the category \mathcal{Opr} of operads carries a model structure with these weak equivalences; nevertheless one can define the term “cofibrant” by a lifting property and there is a good cofibrant replacement functor (see [28] and the discussion in [4, Section 8.1]).

If we replace the Strøm model structure on the category of collections by the Quillen model structure, \mathcal{Opr} has a model structure [3] and the Additivity Theorem also holds in this context.

If we drop the cofibrancy condition in the theorem, then $\mathcal{B} \otimes \mathcal{C}$ is at least E_{k+l} , which means that there is a E_{k+l} operad \mathcal{D} and a map of operads $\mathcal{D} \rightarrow \mathcal{B} \otimes \mathcal{C}$. But there might be an E_r operad \mathcal{D}' with $r > k + l$ and a map of operads $\mathcal{D}' \rightarrow \mathcal{B} \otimes \mathcal{C}$.

Versions of the Additivity Theorem have been known in special cases and have been addressed in other contexts. E.g. Dunn [14] proved that the E_k operads \mathcal{D}_k of decomposable little k -cubes satisfy additivity on the nose:

$$\mathcal{D}_k \otimes \mathcal{D}_l \cong \mathcal{D}_{k+l}.$$

From this he deduced that the n -fold tensor product $\mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_1$ of the little 1-cubes operad is an E_n operad. In his thesis [9] Brinkmeier extended this result to n -fold tensor products of little k -cubes operads: $\mathcal{C}_{k_1} \otimes \dots \otimes \mathcal{C}_{k_n}$ is an $E_{k_1+\dots+k_n}$ operad. These results do not imply the general Additivity Theorem because the tensor product is **not** homotopy invariant, i.e. it does not preserve weak equivalences of operads.

In his book project “Higher Algebra” Lurie introduces a tensor product of ∞ -operads and proves a version of Brinkmeier’s result in his setting [20, Section 5.1.2]. This tensor product seems to be related to ours but the precise relationship has not been worked out yet.

Moerdijk and Weiss extended the tensor product of operads to dendroidal sets [22–24, 29], and suggested an investigation of an Additivity Theorem for n copies of the dendroidal set $N_d(\mathcal{A}ss)$ associated with the operad $\mathcal{A}ss$ of monoid structures. But while $\mathcal{A}ss \otimes \mathcal{A}ss \cong \mathcal{C}om$ (see Corollary 3.9 for more details), the structure of the dendroidal tensor product $N_d(\mathcal{A}ss) \otimes N_d(\mathcal{A}ss)$ is not clear.

Recent work of Cisinski and Moerdijk [11–13] and Heuts, Hinich and Moerdijk [17] relates the dendroidal sets world with that of Lurie. We hope that the methods presented in our paper will be of help in understanding the precise relation of their tensor products with the interchange of structures.

Our tensor product of operads, which encodes interchanging structures on the nose, is quite elusive, and during our work on this paper we often fell into traps. In Section 3 we will list some surprising examples which will give an indication that we have to solve a non-trivial word problem.

The strategy of the proof of the Additivity Theorem is as follows: For a particular choice of universal E_k and E_l operads $\mathcal{B} = W|\mathcal{N}\mathcal{M}_k|$ and $\mathcal{C} = W|\mathcal{N}\mathcal{M}_l|$ we cover the tensor product $\mathcal{B} \otimes \mathcal{C}$ by suitable contractible subsets and relate the nerve of this covering to \mathcal{M}_{k+l} . Here \mathcal{M}_k is the *Cat* operad which was analyzed in [1] and shown to be an E_k operad, \mathcal{N} is the nerve functor, and W is a cofibrant replacement functor.

The paper is organized as follows: In Sections 2 and 3 we recall the definitions of A_∞ and E_k operads, of interchange and of the tensor product of operads. As mentioned above, Section 3 also contains some surprising examples of tensor products. Our main results and a recollection of the operad \mathcal{M}_k follow in Section 4. In Section 5 we explain the strategy of the proof of the Additivity Theorem in greater detail. Section 6 deals with the unary and binary operations in the tensor product of arbitrary operads.

The forgetful functor U from reduced operads to topological monoids has a right adjoint R , and we call a Σ -free operad \mathcal{B} *axial* if the unit $\mathcal{B}(n) \rightarrow RU(\mathcal{B})(n)$ is a closed cofibration for each n . This property is crucial in our proof. Therefore we study the

adjoint pair U and R in Section 7. In Section 8 we recall the definition of the cofibrant replacement functor W . We prove the axiomaticity of $W|\mathcal{N}\mathcal{M}_k|$ and related properties used in the proof of the main result. In the remaining sections we define a cover of $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ by contractible cells, describe the diagram of intersections of these cells, and construct the homotopy equivalences mentioned above.

We dedicate this paper to the memory of Roland Schwänzl [26], our dear friend and collaborator, who started off with us on this project but tragically succumbed to a fatal illness at an early stage of its development.

2. E_n structures and A_∞ structures

E_n structures are closely related to the algebraic structure of an n -fold loop space. They are best described using operads.

2.1 Definition. Let \mathcal{S} be a symmetric monoidal category with multiplication \times . An *operad* \mathcal{B} in \mathcal{S} is a collection $\{\mathcal{B}(k)\}_{k \geq 0}$ of objects equipped with symmetric group actions $\mathcal{B}(k) \times \Sigma_k \rightarrow \mathcal{B}(k)$, composition maps

$$\mathcal{B}(k) \times (\mathcal{B}(j_1) \times \dots \times \mathcal{B}(j_k)) \rightarrow \mathcal{B}(j_1 + \dots + j_k),$$

and a unit $id \in \mathcal{B}(1)$ satisfying the appropriate equivariance, associativity and unitality conditions – see [21] for details.

2.2 Remark. Throughout this paper \mathcal{S} will be \mathcal{Cat} , the category of small categories, \mathcal{Sets} , the category of sets, \mathcal{SSets} , the category of simplicial sets, or \mathcal{Top} , the category of k -spaces and continuous maps.

In all three cases the symmetric monoidal structure is given by the product and the categories are complete and cocomplete. Moreover, they are self-enriched and the product distributes over the coproduct. In particular, we can define the \mathcal{S} -endomorphism operad \mathcal{E}_X of an object X in \mathcal{S} by $\mathcal{E}_X(n) = \mathcal{S}(X^n, X)$ with the obvious Σ_n -action and the obvious composition maps and unit.

2.3 Definition. A topological operad will be called *well-pointed* if $\{id\} \subset \mathcal{B}(1)$ is a closed cofibration.

2.4 . While Definition 2.1 is the most common definition of an operad, it is often helpful to think of it in the following equivalent way, which is the original version from [6]. An operad \mathcal{B} in a symmetric monoidal category \mathcal{S} of Remark 2.2 is an \mathcal{S} enriched symmetric monoidal category $(\mathcal{B}, \oplus, 0)$ such that

- (i) $\text{ob } \mathcal{B} = \mathbb{N}$ and $m \oplus n = m + n$.
- (ii) \oplus is a strictly associative \mathcal{S} -functor with strict unit 0.

(iii)

$$\coprod_{r_1+\dots+r_n=k} \mathcal{B}(r_1, 1) \times \dots \times \mathcal{B}(r_n, 1) \times_{\Sigma_{r_1} \times \dots \times \Sigma_{r_n}} \Sigma_k \rightarrow \mathcal{B}(k, n)$$

$$((f_1, \dots, f_n), \tau) \longmapsto (f_1 \oplus \dots \oplus f_n) \circ \tau$$

is an isomorphism in \mathcal{S} .

In the topological case “well-pointed” translates to the assumption that $\{id\} \subset \mathcal{B}(1, 1)$ is a closed cofibration.

Using (iii), each operad in the sense of [Definition 2.1](#) determines a category as in [2.4](#). Conversely, given a category as in [2.4](#), we obtain an operad by taking the collection $\{\mathcal{B}(k, 1)\}_{k \geq 0}$.

For some inductive arguments we will use the following blown-up version of [2.4](#):

2.5 . Each operad \mathcal{B} gives rise to an \mathcal{S} enriched symmetric monoidal category $(\mathcal{B}, \oplus, 0)$ defined by

- (i) $\text{ob } \mathcal{B} = \{\text{totally ordered finite sets}\}$ and $S \oplus T = S \amalg T$, the ordered disjoint union.
- (ii) \oplus is a strictly associative \mathcal{S} -functor with strict unit $0 = \emptyset$.
- (iii) $\mathcal{B}(S, \{t\}) = \mathcal{B}(|S|)$, where $|S|$ is the cardinality of S .
- (iv)

$$\coprod_{S_1 \amalg \dots \amalg S_n = U} \mathcal{B}(S_1, \{t_1\}) \times \dots \times \mathcal{B}(S_n, \{t_n\}) \times_{\Sigma(S_1) \times \dots \times \Sigma(S_n)} \Sigma(U) \rightarrow \mathcal{B}(U, T)$$

$$((f_1, \dots, f_n), \tau) \longmapsto (f_1 \oplus \dots \oplus f_n) \circ \tau$$

is an isomorphism in \mathcal{S} , where $\Sigma(U)$ is the permutation group of (the underlying set of) U and $T = \{t_1 < \dots < t_n\}$.

We will also find it convenient to use this blown-up version in operad notation; e.g. $\mathcal{B}(S)$ stands for $\mathcal{B}(|S|)$.

2.6 Definition. Let \mathcal{B} and \mathcal{C} be \mathcal{S} -operads.

- (1) \mathcal{B} is called Σ -free if the Σ_n -action on $\mathcal{B}(n)$ is free for each n in the cases $\mathcal{S} = \text{Cat}$, Sets , or \mathcal{SSets} . If $\mathcal{S} = \text{Top}$ we require that $\mathcal{B}(n) \rightarrow \mathcal{B}(n)/\Sigma_n$ is a numerable principal Σ_n -bundle for each n .
- (2) An operad map $\mathcal{B} \rightarrow \mathcal{C}$ is a collection of equivariant maps $\mathcal{B}(n) \rightarrow \mathcal{C}(n)$ in \mathcal{S} , compatible with the operad structure.

- (3) A \mathcal{B} -structure on an object X in \mathcal{S} is an operad map $\mathcal{B} \rightarrow \mathcal{E}_X$ into the endomorphism operad \mathcal{E}_X of X . We say that \mathcal{B} acts on X , or that X is a \mathcal{B} -algebra; if $\mathcal{S} = \mathcal{Top}$ we also call X a \mathcal{B} -space.
- (4) An operad map is called a *weak equivalence* if each map $\mathcal{B}(n) \rightarrow \mathcal{C}(n)$ is an equivariant homotopy equivalence (in \mathcal{Cat} or $\mathcal{S}Sets$ this means that each map is an equivariant homotopy equivalence after applying the classifying space functor, respectively the topological realization functor).
- (5) Two operads are called *equivalent* if there is a finite chain of weak equivalences connecting them.
- (6) A topological operad is called E_n if it is equivalent to the little n -cubes operad \mathcal{C}_n . In particular, it is Σ -free, because \mathcal{C}_n is Σ -free.
- (7) An A_∞ operad is another term for an E_1 operad.
- (8) An operad \mathcal{B} is called *reduced* if $\mathcal{B}(0) = \{0\}$. We denote the categories of operads and reduced operads by \mathcal{Opr} and \mathcal{Opr}_0 respectively. Here we use the same notation for each of our categories \mathcal{S} ; it will be clear from the context which \mathcal{S} we mean.

Recall that $\mathcal{C}_n(k)$ is the space of k -fold configurations of subcubes of the unit cube I^n , whose axes are parallel to those of I^n and whose interiors are disjoint. Any n -fold loop space $\Omega^n Y$ has a natural action by this operad. Conversely, each path-connected space with a \mathcal{C}_n -structure is of the weak homotopy type of an n -fold loop space (cf. [6,7], and [21] for details).

2.7 Remark. As pointed out in the introduction our notion of E_n operads is not shared by all authors dealing with operads. Our basis is Strøm’s model structure on \mathcal{Top} [27], while most authors prefer the cofibrantly generated Quillen model structure. We will call our set-up the *Strøm environment* and – if not stated otherwise – we will work in this environment.

In the *Quillen environment* the weak equivalences are operad maps $f : \mathcal{B} \rightarrow \mathcal{C}$ such that each map $f(n) : \mathcal{B}(n) \rightarrow \mathcal{C}(n)$ is a weak homotopy equivalence of spaces; an operad \mathcal{B} is Σ -free if for each n the right action of Σ_n on $\mathcal{B}(n)$ is free; and \mathcal{B} is called E_n if there is a finite chain of Quillen weak equivalences of Σ -free operads joining \mathcal{B} with \mathcal{C}_n . In particular, each E_n operad in the Strøm environment is E_n in the Quillen environment.

2.8 Definition. Let S and $T = \{t_1, \dots, t_n\}$ be totally ordered finite sets and let \mathcal{B} be a reduced operad. We define *restriction maps*

$$-\cap S : \mathcal{B}(T) \rightarrow \mathcal{B}(S \cap T)$$

by composing with $(\varepsilon_1 \oplus \dots \oplus \varepsilon_n)$, where $\varepsilon_i = id$ if $t_i \in S$ and is 0 otherwise. We will also use this notation in related situations like products, sums, etc., of $\mathcal{B}(T)$ ’s.

We will also make use of the operads $\mathcal{A}ss$ and $\mathcal{C}om$ which encode the structures of a monoid and a commutative monoid respectively. By definition, $\mathcal{A}ss(n) = \Sigma_n$, where $\sigma \in \Sigma_n$ stands for the operation

$$(x_1, \dots, x_n) \mapsto x_{\sigma^{-1}(1)} \cdot \dots \cdot x_{\sigma^{-1}(n)}$$

in a monoid. From this the operad data for $\mathcal{A}ss$ can be deduced.

$\mathcal{C}om(n) = \{\lambda_n\}$ is a single point. Here λ_n stands for the operation

$$(x_1, \dots, x_n) \longmapsto x_1 + \dots + x_n.$$

An n -fold loop space $\Omega^n Y$ has n interchanging loop space structures. Since \mathcal{C}_n acts naturally on $\Omega^n Y$, these n interchanging structures should somehow be encoded in \mathcal{C}_n . Before we can make this precise we need to formally define the notion of interchange.

3. Interchange

Since this paper is about interchange of topological operad structures the operads in this section will be topological operads. The interested reader can easily make the necessary adjustments for our other categories \mathcal{S} .

Let X be a space with actions of operads \mathcal{B} and \mathcal{C} . Then the coproduct $\mathcal{B} \amalg \mathcal{C}$ of \mathcal{B} and \mathcal{C} in the category of operads acts on X .

3.1 Definition. We say that the \mathcal{B} - and \mathcal{C} -actions on X *interchange* if each operation $\beta : X^k \rightarrow X$, $\beta \in \mathcal{B}(k)$, in the \mathcal{B} -structure is a homomorphism of \mathcal{C} -spaces, and vice versa. Explicitly, this means that for each $\beta \in \mathcal{B}(k)$ and each $\gamma \in \mathcal{C}(l)$ the square

$$\begin{array}{ccccc}
 (X^k)^l & \xrightarrow[\cong]{\tau_{k,l}} & (X^l)^k & \xrightarrow{\gamma^k} & X^k \\
 \downarrow \beta^l & & & & \downarrow \beta \\
 X^l & \xrightarrow{\gamma} & & & X
 \end{array}$$

commutes, where $\tau_{kl} \in \Sigma_{k \cdot l}$ is the permutation which reorders the coordinates of $X^{k \cdot l}$ from lexicographical to reserve lexicographical order.

Note that the two composites $\beta \circ \gamma^k \circ \tau_{k,l}$ and $\gamma \circ \beta^l$ are elements in $(\mathcal{B} \amalg \mathcal{C})(k \cdot l)$. If $k > 0$ and $l > 0$, we may interpret the interchange condition as follows. Given a $k \times l$ array $\{x_{ij}\}_{i=1, j=1}^{i=k, j=l}$ of elements of X , we can apply β to the columns of the array and then γ to the resulting products. Alternatively we can apply γ to each row of the array, then β to the resulting products. The interchange condition states that we obtain the same final result either way.

3.2 Definition. The *tensor product* $\mathcal{B} \otimes \mathcal{C}$ of operads \mathcal{B} and \mathcal{C} is obtained from the co-product $\mathcal{B} \amalg \mathcal{C}$ by factoring out the interchange relation (Definition 3.1).

A more detailed description of $\mathcal{B} \amalg \mathcal{C}$ and $\mathcal{B} \otimes \mathcal{C}$ will be given in the beginning of Section 5.

3.3 Remark. In the case when $\beta \in \mathcal{B}(1)$, the interchange relation implies

$$\beta \circ \gamma = \gamma \circ \beta^l.$$

This relation and the dual relation when $\gamma \in \mathcal{C}(1)$ are called *unary interchanges* and will play an important role in our analysis of the tensor product of operads. More generally we shall refer to interchanges involving $\beta \in \mathcal{B}(k)$ and $\gamma \in \mathcal{C}(l)$ as *(k, l)-interchanges*.

By definition, a space X admits a \mathcal{B} -structure interchanging with a \mathcal{C} -structure iff it admits a $(\mathcal{B} \otimes \mathcal{C})$ -structure.

The following two results about tensor products of operads are extant in the literature:

3.4 Proposition. (See Dunn [14].) *There is a canonical weak equivalence of operads*

$$\mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_1 \longrightarrow \mathcal{C}_n \quad (n \text{ tensor factors}).$$

So we “recover” the n interchanging loop space structures in \mathcal{C}_n .

3.5 Proposition. (See [10].) *The operad $\text{Ass} \otimes \mathcal{C}_k$ is equivalent to \mathcal{C}_{k+1} .*

In view of these two results, we might conjecture that:

3.6 Naive Conjecture. *If \mathcal{B} is E_k and \mathcal{C} is E_l , then $\mathcal{B} \otimes \mathcal{C}$ is E_{k+l} .*

Unfortunately, the situation is not that simple: The functor $\mathcal{B} \otimes -$ does not preserve weak equivalences, and the structure of $\mathcal{B} \otimes \mathcal{C}$ is anything but clear. In general, one has to solve a substantial word problem. For instance unary operations in $\mathcal{B}(1)$ and $\mathcal{C}(1)$ may be factored in many different ways and these factors may then be redistributed in a complicated way using the unary interchange relations mentioned in Remark 3.3. The following examples illustrate some rather surprising consequences that these and other interchange relation may imply.

3.7 Proposition. (See [7, Lemma 2.23].) *Let \mathcal{B} and \mathcal{C} be operads such that $\mathcal{B}(0) \neq \emptyset \neq \mathcal{C}(0)$ and let \mathcal{B}' and \mathcal{C}' be their universal quotients with exactly one nullary operation. Then*

- (1) $(\mathcal{B} \otimes \mathcal{C})(0)$ contains exactly one element.
- (2) $\mathcal{B} \otimes \mathcal{C} \cong \mathcal{B}' \otimes \mathcal{C}'$.

Proof. Given $\beta \in \mathcal{B}(0)$ and $\gamma \in \mathcal{C}(0)$, the interchange relation gives

$$\begin{array}{ccccc}
 * = (X^0)^0 & \xrightarrow{\tau_{0,0}=id} & (X^0)^0 & \xrightarrow{\gamma^0=id} & X^0 = * \\
 \downarrow \beta^0=id & & & & \downarrow \beta \\
 * = X^0 & \xrightarrow{\gamma} & & & X
 \end{array}$$

Hence $\beta = \gamma$ in $\mathcal{B} \otimes \mathcal{C}$.

The second part is an immediate consequence of the first part. \square

3.8 Proposition. *Let \mathcal{B} and \mathcal{C} be operads such that $\mathcal{B}(1) = \{id\}$, $\mathcal{C}(1) = \{id\}$ and $\mathcal{B}(0)$, $\mathcal{B}(2)$, $\mathcal{C}(0)$, $\mathcal{C}(2)$ are not empty. Then $\mathcal{B} \otimes \mathcal{C} \cong \mathcal{C}om$.*

Proof. Since $\mathcal{B}(2)$ and $\mathcal{C}(2)$ are not empty, so are $\mathcal{B}(k)$ and $\mathcal{C}(k)$ for $k \geq 2$. Let $\lambda \in (\mathcal{B} \otimes \mathcal{C})(0)$ be the unique nullary operation and let X be a $(\mathcal{B} \otimes \mathcal{C})$ -space. For $\beta \in \mathcal{B}(k)$ and $\gamma \in \mathcal{C}(k)$ consider the following $(k \times k)$ -array of points in X , where $* = \lambda(*)$

$$\begin{array}{cccc}
 x_1 & * & \dots & * \\
 * & x_2 & \dots & * \\
 & & \dots & \\
 * & * & \dots & x_k
 \end{array}$$

Let β act horizontally on this array and γ vertically. By the interchange relation, the results should be the same if we compute the action first horizontally, then vertically or vice versa. Now

$$\beta(*, \dots, *, x_i, *, \dots, *) = \beta \circ (\lambda, \dots, \lambda, id, \lambda, \dots, \lambda)(x_i) = x_i$$

for $i = 1, \dots, k$, since $\beta \circ (\lambda, \dots, \lambda, id, \lambda, \dots, \lambda) \in \mathcal{B}(1) = \{id\}$. Hence computing the action first horizontally and then vertically gives $\gamma(x_1, \dots, x_k)$. Similarly computing it the other way we get $\beta(x_1, \dots, x_k)$, so that $\beta = \gamma$ in $\mathcal{B} \otimes \mathcal{C}$ for all β and γ . \square

3.9 Corollary.

- (i) *Suppose $\mu_1, \mu_2 : X^2 \rightarrow X$ are two (not necessarily associative) H -space structures on a topological space X . Suppose that μ_1 and μ_2 both have the same strict 2-sided unit $*$ and that μ_1 and μ_2 satisfy the $(2, 2)$ -interchange relation*

$$\mu_1(\mu_2(x_1, x_2), \mu_2(x_3, x_4)) = \mu_2(\mu_1(x_1, x_3), \mu_1(x_2, x_4)) \tag{E-H}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then $\mu_1 = \mu_2$ and this multiplication is strictly associative and commutative.

- (ii) More generally, if $\mathcal{B}(2) \times X^2 \rightarrow X$, $\mathcal{C}(2) \times X^2 \rightarrow X$, are continuous nonempty families of (not necessarily associative) multiplications on a topological space X , which have a common 2-sided unit $*$, and which satisfy the (2, 2)-interchange relations (E–H) for all $\mu_1 \in \mathcal{B}(2)$ and $\mu_2 \in \mathcal{C}(2)$, then $\mathcal{B}(2) = \mathcal{C}(2) = \{\mu\}$ and μ defines a commutative monoid structure on X .
- (iii) $Ass \otimes Ass \cong Ass \otimes Com \cong Com \otimes Com \cong Com$.

Corollary 3.9(i) with the additional hypotheses that μ_1 and μ_2 are both associative is well known to topologists as Eckmann–Hilton interchange [16]. Less known is the fact that the associativity hypotheses are superfluous.

Proof of Corollary 3.9. We first prove (i). Let \mathcal{B} be the operad generated by μ_1 and the 2-sided unit. Specifically $\mathcal{B}(0) = \{*\}$, $\mathcal{B}(1) = \{id\}$, $\mathcal{B}(2) = \{\mu_1, \mu_1 \circ (1\ 2)\}$, and for $k > 2$, $\mathcal{B}(k)$ consists of all k -fold iterates of μ_1 and their permutations. (For $k \geq 2$, the elements of $\mathcal{B}(k)$ are in 1–1 correspondence with planar binary trees with k labeled inputs and no stumps.) Similarly let \mathcal{C} be the operad generated by μ_2 and the 2-sided unit. Then \mathcal{B} and \mathcal{C} both act on X . We must show that these operad actions interchange.

The unary interchanges between \mathcal{B} and \mathcal{C} hold trivially since $\mathcal{B}(1) = \{id\} = \mathcal{C}(1)$. Similarly the nullary $(k, 0)$ - and $(0, l)$ -interchanges follow from $\mathcal{B}(0) = \{*\} = \mathcal{C}(0)$. Finally we obtain that all (2, 2)-interchanges hold by applying appropriate permutations to the interchange relation (E–H). Thus it remains to show that (k, l) -interchanges hold when $k > 2$ or $l > 2$.

Let $\beta_2 \in \mathcal{B}(2)$. Then β_2 defines an H -space structure on X and on all products X^l (via coordinatewise multiplication). From the already established (2, 2)-interchange relations, μ_2 is a homomorphism $X^2 \rightarrow X$ with respect to β_2 . Thus iterates of μ_2 are composites of products of β_2 -homomorphisms and thus are also homomorphisms. Clearly permutations of X^l are also homomorphisms. This establishes all $(2, l)$ -interchanges for all $l > 2$.

Now fix an element $\gamma_l \in \mathcal{C}(l)$. Then γ_l defines an l -ary operation on X and coordinatewise on all products X^k . From the already established $(2, l)$ -interchange relations, we obtain that μ_1 , its iterates and permutations thereof determine homomorphisms of these γ_l structures. This establishes all (k, l) -interchanges for $k > 2$.

Thus we obtain that the \mathcal{B} and \mathcal{C} actions on X interchange and thus determine a $\mathcal{B} \otimes \mathcal{C}$ action. But by Proposition 3.8 $\mathcal{B} \otimes \mathcal{C} \cong Com$. Thus $\mu_1 = \mu_2$ is strictly associative and commutative, establishing (i).

Part (ii) is an immediate consequence. For we can pick $\mu_1 \in \mathcal{B}(2)$ and $\mu_2 \in \mathcal{C}(2)$ and apply (i). We obtain that $\mu_1 = \mu_2$ is associative and commutative. Since μ_1 and μ_2 are arbitrary, (ii) follows.

Part (iii) is an immediate consequence of Proposition 3.8. \square

The same argument as in the proof of Corollary 3.9(i) establishes the following result.

3.10 Proposition. *Suppose that two operads \mathcal{B} and \mathcal{C} act on a space X . Suppose that the interchange relations hold between all the generating elements of \mathcal{B} and \mathcal{C} . Then the actions of \mathcal{B} and \mathcal{C} on X interchange.*

3.11 Remark. The tensor product of $\mathcal{A}ss$ with an operad \mathcal{B} satisfying $\mathcal{B}(0) = *$ has been determined in [8]. We will recall this result in Section 7 and deduce from it the structure of $Com \otimes \mathcal{B}$. In the meantime we point out another surprising consequence of that result.

3.12 Proposition. *If the bar construction $B_\bullet M$ on a well-pointed topological monoid M is an H -space in the category of simplicial topological spaces, then the geometric realization $|B_\bullet M|$ is homotopy equivalent to a loop space, as an H -space.*

Proof. As is shown in [7], loop space structures are parametrized by an operad $W\mathcal{A}ss$, a blown-up version of $\mathcal{A}ss$ (which we will discuss in Section 8 below). It is also shown there that H -space structures are parametrized by the suboperad $W^{(2)}\mathcal{A}ss$ of $W\mathcal{A}ss$ generated by the nullary, unary and binary operations $W\mathcal{A}ss(0)$, $W\mathcal{A}ss(1)$, and $W\mathcal{A}ss(2)$. Now a simplicial H -space structure on $B_\bullet M$ amounts to the same thing as an $\mathcal{A}ss \otimes W^{(2)}\mathcal{A}ss$ structure on M . But according to [8], the induced map

$$\mathcal{A}ss \otimes W^{(2)}\mathcal{A}ss \longrightarrow \mathcal{A}ss \otimes W\mathcal{A}ss$$

is an isomorphism of operads. Hence M is an $\mathcal{A}ss \otimes W\mathcal{A}ss$ algebra and it follows that $|B_\bullet M|$ has a $W\mathcal{A}ss$ -structure. Since $|B_\bullet M|$ is a path-connected Dold space [25, Cor. 5.2] this structure admits a homotopy inverse (e.g. [25, Prop. 3.16]). It is well known that a $W\mathcal{A}ss$ -space admitting a homotopy inverse is homotopy equivalent through homotopy homomorphisms to a loop space. \square

Since $\mathcal{A}ss$ is E_1 and Com certainly is not E_2 , Corollary 3.9 provides a counterexample to the Naive Conjecture 3.6. In fact, each connected abelian topological monoid is of the weak homotopy type of an infinite loop space, indeed equivalent to a product of Eilenberg–MacLane spaces. So we adjust the conjecture in the following way:

3.13 Definition. An operad \mathcal{B} is called *at least E_n* if there is an E_n operad \mathcal{C} and a map of operads $\mathcal{C} \rightarrow \mathcal{B}$. (So any \mathcal{B} space has an E_n structure.)

3.14 Conjecture. *If \mathcal{B} is E_k and \mathcal{C} is E_l , then $\mathcal{B} \otimes \mathcal{C}$ is at least E_{k+l} .*

3.15 Remark. Given maps of operads $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{A}$ with \mathcal{A} being E_n , then each \mathcal{B} -space is an \mathcal{A} -space and each \mathcal{A} -space is a \mathcal{B} -space. \mathcal{B} is at least E_n , and one might expect \mathcal{B} to be in fact E_n . This need not be true even if the composite $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{A}$ is the identity, as the following example shows:

$$\mathcal{C}_n \rightarrow \mathcal{C}_n \amalg \mathcal{C}_n \rightarrow \mathcal{C}_n$$

where the first map is the inclusion of the first summand and the second is the folding map. $\mathcal{C}_n \amalg \mathcal{C}_n$ codifies two non-interchanging \mathcal{C}_n -structures and is certainly not E_n : The operad $\pi_0(\mathcal{C}_n)$ of path components of \mathcal{C}_n is isomorphic to Com for $n \geq 2$. Hence there is a surjection (in fact, a bijection) $\pi_0(\mathcal{C}_n \amalg \mathcal{C}_n) \rightarrow Com \amalg Com$. Now $(Com \amalg Com)(2)$ has two elements, while $Com(2)$ has only one. So $\mathcal{C}_n \amalg \mathcal{C}_n$ has too many path components to be E_n . For $n = 1$ the same argument works if one replaces Com by Ass .

4. Main results

4.1 Convention. In view of [Proposition 3.7](#), we only work with *reduced* topological operads unless explicitly stated otherwise. So operad will mean reduced topological operad.

4.2 Definition. An operad \mathcal{B} in \mathcal{Opr}_0 is called *cofibrant* if for any diagram

$$\begin{array}{ccc}
 & & \mathcal{P} \\
 & & \downarrow p \\
 \mathcal{B} & \xrightarrow{f} & \mathcal{C}
 \end{array}$$

of operad maps with p a weak equivalence and each $p : \mathcal{P}(n) \rightarrow \mathcal{C}(n)$ an equivariant fibration, there is a lift $g : \mathcal{B} \rightarrow \mathcal{P}$ such that $p \circ g = f$.

It is shown in [\[7\]](#) that for any Σ -free well-pointed operad \mathcal{B} , there is a cofibrant operad $W\mathcal{B}$ and an operad equivalence $W\mathcal{B} \rightarrow \mathcal{B}$. We call $W\mathcal{B} \rightarrow \mathcal{B}$ a cofibrant resolution of \mathcal{B} . If \mathcal{B} fails to be well-pointed we add a whisker to $\mathcal{B}(1)$ to obtain a well-pointed operad \mathcal{B}' together with a weak equivalence $\mathcal{B}' \rightarrow \mathcal{B}$ (e.g. see [\[21, p. 167\]](#)) and take $W\mathcal{B}' \rightarrow \mathcal{B}$ as cofibrant resolution of \mathcal{B} .

In the Quillen environment \mathcal{Opr} has a model structure [\[3\]](#). As cofibrant replacement functor in this model structure for Σ -free operads in the Quillen sense we can take $WP\mathcal{B} \rightarrow \mathcal{B}$, where W is the construction of [\[4\]](#) or [\[7\]](#) (the two agree in \mathcal{Top}) and P is the usual CW -approximation functor.

The main result of our paper is the following.

4.3 Additivity Theorem. *If \mathcal{B} is an E_k operad and \mathcal{C} an E_l operad and both are cofibrant, then $\mathcal{B} \otimes \mathcal{C}$ is an E_{k+l} operad. This holds in the Strøm and the Quillen environment.*

This verifies the [Naive Conjecture 3.6](#) in the special case of cofibrant operads. It also verifies the modified [Conjecture 3.14](#). Indeed it is equally easy to prove a strengthened version of this conjecture.

4.4 Corollary. *Suppose that \mathcal{A}_i are at least E_{k_i} for $i = 1, 2, \dots, n$. Then the tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \dots \otimes \mathcal{A}_n$ is at least $E_{k_1+k_2+\dots+k_n}$.*

Proof. We proceed by induction on n . The result is trivially true for $n = 1$. Assume it holds for $n - 1$. Then there is an $E_{k_1+k_2+\dots+k_{n-1}}$ operad \mathcal{B} which maps into $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \dots \otimes \mathcal{A}_{n-1}$. Let \mathcal{C} be a E_{k_n} operad which maps into \mathcal{A}_n . Let $W\mathcal{B} \rightarrow \mathcal{B}$ and $W\mathcal{C} \rightarrow \mathcal{C}$ be cofibrant resolutions. Then according to [Theorem 4.3](#), $W\mathcal{B} \otimes W\mathcal{C}$ is $E_{k_1+k_2+\dots+k_n}$ and we have a chain of operad maps

$$W\mathcal{B} \otimes W\mathcal{C} \rightarrow \mathcal{B} \otimes \mathcal{C} \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \dots \otimes \mathcal{A}_n.$$

This completes the induction and the proof. \square

We will now make use of

4.5 Lemma. (*E.g. see [28].*) *Given a diagram of operads*

$$\begin{array}{ccc} & & \mathcal{P} \\ & & \downarrow p \\ \mathcal{B} & \xrightarrow{f} & \mathcal{C} \end{array}$$

with \mathcal{B} cofibrant and p a weak equivalence. Then there is a lift $g : \mathcal{B} \rightarrow \mathcal{P}$ up to homotopy, uniquely up to homotopy. (Here homotopy means homotopy through operad maps.)

4.6 Corollary.

- (i) *Any two cofibrant E_k operads \mathcal{B} and \mathcal{C} are homotopy equivalent in the strong sense; i.e. there are operad maps $f : \mathcal{B} \rightarrow \mathcal{C}$ and $g : \mathcal{C} \rightarrow \mathcal{B}$ such that $g \circ f \simeq id$ and $f \circ g \simeq id$ through operad maps.*
- (ii) *If \mathcal{A} is any operad and \mathcal{B} and \mathcal{C} are cofibrant E_k operads, then $\mathcal{A} \otimes \mathcal{B}$ and $\mathcal{A} \otimes \mathcal{C}$ are homotopy equivalent in the strong sense.*
- (iii) *If \mathcal{B} and \mathcal{C} are E_k and \mathcal{B} is cofibrant, there is a weak equivalence $\mathcal{B} \rightarrow \mathcal{C}$.*

In view of the corollary it suffices to prove [Theorems 4.3](#) for our favorite cofibrant operads. Moreover, the Additivity Theorem in the Quillen environment follows from the one in the Strøm environment: If \mathcal{B} is a cofibrant E_n operad in the Quillen environment then $W\mathcal{P}\mathcal{B}$ is cofibrant in both environments, and the canonical map $W\mathcal{P}\mathcal{B} \rightarrow \mathcal{B}$ is a homotopy equivalence of operads in the strong sense.

Our favorite cofibrant E_k operad will be $W|\mathcal{NM}_k|$, a cofibrant resolution of $|\mathcal{NM}_k|$, the topological realization of the nerve \mathcal{NM}_k of the *Cat*-operad \mathcal{M}_k which parametrizes the algebraic structure of a k -fold monoidal category [\[1\]](#). We can briefly describe $\mathcal{M}_k(m)$ as a poset whose objects are words of length m in the alphabet $\{1, 2, \dots, m\}$ combined together using k binary operations $\square_1, \square_2, \dots, \square_k$, which are strictly associative and have common unit 0. We moreover require that the objects of $\mathcal{M}_k(m)$ are precisely

those words where each generator $\{1, 2, \dots, m\}$ occurs exactly once. The morphisms of $\mathcal{M}_k(m)$ are generated by interchanges

$$\eta_{A,B,C,D}^{ij} : (A \square_j B) \square_i (C \square_j D) \longrightarrow (A \square_i C) \square_j (B \square_i D),$$

where $1 \leq i < j \leq k$ and A, B, C, D are words in $\{1, 2, \dots, m\}$ such that $(A \square_j B) \square_i (C \square_j D)$ (and hence also $(A \square_i C) \square_j (B \square_i D)$) represent objects of $\mathcal{M}_k(m)$. These interchange morphisms are then combined using the binary operations $\square_1, \square_2, \dots, \square_k$ as well as composition of morphisms. The coherence theorem of [1] proves the nonobvious fact that $\mathcal{M}_k(m)$ is a poset, i.e. there is at most one morphism between any two objects, and gives a simple algorithm for determining when there is a morphism between any two objects (cf. the proof of Lemma 5.5 below).

4.7 . The topological operads $|\mathcal{NM}_k|$ and $|\mathcal{NM}_l|$ are far from cofibrant. Indeed $|\mathcal{NM}_k|(1)$ and $|\mathcal{NM}_l|(1)$ are both $\{id\}$ and hence by Proposition 3.8 $|\mathcal{NM}_k| \otimes |\mathcal{NM}_l| = Com$. On the other hand, we will show that $W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|$ is E_{k+l} . In anticipation of this result, it will be convenient to adopt the following convention. We will identify \mathcal{M}_k with the obvious suboperad of \mathcal{M}_{k+l} and we will identify \mathcal{M}_l with the suboperad of \mathcal{M}_{k+l} obtained by shifting the indices on the binary operations from $\square_1, \square_2, \dots, \square_l$ to $\square_{k+1}, \square_{k+2}, \dots, \square_{k+l}$.

In the course of proving Theorem 4.3 we obtain more results about the spaces of unary and binary operations in an arbitrary tensor product of operads. As these results may be of separate interest, we state them below.

4.8 Proposition. *Let \mathcal{A} and \mathcal{B} be arbitrary topological operads with $\mathcal{A}(0) = \{0\}$ and $\mathcal{B}(0) = \{0\}$. Then*

- $(\mathcal{A} \otimes \mathcal{B})(1) \cong \mathcal{A}(1) \times \mathcal{B}(1)$
- $(\mathcal{A} \otimes \mathcal{B})(2)$ is homeomorphic to the pushout of the following diagram

$$\begin{array}{ccc} \mathcal{A}(2) \times \mathcal{B}(2) & \xrightarrow{f} & \mathcal{A}(1)^2 \times \mathcal{B}(2) \\ \downarrow g & & \\ \mathcal{A}(2) \times \mathcal{B}(1)^2 & & \end{array}$$

Here $f(\alpha, \beta) = (\alpha \circ (id \oplus 0), \alpha \circ (0 \oplus id), \beta)$ and $g(\alpha, \beta) = (\alpha, \beta \circ (id \oplus 0), \beta \circ (0 \oplus id))$.

To get a handle on the homotopy type of the space of binary operations, we need to impose additional hypotheses (which are satisfied by $\mathcal{A} = W|\mathcal{NM}_k|$ and $\mathcal{B} = W|\mathcal{NM}_l|$ in particular). We then obtain the following result.

4.9 Corollary. *Let \mathcal{A} and \mathcal{B} be arbitrary topological operads with $\mathcal{A}(0) = \{0\}$ and $\mathcal{B}(0) = \{0\}$. Suppose also that the spaces $\mathcal{A}(1)$ and $\mathcal{B}(1)$ are contractible and the map $\mathcal{A}(2) \rightarrow \mathcal{A}(1)^2$ given by $\alpha \mapsto (\alpha \circ (id \oplus 0), \alpha \circ (0 \oplus id))$ is a cofibration. Then $(\mathcal{A} \otimes \mathcal{B})(2)$ has the homotopy type of the join $\mathcal{A}(2) * \mathcal{B}(2)$.*

Proof. Because of the cofibration hypothesis, the pushout diagram for $(\mathcal{A} \otimes \mathcal{B})(2)$ has the same homotopy type as the homotopy pushout. Because $\mathcal{A}(1)$ and $\mathcal{B}(1)$ are contractible, this in turn has the same homotopy type as the homotopy pushout of

$$\begin{array}{ccc} \mathcal{A}(2) \times \mathcal{B}(2) & \xrightarrow{\text{pr}_2} & \mathcal{B}(2) \\ \downarrow \text{pr}_1 & & \\ \mathcal{A}(2) & & \end{array}$$

which is by definition the join $\mathcal{A}(2) * \mathcal{B}(2)$. \square

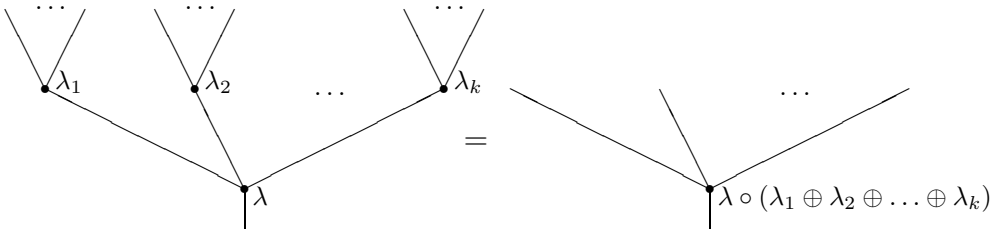
If \mathcal{A} is an E_k operad and satisfies the cofibration hypothesis of [Corollary 4.9](#), while \mathcal{B} is an arbitrary E_l operad, then $\mathcal{A}(2)$ has the homotopy type of S^{k-1} , and $\mathcal{B}(2)$ has the homotopy type of S^{l-1} . Thus by [Corollary 4.9](#), $(\mathcal{A} \otimes \mathcal{B})(2)$ has the homotopy type of $S^{k-1} * S^{l-1} \cong S^{k+l-1}$. This is consistent with $\mathcal{A} \otimes \mathcal{B}$ being an E_{k+l} operad, and suggests that [Theorem 4.3](#) might hold with a weaker hypothesis. However, we will not pursue this further in this paper.

5. Outline of the proofs

Before we discuss the proof of [Theorem 4.3](#), we need a more explicit description of the tensor product of operads. We begin with a description of the coproduct $\mathcal{B} \amalg \mathcal{C}$ of \mathcal{B} and \mathcal{C} . The elements of $(\mathcal{A} \amalg \mathcal{B})(m)$ are equivalence classes of planar trees with m labeled inputs and one output, and whose nodes are labeled by elements of the operads \mathcal{A} and \mathcal{B} . It is required that the arity of each node label correspond to the number of input branches coming into that node. So we have to allow nodes without inputs, which we call *stumps* labeled by elements in $\mathcal{A}(0)$ or $\mathcal{B}(0)$. For a more detailed description see [Definition 8.1](#). The topology imposed is the quotient topology on the evident disjoint union of products of spaces $\mathcal{A}(i)$ and $\mathcal{B}(j)$. Operad composition in $\mathcal{A} \amalg \mathcal{B}$ is given by splicing together trees.

The following equivalence relations are imposed on the trees. First of all if two trees have subtrees of the form shown below but are otherwise identical, then they are identified:

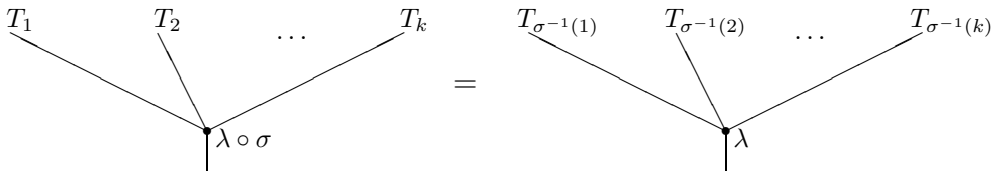
5.1 .



Here we are assuming that the node labels $\lambda, \lambda_1, \lambda_2, \dots, \lambda_k$ all belong to \mathcal{A} or all belong to \mathcal{B} , and $\lambda \circ (\lambda_1 \oplus \lambda_2 \oplus \dots \oplus \lambda_k)$ denotes operad composition.

Additionally we also allow changing unary nodes which are labeled by the unit of \mathcal{A} to unary nodes labeled by the unit of \mathcal{B} and vice versa.

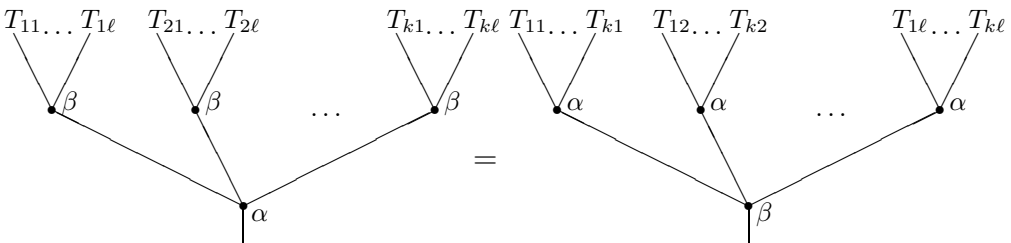
5.2 . We also need to impose an equivariance relation



Here we are assuming that the node label λ is either in $\mathcal{A}(k)$ or in $\mathcal{B}(k)$ and $\sigma \in \Sigma_k$.

The resulting spaces of equivalence classes of trees form the coproduct operad $\mathcal{A} \amalg \mathcal{B}$. The two equivalence relations insure that the images of \mathcal{A} and \mathcal{B} are suboperads of $\mathcal{A} \amalg \mathcal{B}$.

5.3 . To pass from the coproduct $\mathcal{A} \amalg \mathcal{B}$ to the tensor product we need to impose another relation corresponding to the interchange. We identify two trees if they have subtrees of the form shown below, but are otherwise identical:



Here $\alpha \in \mathcal{A}(k), \beta \in \mathcal{B}(\ell)$, and T_{ij} denote the branches of the trees lying above the nodes shown. Note that these branches are permuted on the two sides of the relation. The resulting spaces of equivalence classes of trees form the tensor product $\mathcal{A} \otimes \mathcal{B}$. For future reference, we will define the equivalence relation shown in the picture above as a (k, ℓ) -interchange on trees. Trees related by a sequence of $(k, 1)$ - or $(1, \ell)$ -interchanges will be said to be related by *unary interchanges* (cf. Remark 3.3).

5.4 Remark. According to Proposition 3.7, the nullary operations in \mathcal{A} and \mathcal{B} are identified to a single nullary operation in $\mathcal{A} \otimes \mathcal{B}$. This together with relation 5.1 allows us to remove any stumps from trees representing elements in $\mathcal{A} \otimes \mathcal{B}$. We shall refer to these relations as *nullary interchanges*.

In order to determine the homotopy type of a tensor product of operads, we need to describe its spaces of operations as colimits of “nice” diagrams. Since the operad \mathcal{M}_2 encodes the algebraic structure of a 2-fold monoidal category, i.e. a category with two multiplications which interchange with each other up to coherent natural transformations, it should not be surprising that \mathcal{M}_2 should serve as the basis for constructing such diagrams. However we need to make some adjustments.

First of all, to avoid confusion between the multiplications in \mathcal{M}_2 and the internal multiplications in $W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|$ arising from those in \mathcal{M}_k and \mathcal{M}_l , we will denote the multiplications in \mathcal{M}_2 by \boxtimes_1 and \boxtimes_2 instead of \square_1 and \square_2 .

5.5 Lemma. *The poset operad \mathcal{M}_2 has a quotient poset operad \mathcal{M}_2^{ab} , obtained from \mathcal{M}_2 by making the operations \boxtimes_1 and \boxtimes_2 commutative.*

Proof. In order to verify that making \boxtimes_1 and \boxtimes_2 commutative is compatible with the poset structure of $\mathcal{M}_2(m)$, we recall the criterion for the existence of a morphism $\alpha \rightarrow \beta$ in $\mathcal{M}_2(m)$. If $\{a, b\}$ is any two element subset of $\{1, 2, \dots, m\}$ and $\alpha \cap \{a, b\} = a \boxtimes_i b$, then either $\beta \cap \{a, b\} = a \boxtimes_j b$ for $j \geq i$ or $\beta \cap \{a, b\} = b \boxtimes_j a$ for $j > i$. [Here we use the notation for restriction maps introduced in Definition 2.8.] Using this criterion we see that the number of pairs $\{a, b\}$ for which $\alpha \cap \{a, b\} = a \boxtimes_2 b$ is less than or equal to the number of pairs $\{c, d\}$ for which $\beta \cap \{c, d\} = c \boxtimes_2 d$, and that the counts are equal iff $\alpha = \beta$. Making \boxtimes_1 and \boxtimes_2 commutative does not affect these counts, so the poset structure on $\mathcal{M}_2(m)$ induces a poset structure on $\mathcal{M}_2^{ab}(m)$. It is straightforward to verify that the operad structure on \mathcal{M}_2 passes to an operad structure on \mathcal{M}_2^{ab} . \square

5.6 Definition. We will not need to use the poset structure of \mathcal{M}_2^{ab} until Section 11. However we will need to use the underlying sets of objects of \mathcal{M}_2^{ab} in various constructions before that. Hence we will use the separate notation $\mathfrak{M}_2^{ab}(m)$ to denote the underlying set of objects of $\mathcal{M}_2^{ab}(m)$. Obviously the operad structure on \mathcal{M}_2^{ab} restricts to an operad structure on $\mathfrak{M}_2^{ab}(m)$.

5.7 Definition. We define a collection of simplicial complexes (in the classical sense of the term, rather than simplicial sets) $\{\mathfrak{K}_\bullet(m)\}_{m \geq 0}$ as follows. The vertex set of $\{\mathfrak{K}_\bullet(m)\}_{m \geq 0}$ is $\mathfrak{M}_2^{ab}(m)$. The complexes $\mathfrak{K}_\bullet(0)$ and $\mathfrak{K}_\bullet(1)$ are 0-dimensional consisting of a single vertex while $\mathfrak{K}_\bullet(2)$ is the simplicial complex consisting of the single 1-simplex $\{1 \boxtimes_1 2, 1 \boxtimes_2 2\}$ and its subsimplices. The complex $\mathfrak{K}_\bullet(3)$ has the vertex set

$$\begin{array}{cccc}
 (1 \boxtimes_2 2) \boxtimes_1 3 & & (2 \boxtimes_1 3) \boxtimes_2 1 & \\
 \\
 (1 \boxtimes_1 2 \boxtimes_1 3) & (1 \boxtimes_2 3) \boxtimes_1 2 & (1 \boxtimes_1 3) \boxtimes_2 2 & (1 \boxtimes_2 2 \boxtimes_2 3) \\
 \\
 (2 \boxtimes_2 3) \boxtimes_1 1 & & (1 \boxtimes_1 2) \boxtimes_2 3 &
 \end{array}$$

The simplices are those collections of vertices which contain at most one vertex from each column. For $m \geq 3$ the simplicial complex $\mathfrak{K}_\bullet(m)$ has as r -simplices those $(r + 1)$ -tuples $\{\alpha_0, \alpha_1, \dots, \alpha_r\}$ such that for each subset $\{a < b < c\} \subset \{1, 2, \dots, m\}$ the restrictions $\{\alpha_0 \cap \{a, b, c\}, \alpha_1 \cap \{a, b, c\}, \dots, \alpha_r \cap \{a, b, c\}\}$ form a simplex in $\mathfrak{K}_\bullet(\{a, b, c\})$, which is identified with $\mathfrak{K}_\bullet(3)$ via the isomorphism induced by $1 \mapsto a, 2 \mapsto b$ and $3 \mapsto c$.

5.8 Lemma. *For any topological operads \mathcal{A} and \mathcal{B} , there is a natural map of operads*

$$\varepsilon : \mathcal{A} \amalg \mathcal{B} \longrightarrow \mathfrak{M}_2^{ab}$$

Proof. To define ε , take a tree representative T of an element in $(\mathcal{A} \amalg \mathcal{B})(m)$. Replace each stump by 0 and delete each node with exactly one input by combining its input and output. Regard each node in T with more than 1 input and label coming from \mathcal{A} as an iterated multiplication \boxtimes_1 and label coming from \mathcal{B} as an iterated multiplication \boxtimes_2 . Interpret the edges of the resulting tree as compositions in the operad \mathfrak{M}_2^{ab} . This evaluated tree gives an element of $\mathfrak{M}_2^{ab}(m)$. It is obvious that this construction is compatible with the equivalence relations on trees which define the elements of $(\mathcal{A} \amalg \mathcal{B})(m)$. [Note that this construction does not give a well-defined map $\mathcal{A} \amalg \mathcal{B}$ to the set of objects of \mathfrak{M}_2 , since that would not be compatible with the equivariance relation on trees.] \square

5.9 . We observe that [Lemma 5.8](#) gives a trivial colimit decomposition of $(\mathcal{A} \amalg \mathcal{B})(m)$, namely as a disjoint union of $\varepsilon^{-1}(\alpha)$ indexed over all the elements of $\mathfrak{M}_2^{ab}(m)$. For future reference, we will use the notation $\tilde{G}_m(\alpha)$ to refer to $\varepsilon^{-1}(\alpha)$. The first step in obtaining a colimit decomposition for the tensor product is to observe that trees in $(\mathcal{A} \amalg \mathcal{B})(m)$ which are related by unary interchanges have the same image under ε , since $\mathfrak{M}_2^{ab}(1) = \{1 = id\}$. Thus for any object α in $\mathfrak{M}_2^{ab}(m)$ we define

$$G_m(\alpha) = \tilde{G}_m(\alpha)/\text{nullary and unary interchanges}$$

From now on we will take $\mathcal{A} = W|\mathcal{NM}_k|$ and $\mathcal{B} = W|\mathcal{NM}_l|$.

The proofs of the remaining statements are postponed to Sections [10](#) and [11](#).

5.10 Proposition. *For each element α in $\mathfrak{M}_2^{ab}(m)$, the natural map*

$$G_m(\alpha) \longrightarrow (W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m)$$

is a cofibration.

Thus $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ is a union of subspaces $G_m(\alpha)$ over all objects α in $\mathfrak{M}_2^{ab}(m)$. To determine the homotopy type $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$, we need to analyze the intersections of these closed subspaces. We would expect that these intersections should correspond to nonunary interchanges, which are encoded by the simplices of $\mathfrak{K}_\bullet(m)$.

5.11 Definition. The *interchange diagram* for $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ is the following diagram G_m indexed by $\mathcal{I}(m) = \text{Sd } \mathfrak{K}_\bullet(m)$, the barycentric subdivision of $\mathfrak{K}_\bullet(m)$ with poset structure opposite to the inclusions of the faces of $\mathfrak{K}_\bullet(m)$:

- To each vertex $\alpha \in \mathfrak{K}_0(m) = \mathfrak{M}_2^{ab}(m)$ assign $G_m(\alpha)$ to α .
- To each barycenter of a simplex in $\mathfrak{K}_\bullet(m)$ assign the intersection $\cap G_m(\alpha_i)$, where $\{\alpha_i\}$ are the vertices of that simplex.
- The maps in the diagram are inclusions.

5.12 Proposition. *The interchange diagram for $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ is a diagram of cofibrations, and $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ is the colimit of that diagram. Moreover each space in that diagram has the general form*

$$\left(\prod W|\mathcal{N}\mathcal{M}_k|(r_i) \times \prod W|\mathcal{N}\mathcal{M}_l|(s_j) \right) / \text{unary interchanges},$$

and the maps in the diagram are given by operad compositions (including insertion of the unique constant).

The operad spaces $|\mathcal{N}\mathcal{M}_k|(r)$ and $|\mathcal{N}\mathcal{M}_l|(s)$ can be described as colimits of diagrams of contractible spaces indexed by the posets $\mathcal{M}_k(r)$ and $\mathcal{M}_l(s)$, namely by assigning to each object in $\mathcal{M}_k(r)$ or $\mathcal{M}_l(s)$ the realization of the nerve of the subposet for which that object is terminal. Pulling back these diagrams along the augmentations $W|\mathcal{N}\mathcal{M}_k|(r) \rightarrow |\mathcal{N}\mathcal{M}_k|(r)$ and $W|\mathcal{N}\mathcal{M}_l|(s) \rightarrow |\mathcal{N}\mathcal{M}_l|(s)$, we obtain similar colimit diagrams for $W|\mathcal{N}\mathcal{M}_k|(r)$ and $W|\mathcal{N}\mathcal{M}_l|(s)$. Moreover the maps in these diagrams are cofibrations.

Combining these colimits with the colimit diagram of [Proposition 5.12](#), we obtain a refined iterated colimit decomposition of $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$. By [Proposition 5.2](#) of [\[10\]](#), such an iterated colimit diagram can be reexpressed as a single colimit diagram over an appropriate Grothendieck construction, which we denote $\mathcal{I}(k, l)(m)$. We thus obtain the following result.

5.13 Proposition. *$(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ is the colimit of a diagram G'_m of cofibrations of contractible spaces indexed by the poset $\mathcal{I}(k, l)(m)$, obtained by the Grothendieck construction from the interchange diagram of [Proposition 5.12](#) by replacing the space at each node by the product of posets $\prod \mathcal{M}_k(r_i) \times \prod \mathcal{M}_l(s_j)$ parametrizing the colimit decomposition at that node. The functors in that diagram are given by operad compositions (including insertions of the unique constant).*

Next we recall a relevant definition from [1].

5.14 Definition. By a *cellular decomposition* of a topological space X over a finite poset \mathcal{P} , we will mean a diagram $\{C_\alpha\}_{\alpha \in \mathcal{P}}$ of closed subsets of X indexed by \mathcal{P} satisfying the following conditions:

1. $X = \bigcup_{\alpha \in \mathcal{P}} C_\alpha$.
2. If $\alpha \leq \beta$ in \mathcal{P} , then the inclusion $C_\alpha \rightarrow C_\beta$ is a cofibration.
3. Each C_α is contractible.
4. $C_\alpha \cap C_\beta = \bigcup_{\substack{\gamma \leq \alpha \\ \gamma \leq \beta}} C_\gamma$.

Under these circumstances, as shown in [1], we have a sequence of equivalences and homeomorphisms:

$$|\mathcal{N}\mathcal{P}| \xleftarrow{\simeq} \text{hocolim}_{\alpha \in \mathcal{P}} C_\alpha \xrightarrow{\simeq} \text{colim}_{\alpha \in \mathcal{P}} C_\alpha \xrightarrow{\cong} X,$$

We will say that a topological operad $\mathcal{A} = \{\mathcal{A}(m)\}_{m \geq 0}$ has a cellular decomposition over a poset operad $\mathcal{P} = \{\mathcal{P}(m)\}_{m \geq 0}$, if $\mathcal{A}(m)$ has a cellular decomposition over $\mathcal{P}(m)$ for each m , and the operad structures on \mathcal{A} and \mathcal{P} satisfy the following compatibility conditions.

1. If $a \in C_\alpha \subset \mathcal{A}(m)$ and $b_i \in C_{\beta_i} \subset \mathcal{A}(n_i)$, $i = 1, 2, \dots, m$, then $a \circ (\bigoplus_{i=1}^m b_i) \in C_{\alpha \circ (\bigoplus_{i=1}^m \beta_i)} \subset \mathcal{A}(n_1 + n_2 + \dots + n_m)$.
2. If $a \in C_\alpha \subset \mathcal{A}(m)$ and $\sigma \in \Sigma_m$, then $a\sigma \in C_{\alpha\sigma}$.

Under these circumstances the chain of equivalences

$$|\mathcal{N}\mathcal{P}(m)| \xleftarrow{\simeq} \text{hocolim}_{\alpha \in \mathcal{P}(m)} C_\alpha \xrightarrow{\simeq} \text{colim}_{\alpha \in \mathcal{P}(m)} C_\alpha \xrightarrow{\cong} \mathcal{A}(m),$$

gives an equivalence of operads. (This definition is inspired by, but differs slightly from the one of Berger [2].)

Thus Proposition 5.13 provides a cellular decomposition of $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ over the poset $\mathcal{I}(k, l)(m)$ for each m .

5.15 Proposition. *There is a functor*

$$L : \mathcal{I}(k, l)(m) \longrightarrow \mathcal{M}_{k+l}(m)$$

which satisfies Quillen’s Theorem A, and thus induces an equivalence upon passage to nerves.

Combining these results, we obtain a chain of equivalences

$$\begin{aligned}
 (W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m) &\cong \operatorname{colim}_{\mathcal{I}(k,l)(m)} G'_m \xleftarrow{\simeq} \operatorname{hocolim}_{\mathcal{I}(k,l)(m)} G'_m \\
 &\xrightarrow{\simeq} \operatorname{hocolim}_{\mathcal{I}(k,l)(m)^*} |\mathcal{N}\mathcal{I}(k,l)(m)| \\
 &\xrightarrow{\simeq} |\mathcal{N}\mathcal{M}_{k+l}|(m)
 \end{aligned}$$

Finally we construct an operad structure on $\mathcal{I}(k,l) = \{\mathcal{I}(k,l)(m)\}_{m \geq 0}$, verify the relevant compatibility conditions on the cellular decomposition of $W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|$, and thus show that this chain of equivalences defines a chain of operad equivalences. This completes the proof of [Theorem 4.3](#).

6. Unary and binary operations

In this section we analyze the spaces of unary and binary operations in a tensor product $\mathcal{A} \otimes \mathcal{B}$ of two arbitrary reduced operads \mathcal{A} and \mathcal{B} , and thus prove [Proposition 4.8](#).

First of all observe that in the coproduct operad $\mathcal{A} \amalg \mathcal{B}$, the unary operations are arbitrary compositions of unary operations in \mathcal{A} and \mathcal{B} , with the units of $\mathcal{A}(1)$ and $\mathcal{B}(1)$ identified and with the single relation that adjacent factors both in $\mathcal{A}(1)$ or both in $\mathcal{B}(1)$ can be combined into a single factor using the multiplications in $\mathcal{A}(1)$ or $\mathcal{B}(1)$. In other words, $(\mathcal{A} \amalg \mathcal{B})(1)$ is the coproduct (or free product) $\mathcal{A}(1) * \mathcal{B}(1)$ in the category of topological monoids. The interchange relations in $\mathcal{A} \otimes \mathcal{B}$ restrict on the space of unary operations to the relation that factors from $\mathcal{A}(1)$ commute with factors from $\mathcal{B}(1)$. Thus the factors from $\mathcal{A}(1)$ can be commuted to the front of a word, leaving the factors from $\mathcal{B}(1)$ at the back of the word. Then the factors from $\mathcal{A}(1)$ and from $\mathcal{B}(1)$ can be combined into single factors, using the multiplications in $\mathcal{A}(1)$ and $\mathcal{B}(1)$, leaving a pair of factors, the first from $\mathcal{A}(1)$ and the second from $\mathcal{B}(1)$. This establishes that

$$(\mathcal{A} \otimes \mathcal{B})(1) \cong \mathcal{A}(1) \times \mathcal{B}(1)$$

and thus proves the first part of [Proposition 4.8](#).

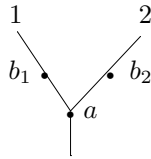
To analyze the binary operations in $\mathcal{A} \otimes \mathcal{B}$, we begin by noting that according to [Lemma 5.8](#) and the follow up discussion in paragraph [5.9](#) we have

$$(\mathcal{A} \amalg \mathcal{B})(2) = \tilde{G}_2(1 \boxtimes_1 2) \amalg \tilde{G}_2(1 \boxtimes_2 2).$$

Now by definition the elements of $\tilde{G}_2(1 \boxtimes_1 2)$ are represented by trees with one binary node labeled by an element a_2 of $\mathcal{A}(2)$ and arbitrary chains of unary nodes labeled by arbitrary elements of $\mathcal{A}(1)$ and $\mathcal{B}(1)$ on both input branches coming into the binary node and on the output branch from that node. Upon dividing out by the unary interchange relations, i.e. passing to $G_2(1 \boxtimes_1 2)$, we may commute the factors coming from $\mathcal{B}(1)$ past factors coming from $\mathcal{A}(1)$ towards the top of each branch and combine them into single factors, using the multiplication in $\mathcal{B}(1)$. Let $b \in \mathcal{B}(1)$ be the factor remaining on the top of the output branch of the binary node labeled by a_2 . We can then use the unary interchange relation

$$b \circ a_2 = a_2 \circ (b \oplus b)$$

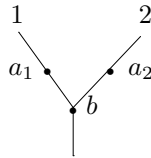
to move b from below the binary node to both branches above that node. We can then commute these copies of b past factors of $\mathcal{A}(1)$ on these branches and then combine them with the factors from $\mathcal{B}(1)$ at the top of these branches. This leaves a tree with one factor of $\mathcal{B}(1)$ at the top of each input branch and factors of $\mathcal{A}(1)$ below them on the input branches, $a_2 \in \mathcal{A}(2)$ on the binary node and additional factors of $\mathcal{A}(1)$ on the output branch below that node. These \mathcal{A} nodes can all be composed together to produce a single binary node with a label in $\mathcal{A}(2)$. Finally we can use the equivariance relation to insure that input 1 is on the left branch of the binary node and input 2 is on the right. Thus each element of $G_2(1 \boxtimes_1 2)$ has a unique tree representative of the form



which we will refer to as the reduced form of that element. This shows that

$$G_2(1 \boxtimes_1 2) \cong \mathcal{A}(2) \times \mathcal{B}(1)^2.$$

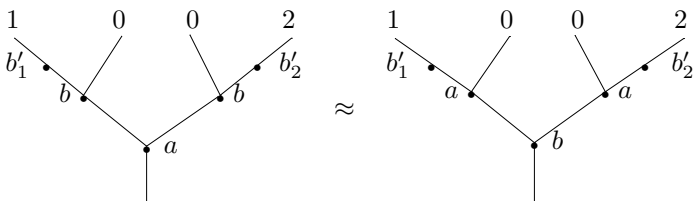
A similar argument, with the roles of \mathcal{A} and \mathcal{B} interchanged, shows that each element of $G_2(1 \boxtimes_2 2)$ has a unique reduced form representative

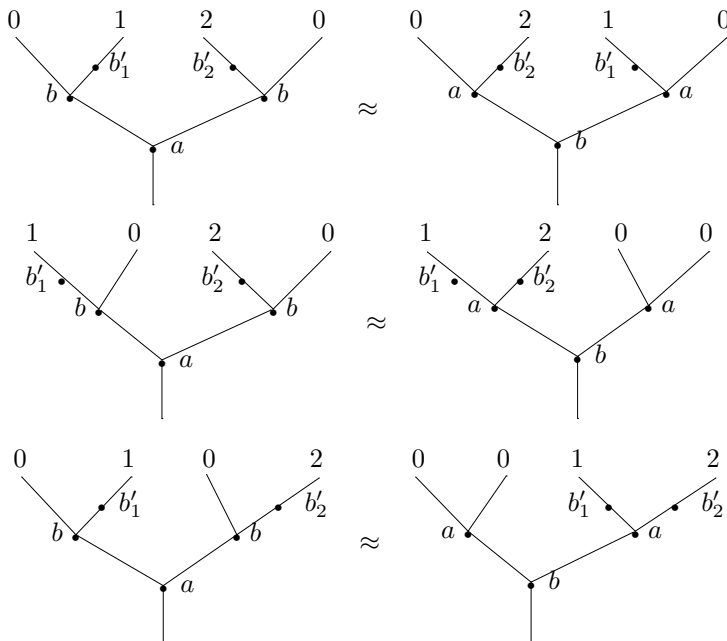


and thus

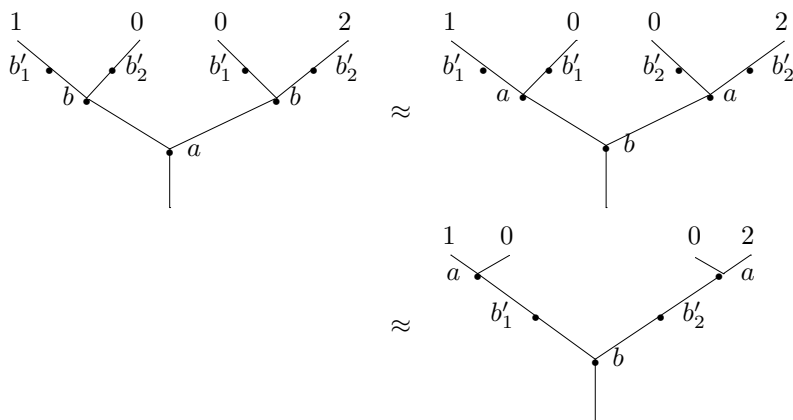
$$G_2(1 \boxtimes_2 2) \cong \mathcal{A}(1)^2 \times \mathcal{B}(2).$$

It now remains to analyze the nonunary interchanges in $(\mathcal{A} \otimes \mathcal{B})(2)$. It is easy to see that any such interchange reduces to a $(2, 2)$ -interchange. A reduced representative in $G_2(1 \boxtimes_1 2)$ can only interchange this way if both b_1 and b_2 factor through a common element in $b \in \mathcal{B}(2)$ by composing with the constant 0. The resulting interchanges are shown below

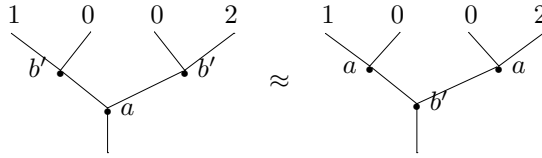




Note that the second interchange follows from the first by replacing b by $b\tau$, where τ is the transposition $(1, 2)$, using the equivariance relation. Similarly the fourth interchange follows from the third. Also note that in the third interchange we can apply the simplification $a \circ (0 \oplus 0) = 0$ (since $\mathcal{A}(0) = \{0\}$). If we then reduce both sides of that interchange (using unary interchanges and composition), we see that both sides become the same. Thus the third (and hence the fourth) interchanges are superfluous. This leaves only the first interchange. Moreover even this interchange can be simplified by noting that $b'_1 \circ 0 = 0$ and $b'_2 \circ 0 = 0$ (since $\mathcal{B}(0) = \{0\}$). This converts the first interchange to the following form



Taking $b' = b \circ (b'_1 \oplus b'_2)$, this in turn simplifies to the interchange



This can be briefly summarized as follows. If an element $(a, b_1, b_2) \in G_2(1 \boxtimes_1 2) \cong \mathcal{A}(2) \times \mathcal{B}(1)^2$ is the image of an element $(a, b) \in \mathcal{A}(2) \times \mathcal{B}(2)$ under the map $(a, b) \mapsto (a, b \circ (id \oplus 0), b \circ (0 \oplus id))$, then it is identified with the element $(a \circ (id \oplus 0), a \circ (0 \oplus id), b)$ in $G_2(1 \boxtimes_2 2) \cong \mathcal{A}(1)^2 \times \mathcal{B}(2)$. A similar analysis of interchanges starting from an element $G_2(1 \boxtimes_2 2)$, shows that the only relation obtained is the reverse of the above relation. It follows that $(\mathcal{A} \times \mathcal{B})(2)$ is a pushout as in the second part of [Proposition 4.8](#).

7. Axial operads

It is technically convenient to prove some of our results in the category \mathcal{SSets} of simplicial sets. So throughout this section we work in the categories \mathcal{Top} and \mathcal{SSets} of k -spaces and simplicial sets respectively. The formulas given in this section make sense in \mathcal{Top} . If we work in \mathcal{SSets} , they are meant to be applied degreewise.

7.1 Observation. If \mathcal{A} and \mathcal{B} are simplicial operads, then

$$|\mathcal{A} \otimes \mathcal{B}| \cong |\mathcal{A}| \otimes |\mathcal{B}|,$$

because $\mathcal{A} \otimes \mathcal{B}$ is a colimit of a diagram involving finite products of simplicial sets $\mathcal{A}(k)$ and $\mathcal{B}(l)$ and the realization functor preserves colimits and finite limits.

7.2 Lemma. (See Igusa [18].) For both categories, the forgetful functor

$$U : \mathcal{Opr}_0 \longrightarrow \mathcal{Monoids}, \quad \mathcal{B} \mapsto \mathcal{B}(1)$$

has a right adjoint $R : \mathcal{Monoids} \longrightarrow \mathcal{Opr}_0$, where $\mathcal{Monoids}$ is the category of monoids in \mathcal{Top} respectively \mathcal{SSets} .

Proof. The right adjoint $R : \mathcal{Monoids} \longrightarrow \mathcal{Opr}_0$ is defined as follows: $RM(k) = M^k$ for a monoid M . The symmetric group permutes the factors, and composition is defined by

$$\begin{aligned} (x_1, \dots, x_n) \circ ((y_{11}, \dots, y_{1k_1}) \oplus \dots \oplus (y_{n1}, \dots, y_{nk_n})) \\ = (x_1 \cdot y_{11}, \dots, x_1 \cdot y_{1k_1}, \dots, x_n \cdot y_{n1}, \dots, x_n \cdot y_{nk_n}). \end{aligned}$$

The unit of the adjunction

$$\xi : \mathcal{B} \longrightarrow RUB$$

is given by the *axial maps*

$$\mathcal{B}(n) \longrightarrow \mathcal{B}(1)^n$$

whose i -th coordinates are the compositions

$$\xi_i : \mathcal{B}(n) \cong \mathcal{B}(n) \times (\mathcal{B}(0)^{i-1} \times \{id\} \times \mathcal{B}(0)^{n-i}) \longrightarrow \mathcal{B}(1).$$

[In other words, the i -th coordinate is the restriction map

$$- \cap \{i\} : \mathcal{B}(n) \longrightarrow \mathcal{B}(\{i\}) \cong \mathcal{B}(1)$$

of [Definition 2.8](#).] The counit $URM \longrightarrow M$ is the identity map. \square

$RU(\mathcal{B})$ is closely related to $\mathcal{A}ss \otimes \mathcal{B}$ and $\mathcal{C}om \otimes \mathcal{B}$. The following result is a reformulation of [\[8, Thm. 5.5\]](#); the topological proof of [\[8\]](#) also works in $\mathcal{S}Sets$.

7.3 Proposition. *In $\mathcal{T}op$ or $\mathcal{S}Sets$, if $\mathcal{B} \in \mathcal{O}pr_0$, then*

$$\mathcal{A}ss \otimes \mathcal{B} \cong (\mathcal{A}ss \times RU(\mathcal{B})) / \sim,$$

where the relation \sim is defined on

$$(\mathcal{A}ss \times RU(\mathcal{B}))(n) = \Sigma_n \times \mathcal{B}(1)^n$$

by $(\pi, (b_1, \dots, b_n)) \sim (\rho, (b_1, \dots, b_n))$ iff $\pi^{-1}(i) < \pi^{-1}(j)$ and $\rho^{-1}(i) > \rho^{-1}(j)$ imply that there is a $c \in \mathcal{B}(2)$ such that $(b_i, b_j) = \xi(c)$. \square

Note that $(\mathcal{A}ss \otimes \mathcal{B})(1) = \mathcal{B}(1)$, so that $RU(\mathcal{A}ss \otimes \mathcal{B}) = RU(\mathcal{B})$.

As a simple consequence we get

7.4 Proposition. *In $\mathcal{T}op$ or $\mathcal{S}Sets$, if $\mathcal{B} \in \mathcal{O}pr_0$, then*

$$\mathcal{C}om \otimes \mathcal{B} \cong RU(\mathcal{B})$$

and the adjunction map $\xi : \mathcal{B} \rightarrow RU(\mathcal{B})$ corresponds to the canonical map $\mathcal{B} \rightarrow \mathcal{C}om \otimes \mathcal{B}$.

Proof. Using [Corollary 3.9](#) and [Proposition 7.32](#) we obtain

$$\mathcal{C}om \otimes \mathcal{B} \cong \mathcal{A}ss \otimes \mathcal{A}ss \otimes \mathcal{B} \cong (\mathcal{A}ss \times RU(\mathcal{A}ss \otimes \mathcal{B})) / \sim \cong (\mathcal{A}ss \times RU(\mathcal{B})) / \sim$$

Now $(\mathcal{A}ss \otimes \mathcal{B})(2) = \Sigma_2 \times \mathcal{B}(1)^2 / \sim$, and a pair $(a, b) \in \mathcal{B}(1)^2$ is the image of $(id, (a, b)) \in (\mathcal{A}ss \otimes \mathcal{B})(2)$ under ξ . Hence

$$(\mathcal{A}ss \times RU(\mathcal{B}) / \sim)(n) = (\Sigma_n \times \mathcal{B}(1)^n) / \Sigma_n = \mathcal{B}(1)^n$$

Let $\beta \in \mathcal{B}(n)$ and let λ_n denote the unique element in $\mathcal{C}om(n)$. Then by the same argument as in the proof of Proposition 3.8, we have $\beta = \lambda_n \circ (\xi_1(\beta) \oplus \dots \oplus \xi_n(\beta))$. Hence the adjunction map ξ corresponds to the canonical map $\mathcal{B} \rightarrow \mathcal{C}om \otimes \mathcal{B}$. \square

7.5 Remark. Proposition 7.4 is a special case of the following general result, which is a simple consequence of Proposition 7.4 and Proposition 4.8: Let \mathcal{A} and \mathcal{B} be reduced operads and let M and N be monoids in $\mathcal{T}op$ or $\mathcal{S}Sets$. Then there are natural isomorphisms

1. $R(M) \otimes \mathcal{A} \cong R(M \times U(\mathcal{A}))$
2. $U(\mathcal{A} \otimes \mathcal{B}) \cong U(\mathcal{A}) \times U(\mathcal{B})$
3. $R(M \times N) \cong R(M) \times R(N) \cong R(M) \otimes R(N)$
4. $RU(\mathcal{A} \otimes \mathcal{B}) \cong RU(\mathcal{A}) \otimes RU(\mathcal{B}) \cong RU(\mathcal{A}) \times RU(\mathcal{B})$

7.6 Definition. A Σ -free topological operad \mathcal{B} is called *axial* if $\xi : \mathcal{B} \rightarrow RUB$ is a closed cofibration. A Σ -free simplicial operad \mathcal{B} is called *axial* if $\xi : \mathcal{B} \rightarrow RU(\mathcal{B})$ is injective. (In these cases we will consider $\mathcal{B}(n)$ as a subspace of $\mathcal{B}(1)^n$ in the sequel.)

8. W -constructions

For the reader’s convenience and for notational reasons we briefly recall the W -constructions we are going to use. They were originally defined for well-pointed topological operads by Boardman and Vogt [7] and extended to other categories including $\mathcal{S}Sets$ by Berger and Moerdijk [4].

8.1 The non-reduced W -construction. In this subsection we allow non-reduced operads.

We define a functor

$$W^u : \mathcal{O}pr \rightarrow \mathcal{O}pr$$

where $\mathcal{O}pr$ is the category of all topological operads, together with a natural transformation $\varepsilon : W^u \rightarrow \text{Id}$.

Let \mathcal{B} be a topological operad. An element in $W^u\mathcal{B}(n)$ is an equivalence class of quadruples (ψ, f, g, h) consisting of

- a finite directed rooted planar tree ψ . Each vertex v has a finite set $\text{In}(v)$ of incoming edges and exactly one outgoing edge. $\text{In}(v) = \emptyset$ is allowed. Thus ψ has a finite set $\text{In}(\psi)$ of *inputs* and exactly one *output*, the *root*. The inputs and the root are called *external* edges, all other edges are called *internal*; they have a vertex at both ends. We require that $|\text{In}(\psi)| = n$, where $|M|$ denotes the cardinality of M .

- a function f assigning to each vertex v a label $f(v) \in \mathcal{B}(|\text{In}(v)|)$.
- a bijection $g : \text{In}(\psi) \rightarrow \underline{n} = \{1, \dots, n\}$, where $n = |\text{In}(\psi)|$. We call $g(i)$ an *input-label* and think of it as label of input i .
- a function h assigning to each internal edge a *length* $t \in [0, 1]$. By convention, the outer edges have length 1.

We usually suppress f, g, h from the notation and think of (ψ, f, g, h) as a tree ψ with labeled inputs, edges, and vertices. We allow the *trivial tree*, i.e. a single edge with label 1 (direction is from top to bottom), and call vertices without an input a *stump*.

8.2 . The equivalence relation between such trees is generated by

1. *The equivariance relation:* The relation 5.2 applied to labeled trees,
2. *The identity relation:*

$$\begin{array}{c} \left| \begin{array}{c} \epsilon_1 \\ \bullet \textit{id} \\ \epsilon_2 \end{array} \right. \sim \left. \begin{array}{c} \left| \right. \\ \max(\epsilon_1, \epsilon_2) \end{array} \right.$$

where ϵ_1 and ϵ_2 are the lengths of the edges.

3. *The shrinking relation:* Edges of length 0 may be shrunk composing the labels of the vertices at their ends using the composition in the operad \mathcal{B} .

The operad structure is given by tree composition: The composite tree $\varphi \circ (\psi_1 \oplus \dots \oplus \psi_n)$ is obtained by grafting ψ_i on the input of φ with input-label i . The newly created inner edges obtain the lengths 1.

The natural transformation

$$\varepsilon : W^u \mathcal{B} \rightarrow \mathcal{B}$$

is defined by shrinking all internal edges to 0. The map $\varepsilon(n) : W^u \mathcal{B}(n) \rightarrow \mathcal{B}(n)$ has a section $\eta(n)$ defined by sending $\alpha \in \mathcal{B}(n)$ to the element represented by the tree with one vertex labeled α and n inputs labeled from left to right in increasing order. The homotopy h_t shrinking the lengths of internal edges by the factor t deforms $W^u \mathcal{B}(n)$ fiberwise into the section. In particular, ε is a weak equivalence.

If the inclusion $\{\textit{id}\} \subset \mathcal{B}(1)$ is a closed cofibration (i.e. the operad is *well-pointed*) then $\varepsilon : W^u \mathcal{B} \rightarrow \mathcal{B}$ is a cofibrant replacement in the sense of Definition 4.2 applied in the category \mathcal{Opr} .

The same construction can be carried out in the category \mathcal{SSets} for simplicial operads with the unit interval $[0, 1]$ being replaced by the standard simplicial 1-simplex. For a

detailed account see [4] where it is also shown that the topological realization of the simplicial W^u -construction applied to a simplicial operad \mathcal{B}_\bullet is naturally homeomorphic to the topological W^u -construction applied to its realization $|\mathcal{B}_\bullet|$:

$$|W^u \mathcal{B}_\bullet| \cong W^u |\mathcal{B}_\bullet|.$$

Since every simplicial operad is well-pointed, $\varepsilon : W^u \mathcal{B}_\bullet \rightarrow \mathcal{B}_\bullet$ is a cofibrant replacement in the category of simplicial operads.

8.3 The reduced W -construction. The inclusion $\mathcal{O}_{pr_0} \subset \mathcal{O}_{pr}$ has a left adjoint L : For an operad \mathcal{B} let $Null(\mathcal{B}) \subset \mathcal{B}$ be the suboperad generated by $id \in \mathcal{B}(1)$ and by $\mathcal{B}(0)$, and let $Null$ be the unique operad with $Null(0)$ and $Null(1)$ consisting of a point. There is a unique map of operads $Null(\mathcal{B}) \rightarrow Null$, and $L(\mathcal{B})$ is the pushout of

$$Null \leftarrow Null(\mathcal{B}) \xrightarrow{c} \mathcal{B}$$

in the category of operads.

The reduced W -construction is the composite functor

$$W : \mathcal{O}_{pr_0} \subset \mathcal{O}_{pr} \xrightarrow{W^u} \mathcal{O}_{pr} \xrightarrow{L} \mathcal{O}_{pr_0}.$$

Hence $W\mathcal{B}$ is obtained from $W^u\mathcal{B}$ by freely adjoining an unlabeled stump and imposing the

8.4 Stump relation. If a labeled tree ψ (unlabeled stumps on edges of length 1 are allowed) has an edge labeled 1 such that the subtree above that edge has no inputs (i.e. it is a subtree topped by stumps) then this subtree may be replaced by the unlabeled stump.

By adjointness W is a cofibrant replacement functor for well-pointed operads in \mathcal{O}_{pr_0} .

There is also a simplicial version of W obtained from W^u in the same way as in the topological case. Since realization preserves quotients, the natural homeomorphism of the W^u -construction induces a natural homeomorphism

$$|W\mathcal{B}_\bullet| \cong W|\mathcal{B}_\bullet|$$

for a reduced simplicial operad \mathcal{B}_\bullet .

Since our favorite cofibrant E_k operad is $W|\mathcal{NM}_k| \cong |W\mathcal{NM}_k|$ we will work both in the topological and the simplicial category. We often find it easier to argue topologically and to transfer the results to the simplicial case using the natural homeomorphisms between the two W -constructions. On the other hand, cofibration conditions in our later constructions reduce to checking injectivity in the simplicial category while they require proofs, which can be quite involved, in the topological category.

8.5 Remark. In [4] and [7] the symbol W is used instead of W^u . We reserve W for the reduced version which corresponds to the W' -construction of [7, p. 159] and is not treated in [4]. In [4] the cofibrancy of the topological as well as the simplicial version of $W^u\mathcal{B}$ requires that \mathcal{B} is Σ -free. The reason is that their notion of a weak equivalence is weaker than ours, but for realizations of Σ -free simplicial operads the two agree.

As pointed out in Lemma 2.4 we often think of $W\mathcal{B}$ as a topologically enriched symmetric monoidal category. Then composition is defined by the operad composition. So composites are represented by trees having inner edges of lengths 1.

8.6 Notation. We call a representing tree of an element in $W\mathcal{B}_\bullet$ or $W\mathcal{C}$ *reduced* if it cannot be further reduced by applying the identity, stump, or, in case of a topological operad \mathcal{C} , the shrinking relation.

An *input path* of a tree is a directed edge path from an input to the root.

8.7 Proposition.

- (1) For a topological operad \mathcal{C} the axial map $\xi : W\mathcal{C}(n) \rightarrow (W\mathcal{C}(1))^n$ is injective.
- (2) $W\mathcal{B}_\bullet$ and $W|\mathcal{B}_\bullet|$ are axial operads for any simplicial operad \mathcal{B}_\bullet .

Proof. (1) Since \mathcal{C} is reduced the space $W^u\mathcal{C}(0)$ has a canonical base point given by the stump. Hence there is an “axial map”

$$\xi^u : W^u\mathcal{C}(n) \rightarrow (W^u\mathcal{C}(1))^n$$

defined by composing with stumps. This map is clearly injective. In the reduced case, we consider the image of a reduced representing tree. Here the stump relation may reduce the components of the axial image of this tree further. Nevertheless one can recover the original tree from these reduced components, because if v is the vertex of the i -th input path of the original tree where it meets another input path the first time, then the subtree above v containing the i -th input path is contained in the i -th coordinate of the axial image.

(2) Since a simplicial map is injective iff its realization is injective (e.g. see [19, II.1.7]) the axial map $\xi : W\mathcal{B}_\bullet(n) \rightarrow (W\mathcal{B}_\bullet(1))^n$ is injective by part (1) so that $W\mathcal{B}_\bullet$ is axial. Since the realization of an injective simplicial map is a closed cofibration, $W|\mathcal{B}_\bullet|$ is axial too. \square

8.8 Definition. A monoid M is called *left-factorial* if any $(x_1, \dots, x_n) \in M^n$ has a maximal left factorization $x_i = y \cdot x'_i$, $i = 1, \dots, n$: if $x_i = z \cdot x''_i$ is any other factorization, $i = 1, \dots, n$, it can be factored further as

$$x''_i = w \cdot x'_i, \quad i = 1, \dots, n \text{ with } y = z \circ w.$$

We call $y = \text{mlf}(x_1, \dots, x_n)$ a *maximal left factor* of (x_1, \dots, x_n) .

An operad \mathcal{B} is called *left-factorial* if the monoid $\mathcal{B}(1)$ is left-factorial.

8.9 Remark. By definition, a maximal left factor is unique up to multiplication by invertible elements. For most of our operads \mathcal{A} , among them $W\mathcal{B}$ for any operad \mathcal{B} , the group of invertible elements in $\mathcal{A}(1)$ is trivial, so that maximal left factors are unique. This justifies the notation $y = \text{mlf}(x_1, \dots, x_n)$.

8.10 Lemma. *Let \mathcal{B} be a topological operad. Then $W\mathcal{B}$ is left factorial.*

Proof. Since $\text{mlf}(\text{mlf}(\alpha_1, \alpha_2), \alpha_3) = \text{mlf}(\alpha_1, \alpha_2, \alpha_3)$ it suffices to show the existence of a maximal left factorization for a pair $(\alpha_1, \alpha_2) \in W\mathcal{B}(1)^2$. Let A be a reduced tree representing α_1 . The left factors of α_1 are represented by subtrees T obtained from A by deleting the subtree on top of an edge of length 1 of the input path. The height of T is defined to be the length of its input path. Then $\text{mlf}(\alpha_1, \alpha_2)$ is represented by the subtree T of A of maximal height which also represents a left factor of α_2 . \square

8.11 Lemma. *Let \mathcal{B} be a topological operad. If $\alpha_1, \dots, \alpha_n \in W\mathcal{B}(1)$, $n \geq 2$, then there is a pair $i < j$ such that*

$$\text{mlf}(\alpha_1, \dots, \alpha_n) = \text{mlf}(\alpha_i, \alpha_j).$$

Proof. This follows from $\text{mlf}(\text{mlf}(\alpha_1, \alpha_2), \alpha_3) = \text{mlf}(\alpha_1, \alpha_2, \alpha_3)$. \square

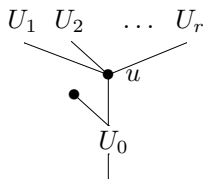
8.12 Lemma. *Let \mathcal{B} be a topological operad such that $\mathcal{B}(1) = \{id\}$. Given $\alpha_1, \dots, \alpha_n \in W\mathcal{B}(1)$ such that $(\alpha_i, \alpha_j) \in W\mathcal{B}(2)$ for each pair $i < j$, then $(\alpha_1, \dots, \alpha_n) \in W\mathcal{B}(n)$.*

Let J denote the category of subsets of $\{1, \dots, n\}$ of cardinality 1 or 2 and inclusions as morphisms. Let P denote the limit of the J^{op} diagram in \mathcal{Top} sending $\{i\}$ to $W\mathcal{B}(1)$, and $\{i, j\}$ to $W\mathcal{B}(2)$. The two maps $W\mathcal{B}(2) \rightarrow W\mathcal{B}(1)$ corresponding to the inclusions of $\{i\}$ and $\{j\}$ into $\{i, j\}$ are given by the components of the axial map. The lemma states that there is a surjection $W\mathcal{B}(n) \rightarrow P$.

Proof. We proceed by induction. For $n = 2$ the statement is trivial. Now suppose that $n \geq 3$.

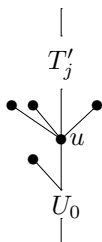
Throughout our proof we will argue with reduced trees representing our elements. Let T_i represent α_i . By assumption there are reduced trees S_{ij} with two inputs, mapped by the axial map to (T_i, T_j) (we may have to reduce the image). Let s_{ij} denote the node where the input paths of S_{ij} meet. The height of s_{ij} is the number of edges between s_{ij} and the root along an input path. By renumbering the α_i we may assume that s_{1n} has the minimal height of all the s_{ij} .

By induction there is a reduced tree U with $n - 1$ inputs whose axial image is (T_1, \dots, T_{n-1}) . Let u be its node of lowest height where two input paths meet. Then U has the form



where the stump stands for a subtree topped by stumps if its outgoing edge has length t , $0 < t < 1$. Of course, there may be more such stumps. We may assume that the input path labeled 1 passes through U_1 . Then T_1 is obtained from U by replacing U_1 by U'_1 , obtained from U_1 by putting stumps on all inputs of U_1 except of input 1 and reducing, and by putting stumps on U_2, \dots, U_r and reducing. In particular, T_1 contains the node u and the subtree of U_0 below it.

By assumption, there is a tree $V \in \mathcal{TB}(2)$ whose axial image is (T_1, T_n) . So V contains the edge path of U from u to the root. By minimality of s_{1n} the two input paths of V , which we denote by 1 and n , meet in this edge path. If they meet in a vertex v below u consider the incoming edge of v which lies on the n -th edge path of V and the subtrees U'_0 of U_0 and V' of V on top of it. Then you obtain U'_0 from V' by putting a stump on its single input and reducing. So if we replace U'_0 in U by V' , we obtain a tree whose axial image is (T_1, \dots, T_n) . We can proceed in the same way if the two input paths meet in u and input path n of V meets u at an input corresponding to a stump in U . This procedure only fails if input path n meets u in an input which is topped by a tree U_i with $1 < i \leq r$. Suppose that the input paths i_1, \dots, i_q of U pass through U_i . Let $(T'_{i_1}, \dots, T'_{i_q})$ be the axial image of U_i and let V' be the subtree of V sitting on top of the input to u below U_i . Then T'_j has the form



for all $j \in \{i_1, \dots, i_q, n\}$ where the stumps again stand for subtrees topped by stumps. The condition $(\alpha_k, \alpha_l) \in \mathcal{WB}(2)$ for all $k \neq l$ therefore implies that (T'_k, T'_l) represents an element in $\mathcal{WB}(2)$ for all $k, l \in \{i_1, \dots, i_q, n\}$, $k \neq l$. By induction there is a tree X with $q + 1$ inputs whose axial image is $(T'_{i_1}, \dots, T'_{i_q}, T'_n)$. If we replace U_i in U by X , we obtain a tree representing an element in $\mathcal{WB}(n)$ whose axial image is (T_1, \dots, T_n) . \square

8.13 Lemma. *Let \mathcal{B} be a Σ -free topological operad. If $\alpha, \beta, \gamma, \delta \in WB(1)$ satisfy that $(\alpha \circ \beta, \gamma)$ and $(\alpha, \gamma \circ \delta)$ are in $WB(2)$, then (α, γ) is in $WB(2)$ (for $(\alpha, \gamma) \in WB(2)$ see our convention in Definition 7.6).*

Proof. Let A, B, G, D be reduced trees representing $\alpha, \beta, \gamma, \delta$. By assumption there are trees S and T whose axial image is $(A \circ B, G)$ respectively $(A, G \circ D)$. In S the input paths corresponding to $A \circ B$ and G can either meet in the A -part or the B -part. If they meet in the A -part we delete the B -part from S and obtain a tree whose axial image is (A, G) . If they meet in the B -part, then G is of the form $G = A \circ G'$. But since $(\alpha \circ \rho, \alpha) \notin WB(2)$ for all $\rho \in WB(1)$, the condition $(\alpha, \gamma \circ \delta) \in WB(2)$ then cannot hold. \square

9. Binodal trees

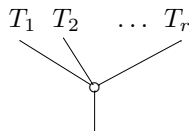
In this section we will discuss a tree notation which is very convenient in analyzing the colimit decomposition of the tensor product $WN\mathcal{M}_k \otimes WN\mathcal{M}_l$.

9.1 Definition. A *binodal tree* is a **nonplanar** rooted tree with labeled inputs, no stumps and no unary nodes. Also all the nodes are colored either black or white and there are no edges connecting two white nodes. [If some construction below gives rise to an edge connecting two white nodes, it is to be understood that such an edge is to be shrunk to a point and the two white nodes merged together.]

When we draw pictures of binodal trees, we will represent black nodes by solid dots \bullet and white nodes by hollow dots \circ . We will also use the following convention: if T is a binodal tree, then $|T|$ will denote the set of inputs for T .

9.2 Definition. Let \mathcal{A} be an operad in one of our categories \mathcal{S} and let T be a binodal tree. We define the object $\mathcal{A}(T)$ with a left $\mathcal{A}(1)$ action recursively as follows.

- If $|T|$ has cardinality 1, then $\mathcal{A}(T) = \mathcal{A}(1)$ with left action given by multiplication.
- If the bottom node of T is colored white and thus T looks like

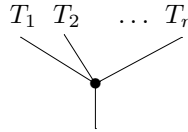


then

$$\mathcal{A}(T) = \mathcal{A}(T_1) \times \mathcal{A}(T_2) \times \dots \times \mathcal{A}(T_r)$$

with the diagonal left action of $\mathcal{A}(1)$ on the right-hand side.

- If the bottom node of T is colored black and thus T looks like



then

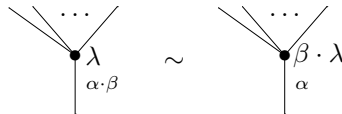
$$\mathcal{A}(T) = \mathcal{A}(r) \times_{\mathcal{A}(1)^r} \mathcal{A}(T_1) \times \mathcal{A}(T_2) \times \dots \times \mathcal{A}(T_r)$$

with left $\mathcal{A}(1)$ action coming from its left action on $\mathcal{A}(r)$.

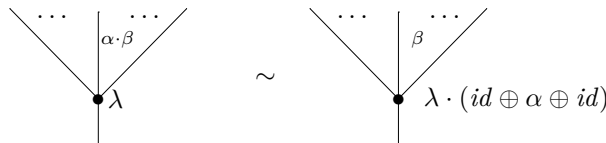
9.3 Remark. If we work in $\mathcal{T}op$ or $\mathcal{S}ets$ it is helpful to think of an object in $\mathcal{A}(T)$ in the following way. We extend this point of view to $\mathcal{S}\mathcal{S}ets$ by applying it degreewise.

An element of $\mathcal{A}(T)$ is a **planar** binodal tree T in which each black node v is decorated with an element of $\mathcal{A}(\text{In}(v))$ and each edge with an element of $\mathcal{A}(1)$ subject to the following relations:

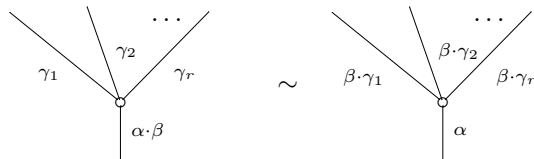
- (1) The equivariance relation 5.2 (here we give each white node a dummy decoration which is invariant under the action of the symmetric group).
- (2) If an edge below a black node has decoration $\alpha \cdot \beta$ then β can be moved into the black node by composing with the node's decoration.



- (3) If an edge above a black node has decoration $\alpha \cdot \beta$ then α can be moved into the black node by composing in the canonical way with the node's decoration.



- (4) If an edge below a white node has decoration $\alpha \cdot \beta$ then β can be moved into all edges above that node simultaneously by composing with the decorations of these edges.



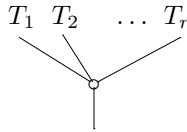
If $\mathcal{S} = \mathcal{C}at$, we can similarly think of the objects and morphisms of $\mathcal{A}(T)$ as binodal trees decorated with appropriate objects or morphisms of \mathcal{A} , subject to the above relations.

9.4 Remark. While we are primarily concerned with the above construction for topological and simplicial operads, we will also need to consider this construction for operads in $\mathcal{C}at$. In particular, we will use the posets $\mathcal{M}_k(T)$ and $\mathcal{M}_l(T)$ as indexing categories in our colimit decomposition of $W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|$. In this case the description in [Remark 9.3](#) simplifies considerably, since \mathcal{M}_k and \mathcal{M}_l have no nontrivial unary operations. Thus there is no need for edge labels or the relations (2)–(4).

9.5 Proposition. *Let \mathcal{A} be an axial operad and T a binodal tree. Then there is a natural imbedding $\mathcal{A}(T) \subseteq RU\mathcal{A}(|T|) = \mathcal{A}(1)^{|T|}$.*

Proof. We proceed inductively on the cardinality of $|T|$. If $|T|$ has cardinality 1, then $\mathcal{A}(T) = \mathcal{A}(1) = \mathcal{A}(1)^{|T|}$ by definition. Assuming we have established the result for binodal trees whose sets of inputs have cardinality less than $|T|$, we consider two cases.

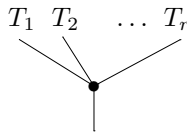
If the bottom node of T is colored white and thus T looks like



then by induction we have

$$\mathcal{A}(T) = \mathcal{A}(T_1) \times \mathcal{A}(T_2) \times \dots \times \mathcal{A}(T_r) \subseteq \prod_{i=1}^r \mathcal{A}(1)^{|T_i|} = \mathcal{A}(1)^{|T|}$$

If the bottom node of T is colored black and thus T looks like



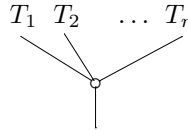
then by induction hypothesis and the axiomaticity of \mathcal{A}

$$\begin{aligned} \mathcal{A}(T) &= \mathcal{A}(r) \times_{\mathcal{A}(1)^r} \mathcal{A}(T_1) \times \mathcal{A}(T_2) \times \dots \times \mathcal{A}(T_r) \\ &\subseteq \mathcal{A}(1)^r \times_{\mathcal{A}(1)^r} \prod_{i=1}^r \mathcal{A}(1)^{|T_i|} \cong \mathcal{A}(1)^{|T|} \end{aligned}$$

where the last map is given by composition in the operad $RU\mathcal{A}$. \square

9.6 Corollary. *Suppose \mathcal{A} is an axial and left factorial topological or simplicial operad. Then for any binodal tree T , $\mathcal{A}(T)$ can be identified with the subspace $\mathcal{A}^*(T)$ of $RU\mathcal{A}(|T|) = \mathcal{A}(1)^{|T|}$ defined recursively as follows:*

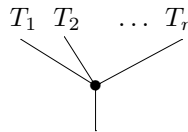
- *If $|T|$ has cardinality 1, we define $\mathcal{A}^*(T) = RU\mathcal{A}(|T|) \cong \mathcal{A}(1)$.*
- *If the bottom node of T is colored white and thus T looks like*



we define

$$\mathcal{A}^*(T) = \mathcal{A}(T_1) \times \mathcal{A}(T_2) \times \dots \times \mathcal{A}(T_r) \subseteq \prod_{i=1}^r \mathcal{A}(1)^{|T_i|} \cong \mathcal{A}(1)^{|T|}$$

- *If the bottom node of T is colored black and thus T looks like*



we define $\mathcal{A}^(T)$ to be the subspace of $\mathcal{A}(T_1) \times \mathcal{A}(T_2) \times \dots \times \mathcal{A}(T_r) \subseteq \prod_{i=1}^r \mathcal{A}(1)^{|T_i|} \cong \mathcal{A}(1)^{|T|}$ consisting of those r -tuples $(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_r)$ such that the corresponding r -tuple of maximal left factors (a_1, a_2, \dots, a_r) is in the image of the axial map $\mathcal{A}(r) \rightarrow \mathcal{A}(1)^r$. Here a_i is the maximal left factor of \bar{a}_i .*

Proof. This follows immediately from the proof of [Proposition 9.5](#). \square

In view of [Proposition 8.7](#) and [Lemma 8.10](#), [Corollary 9.6](#) applies when $\mathcal{A} = W|\mathcal{B}_\bullet|$, where \mathcal{B}_\bullet is a simplicial operad with $\mathcal{B}_\bullet(1) = \{id\}$. Thus in what follows below, $(W|\mathcal{B}_\bullet|)(T)$ will always mean the subspace $(W|\mathcal{B}_\bullet|)^*(T)$ of $(W|\mathcal{B}_\bullet|)(1)^{|T|}$ as described in [Corollary 9.6](#) for any binodal tree T .

9.7 Proposition. *Let \mathcal{B}_\bullet be a simplicial operad with $\mathcal{B}_\bullet(1) = \{id\}$. Let T be a binodal tree. Then $(W|\mathcal{B}_\bullet|)(T)$ can be identified with the subspace of $(W|\mathcal{B}_\bullet|)(1)^{|T|}$ consisting of all tuples $(\alpha_s)_{s \in |T|}$ such that, for any subset $\{a, b, c\} \subseteq |T|$ (of not necessarily distinct elements),*

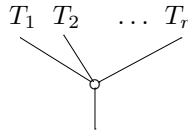
$$(\alpha_s)_{s \in |T|} \cap \{a, b, c\} \in (W|\mathcal{B}_\bullet|)(T \cap \{a, b, c\}).$$

Proof. Let $(W|\mathcal{B}_\bullet|)'(T)$ denote the subspace of $(W|\mathcal{B}_\bullet|)(1)^{|T|}$ consisting of all tuples $(\alpha_s)_{s \in |T|}$ such that, for any subset $\{a, b, c\} \subseteq |T|$ (of not necessarily distinct elements),

$$(\alpha_s)_{s \in |T|} \cap \{a, b, c\} \in (W|\mathcal{B}_\bullet|)(T \cap \{a, b, c\}).$$

The inclusion of $(W|\mathcal{B}_\bullet|)(T) \subseteq (W|\mathcal{B}_\bullet|)'(T)$ follows by naturality. We prove the reverse inclusion by induction on the cardinality of $|T|$. If the cardinality of $|T|$ is ≤ 3 , this holds trivially. So assume the assertion is true for binodal trees T' with $|T'|$ having cardinality less than $|T|$.

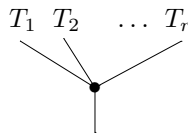
If the bottom node of T is colored white and thus T looks like



then it follows by induction that

$$\begin{aligned} (W|\mathcal{B}_\bullet|)'(T) &= (W|\mathcal{B}_\bullet|)'(T_1) \times (W|\mathcal{B}_\bullet|)'(T_2) \times \dots \times (W|\mathcal{B}_\bullet|)'(T_r) \\ &= (W|\mathcal{B}_\bullet|)(T_1) \times (W|\mathcal{B}_\bullet|)(T_2) \times \dots \times (W|\mathcal{B}_\bullet|)(T_r) \\ &= (W|\mathcal{B}_\bullet|)(T). \end{aligned}$$

If the bottom node of T is colored black and thus T looks like



then it follows by induction that

$$\begin{aligned} (W|\mathcal{B}_\bullet|)'(T) &\subseteq (W|\mathcal{B}_\bullet|)'(T_1) \times (W|\mathcal{B}_\bullet|)'(T_2) \times \dots \times (W|\mathcal{B}_\bullet|)'(T_r) \\ &= (W|\mathcal{B}_\bullet|)(T_1) \times (W|\mathcal{B}_\bullet|)(T_2) \times \dots \times (W|\mathcal{B}_\bullet|)(T_r). \end{aligned}$$

Thus it remains to show that if

$$(\alpha_s)_{s \in S} = (\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_r) \in (W|\mathcal{B}_\bullet|)'(T),$$

then $(\beta_1, \beta_2, \dots, \beta_r)$ is in the image of $W|\mathcal{B}_\bullet|(r) \rightarrow W|\mathcal{B}_\bullet|(1)^r$, where β_i is the maximal left factor of $\bar{\alpha}_i$ for $i = 1, 2, \dots, r$.

By Lemma 8.12 and the induction hypothesis, we can reduce to the case $r = 2$. If both $|T_1|$ and $|T_2|$ have cardinality 1 then $|T|$ has cardinality 2 and we are done by the induction hypothesis. We can assume wlog that $|T_1|$ has cardinality ≥ 2 . By 8.11 we can choose $\{s_1, s_2\} \subseteq |T_1|$ so that the maximal left factor of $(\alpha_{s_1}, \alpha_{s_2})$ is β_1 . If $|T_2| = \{s_3\}$ has cardinality 1, then $\beta_2 = \alpha_{s_3}$ and we have $(\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_3}) \in (W|\mathcal{B}_\bullet|)(T \cap \{s_1, s_2, s_3\})$ which implies that (β_1, β_2) is in the image of $W|\mathcal{B}_\bullet|(2) \rightarrow W|\mathcal{B}_\bullet|(1)^2$. If $|T_2|$ also has cardinality ≥ 2 , pick $\{s_3, s_4\} \subseteq |T_2|$ so that the maximal left factor of $(\alpha_{s_3}, \alpha_{s_4})$ is β_2 . Then we have factorizations in $W|\mathcal{B}_\bullet|(1)$ of the following form:

$$\alpha_{s_1} = \beta_1 \cdot \gamma_1, \quad \alpha_{s_2} = \beta_1 \cdot \gamma_2, \quad \alpha_{s_3} = \beta_2 \cdot \gamma_3, \quad \alpha_{s_4} = \beta_2 \cdot \gamma_4.$$

We then have

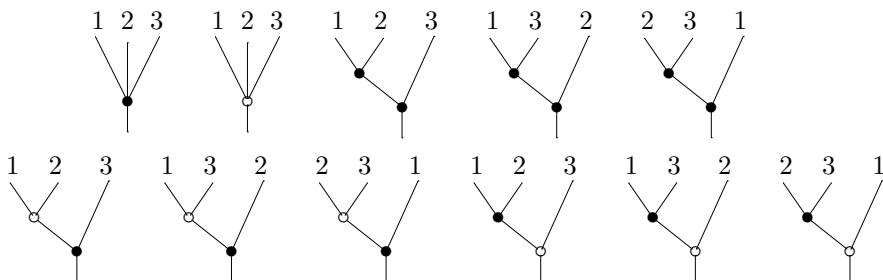
$$(\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_3}) = (\beta_1 \cdot \gamma_1, \beta_1 \cdot \gamma_2, \beta_2 \cdot \gamma_3) \in (W|\mathcal{B}_\bullet|)(T \cap \{s_1, s_2, s_3\}),$$

which implies that $(\beta_1, \beta_2 \cdot \gamma_3)$ is in the image of $W|\mathcal{B}_\bullet|(2) \rightarrow W|\mathcal{B}_\bullet|(1)^2$. Similarly we have

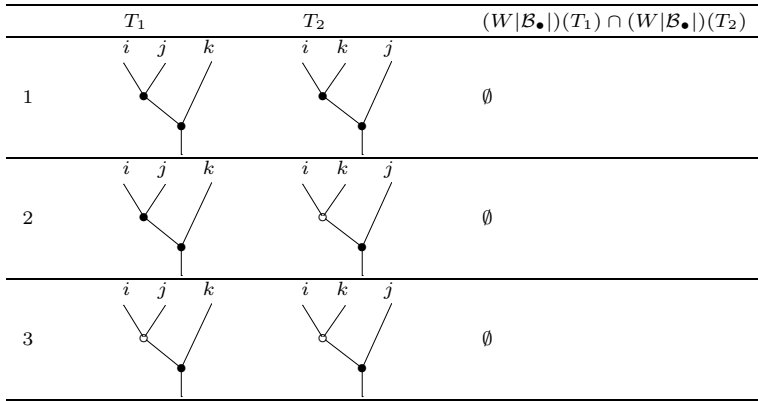
$$(\alpha_{s_1}, \alpha_{s_3}, \alpha_{s_4}) = (\beta_1 \cdot \gamma_1, \beta_2 \cdot \gamma_3, \beta_2 \cdot \gamma_4) \in (W|\mathcal{B}_\bullet|)(T \cap \{s_1, s_3, s_4\}),$$

which implies that $(\beta_1 \cdot \gamma_1, \beta_2)$ is in the image of $W|\mathcal{B}_\bullet|(2) \rightarrow W|\mathcal{B}_\bullet|(1)^2$. By 8.13 this implies that (β_1, β_2) is in the image of $W|\mathcal{B}_\bullet|(2) \rightarrow W|\mathcal{B}_\bullet|(1)^2$. This concludes the induction and proof. \square

Hence we can reduce questions about the spaces $(W|\mathcal{B}_\bullet|)(T)$ to the case of binodal trees with three inputs. For future reference we will need to work out the intersections $(W|\mathcal{B}_\bullet|)(T_1) \cap (W|\mathcal{B}_\bullet|)(T_2)$ in this case. First we note that the following is the complete list of all binodal trees with three inputs.



9.8 Proposition. *Let \mathcal{B}_\bullet be a simplicial operad with $\mathcal{B}_\bullet(1) = \{id\}$. Let T_1 and T_2 be binodal trees with three inputs. Then we have $(W|\mathcal{B}_\bullet|)(T_1) \cap (W|\mathcal{B}_\bullet|)(T_2) = (W|\mathcal{B}_\bullet|)(T_3)$ or \emptyset as shown in the table below and in the table in Appendix A.*



Proof. Let T_{ij} be the tree in row i and column j of our lists. We start with the list in [Appendix A](#).

Row 1 is trivially true because $(W|\mathcal{B}_\bullet|)(T_{11}) = W|\mathcal{B}_\bullet|(1)^3$.

Rows 2 to 4: Since T_{21} is symmetric in 1, 2, 3 we may assume that $(i, j, k) = (1, 2, 3)$. Then

$$\begin{aligned} (W|\mathcal{B}_\bullet|)(T_{21}) &= W|\mathcal{B}_\bullet|(3), \\ (W|\mathcal{B}_\bullet|)(T_{22}) &= W|\mathcal{B}_\bullet|(2) \times_{W|\mathcal{B}_\bullet|(1)^2} (W|\mathcal{B}_\bullet|(2) \times W|\mathcal{B}_\bullet|(1)) \subset W|\mathcal{B}_\bullet|(3), \\ (W|\mathcal{B}_\bullet|)(T_{32}) &= W|\mathcal{B}_\bullet|(2) \times W|\mathcal{B}_\bullet|(1) \supset W|\mathcal{B}_\bullet|(3), \\ (W|\mathcal{B}_\bullet|)(T_{42}) &= W|\mathcal{B}_\bullet|(2) \times_{W|\mathcal{B}_\bullet|(1)^2} (W|\mathcal{B}_\bullet|(1)^2 \times W|\mathcal{B}_\bullet|(1)), \end{aligned}$$

where $W|\mathcal{B}_\bullet|(1)^2$ acts on $W|\mathcal{B}_\bullet|(1)^2 \times W|\mathcal{B}_\bullet|(1)$ by

$$(x, y) \cdot (a_1, a_2, b) = (x \circ a_1, x \circ a_2, y \circ b).$$

Clearly $(W|\mathcal{B}_\bullet|)(T_{43}) \subset (W|\mathcal{B}_\bullet|)(T_{41}) \cap (W|\mathcal{B}_\bullet|)(T_{42})$. Conversely, if you stump k (i.e. you graft a stump on input k) of an element in the intersection, you obtain an element in $W|\mathcal{B}_\bullet|(2)$, i.e. the intersection lies in $(W|\mathcal{B}_\bullet|)(T_{43})$.

Row 5: Since T_{51} is symmetric in i and j we have to consider the cases $(p, q, r) = (i, j, k)$ and $(p, q, r) = (i, k, j)$. In the first case $(W|\mathcal{B}_\bullet|)(T_{51}) \subset (W|\mathcal{B}_\bullet|)(T_{52})$. In the second case we use the equivariance relation to give the inputs of both trees the order k, i, j . We find

$$\begin{aligned} (W|\mathcal{B}_\bullet|)(T_{51}) &= W|\mathcal{B}_\bullet|(2) \times_{W|\mathcal{B}_\bullet|(1)^2} (W|\mathcal{B}_\bullet|(1) \times W|\mathcal{B}_\bullet|(2)) \\ &\subset W|\mathcal{B}_\bullet|(2) \times W|\mathcal{B}_\bullet|(2) = (W|\mathcal{B}_\bullet|)(T_{52}). \end{aligned}$$

The first component of the inclusion is given by stumping j , the second by stumping k and i .

Row 6: $(W|\mathcal{B}_\bullet|)(T_{61}) \subset (W|\mathcal{B}_\bullet|)(T_{62})$.

Row 7: Clearly $(W|\mathcal{B}_\bullet|)(T_{73}) \subset (W|\mathcal{B}_\bullet|)(T_{71}) \cap (W|\mathcal{B}_\bullet|)(T_{72})$. Conversely, by stumping k we see that the intersection has to lie in $(W|\mathcal{B}_\bullet|)(T_{73})$.

Row 8: We use the equivariance relation to give the inputs of both trees the order k, i, j . Then

$$(W|\mathcal{B}_\bullet|)(T_{81}) \subset W|\mathcal{B}_\bullet|(2) \times W|\mathcal{B}_\bullet|(1) = (W|\mathcal{B}_\bullet|)(T_{82}).$$

The first component of the inclusion is given by stumping j , the second by stumping k and i .

Row 9: Again we give the inputs of both trees the order k, i, j and obtain the intersection

$$W|\mathcal{B}_\bullet|(1) \times W|\mathcal{B}_\bullet|(2) \cap W|\mathcal{B}_\bullet|(2) \times W|\mathcal{B}_\bullet|(1)$$

which contains $(W|\mathcal{B}_\bullet|)(3)$, but is not of the form $(W|\mathcal{B}_\bullet|)(T)$.

Now consider the second list: In all three cases we arrange the trees such that the order of inputs is k, i, j .

Row 1: If x is in $(W|\mathcal{B}_\bullet|)(T_{11})$ then the i -th and j -th input paths meet above the i -th and k -th ones, while if x is in $(W|\mathcal{B}_\bullet|)(T_{12})$ it is the other way around. So $(W|\mathcal{B}_\bullet|)(T_{11}) \cap (W|\mathcal{B}_\bullet|)(T_{12}) = \emptyset$.

Rows 2 and 3:

$$(W|\mathcal{B}_\bullet|)(T_{21}) \cap (W|\mathcal{B}_\bullet|)(T_{22}) \subset (W|\mathcal{B}_\bullet|)(T_{31}) \cap (W|\mathcal{B}_\bullet|)(T_{32}).$$

Stumping j or k of an element $y \in (W|\mathcal{B}_\bullet|)(T_{31}) \cap (W|\mathcal{B}_\bullet|)(T_{32})$ makes it an element in $(W|\mathcal{B}_\bullet|)(2)$. Hence

$$(W|\mathcal{B}_\bullet|)(T_{31}) \cap (W|\mathcal{B}_\bullet|)(T_{32}) \subset (W|\mathcal{B}_\bullet|)(T_{11}) \cap (W|\mathcal{B}_\bullet|)(T_{12}) = \emptyset. \quad \square$$

9.9 Proposition. *Let \mathcal{B}_\bullet be a simplicial operad. Then*

- *The imbedding $(W|\mathcal{B}_\bullet|)(T) \subseteq W|\mathcal{B}_\bullet|(1)^{|T|}$ of Proposition 9.5 is a cofibration;*
- *If S and T are binodal trees such that $(W|\mathcal{B}_\bullet|)(S) \subset (W|\mathcal{B}_\bullet|)(T)$, then the inclusion map is a cofibration.*

Proof. It is obvious that the imbedding $(W|\mathcal{B}_\bullet|)(T) \subseteq W|\mathcal{B}_\bullet|(1)^{|T|}$ is the realization of an injective simplicial map. It follows that whenever we have $(W|\mathcal{B}_\bullet|)(S) \subset (W|\mathcal{B}_\bullet|)(T)$, the inclusion is also the realization of an injective simplicial map. Since realizations of injective simplicial maps are cofibrations, the result follows. \square

9.10 Proposition. *Let \mathcal{B}_\bullet be a simplicial operad such that $\mathcal{B}_\bullet(1) = \{id\}$. and let $\varepsilon : W|\mathcal{B}_\bullet| \longrightarrow |\mathcal{B}_\bullet|$ be the augmentation map. Then*

- *For any binodal tree T , there is an induced map $\varepsilon : (W|\mathcal{B}_\bullet|)(T) \longrightarrow |\mathcal{B}_\bullet(T)|$;*
- *For any subcomplex $K_\bullet \subset \mathcal{B}_\bullet(T)$, the restriction $\varepsilon : \varepsilon^{-1}(|K_\bullet|) \longrightarrow |K_\bullet|$ is an equivalence.*

Proof. Since $\mathcal{B}_\bullet(1) = \{id\}$, $\mathcal{B}_\bullet(T)$ is simply a product of the form $\prod_i \mathcal{B}_\bullet(k_i)$. Thus ε induces a map of products $\prod_i W|\mathcal{B}_\bullet|(k_i) \longrightarrow \prod_i |\mathcal{B}_\bullet(k_i)| = |\mathcal{B}_\bullet(T)|$ which factors through a map

$$\varepsilon : (W|\mathcal{B}_\bullet|)(T) \longrightarrow |\mathcal{B}_\bullet(T)|.$$

Let $W'|\mathcal{B}_\bullet|$, denote the suboperad of $W|\mathcal{B}_\bullet|$ consisting of trees without any stumps. Then $\mathcal{B}_\bullet(1) = \{id\}$ and the identity relation on trees implies that $W'|\mathcal{B}_\bullet|(1) = \{id\}$. It follows that $(W'|\mathcal{B}_\bullet|)(T) \subset (W|\mathcal{B}_\bullet|)(T)$, is also a product $\prod_i W'|\mathcal{B}_\bullet|(k_i)$. There is a strong deformation retraction $(W|\mathcal{B}_\bullet|)(T) \longrightarrow (W'|\mathcal{B}_\bullet|)(T)$ given by shrinking stumps. The product of the canonical sections $\prod_i \eta(k_i) : \prod_i |\mathcal{B}_\bullet(k_i)| \longrightarrow \prod_i W|\mathcal{B}_\bullet|(k_i)$ followed by the quotient map into $(W\mathcal{B}_\bullet)(T)$ is contained in $(W'\mathcal{B}_\bullet)(T) = \prod_i W'|\mathcal{B}_\bullet|(k_i)$. Shrinking the internal edges of $(W'\mathcal{B}_\bullet)(T)$ provides a strong deformation retraction onto this section. Composing these two deformation retractions, we obtain that

$$\varepsilon : (W|\mathcal{B}_\bullet|)(T) \longrightarrow |\mathcal{B}_\bullet(T)|$$

is an equivalence, which restricts to an equivalence

$$\varepsilon : \varepsilon^{-1}(|K_\bullet|) \longrightarrow |K_\bullet|$$

for any subcomplex $K_\bullet \subset \mathcal{B}_\bullet(T)$. \square

9.11 Corollary. *For any binodal tree T , $|\mathcal{NM}_k(T)|$ has a cellular decomposition (recall Definition 5.14) over the poset $\mathcal{M}_k(T)$ which lifts to a cellular decomposition of $(W|\mathcal{NM}_k|)(T)$ over $\mathcal{M}_k(T)$.*

Proof. Since $\mathcal{NM}(1) = \{id\}$, $|\mathcal{NM}_k(T)|$ is simply a product of the form $\prod_i \mathcal{NM}(k_i)$, which has a cellular decomposition over $\prod_i \mathcal{M}(k_i) = \mathcal{M}_k(T)$. Since this cellular decomposition is the realization of a simplicial cellular decomposition, by the second part of Proposition 9.10, the inverse images of the cells under the augmentation $\varepsilon : (W|\mathcal{NM}_k|)(T) \longrightarrow |\mathcal{NM}(T)|$ provide the required cellular decomposition. \square

In order to analyze the behavior of the colimit decomposition of the tensor product $W\mathcal{NM}_k \otimes W\mathcal{NM}_l$ with respect to its operad structure, we will need the following constructions of operads associated to binodal trees.

9.12 Definition. The operad \mathcal{BT} is the operad in *Sets* with $\mathcal{BT}(m)$ being the set of all binodal trees T with $|T| = \{1, 2, \dots, m\}$. Composition in \mathcal{BT} is defined in the usual way by grafting trees, followed by shrinking any edges connecting two nodes labeled by \circ . The symmetric group action is the one induced by its action on $|T|$.

9.13 Remark. The operad \mathcal{BT} is a slight variant of the operad of grown trees TX of [28], where $X(0) = X(1) = \emptyset$ and $X(i) = \{\bullet, \circ\}$ for all $i \geq 2$, with the symmetric group acting trivially on $X(i)$. Indeed \mathcal{BT} is the quotient of TX obtained by shrinking all edges joining nodes labeled by \circ .

9.14 Definition. Let \mathcal{A} be an operad in one of our categories \mathcal{S} . Then we define $\mathcal{A}(\mathcal{BT})$ to be the \mathcal{S} -operad with

$$\mathcal{A}(\mathcal{BT})(m) = \coprod_{|T|=\{1,2,\dots,m\}} \mathcal{A}(T),$$

where we think of elements of $\mathcal{A}(T)$ as binodal trees T decorated with appropriate elements of \mathcal{A} as in Remark 9.3. Composition in $\mathcal{A}(\mathcal{BT})$ is defined by grafting trees, followed by shrinking any edges connecting two nodes labeled by \circ . The map $\varepsilon : \mathcal{A}(\mathcal{BT}) \rightarrow \mathcal{BT}$ given by forgetting the decorations from \mathcal{A} is evidently an operad map. [Note that $\mathcal{Sets} \subset \mathcal{S}$.] If \mathcal{A} is an axial operad, then the imbeddings $\mathcal{A}(T) \subset RUA(|T|)$ of Proposition 9.5 define an imbedding of $\mathcal{A}(\mathcal{BT})$ as a suboperad of RUA .

9.15 Remark. If \mathcal{A} is a topological operad which has a cellular decomposition over a poset operad \mathcal{P} , then it is easy to see that $\mathcal{A}(\mathcal{BT})$ has a cellular decomposition over the poset operad $\mathcal{P}(\mathcal{BT})$.

10. The interchange diagram for a tensor product

We begin by showing how to associate a pair of binodal trees (S_α, T_α) to any element α in $\mathfrak{M}_2^{ab}(m)$. First of all we can represent α by a nonplanar rooted tree with inputs labeled by $\{1, 2, \dots, m\}$ with nodes marked by \boxtimes_1 and \boxtimes_2 . Given such a tree, the corresponding object α is obtained by interpreting the edges as compositions in the operad \mathfrak{M}_2^{ab} . There is more than one tree representation of α , but there is a unique reduced form obtained by removing all univalent edges (which correspond to composing with the identity of \mathfrak{M}_2^{ab}) and by shrinking all edges connecting nodes which are both marked by \boxtimes_1 and or both marked by \boxtimes_2 (which corresponds to the strict associativity of \boxtimes_1 and \boxtimes_2). Now given the reduced form representation of α we obtain the binodal tree S_α by replacing each node decorated with \boxtimes_1 by a black node and each node decorated with \boxtimes_2 by a white node. The binodal tree T_α is obtained similarly by reversing the roles of \boxtimes_1 and \boxtimes_2 . Thus

we may choose planar representatives of the binodal trees S_α and T_α having exactly the same shape and input labels, but each white node in S_α corresponds to a black node in T_α and vice versa. We call such a pair of representatives compatible. Clearly α can be uniquely reconstructed from a compatible pair (S_α, T_α) .

As a first step in our analysis of $(W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m)$ we prove [Proposition 5.10](#) by proving a stronger result.

10.1 Proposition. *Let α be an element of $\mathfrak{M}_2^{ab}(m)$. Then the composite map*

$$\begin{aligned} G_m(\alpha) &\longrightarrow (W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m) \xrightarrow{\xi} RU(W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m) \\ &= (W|\mathcal{NM}_k|(1))^m \times (W|\mathcal{NM}_l|(1))^m \end{aligned}$$

is an imbedding and its image is $(W|\mathcal{NM}_k|)(S_\alpha) \times (W|\mathcal{NM}_l|)(T_\alpha)$.

Proof. We define a map

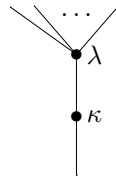
$$\zeta_\alpha : (W|\mathcal{NM}_k|)(S_\alpha) \times (W|\mathcal{NM}_l|)(T_\alpha) \longrightarrow G_m(\alpha)$$

using the descriptions of $(W|\mathcal{NM}_k|)(S_\alpha)$ and $(W|\mathcal{NM}_l|)(T_\alpha)$ given in [Remark 9.3](#). Choose compatible planar representatives of the binodal trees S_α and T_α , and then pick corresponding labeled planar tree representatives $S \in (W|\mathcal{NM}_k|)(S_\alpha)$ and $T \in (W|\mathcal{NM}_l|)(T_\alpha)$. Now for any unlabeled white node in S there is a unique matching black node in T , labeled by an element of $W|\mathcal{NM}_l|$. Replace the white node in S by the corresponding labeled black node in T . Also combine the edge labels from S and T by putting a univalent node marked with the edge label (in $W|\mathcal{NM}_l|(1)$) from T above another univalent node marked with the edge label (in $W|\mathcal{NM}_k|(1)$) from S . The resulting tree $\zeta_\alpha(S, T)$ represents an element of $G_m(\alpha)$. The map ζ_α is well-defined, since the relations (1)–(4) in [Remark 9.3](#) on the domain of ζ_α correspond to the equivariance and unary interchange relations on $G_m(\alpha)$. By [Proposition 9.5](#) and [Corollary 9.6](#) it follows that the composite

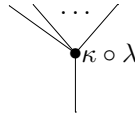
$$\begin{aligned} &(W|\mathcal{NM}_k|)(S_\alpha) \times (W|\mathcal{NM}_l|)(T_\alpha) \\ &\xrightarrow{\zeta_\alpha} G_m(\alpha) \longrightarrow (W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m) \\ &\xrightarrow{\xi} RU(W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m) \\ &= (W|\mathcal{NM}_k|(1))^m \times (W|\mathcal{NM}_l|(1))^m \end{aligned}$$

is the natural imbedding. It follows that ζ_α is injective.

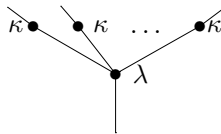
It remains to show that ζ_α is surjective. Let B be a tree representative of an element in $G_m(\alpha)$. We may reduce this tree using quotient relations in $G_m(\alpha)$ as follows. First of all using the nullary interchange relations (cf. Remark 5.4), we remove any stumps in B . We then eliminate any unary nodes which are below a multivalent node. For given a subtree of B having the form



we may replace it by the equivalent tree



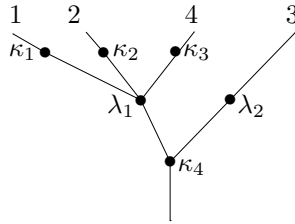
if κ and λ are both in $W|\mathcal{NM}_k|$ or both in $W|\mathcal{NM}_l|$, and by



if one of κ, λ is in $W|\mathcal{NM}_k|$ and the other is in $W|\mathcal{NM}_l|$. Thus B is equivalent to a tree where all the unary nodes are at the top of the tree, just below the input edges. We further reduce the tree by interpreting any edge connecting multivalent nodes both labeled by elements of $W|\mathcal{NM}_k|$ or both labeled by elements of $W|\mathcal{NM}_l|$ as compositions within that operad. After these reductions, we obtain a tree $S \in (W|\mathcal{NM}_k|)(S_\alpha)$ by replacing all multivalent nodes labeled with elements of $W|\mathcal{NM}_l|$ by unlabeled white nodes, dropping all univalent nodes, and transferring the univalent node labels coming from $W|\mathcal{NM}_k|$ to the corresponding edges. Any edge which remains unlabeled is considered to be labeled by the unit. Similarly we obtain a tree $T \in (W|\mathcal{NM}_l|)(T_\alpha)$ by reversing the roles of $W|\mathcal{NM}_k|$ and $W|\mathcal{NM}_l|$. It is clear that $\zeta_\alpha(S, T) = B$. Thus ζ_α is bijective and hence $G_m(\alpha)$ imbeds as claimed. \square

We illustrate the proof of Proposition 10.1 with a simple example.

10.2 Example. Let $\alpha = (1 \boxtimes_2 2 \boxtimes_2 4) \boxtimes_1 3$. Then after completing the reduction process described in the proof, any element B of $G_4(\alpha)$ has a representative of the form



Here $\kappa_i \in W|\mathcal{NM}_k|(1)$ for $i = 1, 2, 3$, $\kappa_4 \in W|\mathcal{NM}_k|(2)$, $\lambda_1 \in W|\mathcal{NM}_l|(3)$, and $\lambda_2 \in W|\mathcal{NM}_l|(1)$. Then $\zeta_\alpha(S, T) = B$, where

$$(S, T) = \left(\begin{array}{c} \begin{array}{cccc} 1 & 2 & 4 & 3 \\ \kappa_1 & \kappa_2 & \kappa_3 & \\ & & & \\ & & & \kappa_4 \end{array} \\ \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \kappa_1 \quad \kappa_2 \quad \kappa_3 \\ \diagdown \quad \diagup \\ \kappa_4 \end{array} \end{array} , \begin{array}{c} \begin{array}{cccc} 1 & 2 & 4 & 3 \\ & & & \\ & & & \\ & & & \lambda_2 \end{array} \\ \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \lambda_1 \quad \lambda_2 \\ \diagdown \quad \diagup \\ \circ \end{array} \end{array} \right)$$

Here the unlabeled edges are considered to be labeled by the unit of the operad.

Thus we can think of $G_m(\alpha)$ as a subspace, either of $(W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m)$ or of $W|\mathcal{NM}_k|(1)^m \times W|\mathcal{NM}_l|(1)^m$, and it makes sense to talk about intersections of the form $\bigcap_i G_m(\alpha_i)$. However at this point we cannot determine if these intersections are the same in $(W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m)$ as they are in $W|\mathcal{NM}_k|(1)^m \times W|\mathcal{NM}_l|(1)^m$ because we so far have not proved that $(W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)$ is axial. To distinguish between the two types of intersections, we will write $G_m^{ax}(\alpha)$ (i.e. axial image) when we think of $G_m(\alpha)$ as a subset of the latter instead of the former.

Moreover since the map $G_m(\alpha) \rightarrow (W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m)$ is the realization of a simplicial map, it follows that this map is cofibration, thus establishing Proposition 5.10.

Our next goal is to show that there are no nonempty intersections in either $(W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m)$ or $W|\mathcal{NM}_k|(1)^m \times W|\mathcal{NM}_l|(1)^m$ except those encoded by the simplices of $\mathfrak{K}_\bullet(m)$.

10.3 Proposition. *If $\alpha_0, \dots, \alpha_r$ are not vertices of a simplex of $\mathfrak{K}_\bullet(m)$, then $\bigcap_{i=0}^r G_m(\alpha_i) = \bigcap_{i=0}^r G_m^{ax}(\alpha_i) = \emptyset$.*

Proof. By assumption there is a triple $\{a, b, c\} \subset \{1, \dots, m\}$ such that the set of vertices of $\{\alpha_0 \cap \{a, b, c\}, \dots, \alpha_n \cap \{a, b, c\}\}$ is not a simplex in $\mathfrak{K}_\bullet(\{a, b, c\})$. Wolog $\{a, b, c\} = \{1, 2, 3\}$. Hence this collection of vertices must contain two different objects β_1 and β_2 in the second or third column of the table of Definition 5.7. By Proposition 9.8, if β_1 and β_2 are in the second column, then $(W|\mathcal{B}_\bullet|)(S_{\beta_1}) \cap (W|\mathcal{B}_\bullet|)(S_{\beta_2}) = \emptyset$, and if β_1 and β_2 are in the third column, then $(W|\mathcal{B}_\bullet|)(T_{\beta_1}) \cap (W|\mathcal{B}_\bullet|)(T_{\beta_2}) = \emptyset$. Since the restriction into $\mathfrak{K}_\bullet(3)$ corresponds to looking at the first three components of the axial image, we

conclude that $\bigcap_{i=0}^n G_m^{ax}(\alpha_i) = \emptyset$. Since the image of a nonempty set must be nonempty, it follows that $\bigcap_{i=0}^n G_m(\alpha_i) = \emptyset \quad \square$

Our next aim is to show that the intersections in $W|\mathcal{N}\mathcal{M}_k|(1)^m \times W|\mathcal{N}\mathcal{M}_l|(1)^m$ along a simplex are not empty.

10.4 Lemma. *If $\sigma = \{\alpha_0, \dots, \alpha_r\}$ is an r -simplex, $r \geq 1$, in $\mathfrak{K}_\bullet(m)$ and all vertices α_i have outermost operation \boxtimes_1 , then for all i*

$$\alpha_i = \alpha_{i1} \boxtimes_1 \alpha_{i2}$$

with $|\alpha_{i1}| = |\alpha_{j1}|$ and $|\alpha_{i2}| = |\alpha_{j2}|$ for all i, j , where $|\alpha|$ is the underlying set of generators of α .

The same holds if we interchange \boxtimes_1 and \boxtimes_2 .

Proof. *Special case:* There is a vertex $\alpha = 1 \boxtimes_1 \alpha'$ in σ where α' has outermost operation \boxtimes_2 . We show that any other vertex β of σ has the form $\beta = 1 \boxtimes_1 \beta'$.

We proceed by induction. For $m = 3$ the statement is true (see table in [Definition 5.7](#)).

Assume that $m > 3$. We can find a generator (wolog 2) such that replacing 2 by 0 leaves an outermost \boxtimes_2 in $\alpha' \cap \{1, 3, \dots, m\}$. Then by induction, $\beta \cap \{1, 3, \dots, m\} = 1 \boxtimes_1 \beta''$. If $\beta \neq 1 \boxtimes_1 \beta'$ it must have the form $\beta = (1 \boxtimes_2 2) \boxtimes_1 \beta''$. There is a generator (wolog 3) such that $\alpha' \cap \{2, 3\} = 2 \boxtimes_2 3$. Then

$$\alpha \cap \{1, 2, 3\} = 1 \boxtimes_1 (2 \boxtimes_2 3) \quad \text{and} \quad \beta \cap \{1, 2, 3\} = (1 \boxtimes_2 2) \boxtimes_1 3 = 3 \boxtimes_1 (1 \boxtimes_2 2).$$

But then $\{\alpha, \beta\}$ is not a 1-simplex in $\mathfrak{K}_\bullet(m)$.

General case: Let

$$\alpha_i = \alpha_{i1} \boxtimes_1 \alpha_{i2} \boxtimes_1 \dots \boxtimes_1 \alpha_{iq_i}$$

with q_i maximal for such a decomposition. Among the α_{is} , $0 \leq i \leq n$, $1 \leq s \leq q_i$ there is a not necessarily unique block with a maximal number of generators. Wolog we may assume that this block is α_{01} and that $|\alpha_{01}| = \{1, \dots, p\}$. Then $\alpha_0 = \alpha_{01} \boxtimes_1 \alpha'_0$ and $|\alpha'_0| = \{p+1, \dots, m\}$. For each $j \in \{p+1, \dots, m\}$ define $\alpha_i(j) = \alpha_i \cap \{1, 2, \dots, p, j\}$. Then $\alpha_i(j)$ has outermost operation \boxtimes_1 by maximality of α_{01} . Since $\alpha_0(j) = j \boxtimes_1 \alpha_{01}$ and α_{01} has outermost operation \boxtimes_2 we obtain that each $\alpha_i(j)$ has the form $\alpha_i(j) = j \boxtimes_1 \alpha_i(j)'$ by the special case above. Since this holds for each $j \in \{p+1, \dots, m\}$ we obtain a decomposition

$$\alpha_i = (\alpha_i \cap \{1, \dots, p\}) \boxtimes_1 (\alpha_i \cap \{p+1, \dots, m\})$$

which proves the lemma. \square

10.5 Remark. Note that the proof of Lemma 10.4 is constructive. That is, we give an explicit algorithm for constructing the decomposition in question.

By iterating Lemma 10.4 we obtain

10.6 Corollary. If $\sigma = \{\alpha_0, \dots, \alpha_r\}$ is an r -simplex, $r \geq 1$, in $\mathfrak{K}_\bullet(m)$ and all vertices α_i have outermost operation \boxtimes_1 , then there is a maximal decomposition such that for all i

$$\alpha_i = \alpha_{i1} \boxtimes_1 \alpha_{i2} \boxtimes_1 \dots \boxtimes_1 \alpha_{ip}$$

and for all $1 \leq j \leq p$

$$|\alpha_{0j}| = |\alpha_{1j}| = \dots = |\alpha_{rj}|.$$

Maximal means that for all $1 \leq j \leq p$, either $|\alpha_{0j}| = |\alpha_{1j}| = \dots = |\alpha_{rj}|$ consists of a single generator, or $|\alpha_{ij}|$ has outermost operation \boxtimes_2 for at least one i .

The same holds if we interchange \boxtimes_1 and \boxtimes_2 .

We shall refer to the decomposition specified in Corollary 10.6 as the maximal common \boxtimes_1 decomposition of the vertices of a simplex of the first kind or the maximal common \boxtimes_2 decomposition of the vertices of a simplex of the second kind.

10.7 Proposition. Suppose that

$$\{\alpha_0, \alpha_1, \dots, \alpha_r\}$$

is an r -simplex in $\mathfrak{K}_\bullet(m)$ (which we denote $\bar{\alpha}$). Then $\bigcap_{i=0}^r G_m^{ax}(\alpha_i)$ is nonempty and we may find a pair of binodal trees $(S_{\bar{\alpha}}, T_{\bar{\alpha}})$ such that

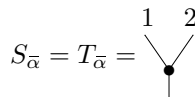
$$\bigcap_{i=0}^r G_m^{ax}(\alpha_i) = W|\mathcal{NM}_k|(S_{\bar{\alpha}}) \times W|\mathcal{NM}_l|(T_{\bar{\alpha}}).$$

Proof. We will first describe a recursive algorithm for constructing the trees $S_{\bar{\alpha}}$ and $T_{\bar{\alpha}}$. We defer proving that

$$\bigcap_{i=0}^r G_m^{ax}(\alpha_i) = (W|\mathcal{NM}_k|)(S_{\bar{\alpha}}) \times (W|\mathcal{NM}_l|)(T_{\bar{\alpha}})$$

until later.

The first case of a simplex of positive dimension occurs when $m = 2$, where we have the 1-simplex $\bar{\alpha} = \{1 \boxtimes_1 2, 1 \boxtimes_2 2\}$. In this case we take



Now assume we have constructed the trees $S_{\bar{\beta}}, T_{\bar{\beta}}$ for all simplices $\bar{\beta}$ in $\mathfrak{K}_{\bullet}(a)$, for $a < m$. For a simplex $\bar{\alpha}$ in $\mathfrak{K}_{\bullet}(m)$, we consider three possible cases.

Case 1: Some of the objects α_i have outermost operation \boxtimes_1 while others have outermost operation \boxtimes_2 .

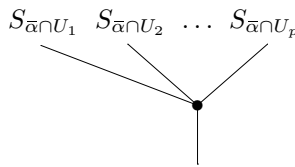
In this case reindex the vertices so that $\alpha_0, \alpha_1, \dots, \alpha_s$ have outermost operation \boxtimes_1 , while $\alpha_{s+1}, \dots, \alpha_r$ have outermost operation \boxtimes_2 . Let

$$\begin{aligned} \alpha_i &= \alpha_{i1} \boxtimes_1 \alpha_{i2} \boxtimes_1 \dots \boxtimes_1 \alpha_{ip} & 0 \leq i \leq s \\ \alpha_i &= \alpha_{i1} \boxtimes_2 \alpha_{i2} \boxtimes_2 \dots \boxtimes_2 \alpha_{iq} & s+1 \leq i \leq r \end{aligned}$$

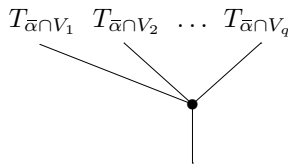
be the maximal common decompositions as in [Corollary 10.6](#). Let

$$\begin{aligned} U_j &= |\alpha_{0j}| = |\alpha_{1j}| = \dots = |\alpha_{sj}| & 1 \leq j \leq p \\ V_j &= |\alpha_{s+1,j}| = |\alpha_{s+2,j}| = \dots = |\alpha_{rj}| & 1 \leq j \leq q \end{aligned}$$

By recursion, we have already defined pairs of binodal trees $(S_{\bar{\alpha} \cap U_j}, T_{\bar{\alpha} \cap U_j})$ for each $j = 1, 2, \dots, p$. Similarly for each $j = 1, 2, \dots, q$ we have already defined pairs of binodal trees $(S_{\bar{\alpha} \cap V_j}, T_{\bar{\alpha} \cap V_j})$. We define $S_{\bar{\alpha}}$ to be the binodal tree



and we define $T_{\bar{\alpha}}$ to be the binodal tree



Case 2: All of the objects α_i have outermost operation \boxtimes_1 .

Let

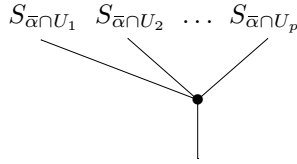
$$\alpha_i = \alpha_{i1} \boxtimes_1 \alpha_{i2} \boxtimes_1 \dots \boxtimes_1 \alpha_{ip} \quad 0 \leq i \leq r$$

and

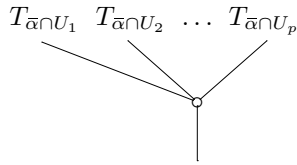
$$U_j = |\alpha_{0j}| = |\alpha_{1j}| = \dots = |\alpha_{rj}| \quad 1 \leq j \leq p$$

be as in Case 1.

By recursion, for each $j = 1, 2, \dots, p$ we have already defined pairs of binodal trees $(S_{\bar{\alpha} \cap U_j}, T_{\bar{\alpha} \cap U_j})$. We define $S_{\bar{\alpha}}$ to be the binodal tree



and $T_{\bar{\alpha}}$ to be the binodal tree



Case 3: All of the objects α_i have outermost operation \boxtimes_2 .
Let

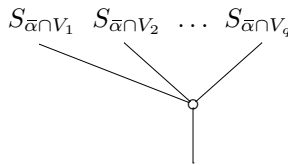
$$\alpha_i = \alpha_{i1} \boxtimes_2 \alpha_{i2} \boxtimes_2 \dots \boxtimes_2 \alpha_{iq} \quad 0 \leq i \leq r$$

and

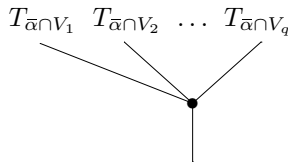
$$V_j = |\alpha_{0j}| = |\alpha_{1j}| = \dots = |\alpha_{rj}| \quad 1 \leq j \leq q$$

be as in Case 1.

By recursion, for each $j = 1, 2, \dots, q$ we have already defined pairs of binodal trees $(S_{\bar{\alpha} \cap V_j}, T_{\bar{\alpha} \cap V_j})$. We define $S_{\bar{\alpha}}$ to be the binodal tree



and $T_{\bar{\alpha}}$ to be the binodal tree



To check that

$$\bigcap_{i=1}^r G_m^{ax}(\alpha_i) = (W|\mathcal{NM}_k|)(S_{\bar{\alpha}}) \times (W|\mathcal{NM}_l|)(T_{\bar{\alpha}})$$

we observe that it is true for the unique 1-simplex in the case $m = 2$ (compare the discussion of binary operations in Section 6). If $m > 2$, then by 9.7 we can reduce checking this to the case $m = 3$. Up to permutations, there are two maximal simplices in $\mathfrak{M}_2^{ab}(3)$:

$$\{1 \boxtimes_1 2 \boxtimes_1 3, (1 \boxtimes_2 2) \boxtimes_1 3, (2 \boxtimes_1 3) \boxtimes_2 1, 1 \boxtimes_2 2 \boxtimes_2 3\}$$

$$\{1 \boxtimes_1 2 \boxtimes_1 3, (1 \boxtimes_2 2) \boxtimes_1 3, (1 \boxtimes_1 2) \boxtimes_2 3, 1 \boxtimes_2 2 \boxtimes_2 3\}$$

We index the vertices of the first simplex by the indices 0, 1, 2, 3 and we use the notations $\bar{\alpha}_{[i]}$, $\bar{\alpha}_{[i,j]}$, $\bar{\alpha}_{[i,j,k]}$, $\bar{\alpha}_{[0,1,2,3]}$ to indicate the subsimplices spanned by the vertices with those indices. We list below the pairs $(S_{\bar{\alpha}}, T_{\bar{\alpha}})$ assigned to these simplices by our recursive algorithm.

$$(S_{\bar{\alpha}_{[0]}}, T_{\bar{\alpha}_{[0]}}) = \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad | \quad / \\ \bullet \\ | \end{array} , \begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad | \quad / \\ \circ \\ | \end{array} \right)$$

$$(S_{\bar{\alpha}_{[1]}}, T_{\bar{\alpha}_{[1]}}) = \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad \diagup \quad / \\ \circ \quad \bullet \\ | \end{array} , \begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad \bullet \quad / \\ \circ \\ | \end{array} \right)$$

$$(S_{\bar{\alpha}_{[2]}}, T_{\bar{\alpha}_{[2]}}) = \left(\begin{array}{c} 2 \quad 3 \quad 1 \\ \diagdown \quad \bullet \quad / \\ \circ \\ | \end{array} , \begin{array}{c} 2 \quad 3 \quad 1 \\ \diagdown \quad \diagup \quad / \\ \circ \quad \bullet \\ | \end{array} \right)$$

$$(S_{\bar{\alpha}_{[3]}}, T_{\bar{\alpha}_{[3]}}) = \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad | \quad / \\ \circ \\ | \end{array} , \begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad | \quad / \\ \bullet \\ | \end{array} \right)$$

$$(S_{\bar{\alpha}_{[0,1]}}, T_{\bar{\alpha}_{[0,1]}}) = \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad \bullet \quad / \\ \circ \quad \bullet \\ | \end{array} , \begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad \bullet \quad / \\ \bullet \quad \circ \\ | \end{array} \right)$$

$$(S_{\bar{\alpha}_{[0,2]}}, T_{\bar{\alpha}_{[0,2]}}) = \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad | \quad / \\ \bullet \\ | \end{array} , \begin{array}{c} 2 \quad 3 \quad 1 \\ \diagdown \quad \diagup \quad / \\ \circ \quad \bullet \\ | \end{array} \right)$$

$$(S_{\bar{\alpha}_{[0,3]}}, T_{\bar{\alpha}_{[0,3]}}) = \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad | \quad / \\ \bullet \\ | \end{array} , \begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad | \quad / \\ \bullet \\ | \end{array} \right)$$

$$\begin{aligned}
 (S_{\bar{\alpha}_{[1,2]}}, T_{\bar{\alpha}_{[1,2]}}) &= \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad \diagup \\ \circ \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array}, \begin{array}{c} 2 \quad 3 \quad 1 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array} \right) \\
 (S_{\bar{\alpha}_{[1,3]}}, T_{\bar{\alpha}_{[1,3]}}) &= \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad \diagup \\ \circ \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array}, \begin{array}{c} 1 \quad 2 \quad 3 \\ | \quad | \quad | \\ \bullet \\ | \end{array} \right) \\
 (S_{\bar{\alpha}_{[2,3]}}, T_{\bar{\alpha}_{[2,3]}}) &= \left(\begin{array}{c} 2 \quad 3 \quad 1 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \circ \\ | \end{array}, \begin{array}{c} 2 \quad 3 \quad 1 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array} \right) \\
 (S_{\bar{\alpha}_{[0,1,2]}}, T_{\bar{\alpha}_{[0,1,2]}}) &= \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array}, \begin{array}{c} 2 \quad 3 \quad 1 \\ \diagdown \quad \diagup \\ \circ \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array} \right) \\
 (S_{\bar{\alpha}_{[0,1,3]}}, T_{\bar{\alpha}_{[0,1,3]}}) &= \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array}, \begin{array}{c} 1 \quad 2 \quad 3 \\ | \quad | \quad | \\ \bullet \\ | \end{array} \right) \\
 (S_{\bar{\alpha}_{[0,2,3]}}, T_{\bar{\alpha}_{[0,2,3]}}) &= \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ | \quad | \quad | \\ \bullet \\ | \end{array}, \begin{array}{c} 2 \quad 3 \quad 1 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array} \right) \\
 (S_{\bar{\alpha}_{[1,2,3]}}, T_{\bar{\alpha}_{[1,2,3]}}) &= \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad \diagup \\ \circ \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array}, \begin{array}{c} 2 \quad 3 \quad 1 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array} \right) \\
 (S_{\bar{\alpha}_{[0,1,2,3]}}, T_{\bar{\alpha}_{[0,1,2,3]}}) &= \left(\begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array}, \begin{array}{c} 2 \quad 3 \quad 1 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \\ | \end{array} \right)
 \end{aligned}$$

We can check that

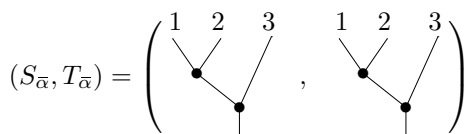
$$\bigcap_{i=0}^r G_m^{ax}(\alpha_i) = (W|\mathcal{NM}_k|)(S_{\bar{\alpha}}) \times (W|\mathcal{NM}_l|)(T_{\bar{\alpha}})$$

holds for each of these simplices by repeatedly applying [Proposition 9.8](#).

For the second simplex

$$\{1 \boxtimes_1 2 \boxtimes_1 3, (1 \boxtimes_2 2) \boxtimes_1 3, (1 \boxtimes_1 2) \boxtimes_2 3, 1 \boxtimes_2 2 \boxtimes_2 3\}$$

any subsimplex not containing the edge $\{(1 \boxtimes_2 2) \boxtimes_1 3, (1 \boxtimes_1 2) \boxtimes_2 3\}$ is, up to permutation, a subsimplex of the first maximal simplex and thus has already been analyzed. For any subsimplex containing this edge, the recursive algorithm assigns



Again we can check that

$$\bigcap_{i=0}^r G_m^{ax}(\alpha_i) = (W|\mathcal{NM}_k|)(S_{\bar{\alpha}}) \times (W|\mathcal{NM}_l|)(T_{\bar{\alpha}})$$

holds for each of these simplices by repeatedly applying [Proposition 9.8](#). \square

10.8 Proposition. *Let $\bar{\alpha} = \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ be an r -simplex in $\mathfrak{K}_\bullet(m)$ and let*

$$\bigcap_{i=0}^r G_m^{ax}(\alpha_i) = (W|\mathcal{NM}_k|)(S_{\bar{\alpha}}) \times (W|\mathcal{NM}_l|)(T_{\bar{\alpha}}).$$

Then

- (1) $\bigcap_{i=0}^r G_m^{ax}(\alpha_i)$ has a cellular decomposition over the poset $\mathcal{M}_k(S_{\bar{\alpha}}) \times \mathcal{M}_l(T_{\bar{\alpha}})$.
- (2) If $\bar{\alpha}$ is a subsimplex of $\bar{\beta}$, then the inclusion $\bigcap_j G_m^{ax}(\beta_j) \subset \bigcap_i G_m^{ax}(\alpha_i)$ is compatible with the cellular decomposition.
- (3) The axial map sends $\bigcap_{i=0}^r G_m(\alpha_i)$ homeomorphically onto $\bigcap_{i=0}^r G_m^{ax}(\alpha_i)$.

Proof. Part (1) is an immediate consequence of [Corollary 9.11](#). To establish the second part, we proceed by induction on m and refer back to the proof of [Proposition 10.7](#). If $m = 1$, the diagram is trivial and there is nothing to prove. So let us assume we have established this for all $a < m$. It suffices to check the compatibility with the cellular decompositions for inclusions $G'_m(\bar{\beta}) \subset G'_m(\bar{\alpha})$, where the simplex $\bar{\beta}$ is obtained by adding a single vertex λ to the simplex $\bar{\alpha}$.

We refer back to the recursive algorithm of [Proposition 10.7](#) and analyze several different cases. For convenience sake we will call vertices of our simplices as \boxtimes_1 -vertices if their outermost operation is \boxtimes_1 , and \boxtimes_2 -vertices if their outermost operation is \boxtimes_2 .

Case (a): Suppose $\bar{\alpha}$ contains both \boxtimes_1 -vertices and \boxtimes_2 -vertices, the new vertex λ is a \boxtimes_1 -vertex and has a \boxtimes_1 decomposition compatible with the maximal common \boxtimes_1 decomposition of the \boxtimes_1 -vertices of $\bar{\alpha}$.

In this case both simplices $\bar{\alpha}$ and $\bar{\beta}$ are in Case 1 of the proof of Proposition 10.7. Thus the inclusion $(W|\mathcal{N}\mathcal{M}_k|)(S_{\bar{\beta}}) \subset (W|\mathcal{N}\mathcal{M}_k|)(S_{\bar{\alpha}})$ takes the form

$$\begin{aligned} & W|\mathcal{N}\mathcal{M}_k|(p) \times_{W|\mathcal{N}\mathcal{M}_k|(1)^p} \prod_{j=1}^p (W|\mathcal{N}\mathcal{M}_k|)(S_{\bar{\beta} \cap U_j}) \\ & \subset W|\mathcal{N}\mathcal{M}_k|(p) \times_{W|\mathcal{N}\mathcal{M}_k|(1)^p} \prod_{j=1}^p (W|\mathcal{N}\mathcal{M}_k|)(S_{\bar{\alpha} \cap U_j}) \end{aligned}$$

and is thus compatible with the cellular decomposition by the induction hypothesis. Similarly the inclusion $(W|\mathcal{N}\mathcal{M}_l|)(T_{\bar{\beta}}) \subset (W|\mathcal{N}\mathcal{M}_l|)(T_{\bar{\alpha}})$ takes the form

$$\begin{aligned} & W|\mathcal{N}\mathcal{M}_l|(q) \times_{W|\mathcal{N}\mathcal{M}_l|(1)^q} \prod_{j=1}^q (W|\mathcal{N}\mathcal{M}_l|)(T_{\bar{\beta} \cap V_j}) \\ & \subset W|\mathcal{N}\mathcal{M}_l|(q) \times_{W|\mathcal{N}\mathcal{M}_l|(1)^q} \prod_{j=1}^q (W|\mathcal{N}\mathcal{M}_l|)(T_{\bar{\alpha} \cap V_j}) \end{aligned}$$

which again is compatible with the cellular decompositions for the same reason.

A similar analysis, reducing the proof to the induction hypothesis, applies in Cases (b), (c) and (d) below.

Case (b): Suppose $\bar{\alpha}$ contains both \boxtimes_1 -vertices and \boxtimes_2 -vertices, the new vertex λ is a \boxtimes_2 -vertex and has a \boxtimes_2 decomposition compatible with the maximal common \boxtimes_2 decomposition of the \boxtimes_2 -vertices of $\bar{\alpha}$.

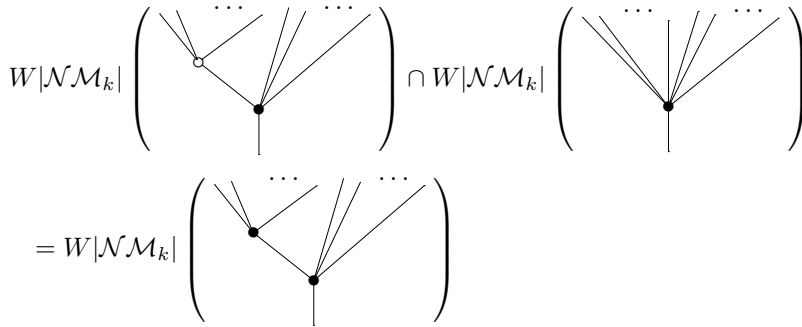
Case (c): Suppose $\bar{\beta}$ contains only \boxtimes_1 -vertices and the new vertex λ has a \boxtimes_1 decomposition compatible with the maximal common \boxtimes_1 decomposition of the vertices of $\bar{\alpha}$.

Case (d): Suppose $\bar{\beta}$ contains only \boxtimes_2 -vertices and the new vertex λ has a \boxtimes_2 decomposition compatible with the maximal common \boxtimes_2 decomposition of the vertices of $\bar{\alpha}$.

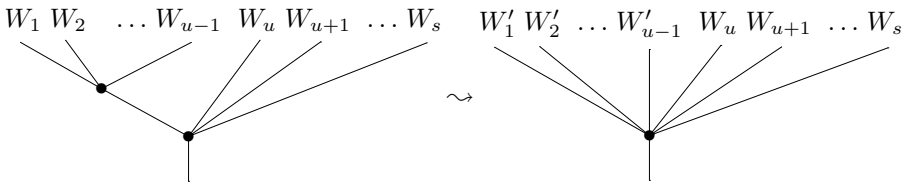
Case (e): Suppose $\bar{\alpha}$ contains both \boxtimes_1 -vertices and \boxtimes_2 -vertices and the new vertex λ is a \boxtimes_1 -vertex which does not have a \boxtimes_1 decomposition compatible with the maximal common \boxtimes_1 decomposition of the \boxtimes_1 -vertices of $\bar{\alpha}$.

Again both simplices $\bar{\alpha}$ and $\bar{\beta}$ are in Case 1 of the proof of Proposition 10.7 but now the maximal common \boxtimes_1 -decomposition of the \boxtimes_1 -vertices of $\bar{\alpha}$ is finer (i.e. has more summands) than the maximal common \boxtimes_1 -decomposition of the \boxtimes_1 -vertices of $\bar{\beta}$.

In this case the inclusion $W|\mathcal{N}\mathcal{M}_k|(S_{\bar{\beta}}) \subset W|\mathcal{N}\mathcal{M}_k|(S_{\bar{\alpha}})$ corresponds to a sequence of intersections of the type



Hence $S_{\bar{\alpha}}$ can be obtained from $S_{\bar{\beta}}$ by a sequence of the following types of moves:



where W_j and W'_j are obtained by restricting $S_{\bar{\beta}}$ and $S_{\bar{\alpha}}$, respectively, to the same set of inputs. The induced map on $W|\mathcal{NM}_k|(-)$ is the product of the composition map:

$$W|\mathcal{NM}_k|(s - u + 1) \times W|\mathcal{NM}_k|(u - 1) \longrightarrow W|\mathcal{NM}_k|(s)$$

$$(a, b) \mapsto a \circ (b \oplus id \oplus \dots \oplus id),$$

which is obviously compatible with the cellular decomposition, and the inclusion map:

$$\prod_{j=1}^s W|\mathcal{NM}_k|(W_j) \subset \prod_{j=1}^s W|\mathcal{NM}_k|(W'_j)$$

which is compatible with the cellular decomposition by induction hypothesis. The induced map $(W|\mathcal{NM}_l|)(T_{\bar{\beta}}) \subset (W|\mathcal{NM}_l|)(T_{\bar{\alpha}})$ takes the form

$$W|\mathcal{NM}_l|(q) \times_{W|\mathcal{NM}_l|(1)^q} \prod_{j=1}^q (W|\mathcal{NM}_l|)(T_{\bar{\beta} \cap V_j})$$

$$\subset W|\mathcal{NM}_l|(q) \times_{W|\mathcal{NM}_l|(1)^q} \prod_{j=1}^q (W|\mathcal{NM}_l|)(T_{\bar{\alpha} \cap V_j}),$$

for which compatibility with the cellular decomposition follows by induction hypothesis.

Case (f): Suppose $\bar{\alpha}$ contains both \boxtimes_1 -vertices and \boxtimes_2 -vertices and the new vertex λ is a \boxtimes_2 -vertex which does not have a \boxtimes_2 decomposition compatible with the maximal common \boxtimes_2 decomposition of the \boxtimes_2 -vertices of $\bar{\alpha}$.

In this case, a similar analysis as in Case (e) applies, with the roles of the inclusions $(W|\mathcal{NM}_k|)(S_{\bar{\beta}}) \subset (W|\mathcal{NM}_k|)(S_{\bar{\alpha}})$ and $(W|\mathcal{NM}_l|)(T_{\bar{\beta}}) \subset (W|\mathcal{NM}_l|)(T_{\bar{\alpha}})$ reversed.

Case (g): Suppose $\bar{\alpha}$ contains only \boxtimes_1 -vertices and the new vertex λ is a \boxtimes_1 -vertex which does not have a \boxtimes_1 decomposition compatible with the maximal common \boxtimes_1 decomposition of the vertices of $\bar{\alpha}$.

In this case the argument that the resulting inclusion is compatible with the cellular decomposition is similar to that in Case (e).

Case (h): Suppose $\bar{\alpha}$ contains only \boxtimes_2 -vertices and the new vertex λ is a \boxtimes_2 -vertex which does not have a \boxtimes_2 decomposition compatible with the maximal common \boxtimes_2 decomposition of the vertices of $\bar{\alpha}$.

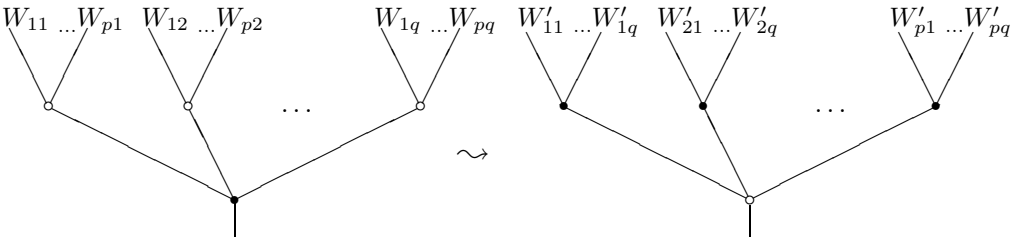
In this case the argument that the resulting inclusion is compatible with the cellular decomposition is similar to that in Case (f).

Case (i): Suppose $\bar{\alpha}$ contains only \boxtimes_1 -vertices and the new vertex λ is \boxtimes_2 -vertex.

In this case the induced inclusion $(W|\mathcal{NM}_k|)(S_{\bar{\beta}}) \subset (W|\mathcal{NM}_k|)(S_{\bar{\alpha}})$ takes the form

$$\begin{aligned} &W|\mathcal{NM}_k|(p) \times_{W|\mathcal{NM}_k|(1)^p} \prod_{j=1}^p (W|\mathcal{NM}_k|)(S_{\bar{\beta} \cap U_j}) \\ &\subset W|\mathcal{NM}_k|(p) \times_{W|\mathcal{NM}_k|(1)^p} \prod_{j=1}^p (W|\mathcal{NM}_k|)(S_{\bar{\alpha} \cap U_j}) \end{aligned}$$

and is thus compatible with the cellular decomposition by the induction hypothesis. The inclusion $(W|\mathcal{NM}_l|)(T_{\bar{\beta}}) \subset (W|\mathcal{NM}_l|)(T_{\bar{\alpha}})$ is more complicated. The trees $T_{\bar{\beta}}$ and $T_{\bar{\alpha}}$ are related as shown below:



Here W_{ij} and W'_{ij} have the same sets of inputs, some of which may be empty, in which case the trees are understood to be empty as well. The induced inclusion $(W|\mathcal{NM}_l|)(T_{\bar{\beta}}) \subset (W|\mathcal{NM}_l|)(T_{\bar{\alpha}})$ is then the product of the maps

$$\mathcal{NM}_l(q) \longrightarrow \prod_{j=1}^p \mathcal{NM}_l(a_j)$$

and

$$\prod_{i,j} \mathcal{NM}_l(W_{ij}) \subset \prod_{i,j} \mathcal{NM}_l(W'_{ij}).$$

Here a_i is the cardinality of $\bigcup_{j=1}^q |W'_{ij}|$ and the first map is given by

$$\phi \mapsto (\phi \cdot (\epsilon_{11} \oplus \dots \oplus \epsilon_{1q}), \phi \cdot (\epsilon_{21} \oplus \dots \oplus \epsilon_{2q}), \dots, \phi \cdot (\epsilon_{p1} \oplus \dots \oplus \epsilon_{pq})),$$

where $\epsilon_{ij} = id \in \mathcal{NM}_l(1)$ if $|W'_{ij}|$ is nonempty or $\epsilon_{ij} = 0 \in \mathcal{NM}_l(0)$ otherwise. Clearly this is compatible with the cellular decompositions while the same is true for the second map by the induction hypothesis.

Case (j): Suppose $\bar{\alpha}$ contains only \boxtimes_2 -vertices and the new vertex λ is \boxtimes_1 -vertex.

In this case we argue as in Case (i), except with the roles of the inclusions $(W|\mathcal{NM}_k|)(S_{\bar{\beta}}) \subset (W|\mathcal{NM}_k|)(S_{\bar{\alpha}})$ and $(W|\mathcal{NM}_l|)(T_{\bar{\beta}}) \subset (W|\mathcal{NM}_l|)(T_{\bar{\alpha}})$ interchanged.

This concludes the proof of part (2).

To prove part (3) we proceed by a double induction on m and r . If $m = 1$ or $r = 1$, there is nothing to prove. So assume we have established the result for $m' < m$ and all r' as well as for $m' = m$ and $r' \leq r$. Now consider an $(r + 1)$ -simplex $\bar{\beta}$ in \mathfrak{R}_\bullet obtained by adding a new vertex λ to an r -simplex $\bar{\alpha}$. Suppose we are given an element $x \in \bigcap_{i=0}^r G_m^{ax}(\alpha_i) \cap G_m^{ax}(\lambda)$, and let $y \in G_m(\lambda)$ be the unique preimage of x . It suffices to find a single vertex α_j such that the preimage $z_j \in G_m(\alpha_j)$ of x is the same element of $(W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m)$ as y . For then we would have $y = z_j \in G_m(\alpha_j) \cap G_m(\lambda)$ and by the second induction hypothesis $z_j \in \bigcap_{i=0}^r G_m^{ax}(\alpha_i)$. Thus we would have $y = z_j \in \bigcap_{i=0}^r G_m(\alpha_i) \cap G_m(\lambda)$ and we would be done.

To establish this fact, we do the same case by case analysis as in the proof of part (2). In Cases (a)–(d), that analysis establishes that we can find a vertex α_j such that y and z_j have tree representatives whose root nodes are identical, and all the axial images of the corresponding tree branches of y and z_j above the common root node are the same. By the primary induction hypothesis, these tree branches of y and z_j are also the same in $W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|$. Hence $y = z_j$.

In Cases (e)–(h), the analysis of the proof of part (2) shows that y has a tree representative which can be converted by series of compositions, either in $W|\mathcal{NM}_k|$ or in $W|\mathcal{NM}_l|$, into a tree which is related to some $z_j \in G_m(\alpha_j)$ in the same way as in Cases (a)–(d). By the same argument as above $y = z_j$. In Cases (i) and (j), that analysis shows y has a tree representative which can be converted by an interchange into a tree which is related to some $z_j \in G_m(\alpha_j)$ in the same way as in Cases (a)–(d). Again we can conclude that $y = z_j$. \square

By combining the above results, we obtain the following more precise version of [Proposition 5.12](#).

10.9 Corollary.

- (1) $(W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m)$ is the colimit of the diagram $G_m : \mathcal{I}(m) \rightarrow \text{Top}$.
- (2) The maps in this diagram are all cofibrations.

- (3) Each space $G_m(\bar{\alpha})$ has a cellular decomposition over the poset $\mathcal{M}_k(S_{\bar{\alpha}}) \times \mathcal{M}_l(T_{\bar{\alpha}})$ and the maps in the diagram are compatible with these cellular decompositions.
- (4) $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)$ is axial and left factorial.

Proof. Part (1) is tautological. In view of Proposition 10.8, we have for any object $\bar{\alpha}$ in $\mathcal{I}(m)$, or equivalently any simplex $\bar{\alpha} = \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ in $\mathfrak{R}_\bullet(m)$

$$G_m(\bar{\alpha}) = \prod_{i=0}^r G_m(\alpha_i) \cong \prod_{i=0}^r G_m^{ax}(\alpha_i) = (W|\mathcal{N}\mathcal{M}_k|)(S_{\bar{\alpha}}) \times (W|\mathcal{N}\mathcal{M}_l|)(T_{\bar{\alpha}}).$$

Thus the maps in the diagram are realizations of simplicial injections and thus are cofibrations. Part (3) follows from parts (1) and (2) of Proposition 10.8. Axiality follows from part (3) of Proposition 10.8, whereas left factoriality immediately follows from the obvious observation that the product of two left factorial monoids is left factorial. \square

11. The Grothendieck construction for interchanges

According to Corollary 10.9, $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ is homeomorphic to an iterated colimit of diagrams of cofibrations. We can now apply Proposition 5.2 of [10] which identifies such an iterated colimit as a simple colimit of the Grothendieck construction of the diagram of categories which to each node $\bar{\alpha}$ of the outer diagram assigns the category indexing the inner colimit at that node.

11.1 Definition. Let $F : \mathcal{A} \rightarrow \mathcal{C}at$ be any functor. Then the Grothendieck construction $\mathcal{A} \int F$ is the category whose objects are pairs (A, B) with $A \in \text{Ob } \mathcal{A}$ and with $B \in \text{Ob } F(A)$. A morphism $(A_1, B_1) \rightarrow (A_2, B_2)$ is a pair (α, β) where $\alpha : A_1 \rightarrow A_2$ and $\beta : F(\alpha)(B_1) \rightarrow B_2$.

11.2 Definition. In Corollary 10.9 the appropriate diagram for the Grothendieck construction is the diagram of posets $\hat{G}'_m : \mathcal{I}(m) \rightarrow \mathcal{C}at$ given by $\hat{G}'_m(\bar{\alpha}) = \mathcal{M}_k(S_{\bar{\alpha}}) \times \mathcal{M}_l(T_{\bar{\alpha}})$. We denote the Grothendieck construction $\mathcal{I}(m) \int \hat{G}'_m$ by $\mathcal{I}(k, l)(m)$.

Thus we have established that

11.3 Proposition. $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ has a cellular decomposition indexed by $\mathcal{I}(k, l)(m)$. This decomposition assigns to an object $(\bar{\alpha}, \beta)$ the cell $G'_m(\bar{\alpha}, \beta)$ in $G_m(\bar{\alpha}) \cong (W|\mathcal{N}\mathcal{M}_k|)(S_{\bar{\alpha}}) \times (W|\mathcal{N}\mathcal{M}_l|)(T_{\bar{\alpha}})$ indexed by $\beta \in \mathcal{M}_k(S_{\bar{\alpha}}) \times \mathcal{M}_l(T_{\bar{\alpha}})$

We now prove Proposition 5.15. We first need the following result.

11.4 Lemma. Let $\bar{\alpha}$ be a simplex in $\mathfrak{R}_\bullet(m)$. Consider the set $\bar{\gamma}$ of all objects $\lambda \in \mathfrak{M}_2^{ab}(m) = \text{Ob } \mathcal{M}_2^{ab}(m)$ such that $G_m(\bar{\alpha}) \subseteq G_m(\lambda)$. Then $\bar{\gamma}$ contains a unique minimal element

$L'(\bar{\alpha})$ with respect to the order relation on the poset $\mathcal{M}_2^{ab}(m)$. The mapping $\bar{\alpha} \mapsto L'(\bar{\alpha})$ defines a poset map $L' : \mathcal{I}(m) \rightarrow \mathcal{M}_2^{ab}(m)$.

Proof. By construction and Proposition 10.3, $\bar{\gamma}$ is also a simplex in $\mathfrak{K}_\bullet(m)$ and we have $G_m(\bar{\alpha}) = G_m(\bar{\gamma})$. Hence wolog we may take $\bar{\alpha} = \bar{\gamma}$, i.e. that $G_m(\bar{\alpha}) \subseteq G_m(\lambda)$ implies $\lambda \in \bar{\alpha}$.

Let $(S_{\bar{\alpha}}, T_{\bar{\alpha}})$ be the pair of binodal trees corresponding to $\bar{\alpha}$ in Proposition 10.7. We define $L'(\bar{\alpha})$ as follows. In the tree $S_{\bar{\alpha}}$ replace each node label \bullet by node label \boxtimes_1 and each node label \circ by node label \boxtimes_2 . Let $L'(\bar{\alpha})$ be the element of $\mathcal{M}_2^{ab}(m)$ corresponding to this relabeled tree.

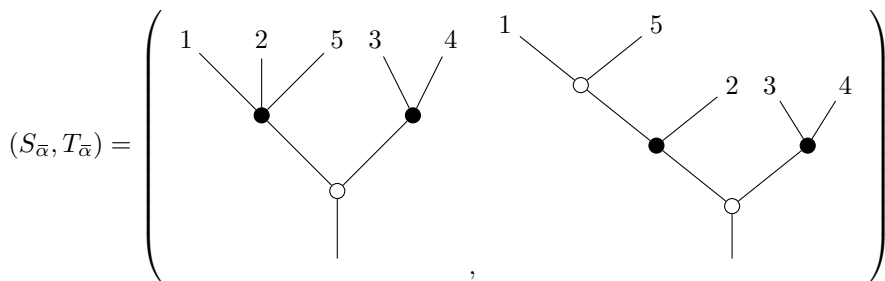
To check that $G_m(\bar{\alpha}) \subseteq G_m(L'(\bar{\alpha}))$, it suffices to check that $G_m(\bar{\alpha} \cap \{a, b, c\}) \subseteq G_m(L'(\bar{\alpha}) \cap \{a, b, c\})$ for all $\{a < b < c\} \subseteq \{1, 2, \dots, m\}$, which is easily verified. To check that $L'(\bar{\alpha}) \leq \alpha_j$ for all vertices α_j in $\bar{\alpha} = \bar{\gamma}$, it suffices to check that $L'(\bar{\alpha}) \cap \{a, b\} \leq \alpha_j \cap \{a, b\}$ for all $\{a < b\} \subseteq \{1, 2, \dots, m\}$, which is even easier to verify.

Finally if $\bar{\beta} \subset \bar{\alpha}$ is an inclusion of simplices in $\mathfrak{K}_\bullet(m)$, then it easily follows that $L'(\bar{\alpha}) \cap \{a, b\} \leq L'(\bar{\beta}) \cap \{a, b\}$ for all $\{a < b\} \subseteq \{1, 2, \dots, m\}$. This implies $L'(\bar{\beta}) \leq L'(\bar{\alpha})$, and so $L' : \mathcal{I}(m) \rightarrow \mathcal{M}_2^{ab}$ is a map of posets. (Recall that the poset structure on $\mathcal{I}(m)$ is opposite to inclusion of simplices, cf. Definition 5.11.) \square

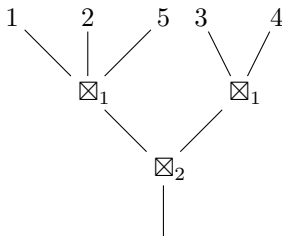
11.5 Example. Consider the following 1-simplex in $\mathfrak{K}_\bullet(5)$

$$\bar{\alpha} = \{(1 \boxtimes_1 2 \boxtimes_1 5) \boxtimes_2 3 \boxtimes_2 4, (1 \boxtimes_1 5) \boxtimes_2 2 \boxtimes_2 (3 \boxtimes_1 4)\}.$$

The pair of binodal trees associated to this simplex is



We obtain $L'(\bar{\alpha})$ by taking the first component of this pair and replacing \bullet by \boxtimes_1 and \circ by \boxtimes_2 . This gives the tree



which corresponds to the element $L'(\bar{\alpha}) = (1 \boxtimes_1 2 \boxtimes_1 5) \boxtimes_2 (3 \boxtimes_1 4) \in \mathcal{M}_2^{ab}(5)$. The set $\bar{\gamma}$ is the 3-simplex

$$\{(1 \boxtimes_1 2 \boxtimes_1 5) \boxtimes_2 (3 \boxtimes_1 4), (1 \boxtimes_1 2 \boxtimes_1 5) \boxtimes_2 3 \boxtimes_2 4, (1 \boxtimes_1 5) \boxtimes_2 2 \boxtimes_2 (3 \boxtimes_1 4), (1 \boxtimes_1 5) \boxtimes_2 2 \boxtimes_2 3 \boxtimes_2 4\}.$$

Proof of Proposition 5.15. We construct

$$L : \mathcal{I}(k, l)(m) \longrightarrow \mathcal{M}_{k+l}(m)$$

so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{I}(k, l)(m) & \xrightarrow{L} & \mathcal{M}_{k+l}(m) \\ \downarrow P & & \downarrow P' \\ \mathcal{I}(m) & \xrightarrow{L'} & \mathcal{M}_2^{ab}(m) \end{array}$$

Here $P : \mathcal{I}(k, l)(m) = \mathcal{I}(m) \int \hat{G}'_m \longrightarrow \mathcal{I}(m)$ is the natural map in the Grothendieck construction and $P' : \mathcal{M}_{k+l}(m) \longrightarrow \mathcal{M}_2^{ab}(m)$ is the poset map which replaces \square_i by \boxtimes_1 if $i \leq k$ and by \boxtimes_2 if $i > k$.

The construction of L goes as follows. An object in $\mathcal{I}(k, l)(m)$ is a pair $(\bar{\alpha}, \beta)$, where $\bar{\alpha}$ is an object in $\mathcal{I}(m) = \text{Sd } \mathfrak{K}_\bullet(m)$, or equivalently a simplex $\{\alpha_0, \alpha_1, \dots, \alpha_r\}$ in $\mathfrak{K}_\bullet(m)$, and β is an object in the poset $\mathcal{M}_k(S_{\bar{\alpha}}) \times \mathcal{M}_l(T_{\bar{\alpha}})$. As in the proof of Lemma 11.4 wolog we may assume that $\bar{\alpha} = \bar{\gamma}$.

There is then a unique morphism $\bar{\alpha} \longrightarrow L'(\bar{\alpha})$ in $\mathcal{I}(m)$. Let β' denote the image of β under the corresponding functor $\hat{G}'_m(\bar{\alpha}) \longrightarrow \hat{G}'_m(L'(\bar{\alpha}))$. Then β' is an object in the poset $\mathcal{M}_k(S_{L'(\bar{\alpha})}) \times \mathcal{M}_l(T_{L'(\bar{\alpha})})$. But $S_{L'(\bar{\alpha})}$ and $T_{L'(\bar{\alpha})}$ are trees of the same shape with black nodes in $S_{L'(\bar{\alpha})}$ corresponding to white nodes in $T_{L'(\bar{\alpha})}$ and vice versa. Now an element β' in $\mathcal{M}_k(S_{L'(\bar{\alpha})}) \times \mathcal{M}_l(T_{L'(\bar{\alpha})})$ can be identified with a pair of planar trees (S', T') with S' looking exactly like $S_{L'(\bar{\alpha})}$, except that every black node is decorated with an object of \mathcal{M}_k of the appropriate arity and with T' looking exactly like $T_{L'(\bar{\alpha})}$, except that every black node is decorated with an object of \mathcal{M}_l of the appropriate arity. Putting together the decorations on S' and T' , we obtain a tree all of whose nodes are decorated with objects either of \mathcal{M}_k or of \mathcal{M}_l . Regarding \mathcal{M}_k and \mathcal{M}_l as suboperads of \mathcal{M}_{k+l} as in paragraph 4.7, we can interpret the resulting tree as an object β'' in \mathcal{M}_{k+l} , by considering the edges of the tree to be compositions in the operad \mathcal{M}_{k+l} . We define $L(\bar{\alpha}, \beta) = \beta''$. This is easily checked to be a map of posets.

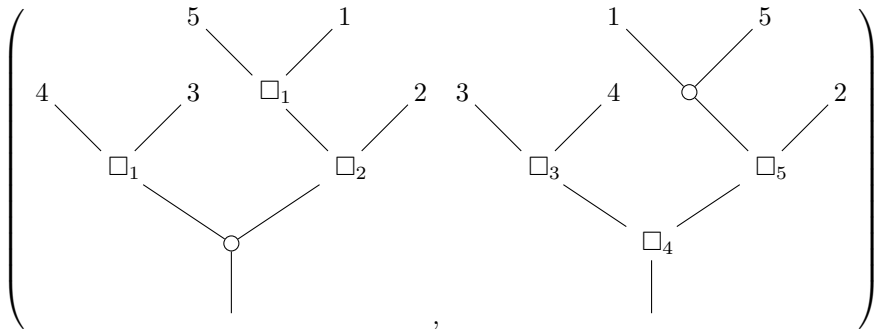
To check Quillen’s Theorem A for the functor L , we show that the over categories L/α have contractible nerves for any object α in $\mathcal{M}_{k+l}(m)$. We note that we can obtain

an object α' in $\overline{\mathcal{M}}_2(m)$ by replacing each \square_i in α by \boxtimes_1 if $i \leq k$ and by \boxtimes_2 if $i > k$. This resulting object α' can also be regarded as an object in $\mathcal{I}(m)$. It is easy to see that there is a unique object β' in $\mathcal{M}_k(S_{\alpha'}) \times \mathcal{M}_l(T_{\alpha'})$ such that $L(\alpha', \beta') = \alpha$ and that $id : L(\alpha', \beta') \rightarrow \alpha$ is the terminal object in L/α . \square

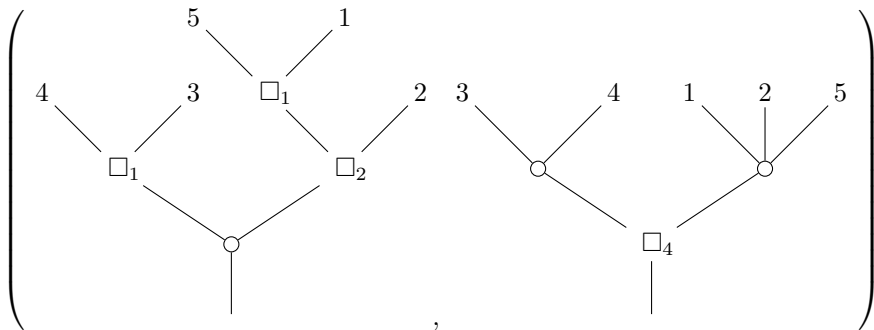
11.6 Example. We illustrate the construction L with an example based on [Example 11.5](#). Let us take $k = 2$ and $l = 3$. Now consider an object $(\bar{\alpha}, \beta) \in \mathcal{I}(2, 3)(5)$. The second component β is an element in

$$\mathcal{M}_2(S_{\bar{\alpha}}) \times \mathcal{M}_3(T_{\bar{\alpha}}) \cong \mathcal{M}_2(3) \times \mathcal{M}_2(2) \times \mathcal{M}_3(2)^3.$$

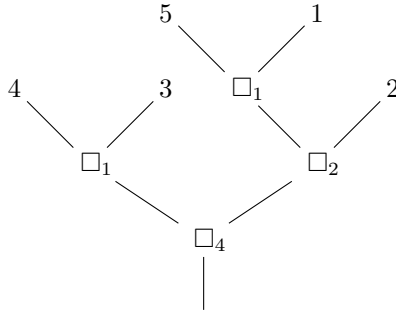
The object $(\bar{\alpha}, \beta)$ can be represented by a pair of trees $(S_{\bar{\alpha}, \beta}, T_{\bar{\alpha}, \beta})$, obtained from the pair of binodal trees $(S_{\bar{\alpha}}, T_{\bar{\alpha}})$ in [Example 11.5](#) as follows. We replace each black node v of $S_{\bar{\alpha}}$ by the tree representing the object in $\mathcal{M}_2(\text{In}(v))$, specified by the component β , to obtain $S_{\bar{\alpha}, \beta}$. Likewise we replace each black node w of $T_{\bar{\alpha}}$ by the tree representing the object in $\mathcal{M}_3(\text{In}(w))$, specified by the component β , to obtain $T_{\bar{\alpha}, \beta}$. Thus a typical object $(\bar{\alpha}, \beta)$ might be represented by the pair of trees



Then $(L'(\bar{\alpha}), \beta')$ is represented by



and $L(\bar{\alpha}, \beta)$ is represented by the tree



Thus $L(\bar{\alpha}, \beta) = (4 \square_1 3) \square_4 ((5 \square_1 1) \square_2 2)$.

Thus we have established a chain of equivalences

$$\begin{aligned} (W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m) &\cong \operatorname{colim}_{\mathcal{I}(k,l)(m)} G'_m \xleftarrow{\cong} \operatorname{hocolim}_{\mathcal{I}(k,l)(m)} G'_m \\ &\xrightarrow{\cong} \operatorname{hocolim}_{\mathcal{I}(k,l)(m)^*} = |\mathcal{NI}(k,l)(m)| \\ &\xrightarrow{\cong} |\mathcal{NM}_{k+l}|(m). \end{aligned}$$

We now need to show that this chain of maps is compatible with the operad structures on the two ends of the chain. Thus we need to construct operad structures on all the intervening spaces and to show that the maps are homomorphisms between these structures.

We begin with a simple construction.

11.7 Lemma. *Let \mathcal{C} be an operad in Sets such that $\mathcal{C}(m)$ is finite for each $m \geq 0$. Let $\mathfrak{D}(m)$ be the simplex whose vertex set is $\mathcal{C}(m)$, regarded as an abstract simplicial complex in the classical sense. Then the barycentric subdivisions $\operatorname{Sd} \mathfrak{D} = \{\operatorname{Sd} \mathfrak{D}(m)\}_{m \geq 0}$ have a natural structure of an operad in $\mathcal{C}at$.*

Proof. Define the operad composition

$$\operatorname{Sd} \mathfrak{D}(k) \times \prod_{i=1}^k \operatorname{Sd} \mathfrak{D}(m_i) \longrightarrow \operatorname{Sd} \mathfrak{D}(m_1 + m_2 + \dots + m_k)$$

by

$$\{a_i\} \circ (\oplus \{b_{ij}\}) = \{a_i \circ (b_{1j_1} \oplus b_{2j_2} \oplus \dots \oplus b_{kj_k})\}.$$

In other words, take the barycenter of the simplex whose vertices are all possible compositions of the vertices of the inputs. The symmetric group actions are induced by the symmetric group actions on the vertices. It is straightforward to check that this defines an operad structure in $\mathcal{C}at$. \square

11.8 Proposition.

- (i) $\mathcal{I} = \{\mathcal{I}(m)\}_{m \geq 0}$ has a *Cat-operad structure*.
- (ii) The colimit decomposition of the operad $W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|$ over \mathcal{I} , given in [Proposition 5.12](#) is compatible with this operad structure.
- (iii) The functor $L' : \mathcal{I} \rightarrow \mathcal{M}_2^{ab}$ of [Lemma 11.4](#) is a map of *Cat-operads*.

Proof. In [Lemma 11.7](#) take the set operad \mathcal{C} to be \mathfrak{M}_2^{ab} . Then by definition $\mathcal{I}(m)$ is a subset of $\text{Sd } \mathfrak{D}(m)$ (where we take the order relation in both cases to be opposite to inclusion of faces). Thus to prove (i), we have to show that the operad structure on $\text{Sd } \mathfrak{D}$ restricts to \mathcal{I} .

Suppose that $\bar{\alpha} \in \mathcal{I}(m)$ and that $\bar{\beta}_i \in \mathcal{I}(n_i)$, for $i = 1, 2, \dots, m$. Then it is clear from [Definition 5.11](#) that the operad composition

$$\begin{aligned} & (W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(m) \times \prod_{i=1}^m (W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(n_i) \\ & \longrightarrow (W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|)(n_1 + n_2 + \dots + n_m) \end{aligned}$$

restricts to

$$G(\bar{\alpha}) \times \prod_{i=1}^m G(\bar{\beta}_i) \longrightarrow \bigcap_{\lambda \in \bar{\alpha} \circ (\bigoplus_{i=1}^m \bar{\beta}_i)} G(\lambda),$$

where $\bar{\alpha} \circ (\bigoplus_{i=1}^m \bar{\beta}_i)$ denotes operad composition in $\text{Sd } \mathfrak{D}$. It follows that $\bigcap_{\lambda \in \bar{\alpha} \circ (\bigoplus_{i=1}^m \bar{\beta}_i)} G(\lambda)$ is nonempty. By [Proposition 5.12](#) it follows that $\bar{\alpha} \circ (\bigoplus_{i=1}^m \bar{\beta}_i) \in \mathcal{I}(n_1 + n_2 + \dots + n_k)$ and $\bigcap_{\lambda \in \bar{\alpha} \circ (\bigoplus_{i=1}^m \bar{\beta}_i)} G(\lambda) = G(\bar{\alpha} \circ (\bigoplus_{i=1}^m \bar{\beta}_i))$. This proves (i) and also (ii), since the additional condition that the colimit decomposition behaves appropriately under the symmetric group actions is obviously true.

Finally to prove (iii) we first note that since both \mathcal{I} and \mathcal{M}_2^{ab} are operads in the category of posets, it suffices that L' induces a map of operads on objects, which are operads in *Sets*. To check this, we observe that this map decomposes as a composite

$$\text{Ob } \mathcal{I} \xrightarrow{S} \mathcal{BT} \xrightarrow{V} \text{Ob } \mathcal{M}_2^{ab} = \mathfrak{M}_2^{ab},$$

where \mathcal{BT} is the operad of binodal trees (cf. [Definition 9.12](#)), $S(\bar{\alpha}) = S_{\bar{\alpha}}$, and V replaces each node label \bullet by node label \boxtimes_1 and each node label \circ by node label \boxtimes_2 . Both S and V are easily seen to be operad maps. \square

The following result establishes the desired operad equivalence between $W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|$ and \mathcal{M}_{k+l} and shows that the former is an E_{k+l} -operad.

11.9 Theorem.

- (i) $\mathcal{I}(k, l) = \{\mathcal{I}(k, l)(m)\}_{m \geq 0}$ has a *Cat-operad structure*.

- (ii) The cellular decomposition of the operad $W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|$ over $\mathcal{I}(k, l)$, given in [Proposition 11.3](#) is compatible with this operad structure.
- (iii) The equivalence $L : \mathcal{I}(k, l) \rightarrow \mathcal{M}_{k+l}$ of [Proposition 5.15](#) is a map of *Cat*-operads.

Proof. We define the operad structure on $\text{Ob}\mathcal{I}(k, l)$ by observing that

$$\text{Ob}\mathcal{I}(k, l) \subset \text{Ob}\mathcal{I} \times \text{Ob}(\mathcal{M}_k)(\mathcal{BT}) \times \text{Ob}(\mathcal{M}_l)(\mathcal{BT})$$

and noting that the operad structure on $\text{Ob}\mathcal{I} \times \text{Ob}(\mathcal{M}_k)(\mathcal{BT}) \times \text{Ob}(\mathcal{M}_l)(\mathcal{BT})$ restricts to $\text{Ob}\mathcal{I}(k, l)$. This follows from the evident fact that

$$S : \text{Ob}\mathcal{I} \rightarrow \mathcal{BT}, \quad T : \text{Ob}\mathcal{I} \rightarrow \mathcal{BT}$$

given by $S(\bar{\alpha}) = S_{\bar{\alpha}}$, $T(\bar{\alpha}) = T_{\bar{\alpha}}$ respectively, are operad maps. It is straightforward to check that this operad structure on objects extends to a *Cat*-operad structure on the poset operad $\mathcal{I}(k, l)$. This proves part (i).

To prove part (ii), we note that according to [Proposition 11.8\(ii\)](#), operad composition in $W|\mathcal{NM}_k| \otimes W|\mathcal{NM}_l|$ restricts to maps of the form

$$G(\bar{\alpha}) \times \prod_{i=1}^m G(\bar{\beta}_i) \rightarrow G\left(\bar{\alpha} \circ \left(\bigoplus_{i=1}^m \bar{\beta}_i\right)\right).$$

We must show that these restrictions are compatible with the cellular decompositions on both sides. This follows from the commutativity of the diagram

$$\begin{array}{ccc}
 G(\bar{\alpha}) \times \prod_{i=1}^m G(\bar{\beta}_i) & \xrightarrow{\cong} & W|\mathcal{NM}_k(S_{\bar{\alpha}}) \times W|\mathcal{NM}_l(T_{\bar{\alpha}}) \\
 & & \times \prod_{i=1}^m W|\mathcal{NM}_k(S_{\bar{\beta}_i}) \times W|\mathcal{NM}_l(T_{\bar{\beta}_i}) \\
 \downarrow & & \downarrow \\
 & & W|\mathcal{NM}_k(S_{\bar{\alpha} \circ (\bigoplus_{i=1}^m \bar{\beta}_i)}) \times W|\mathcal{NM}_l(T_{\bar{\alpha} \circ (\bigoplus_{i=1}^m \bar{\beta}_i)}) \\
 & & \parallel \\
 G(\bar{\alpha} \circ (\bigoplus_{i=1}^m \bar{\beta}_i)) & \xrightarrow{\cong} & W|\mathcal{NM}_k(S_{\bar{\alpha} \circ (\bigoplus_{i=1}^m \bar{\beta}_i)}) \times W|\mathcal{NM}_l(T_{\bar{\alpha} \circ (\bigoplus_{i=1}^m \bar{\beta}_i)})
 \end{array}$$

























Here the horizontal homeomorphisms are those given in [Proposition 10.8](#) and the right-hand vertical map is operad composition in the operad $(W|\mathcal{NM}_k|)(\mathcal{BT}) \times (W|\mathcal{NM}_l|)(\mathcal{BT})$, which as noted in [Remark 9.15](#), has a cellular decomposition over the poset operad $(\mathcal{M}_k)(\mathcal{BT}) \times (\mathcal{M}_l)(\mathcal{BT})$.

Part (iii) is an immediate consequence of the definition of the operad structure on $\mathcal{I}(k, l)$ and the fact that L' is a map of operads, as shown in [Proposition 11.8](#) part (iii). \square

Acknowledgments

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Appendix A. Intersection table for Proposition 9.8

	T_1	T_2	T_3
1		T_2 arbitrary	T_2
2			
3			
4			
5			
6			
7			
8			
9			nonempty but not of the form $(WB_\bullet)(T_3)$

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