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Non-connective Delooping of K-Theory of an A_∞ Ring Space

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1. Introduction

An A_∞ ring space (May [4], Schwänzl and Vogt [6]) is a space which is a semiring up to homotopy in a specifically coherent way. (A semiring is like a ring with one except that additive inverses are not required.) In some ways an A_∞ ring space X behaves like an algebraic ring. In particular, if X is ringlike (the semiring $\pi_0 X$ of path-components of X is actually a ring), then one can imitate the construction of algebraic K-theory. There is a space glX, which serves as substitute for an infinite general linear group, and one can form the plus-construction $(B\mathrm{gl}X)^+$ of its classifying space $B\mathrm{gl}X$ (May [4], Steiner [9]). The space $(B\mathrm{gl}X)^+$ can be regarded as the algebraic K-theory of X; one sets $K_q X = \pi_q[(B\mathrm{gl}X)^+]$ $(q \geqq 1)$. These ideas are used in geometric topology (Waldhausen [12]).

In the same way as for rings, one can use infinite loop space machinery to show that $(B\mathrm{gl}X)^+$ is an infinite loop space (Steiner [9], Schwänzl and Vogt [7]). In fact there are connected spaces $E_1, E_2, \ldots$ such that

$$\mathbb{Z} \times (B\mathrm{gl}X)^+ \simeq \Omega E_1,\ E_1 \simeq \Omega E_2, \ldots .$$

This delooping has some disadvantages. Very little is known about the spaces E_n. The factor $\mathbb{Z}$ appears because the method works with the analogue of free rather than projective modules.

There is however a different method of delooping the algebraic K-theory of a ring, due to Wagoner [11]. Given a ring R, he constructs another ring SR, the *suspension* of R (in Wagoner [11] SR is denoted μR), such that

$$K_0 R \times (BGLR)^+ \simeq \Omega[K_0 SR \times (BGLSR)^+] .$$

The object of this paper is to generalise Wagoner's delooping to A_∞ ring spaces. The result is as follows.

Theorem 1.1. *There is a suspension construction* s *passing from an* A_∞ *ring space* X *to*

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an A_∞ *ring space* sX. *If* X *is a ringlike* A_∞ *ring space, then* sX *is ringlike and*

$$K_0\pi_0 X \times (BglX)^+ \simeq \Omega[K_0\pi_0 sX \times (BglsX)^+] .$$

This shows that if X is a ringlike A_∞ ring space, then one may take $K_q X$ as $K_0\pi_0 S^{-q}X$ for $q < 0$. In fact one has a nonconnective spectrum KX with

$$\pi_q KX \cong K_q\pi_0 X \quad \text{for } q \leqq 0 .$$

(To see that $\pi_q KX$ is as stated for $q \leqq 0$, note that $\pi_q KX = K_0\pi_0 s^{-q}X$ by 1.1, that $K_0\pi_0 s^{-q}X \cong K_0 S^{-q}\pi_0 X$ by 5.1 below, and that $K_0 S^{-q}\pi_0 X \cong K_q\pi_0 X$ by [Wagoner, end of Section 3].)

Thus the suspension construction produces a definition $K_q X = K_q\pi_0 X$ $(q \leqq 0)$ for a ringlike A_∞ ring space in a natural way. The construction is also being used in the work of Fiedorowicz, Schwänzl and Vogt on the Hermitian K-theory of A_∞ ring spaces with involution, which has applications to spaces of self homotopy equivalences of manifolds.

Section 2 describes a general method for dealing with A_∞ ring spaces. The suspension sX is constructed in Section 3 and proved to be an A_∞ ring space in Section 4. In Section 5 it is shown that sX is ringlike if X is, and the delooping result is proved in Section 6. Sections 7–9 give the proofs of subsidiary results used in Section 6.

2. Constructions on Rings and A_∞ Ring Spaces

Throughout the paper we shall be making constructions on semirings and transferring them to A_∞ ring spaces by the method of Steiner [9]. We first describe this informally; there is a more detailed and more formal description at the end of the section, which may be helpful.

The constructions to which the method applies are functors taking a semiring R to a diagram of sets and functions in which the sets are disjoint unions of powers of R. A function

$$f\colon \coprod_{E\in\mathscr{E}} R^E \to \coprod_{E'\in\mathscr{E}'} R^{E'}$$

between two such disjoint unions must have the following form: there is a function $\theta\colon \mathscr{E} \to \mathscr{E}'$ and f maps each R^E into $R^{\theta(E)}$ by a natural transformation. Such natural transformations are called *semiring operations*; they must have the form

$$(x_e\colon e\in E) \to (y_{e'}\colon e'\in E')$$

where each $y_{e'}$ is a finite sum of non-commutative monomials in the x_e. The operation is called *simple* if each $y_{e'}$ is a sum of distinct monomials. The result of Steiner ([9], 1.4) is that if all the operations involved in such a diagram are simple, then one can construct a similar diagram of spaces and maps for an A_∞ ring space X. The spaces are homotopy equivalent to the appropriate disjoint unions of powers of X and the maps have the appropriate homotopy classes in terms of the homotopy semiring structure of X.

There is a naturality statement: if the construction is applied to a diagram D and to a subdiagram D′, then the diagram resulting from D′ is naturally homotopy equivalent to the appropriate subdiagram of that resulting from D. Because of this we can use different diagrams in different parts of the paper and get compatible results.

In fact we need a slightly more general result than that stated in Steiner [9]: we have to allow infinite powers; that is in $\coprod_E R^E$ the sets E have to be permitted to be infinite. However the proofs of Steiner [9] still work as given.

In the applications of this method the semiring operations involved are mostly addition and multiplication of distinct matrices, and they are easily checked to be simple. We shall not usually mention simplicity explicitly.

As an example we consider the space $B\mathrm{gl}X$ that appears in Theorem 1.1. Let R be a semiring. We have a chain of monoids

$$M_1^\times R \xrightarrow{\subset} M_2^\times \xrightarrow{\subset} \dots,$$

where $M_d R = R^{d^2}$ is the $d \times d$ matrices over R, the superscript $\times$ signifies that one is using matrix multiplication, and the inclusions are direct sums with 1. By applying the bar construction one gets a chain of simplicial sets

$$B_\bullet M_1^\times R \to B_\bullet M_2^\times R \to \dots;$$

when drawn in full this is a diagram containing sets

$$B_q M_d^\times R = (M_d^\times R)^q .$$

The general method now gives a similar chain of simplicial spaces

$$B_\bullet M_1^\times X \to B_\bullet M_2^\times X \to \dots$$

for an A_∞ ring space X. If X is ringlike, then there is a subdiagram

$$B_\bullet \mathrm{gl}_1 X \to B_\bullet \mathrm{gl}_2 X \to \dots$$

got by taking points in the path components corresponding to elements of $GL_d \pi_0 X$ inside $M_d^\times \pi_0 X \cong \pi_0 M_d^\times X$. We can now use geometric realisation and a homotopy colimit to get

$$B\mathrm{gl}X = \mathrm{hocolim}_d B\mathrm{gl}_d X = \mathrm{hocolim}_d |B_\bullet \mathrm{gl}_d X| .$$

Note that by using a *homotopy* colimit we get the correct fundamental group:

$$\pi_1 B\mathrm{gl}X \cong \mathrm{colim}_d \pi_1 B\mathrm{gl}_d X \cong \mathrm{colim}_d \pi_0 \mathrm{gl}_d X$$
$$\cong \mathrm{colim}_d GL_d \pi_0 X \cong GL\pi_0 X .$$

The method described so far produces results up to homotopy. It is adequate for all the results of this paper except the result of section 4: If X is an A_∞ ring space then sX is an A_∞ ring space. This requires the methods of Schwänzl and Vogt [6].

In order to explain this we describe A_∞ ring spaces more formally and in more detail. Let θ_{sr} be the *theory of semi-rings*; this is the category with objects $0, 1, 2, \dots$ and with morphisms from m to n the semiring operations $R^m \to R^n$. Let θ_{sr}^∞ be the corresponding "infinitary" theory; its objects are sets E, finite or infinite, and the morphisms from E to E′ are the semiring operations from R^E to $R^{E'}$, note that there is no question of infinite sums or products. Finally let $\Sigma \int \theta_{sr}^\infty$ be a "wreath product with sets"; its objects are families $\mathscr{E}$ of sets and the morphisms from $\mathscr{E}$ to $\mathscr{E}'$ consist of a function $\theta: \mathscr{E} \to \mathscr{E}'$ and a family of semiring operations $R^E \to R^{\theta(E)}$ ($E \in \mathscr{E}$). We can now give a formal description of the diagrams of disjoint unions of powers of semirings described at the beginning of the section; they are functors $\Phi \to \Sigma \int \theta_{sr}^{\infty}$.

Recall Schwänzl–Vogt ([6], 1.7) that a topological theory θ together with an augmentation $\varepsilon: \theta \to \theta_{sr}$ which is the identity on objects and a homotopy equivalence over simple morphisms (since θ_{sr} is discrete this means that $\varepsilon^{-1}(\varphi)$ is to be contractible if φ is a simple semiring operation) is an A_∞ *ring theory*. A space X is an A_∞ *ring space* if it is a θ-space for some A_∞ ring theory θ; that is, if there is a continuous product-preserving functor $X: \theta \to \mathrm{Top}$ with $X(m) = X^m$ for $m = 0, 1, 2, \ldots$.

The application of a construction $F: \Phi \to \Sigma \int \theta_{sr}^{\infty}$ on semirings to an A_∞ ring space X can now be described as follows. In an evident notation we have functors

$$\begin{array}{ccc} & & \Sigma\int\theta^{\infty} \xrightarrow{\Sigma\int X^{\infty}} \mathrm{Top}. \\ & & \downarrow{\scriptstyle \Sigma\int\varepsilon^{\infty}} \\ \Phi & \xrightarrow{F} & \Sigma\int\theta_{sr}^{\infty} \end{array}$$

Form the pullback

$$\begin{array}{ccc} \not{h} & \xrightarrow{\bar F} & \Sigma\int\theta^{\infty}. \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \Sigma\int\varepsilon^{\infty}} \\ \Phi & \xrightarrow{F} & \Sigma\int\theta_{sr}^{\infty} \end{array}$$

If F involves only simple operations, then ν is an equivalence, so by Segal's construction (Segal [8], Appendix B) one gets a functor $\nu_*[(\Sigma\int X^{\infty}) \circ \bar F]: \Phi \to \mathrm{Top}$ homotopy equivalent to $(\Sigma\int X^{\infty}) \circ \bar F$. This is the required diagram of spaces homotopy equivalent to disjoint unions of powers of X.

In Section 4 we have a case in which $\Phi = \theta_{sr} \times \Delta^{op} \times \Delta^{op}$, where Δ is the simplicial category; thus F corresponds to a construction of bisimplicial semirings from semirings. The function ν is a homotopy equivalence only over morphisms of the form (φ, α, β) with φ simple.

The lifting theorem of Schwänzl and Vogt ([6], 2.16) now gives a commutative diagram of theories

$$\begin{array}{ccc} W(\theta(\mathscr{H}, \mathscr{L}) \times \Delta^{op} \times \Delta^{op}) & \xrightarrow{\eta} & \not{h} \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \nu} \\ \theta(\mathscr{H}, \mathscr{L}) \times \Delta^{op} \times \Delta^{op} & \xrightarrow{\varepsilon \times 1 \times 1} & \theta_{sr} \times \Delta^{op} \times \Delta^{op} \end{array}$$

where $\theta(\mathscr{H}, \mathscr{L})$ is the A_∞ ring theory corresponding to the canonical A_∞ *operad pair* of Steiner [10] and W( ) and μ are such that μ is a homotopy equivalence over *all* morphisms. One can now use Segal's construction as before (or rather a product preserving version, e.g. (Schwänzl and Vogt [6], 4.12), which is compatible with Segal's up to homotopy), to get

$$\mu_*[(\Sigma \textstyle\int X^\infty) \circ \bar{F} \circ \eta]: \theta(\mathscr{H}, \mathscr{L}) \times \Delta^{op} \times \Delta^{op} \to \mathrm{Top} .$$

This is a bisimplicial $\theta(\mathscr{H}, \mathscr{L})$-space, and gives a $\theta(\mathscr{H}, \mathscr{L})$-space (that is, an A_∞ ring space) by double geometric realisation.

3. The Construction of sX

Let R be a ring. Then Wagoner's construction of the suspension SR is as follows (Wagoner [11]). Let CR be the ring of *locally finite* matrices over R; these are infinite matrices with rows and columns indexed by $\mathbb{N} = \{1, 2, \ldots\}$ and with only finitely many non-zero elements in each row and column. Inside CR there is the two-sided ideal MR of *finite matrices* (only finitely many non-zero entries altogether). Then SR is the quotient ring CR/MR. Clearly this construction can also be applied to a semi-ring R and gives a semiring SR.

We now adapt this construction so that the method of Section 2 can be applied. Because of the delicacy of Section 4 we give more details than usual. Let R be a semiring. Call a set A *locally finite* if $A \subset \mathbb{N} \times \mathbb{N}$ and for each $k \in \mathbb{N}$ the sets

$$\{i \in \mathbb{N}: (i, k) \in A\} \quad \text{and} \quad \{j \in \mathbb{N}: (k, j) \in A\}$$

are finite. By a *finite set* we will mean a finite subset of $\mathbb{N} \times \mathbb{N}$. Evidently

$$CR = \operatorname{colim}_{A \textit{ locally finite}} R^A, \; MR = \operatorname{colim}_{A \textit{ finite}} R^A.$$

To get the correct homotopy groups we replace the colimits by homotopy colimits; thus we set

$$cR = \operatorname{hocolim}_{A \textit{ locally finite}} R^A, \; mR = \operatorname{hocolim}_{A \textit{ finite}} R^A.$$

From the definition (Bousfield and Kan [3], XII.2) this means that cR is the realisation of a simplicial set $c_\bullet R$ for which

$$c_q R = \{(x, A(0), \ldots, A(q)): A(0) \subset A(1) \subset \ldots \subset A(q) \text{ are \textit{locally finite sets} and } x \in R^{A(0)}\} .$$

The faces in $c_\bullet R$ are got by omitting $A(i)$'s and the degeneracies by duplicating them. Similarly mR is the realisation of a simplicial set $m_\bullet R$ defined using finite sets. We can now apply the methods of Section 2 to get spaces cX and mX for X an A_∞ ring space such that

$$cX \simeq \operatorname{hocolim}_{A \textit{ locally finite}} X^A, \; mX \simeq \operatorname{hocolim}_{A \textit{ finite}} X^A .$$

To get the suspension we need some analogue for the quotient CR/MR (R a semiring). We note that $c_\bullet R$ is a simplicial commutative monoid, essentially because CR is a filtered commutative monoid; the addition on $C_q R$ is given by

$$(x, A(0), \ldots, A(q)) + (x', A'(0), \ldots, A'(q))$$
$$= ((x+x'), A(0) \cup A'(0), \ldots, A(q) \cup A'(q)).$$

Also $m_\bullet R$ is a simplicial submonoid. We can therefore form the bar construction (May [4], 2.7). It is a bisimplicial set $s_{\bullet\bullet}R = B_\bullet(c_\bullet R, m_\bullet R, o)$ with

$$B_q(c_\bullet R, m_\bullet R, o) = c_\bullet R \times (m_\bullet R)^q \times \{o\} ;$$

the faces are got by adding consecutive entries and the degeneracies by inserting zeros.

By the method of Section 2 we get a bisimplicial space $s_{\bullet\bullet}X$ from an A_∞ ring space X; we define the *suspension* sX of X as its double geometric realisation. Thus

$$sX \simeq B(cX, mX, o)$$

is a plausible substitute for a quotient of cX by mX.

4. The A_∞ Ring Structure on sX

In this section we show that sX is an A_∞ ring space for any A_∞ ring space X, thus proving the first part of Theorem 1.1. By the argument at the end of Section 2, it suffices to construct a bisimplicial semiring structure on $s_{\bullet\bullet}R$ (R a semiring).

We first note that $c_\bullet R$ is a simplicial semiring, essentially because CR is a filtered semiring. The addition of $c_q R$ was given in the last section; the multiplication is given by

$$(x, A(0), \ldots, A(q)) \circ (x', A'(0), \ldots, A'(q))$$
$$= (xx', A(0) \circ A'(0), \ldots, A(q) \circ A'(q)) ,$$

where for locally finite sets A and A′ the locally finite set $A \circ A'$ is defined by

$$A \circ A' = \{(i, j) \colon \text{there exists } k \text{ such that } (i, k) \in A \text{ and } (k, j) \in A'\}.$$

One also checks that $m_\bullet R$ is a simplicial two-sided ideal of $c_\bullet R$; essentially this is because MR is an ideal of CR. To complete the proof that $s_{\bullet\bullet}R$ is a simplicial ring one uses the following result.

Proposition 4.1. *Let* S *be a semiring and* I *be a two-sided ideal. Then* $B_\bullet(S, I, o)$ *is a simplicial semiring.*

Proof. The addition on $B_q(S, I, o) = S \times I^q$ is componentwise. The multiplication is given by

$$(x_0, \ldots, x_q)(y_0, \ldots, y_q) =$$
$$(x_0 y_0, (x_0 + x_1)(y_0 + y_1) - x_0 y_0, \ldots, (x_0 + \cdots + x_q)(y_0 + \cdots + y_q)$$
$$- (x_0 + \cdots + x_{q-1})(y_0 + \cdots + y_{q-1})) .$$

One can check that this makes sense over a semiring and makes $B_\bullet(S, I, o)$ a simplicial semiring.

5. sX is Ringlike

For the rest of this paper X will be a ringlike A_∞ ring space. In this section we prove that sX is ringlike, that is $\pi_0 sX$ is a ring. In fact we describe all the homotopy groups of sX. To do this, we extend the functors M, C and S to abelian groups. Thus, if G is an abelian group, then there are abelian groups

$$MG = \operatorname{colim}_{A\ finite} G^A ,$$
$$CG = \operatorname{colim}_{A\ locally\ finite} G^A ,$$
$$SG = CG/MG .$$

Proposition 5.1. *For all q,*

$$\pi_q mX \cong M\pi_q X,\ \pi_q cX \cong C\pi_q X,\ \pi_q sX \cong S\pi_q X .$$

In particular $\pi_0 sX$ *is the ring* $S\pi_0 X$, *so* sX *is ringlike.*

Proof. Since $mX = \operatorname{hocolim}_{A\ finite} X^A$,

$$\pi_q mX \cong \operatorname{colim}_{A\ finite} \pi_q(X^A) \cong \operatorname{colim}_{A\ finite} (\pi_q X)^A \cong M\pi_q X .$$

The same argument works for cX as π_* commutes with arbitrary products. In particular $\pi_0(mX)$ is an abelian group. Imitating the theory of the two-sided bar construction for monoids (May [5], 7.9, 8.6) using (Schwänzl and Vogt [6], 2.16; Boardman and Vogt [1], 4.58, 4.23) we get a fibration

$$\begin{array}{c} cX \to B(cX, mX, o) \to B(o, mX, o) , \\ \wr\!\! \mid \\ sX \end{array}$$

where $B(o, mX, o)$ is constructed in a similar way to sX. Now $\Omega B(o, mX, o) \simeq mX$, so we get a fibration

$$mX \to cX \to sX . \tag{5.2}$$

Since $\pi_q mX \to \pi_q cX$ is the inclusion $M\pi_q X \to C\pi_q X$ for each q, the homotopy exact sequence of (5.2) splits into short exact sequences and

$$\pi_q sX \cong \pi_q cX/\pi_q mX \cong C\pi_q X/M\pi_q X = S\pi_q X ,$$

as required.

6. The Delooping Result

In this section we outline the proof of the delooping result

$$K_0\pi_0 X \times (BglX)^+ \simeq \Omega[K_0\pi_0 sX \times (BglsX)^+] .$$

The general plan is a close imitation of Wagoner [11], but many of the proofs use Berrick [2]. Roughly speaking, one tries to get a fibration by applying $(B\mathrm{gl})^+$ to the fibration (5.2). However mX corresponds to MR (R a ring), which is a ring without one, so GLMR does not make sense. We therefore consider a new multiplication on $\mathrm{M_d S}$ (S a semiring without one) defined by

$$x \circ y = x \cdot y + x + y$$

($x \circ y$ is the new product, $x \cdot y$ is the original product). Under this product $\mathrm{M_d S}$ is a monoid, which we denote $\mathrm{M^\circ_d S}$. It has 0 as its identity element. If S is a ring (still without one) then we denote the group of invertible elements by $\mathrm{GL^\circ_d S}$. We embed $\mathrm{M^\circ_d S}$ in $\mathrm{M^\circ_{d+1} S}$ by taking direct sums with 0; in this way we get $\mathrm{M^\circ S}$ (and $\mathrm{GL^\circ S}$). If S actually has a one, then there is a homomorphism $\mathrm{M^\circ_d S} \to \mathrm{M^\times_d S}$ (recall that $\mathrm{M^\times_d S}$ is $\mathrm{M_d S}$ with its usual product) given by $x \to x+1$. This induces a homomorphism $\mathrm{M^\circ S} \to \mathrm{M^\times S}$, and if S is a ring with one it gives an isomorphism $\mathrm{GL^\circ S} \to \mathrm{GLS}$.

We now apply these constructions to X a ring-like A_∞-ring space. Using the method of Section 2, we obtain spaces and maps

$$B\mathrm{glX} \leftarrow B\mathrm{gl^\circ X} \to B\mathrm{gl^\circ mX}$$

(the map $B\mathrm{gl^\circ X} \to B\mathrm{gl^\circ mX}$ comes from the stabilisation homomorphism

$$x \to \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} : \mathrm{R} \to \mathrm{MR} \quad (\mathrm{R}\text{ a semiring})),$$

$$B\mathrm{gl^\circ cX} \to B\mathrm{glcX},$$

$$B\mathrm{gl^\circ sX} \to B\mathrm{glsX}.$$

Proposition 6.1. *The maps*

$$B\mathrm{glX} \leftarrow B\mathrm{gl^\circ X} \to B\mathrm{gl^\circ}m\mathrm{X},$$

$$B\mathrm{gl^\circ cX} \to B\mathrm{glcX},$$

$$B\mathrm{gl^\circ sX} \to B\mathrm{glsX} \textit{ are homotopy equivalences}.$$

Proof. We compute the homotopy groups. All the spaces are path-connected. For $B\mathrm{glX}$ we have

$$\pi_1 B\mathrm{glX} \cong \pi_0 \mathrm{glX} \cong \mathrm{GL}\pi_0 \mathrm{X}$$

and for $q \geqq 1$,

$$\pi_{q+1} B\mathrm{glX} \cong \pi_q \mathrm{glX} \cong \pi_q \mathrm{mX} \cong \mathrm{M}\pi_q \mathrm{X}$$

(recall that glX is got from mX by deleting some path-components and X is ring-like).

Similarly the homotopy groups of $B\mathrm{gl}^\circ X$ are $\mathrm{GL^\circ}\pi_0\mathrm{X}$ and $\mathrm{M}\pi_q\mathrm{X}$ ($q \geqq 1$). Since $\mathrm{GL^\circ}\pi_0\mathrm{X} \cong \mathrm{GL}\pi_0\mathrm{X}$, we have a homotopy equivalence $B\mathrm{gl^\circ X} \to B\mathrm{glX}$. Similarly the homotopy groups of $B\mathrm{gl^\circ mX}$ are $\mathrm{GL^\circ M}\pi_0\mathrm{X} \cong \mathrm{GL^\circ}\pi_0\mathrm{X}$ and $\mathrm{MM}\pi_q\mathrm{X} \cong \mathrm{M}\pi_q\mathrm{X}$ ($q \geqq 1$), so $B\mathrm{gl^\circ X} \simeq B\mathrm{gl^\circ mX}$. The other homotopy equivalences are proved in the same way.

The next step in the proof is the following result.

Lemma 6.2. *The spaces* $(B\mathrm{glX})^+$ *and* $(B\mathrm{gl}^\circ\mathrm{mX})^+$ *are nilpotent.*

Indeed $(B\mathrm{glX})^+$ is nilpotent because it is an infinite loop space (Steiner [9]), and the same method works for $(B\mathrm{gl}^\circ\mathrm{mX})^+$ (which is equivalent to $(B\mathrm{glX})^+$ by 6.1). We shall sketch a proof of 6.2 independent of Steiner [9] in Section 9.

The next step is the following.

Proposition 6.3. *For any field of coefficients,* $\tilde{H}_*(B\mathrm{glcX}) = 0$. *Hence* $(B\mathrm{gl}^\circ\mathrm{cX})^+$ *is contractible.*

The homology calculation will be given in Section 7. To deduce that $(B\mathrm{gl}^\circ\mathrm{cX})^+$ is contractible we note that by 6.1

$$\tilde{H}_*((B\mathrm{gl}^\circ\mathrm{cX})^+) \cong \tilde{H}_*((B\mathrm{glcX})^+) \cong \tilde{H}_*(B\mathrm{glcX}) = 0\,.$$

To proceed further, we recall (Wagoner, [11], 1) that if R is a ring then the maximal perfect subgroup of $\mathrm{GL}^\circ\mathrm{R}$ is the group $\mathrm{E}^\circ\mathrm{R}$ generated by elementary matrices, that is those with zeros on the main diagonal and at most one non-zero entry elsewhere. We recall also that $K_1\mathrm{R}$ is by definition the abelian group $\mathrm{GL}^\circ\mathrm{R}/\mathrm{E}^\circ\mathrm{R}$.

By the method of Section 2 we can form $\mathrm{e}^\circ\mathrm{sX}$ inside $\mathrm{gl}^\circ\mathrm{sX}$ and we get a map $B\mathrm{e}^\circ\mathrm{sX} \to B\mathrm{gl}^\circ\mathrm{sX}$.

Proposition 6.4. *There is a fibration*

$$(B\mathrm{e}^\circ\mathrm{sX})^+ \to (B\mathrm{gl}^\circ\mathrm{sX})^+ \to K(K_0\pi_0\mathrm{X}, 1)\,.$$

Proof. As in the proof of 6.1 we find that the homotopy groups of $B\mathrm{e}^\circ\mathrm{sX}$ are $\mathrm{E}^\circ\mathrm{S}\pi_0\mathrm{X}$ and $\mathrm{MS}\pi_q\mathrm{X}$ ($d \geqq 1$), while the homotopy groups of $B\mathrm{gl}^\circ\mathrm{sX}$ are $\mathrm{GL}^\circ\mathrm{S}\pi_0\mathrm{X}$ and $\mathrm{MS}\pi_q\mathrm{X}$ ($q \geqq 1$). Now

$$\mathrm{GL}^\circ\mathrm{S}\pi_0\mathrm{X}/\mathrm{E}^\circ\mathrm{S}\pi_0\mathrm{X} = K_1\mathrm{S}\pi_0 X \cong K_0\pi_0\mathrm{X}$$

(Wagoner [11], 3.9), so there is a fibration

$$B\mathrm{e}^\circ\mathrm{sX} \to B\mathrm{gl}^\circ\mathrm{sX} \to K(K_0\pi_0\mathrm{X}, 1)\,.$$

Since the base has abelian fundamental group, the fibration remains a fibration after applying the plus construction (Berrick [2], 6.4(a)). This completes the proof.

We can apply similar considerations to cX. We get fibrations

$$B\mathrm{e}^\circ\mathrm{cX} \to B\mathrm{gl}^\circ\mathrm{cX} \to K(K_1\mathrm{C}\pi_0\mathrm{X}, 1)\,,$$

$$(B\mathrm{e}^\circ\mathrm{cX})^+ \to \mathrm{c}B\mathrm{gl}^\circ\mathrm{cX})^+ \to K(K_1\mathrm{C}\pi_0\mathrm{X}, 1)\,.$$

However $K_1\mathrm{C}\pi_0\mathrm{X} \cong \pi_1[(B\mathrm{gl}^\circ\mathrm{cX})^+]$ is trivial by 6.3, that is $\mathrm{E}^\circ\mathrm{C}\pi_0\mathrm{X} = \mathrm{GL}^\circ\mathrm{C}\pi_0\mathrm{X}$. (These results also follow from (Wagoner [11], 2).) We get the following.

Proposition 6.5. *There are equalities* $\mathrm{E}^\circ\mathrm{C}\pi_0\mathrm{X} = \mathrm{GL}^\circ\mathrm{C}\pi_0\mathrm{X}$ *and* $B\mathrm{e}^\circ\mathrm{cX} = B\mathrm{gl}^\circ\mathrm{cX}$.

We can now make precise what we mean by applying $(B\mathrm{gl})^+$ to the fibration $\mathrm{mX} \to \mathrm{cX} \to \mathrm{sX}$.

Proposition 6.6. *There is a fibration*

$$B\mathrm{gl}^\circ\mathrm{mX} \to B\mathrm{gl}^\circ\mathrm{cX} \to B\mathrm{e}^\circ\mathrm{sX}\,.$$

Proof. Applying π_1 gives the sequence

$$\mathsf{GL}^\circ\mathsf{M}\pi_0\mathsf{X} \to \mathsf{GL}^\circ\mathsf{C}\pi_0\mathsf{X} \to \mathsf{E}^\circ\mathsf{S}\pi_0\mathsf{X}\,,$$

which is short exact by 6.5. Applying π_{q+1} for $q \geqq 1$ gives the sequence

$$\mathsf{MM}\pi_q\mathsf{X} \to \mathsf{MC}\pi_q\mathsf{X} \to \mathsf{MS}\pi_q\mathsf{X}\,,$$

which is evidently short exact.

We now apply the plus construction.

Proposition 6.7. *Applying the plus construction to the fibration in* 6.6 *gives a fibration*

$$(B\mathsf{gl}^\circ\mathsf{mX})^+ \to (B\mathsf{gl}^\circ\mathsf{cX})^+ \to (B\mathsf{e}^\circ\mathsf{sX})^+\,.$$

Proof. We want to apply (Berrick [2], 6.4(b)). This requires $(B\mathsf{gl}^\circ\mathsf{mX})^+$ to be nilpotent, what is true by 6.2, and the following result, which will be proved in Section 8.

Proposition 6.8. *In the fibration of* 6.6, *the action of* $\pi_1(B\mathsf{e}^\circ\mathsf{sX})$ *on* $H_*(B\mathsf{gl}^\circ\mathsf{mX})$ *is trivial.*

We can now prove that

$$K_0\pi_0\mathsf{X} \times (B\mathsf{glX})^+ \simeq \Omega[K_0\pi_0\mathsf{sX} \times (B\mathsf{glsX})^+]\,,$$

and so complete the proof of Theorem 1.1. Indeed

$$\begin{aligned} K_0\pi_0\mathsf{X} \times (B\mathsf{glX})^+ &\simeq K_0\pi_0\mathsf{X} \times (B\mathsf{gl}^\circ\mathsf{mX})^+ && (6.1)\\ &\simeq K_0\pi_0\mathsf{X} \times \Omega[(B\mathsf{e}^\circ\mathsf{sX})^+] && (6.7 \text{ and } 6.3)\\ &\simeq \Omega[(B\mathsf{gl}^\circ\mathsf{sX})^+] && (6.4)\\ &\simeq \Omega[(B\mathsf{glsX})^+] && (6.1)\\ &= \Omega[K_0\pi_0\mathsf{sX} \times (B\mathsf{glsX})^+] && (\text{trivially})\,. \end{aligned}$$

7. Proof of 6.3

We prove that $\tilde{H}_*(B\mathsf{glcX}) = 0$ for any field of coefficients by the method of (Wagoner [11], 2).

Here we use the ordinary product on matrices. For any semiring R there are homomorphisms

$$\phi\colon \mathsf{M}_d^\times\mathsf{C}R \times \mathsf{M}_d^\times\mathsf{C}R \to \mathsf{M}_d^\times\mathsf{C}R\,,$$

$$\tau\colon \mathsf{M}_d^\times\mathsf{C}R \to \mathsf{M}_d^\times\mathsf{C}R \quad \text{and}$$

$$\tau\colon \mathsf{M}_{d+1}^\times\mathsf{C}R \to \mathsf{M}_{d+1}^\times\mathsf{C}R$$

induced by ϕ: CR × CR → CR ,

τ: CR → CR, where

$$\phi\left(\begin{bmatrix} x_{11} & x_{12} & * \\ x_{21} & x_{22} & \\ & * & * \end{bmatrix}, \begin{bmatrix} y_{11} & y_{12} & * \\ y_{21} & y_{22} & \\ & * & * \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_{11} & 0 & x_{12} & 0 & \\ 0 & y_{11} & 0 & y_{12} & * \\ x_{21} & 0 & x_{22} & 0 & \\ 0 & y_{21} & 0 & y_{22} & \\ & & * & & * \end{bmatrix}$$

$$\tau([x_{ij}])_{kl} = \begin{cases} x_{st} & \text{if } k = 2^r(2s-1),\ l = 2^r(2t-1) \\ & \text{for some r, s, t} \\ 0 & \text{otherwise} \end{cases}$$

One can think of $\phi(x, y)$ as $x \oplus y$ and $\tau(x)$ as $x \oplus x \oplus x \ldots$. We have $\phi(1 \times \tau)\Delta(x) = \phi(x, \tau x) = \tau x$.

If $j: M_d^\times CR \to M_{d+1}^\times CR$ is defined by

$$x \to \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}$$

then $\tau j = j\tau: M_d^\times CR \to M_{d+1}^\times CR$. Also if

$$\eta_1, \eta_2: M_{d+1}^\times CR \to M_{d+1}^\times CR \times M_{d+1}^\times CR$$

are the inclusions, then there are permutation matrices a, b in $M_{d+1}^\times CR$ such that

$$a(\phi\eta_1 j)a^{-1}(x) = a\phi(jx, 1)a^{-1} = b\phi(1, jx)b^{-1} = b(\phi\eta_2 j)b^{-1}(x) = jx$$

In (Wagoner [11], 2.5) the index is 3d, but $M_2 CR$ is isomorphic to CR, so d + 1 will do the same job.

These induce simplicial homotopies of bar constructions

$$(M_d^\times CR)^q \times \mathrm{Hom}_\Delta(q, 1) \to (M_{d+1}^\times CR)^q$$

between $x \to \phi(jx, 1)$ and j, and between
$x \to \phi(1, jx)$ and j. The first one given by

$$y_u = \begin{cases} jx_u\ ; & \alpha(u) = 0, \\ a\phi(jx_u, 1)\ ; & \alpha(u-1) = 0,\ \alpha(u) = 1\ , \\ \phi(jx_u, 1)\ ; & \alpha(u-1) = \alpha(u) = 1\ . \end{cases}$$

We now apply these constructions to X by the method of Section 2, restrict to appropriate path-components, and take classifying spaces. Up to homotopy we get maps

$$\tau\colon B\mathrm{gl}_d\mathrm{cX} \to B\mathrm{gl}_d\mathrm{cX}\,,$$

$$\tau\colon B\mathrm{gl}_{d+1}\mathrm{cX} \to B\mathrm{gl}_{d+1}\mathrm{cX}\,,$$

$$\phi\colon B\mathrm{gl}_{d+1}\mathrm{cX} \times B\mathrm{gl}_{d+1}\mathrm{cX} \to B\mathrm{gl}_{d+1}\mathrm{cX}$$

such that

$$\tau \mathrm{j} = \mathrm{j}\tau\colon B\mathrm{gl}_d\mathrm{cX} \to B\mathrm{gl}_{d+1}\mathrm{cX}\,,$$

$$\phi(1\times\tau)\Delta = \tau\colon B\mathrm{gl}_{d+1}\mathrm{cX} \to B\mathrm{gl}_{d+1}\mathrm{cX}\,.$$

We also have non-base-point-preserving homotopies

$$\phi\eta_1\mathrm{j} \simeq \phi\eta_2\mathrm{j} \simeq \mathrm{j}\colon B\mathrm{gl}_d\mathrm{cX} \to B\mathrm{gl}_{d+1}\mathrm{cX}\,.$$

This suffices by a standard argument to show that $\mathrm{j}_*\colon \tilde H_*(B\mathrm{gl}_d\mathrm{cX}) \to \tilde H_*(B\mathrm{gl}_{d+1}\mathrm{cX})$ is zero, which gives $\tilde H_*(B\mathrm{gl}\,\mathrm{cX}) = 0$ as required, since $H_*(B\mathrm{gl}\,\mathrm{cX}) \cong \mathrm{colim}_d H_*(B\mathrm{gl}_d\mathrm{cX})$. Indeed let $z \in H_n(B\mathrm{gl}_d\mathrm{cX})$ with $n > 0$. We show that $\mathrm{j}_* z = 0$ by induction on n. The inductive hypothesis shows that

$$(\mathrm{j}\times\mathrm{j})_*\Delta_* z = \mathrm{j}_* z \times 1 + 1 \times \mathrm{j}_* z\,,$$

so

$$\begin{aligned}
\tau_*\mathrm{j}_* z &= \phi_*(1\times\tau)_*\Delta_*\mathrm{j}_* z\\
&= \phi_*(1\times\tau)_*(\mathrm{j}\times\mathrm{j})_*\Delta_* z\\
&= \phi_*(\mathrm{j}_* z \times 1 + 1 \times \tau_*\mathrm{j}_* z)\\
&= \phi_*\eta_{1*}\mathrm{j}_* z + \phi_*\eta_{2*}\tau_*\mathrm{j}_* z\\
&= \phi_*\eta_{1*}\mathrm{j}_* z + \phi_*\eta_{2*}\mathrm{j}_*\tau_* z\\
&= \mathrm{j}_* z + \mathrm{j}_*\tau_* z\\
&= \mathrm{j}_* z + \tau_*\mathrm{j}_* z\,;
\end{aligned}$$

thus $\mathrm{j}_* z = 0$ as required.

8. Proof of 6.8

We must show that $\pi_1(B\mathrm{e}^\circ\mathrm{sX})$ acts trivially on $H_*(B\mathrm{gl}^\circ\mathrm{mX})$. Now $B\mathrm{gl}^\circ\mathrm{mX} \simeq \mathrm{hocolim}_d B\mathrm{gl}^\circ\mathrm{m}_d\mathrm{X}$, so it suffices to show that $\pi_1(B\mathrm{e}^\circ\mathrm{sX})$ acts trivially on elements in the image of $H_*(B\mathrm{gl}^\circ\mathrm{m}_d\mathrm{X})$ for each d. We shall find subgroups generating $\pi_1(B\mathrm{e}^\circ\mathrm{sX})$ which act trivially on the elements of $H_*(B\mathrm{gl}^\circ\mathrm{m}_d\mathrm{X})$; this is sufficient.

From 6.5 and the proof of 6.6,

$$\pi_1(B\mathrm{e}^\circ\mathrm{sX}) \cong \mathrm{E}^\circ\mathrm{S}\pi_0\mathrm{X} \cong \mathrm{E}^\circ\mathrm{C}\pi_0\mathrm{X}/\mathrm{GL}^\circ\mathrm{M}\pi_0\mathrm{X}\,.$$

For any semiring R and for distinct natural numbers k, l and a locally finite set A there is a monoid monomorphism

$$R^A \to M^\circ CR$$

sending $x \in R^A \subset CR$ to the elementary matrix with x in position (k, l) and with zeros elsewhere. Write $E^\circ_{k,l,A}CR$ for the image of this monomorphism. By definition, $E^\circ C\pi_0 X$ is generated by the $E^\circ_{k,l,A}C\pi_0 X$, so $\pi_1(Be^\circ sX)$ is generated by the images of the $E^\circ_{k,l,A}C\pi_0 X$. Since moreover an element of $C\pi_0 X$ differs by an element of $M\pi_0 X$ from an element of $C\pi_0 X$ with zeros in its first d rows and columns, we see that it suffices to take A such that $i, j > d$ for all $(i, j) \in A$. Thus it suffices to show that

$$\mathrm{Im}\,[E^\circ_{k,l,A}C\pi_0 X \to \pi_1(Be^\circ sX)]$$

acts trivially on the elements of

$$\mathrm{Im}\,[H_*(Bgl^\circ m_d X) \to H_*(Bgl^\circ mX)]$$

whenever A is such that $i, j > d$ for all $(i, j) \in A$.

We shall deduce this from the fact that if R is a semiring then elements of $E^\circ_{k,l,A}CR$ commute with elements of $M^\circ M_d R$ inside $M^\circ CR$. Let us think of the monoid $M^\circ CR$ and its submonoid $M^\circ M_d R$ as categories with one object. We can think of an element α of $M^\circ CR$ which commutes with the elements of $M^\circ M_d R$ as a natural transformation from the inclusion functor $M^\circ M_d R \to M^\circ CR$ to itself. We can also regard $M^\circ M_d R$ and $M^\circ CR$ as the 0-cells of the simplicial categories $M^\circ B_\bullet(M_d R, M_d R, o)$ and $M^\circ B_\bullet(CR, MR, o)$; then $\alpha \in M^\circ B_0(CR, MR, o)$ and its degeneracies commute with the elements of $M^\circ B_0(M_d R, M_d R, o)$, so α extends to a self natural transformations of the inclusion functor

$$M^\circ B_\bullet(M_d R, M_d R, o) \to M^\circ B_\bullet(CR, MR, o)\,.$$

Now a natural transformation between functors from a category $\mathscr{C}$ to a category $\mathscr{D}$ induces a homotopy

$$|\mathscr{C}| \times I \to |\mathscr{D}|$$

after geometric realisation, so a self natural transformation induces a map

$$|\mathscr{C}| \times S^1 \to |\mathscr{D}|\,.$$

So if one applies the constructions of the last paragraph to X by the method of Section 2, restricts to the appropriate path-components, and takes geometric realisation, then one gets a commutative diagram with spaces of the homotopy types given by

$$\begin{array}{ccc}
* \times Bgl^\circ m_d X \times * & \xrightarrow{\subset} & Bgl^\circ mX \\
\downarrow{\scriptstyle \cap} & & \downarrow \\
e^\circ_{k,l,A}cX \times Bgl^\circ m_d X \times S^1 & \longrightarrow & Bgl^\circ cX \\
{\scriptstyle \mathrm{proj}}\downarrow & & \downarrow \\
e^\circ_{k,l,A}cX \times * \times S^1 & \longrightarrow & Be^\circ sX
\end{array}$$

(recall that $B_\bullet(\mathsf{CR}, \mathsf{MR}, \circ)$ yields SR and note that $B_\bullet(\mathsf{M_d R}, \mathsf{M_d R}, \circ)$ yields a point because it has a standard contracting simplicial homotopy).

In this commutative diagram the right-hand column is the fibration of 6.6 while the left-hand column is a trivial fibration. So $\mathrm{Im}[\pi_1(\mathsf{e}^{\circ}_{\mathsf{k,l,A}}\mathsf{cX} \times * \times \mathsf{S}^1) \to \pi_1(B\mathsf{e}^{\circ}\mathsf{sX})]$ acts trivially on elements of $\mathrm{Im}[H_*(* \times B\mathsf{gl}^{\circ}m_{\mathsf{d}}\mathsf{X} \times *) \to H_*(B\mathsf{gl}^{\circ}\mathsf{mX})]$. But up to homotopy the map $\mathsf{e}^{\circ}_{\mathsf{k,l,A}}\mathsf{cX} \times * \times \mathsf{S}^1 \to B\mathsf{e}^{\circ}\mathsf{sX}$ is the composite of the standard map $\mathsf{e}^{\circ}_{\mathsf{k,l,A}}\mathsf{cX} \times \mathsf{S}^1 \to B\mathsf{e}^{\circ}_{\mathsf{k,l,A}}\mathsf{cX}$ with the obvious map $B\mathsf{e}^{\circ}_{\mathsf{k,l,A}}\mathsf{cX} \to B\mathsf{e}^{\circ}\mathsf{sX}$, so the image of the induced map of fundamental groups contains those elements of $\pi_1(B\mathsf{e}^{\circ}\mathsf{sX})$ which come from elements of $\pi_0(\mathsf{e}^{\circ}_{\mathsf{k,l,A}}\mathsf{cX}) \cong \mathsf{E}^{\circ}_{\mathsf{k,l,A}}\mathsf{C}\pi_0\mathsf{X}$. Thus $\mathrm{Im}[\mathsf{E}^{\circ}_{\mathsf{k,l,A}}\mathsf{C}\pi_0\mathsf{X} \to \pi_1(B\mathsf{e}^{\circ}\mathsf{sX})]$ acts trivially on elements of $\mathrm{Im}[H_*(B\mathsf{gl}^{\circ}m_{\mathsf{d}}\mathsf{X}) \to H_*(B\mathsf{gl}^{\circ}\mathsf{mX})]$, as required.

9. Proof of 6.2

In this section we indicate how $(B\mathsf{glX})^+$ can be proved nilpotent by the method of Section 8. In fact we work with $(\mathsf{Bgl}^{\circ}\mathsf{X})^+$, which is equivalent to $(B\mathsf{glX})^+$ by 6.1.

By (Berrick [2], 10.3) it suffices to show that if $\pi = \pi_1 B\mathsf{gl}^{\circ}\mathsf{X}$ and $\mathscr{P}\pi$ is its maximal perfect subgroup, then $\pi/\mathscr{P}\pi$ is nilpotent and acts trivially on the homology of the fibre of $B\mathsf{gl}^{\circ}\mathsf{X} \to K(\pi/\mathscr{P}\pi, 1)$. As in 6.4 we find that $\pi \cong \mathsf{GL}^{\circ}\pi_0\mathsf{X}$; hence $\mathscr{P}\pi \cong \mathsf{E}^{\circ}\,\pi_0\mathsf{X}$ and $\pi/\mathscr{P}\pi \cong \mathsf{GL}^{\circ}\pi_0\mathsf{X}/\mathsf{E}^{\circ}\pi_0\mathsf{X} = K_1\pi_0\mathsf{X}$ is certainly nilpotent, in fact abelian. Also the fibration we have to study is

$$B\mathsf{e}^{\circ}\mathsf{X} \to B\mathsf{gl}^{\circ}\mathsf{X} \to K(\pi/\mathscr{P}\pi, 1)$$

(compare 6.4).

One can prove that $\pi/\mathscr{P}\pi$ acts trivially on $H_*(B\mathsf{e}^{\circ}\mathsf{X})$ by the method of the last section. Note that for each fixed d the group $\pi/\mathscr{P}\pi \cong \mathsf{GL}^{\circ}\pi_0\mathsf{X}/\mathsf{E}^{\circ}\pi_0\mathsf{X}$ is generated by the images of matrices over $\pi_0\mathsf{X}$ which have only zeros in their first d rows and columns.

For $\mathsf{GL}^{\circ}\pi_0\mathsf{X}$ is generated by matrices of the form $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ ($a \in \mathsf{GL}^{\circ}_{\mathsf{e}}\pi_0\mathsf{X}$ with $\mathsf{e} \geqq \mathsf{d}$), and this represents the same element of $\pi/\mathscr{P}\pi$ as

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

since $\begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is in $\mathsf{E}^{\circ}\pi_0\mathsf{X}$

(Berrick [2], 1.9). Now if R is a semiring, then matrices in $\mathsf{M}^{\circ}\mathsf{R}$ which have only zeros in their first d rows and columns commute with matrices in $\mathsf{M}^{\circ}_{\mathsf{d}}\mathsf{R}$. By the method of the last section it follows that $\pi/\mathscr{P}\pi$ acts trivially on $H_*(B\mathsf{e}^{\circ}\mathsf{X})$, as required.

References

1. Boardman, J.M., Vogt, R.M.: Homotopy invariant algebraic structures on topological spaces. (Lect. Notes Math., vol. 347) Berlin Heidelberg New York: Springer 1973
2. Berrick, A.J.: An approach to algebraic K-theory. Res. Notes Math., 56 (1982)
3. Bousfield, A.K., Kan, D.M.: Homotopy limits, completions and localizations. (Lect. Notes Math., vol. 304) Berlin Heidelberg New York: Springer 1972
4. May, J.P.: A_∞ ring spaces and algebraic K-theory. In: Barratt, M.G., Mahowald, M.E. (eds.) Geometric applications of homotopy theory II. Proceedings, Evanston 1977, (Lect. Notes Math., vol. 658, pp. 240–315) Berlin Heidelberg New York: Springer 1978
5. May, J.P.: Classifying Spaces and Fibrations, Mem. Am. Math. Soc. **155** (1975)
6. Schwänzl, R., Vogt, R.: Homotopy Invariance of A_∞- and E_∞-ring spaces. In: Madsen, I., Oliver, B. (eds.) Algebraic topology. Aarhus 1982. (Lect. Notes Math., vol. 1051, pp. 442–481) Berlin Heidelberg New York: Springer 1984
7. Schwänzl, R., Vogt, R.: Matrices over homotopy ring spaces and algebraic K-theory, OSM, P73 (1984), Universität Osnabrück (preprint)
8. Segal, G.: Categories and cohomology theories, Topology **13**, 293–312 (1974)
9. Steiner, R.: Infinite loop structures on the algebraic K-theory of spaces. Math. Proc. Camb. Phil. Soc. **90**, 85–111 (1981)
10. Steiner, R.: A canonical operad pair. Math. Proc. Camb. Philos. Soc. **86** 443–449 (1979)
11. Wagoner, J.B.: Delooping classifying spaces in algebraic K-theory. Topology **11**, 349–370 (1972)
12. Waldhausen, F.: Algebraic K-theory of topological spaces I. Proc. Symp. Pure Math. **32**, 35–60 (1978)

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