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# Stable K-theory and topological Hochschild homology

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## Introduction

Tom Goodwillie conjectured that there existed a construction for rings like that used to compute its Hochschild homology in which one replaced the tensor products over a ground ring by the "tensor product" over the ring up to homotopy $QS^0 = \lim_{n\to\infty} \Omega^n S^n$. He conjectured further what the weak homotopy type of this new theory would be for the integers and $\mathbf{Z}/p\mathbf{Z}$ for any prime $p$. Lastly, he conjectured that this theory would be homotopy equivalent to stable K-theory for any ring.

This new theory was constructed by Marcel Bökstedt in [3] and is called the *topological Hochschild homology* for a ring $R$. It is generally denoted $\mathrm{THH}(R)$. He computed $\mathrm{THH}(\mathbf{Z})$ and $\mathrm{THH}(\mathbf{Z}/p\mathbf{Z})$ in [4] and showed they agreed with Goodwillie's original conjecture. In this paper, we give a solution to Goodwillie's third conjecture that $\mathrm{THH}(R)$ is weakly homotopic to the stable K-theory of $R$ for any ring. A proof of this conjecture including the $A_\infty$-case is expected to appear in [18]. That proof derives the conjecture from an analysis of Nil-term phenomena arising in an $A_\infty$-version of a generalized free product situation [20]. An outline of the program was indicated in [23]. The two proofs are completely different; the one in [18] will require the techniques of $A_\infty$-rings even to obtain the conjecture for simplicial rings, which is not the case with the present approach. The following is Theorem 5.3.

THEOREM. *For any simplicial ring $R$ and simplicial $R$-bimodule $M$, there is a natural weak homotopy equivalence between* $\mathrm{K}^s(R,M)$ *and* $\mathrm{THH}\,(R;M)$.

The general scheme for this proof goes as follows. It was shown in [17] that $\mathrm{THH}(R)$ is naturally equivalent to the homology of the category of finitely generated projective right $R$-modules with coefficients in the bi-functor $\mathrm{Hom}_R$. Since we want to compare this theory to algebraic K-theory we "twist" the two

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theories together to obtain a complex which collapses in two different ways. One of these ways recovers our original model for THH($R$) and the other way defines a "new" equivalent construction which formally appears similar to algebraic K-theory.

For an $R$-bimodule $M$, we define $\mathrm{K}(R;M)$ to be the algebraic K-theory of the exact category with objects $(P,\alpha)$, where $P$ is a finitely generated projective module and $\alpha$ is an $R$-module homomorphism from $P$ to $P \otimes_R M$. We extend this definition to simplicial $R$-bimodules degree-wise and we show that our "new" construction for THH is equivalent to the underlying space of the Goodwillie derivative of the functor which sends a pointed simplicial set $X$ to $\mathrm{K}(R; M[X]/M[*])$. We let $R \oplus M$ denote the new ring with multiplication defined by $(r,m)(r',m') = (rr, rm' + mr')$. One can define $\mathrm{K}^s(R,M)$ as the Goodwillie derivative of the functor which sends $X$ to $\Omega\mathrm{K}(R \oplus M[X]/M[*])$ where we use F. Waldhausen's definition for the algebraic K-theory of a simplicial ring.

Thus, the key to comparing $\mathrm{K}^s$ to THH is a comparison between $\mathrm{K}(R;\ )$ and $\mathrm{K}(R \bigoplus\ )$. We prove there is a natural homotopy equivalence from $\mathrm{K}(R; M[S^1]/M[*])$ to $\mathrm{K}(R \oplus M)$ for any bimodule $M$. The idea for this equivalence comes from the fact that a morphism from $(R \oplus M)^m$ to $(R \oplus M)^n$ as $R \oplus M$-modules is uniquely determined by $R$-module homomorphisms $\beta$ from $R^m$ to $R^n$ and $\alpha$ from $R^m$ to $M^n$. We write such a map as $(\beta, \alpha)$. One can define a map from the bar construction of $\mathrm{Hom}_R(R^n, M^n)$ (as an abelian group) to the nerve of the category of free $R \oplus M$-modules with respect to isomorphisms by sending $\alpha$ to $(\mathrm{id}, \alpha)$ because $(\mathrm{id}, \alpha) \circ (\mathrm{id}, \alpha') = (\mathrm{id}, \alpha + \alpha')$. This induces a map which we show to be a homotopy equivalence.

The paper is organized as follows. In Section 1 we define a specific resolution $F.$ for the homology of the category $\mathcal{P}_R$ (finitely generated projective right $R$-modules) with coefficients in the bi-functor $\mathrm{Hom}_R$. This is a functor from linear categories to simplicial abelian groups and we derive a few simple facts about it which we will want to use. In Section 2 we show how one can incorporate the $S$ construction from [22] in such a way that the composite functor $\Omega|F.S.\mathcal{P}_R|$ is again naturally THH($R$). The main theorem of Section 2 is that the inclusion by degeneracies produces a homotopy equivalence of $F_0S.\mathcal{P}_R$ with $F.S.\mathcal{P}_R$. Using this model $F_0S.\mathcal{P}_R$, we show by a series of reductions how one can interpret THH($R$) as the underlying space of the Goodwillie derivative of the functor sending a pointed simplicial set $X$ to the space $\mathrm{K}(R; R[X]/R[*])$ defined above. We prove our comparison theorem between the theories $\mathrm{K}(R;M)$ and $\mathrm{K}(R \oplus M)$ in Section 4. From this fact and the observation of Goodwillie that the relative K-theory of a surjective map of rings with square zero ideal can be computed degree-wise, we deduce our main result in Section 5.

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## 1. The simplicial functor $F$

In this section we quickly recall a functor from small linear categories to simplicial abelian groups. This construction can be found in [2] and is an example of a construction by B. Mitchell in [16].

*Definition* 1.1. We will call a category *linear* if it has at least one object, its Hom sets are abelian groups and composition is bi-linear. A *linear functor* between linear categories is a functor such that the induced maps of Hom-sets are homomorphisms of abelian groups. For $\mathcal{C}$ a small linear category, we define $F.(\mathcal{C})$ to be the following simplicial abelian group:

$$F_n(\mathcal{C}) = \bigoplus_{\vec{C}\in N_n(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(C_1, C_0), \quad \vec{C} = C_1 \leftarrow \cdots \leftarrow C_n \leftarrow C_0$$

where the direct sum runs over all $C_1 \leftarrow C_2 \leftarrow \cdots \leftarrow C_n \leftarrow C_0 \in N_n(\mathcal{C})$ and $N.(\mathcal{C})$ denotes the simplicial *nerve* of $\mathcal{C}$. Face and degeneracy operators applied to an element like $(f_0; f_1, \dots, f_n) \in \mathrm{Hom}_{\mathcal{C}} \times N_n(\mathcal{C})$ are defined by:

$$\mathrm{d}_i(f_0; f_1, \dots, f_n) = \begin{cases} (f_0 \circ f_1; f_2, \dots, f_n) & \text{if } i = 0 \\ (f_0; f_1, \dots, f_i \circ f_{i+1}, \dots, f_n) & \text{if } 1 \le i \le n-1 \\ (f_n \circ f_0; f_1, \dots, f_{n-1}) & \text{if } i = n \end{cases}$$

$$\mathrm{s}_i(f_0; f_1, \dots, f_n) = \begin{cases} (f_0; f_1, \dots, f_i, \mathrm{id}_{C_{i+1}}, f_{i+1}, \dots, f_n) & \text{if } 0 \le i \le n-1 \\ (f_0; f_1, \dots, f_n, id_{C_0}) & \text{if } i = n. \end{cases}$$

One sees that $F.(\ )$ is a covariant functor from the category of small linear categories to simplicial abelian groups. If $G$ and $G'$ are naturally isomorphic linear functors then $F.(G)$ is naturally simplicial homotopic to $F.(G')$ by (1.11) of [2]. Thus, if $\mathcal{C}'$ is a linear category which is equivalent to a small linear category $\mathcal{C}$, we will define $F.(\mathcal{C}')$ to be $F.(\mathcal{C})$ which is well-defined up to homotopy.

THEOREM 1.2 ([11],[17]). *Let $R$ denote a ring and $\mathcal{P}$ the linear category of finitely generated projective right $R$-modules. Then there is a natural sequence of isomorphisms*

$$\pi_*(F.(\mathcal{P})) \cong \mathrm{H}_*^{\mathrm{Mac}}(R, R) \cong \mathrm{THH}_*(R)$$

*where* $\mathrm{H}_*^{\mathrm{Mac}}(R,R)$ *denotes the MacLane homology of the ring* $R$ *with coefficients in the bimodule* $R$ *(see* [13]) *and* $\mathrm{THH}(R)$ *denotes the topological Hochschild homology for the (discrete) ring* $R$ *(see* [3]). *The first isomorphism is by* [11] *Section* 1 *and the second isomorphism is by* [17].

*Definition* 1.3. Given a linear functor $G$ from $\mathcal{A}$ to $\mathcal{B}$, we define the "twisted" product category $\mathcal{A}_G\mathcal{B}$ as follows. We set $\mathrm{Obj}(\mathcal{A}_G\mathcal{B})$ to be $\mathrm{Obj}(\mathcal{A})\times\mathrm{Obj}(\mathcal{B})$ and

$$\mathrm{Hom}_{\mathcal{A}_G\mathcal{B}}((A,B),(A',B')) = \mathrm{Hom}_{\mathcal{A}}(A,A')\oplus\mathrm{Hom}_{\mathcal{B}}(B,B')\oplus\mathrm{Hom}_{\mathcal{B}}(G(A),B')$$

with composition defined by $(f,g,h)\circ(f',g',h') = (f\circ f', g\circ g', h\circ G(f')+g\circ h')$.

PROPOSITION 1.4. *The functor from* $\mathcal{A}_G\mathcal{B}$ *to* $\mathcal{A}\times\mathcal{B}$ *which is the identity on objects (and sends* $(f,g,h)$ *to* $f\times g$*) induces a homotopy equivalence from* $F.(\mathcal{A}_G\mathcal{B})$ *to* $F.(\mathcal{A}\times\mathcal{B})$.

*Proof*. We will be defining three chain homotopies which arise from semi-simplicial homotopies (they satisfy the simplicial homotopy identities with respect to the face maps; see [14], Section 5). Given a semi-simplicial homotopy $\{H_i\}$ one can construct a chain homotopy $h$ by setting $h_n = \Sigma_{i=0}^n(-1)^iH_i$.

*First reduction*. The subcomplex of $F.\mathcal{A}_G\mathcal{B}$ generated by elements of the form $(\alpha_0;\alpha_1,\ldots,\alpha_n)$ such that $\alpha_0=(0,0,h_0)$ is acyclic.

*Proof*. We define a semi-simplicial homotopy from 0 to the identity as follows (we let $\alpha_i=(f_i,g_i,h_i)$)

$$H_i((0,0,h_0);\alpha_1,\ldots,\alpha_n) = ((0,0,h_0);\alpha_1,\ldots,\alpha_i,(0,\mathrm{id}_{B_{i+1}},0),(0,g_{i+1},0),\ldots,(0,g_n,0)).\quad\square$$

Now we take the quotient of $F.\mathcal{A}_G\mathcal{B}$ by the above acyclic subcomplex to get a new complex $F^*_.\mathcal{A}_G\mathcal{B}$. We will write a generating element of this complex as $((f_0,g_0,\star);\alpha_1,\ldots,\alpha_n)$. The complex $F^*_.\mathcal{A}_G\mathcal{B}$ splits into two subcomplexes $A.$ and $B.$ generated by elements of the form $((f_0,0,\star);\alpha_1,\ldots,\alpha_n)$ and $((0,g_0,\star);\alpha_1,\ldots,\alpha_n)$ respectively.

*Second reduction*. The projection from $A.$ to $F.\mathcal{A}$ generated by sending $((f_0,0,\star);\alpha_1,\ldots,\alpha_n)$ to $(f_0;f_1,\ldots,f_n)$ is a homotopy equivalence.

*Proof*. Choose some element $b$ of $\mathcal{B}$. The projection has a section defined by sending $(f_0;f_1,\ldots,f_n)$ to the equivalence class containing $((f_0,0_b,\star);(f_1,0_b,0),\ldots,(f_n,0_b,0))$ where we let $0_b$ denote the zero endomorphism of $b$.

A simplicial homotopy from the identity to the composite can be defined by setting $H_i$ of $((f_0, 0, \star); \alpha_1, \ldots, \alpha_n)$ to be the class of

$$((f_0, 0, \star); (f_1, 0_b, 0), \ldots, (f_i, 0_b, 0), (\mathrm{id}_{A_{i+1}}, 0, 0), \alpha_{i+1}, \ldots, \alpha_n). \qquad \square$$

*Third reduction.* The projection from $B.$ to $F.\mathcal{B}$ generated by sending $((0, g_0, \star); \alpha_1, \ldots, \alpha_n)$ to $(g_0; g_1, \ldots, g_n)$ is a homotopy equivalence.

*Proof.* Choose some element $a$ of $\mathcal{A}$. The projection has a section defined by sending $(g_0; g_1, \ldots, g_n)$ to the equivalence class containing $((0_a, g_0, \star); (0_a, g_1, 0), \ldots, (0_a, g_n, 0))$ where we let $0_a$ denote the zero endomorphism of $a$. A simplicial homotopy from the composite to the identity can be defined by setting $H_i$ of $((0, g_0, \star); \alpha_1, \ldots, \alpha_n)$ to be the class of

$$((0, g_0, \star); \alpha_1, \ldots, \alpha_i, (0, \mathrm{id}_{B_{i+1}}, 0), (0_a, g_{i+1}, 0), \ldots, (0_a, g_n, 0)).$$

We have constructed a diagram of complexes

$$\begin{array}{ccc} F.\mathcal{A} \times F.\mathcal{B} & \xrightarrow{\text{inc}} & F.\mathcal{A}_G\mathcal{B} \\ \uparrow \simeq & & \downarrow \simeq \\ A. \times B. & \xleftarrow{\cong} & F^*_.\mathcal{A}_G\mathcal{B} \end{array}$$

with the maps up and down quasi-isomorphisms by reductions 1, 2 and 3 above. Since the composite around the square is the identity on $F.\mathcal{A} \times F.\mathcal{B}$ we see that inc is a quasi-isomorphism. Since the inclusion inc is a section to our map we are done. $\square$

*Definition* 1.5. For $\mathcal{A}$ a small linear category and $n$ a natural number, we let $T_n(\mathcal{A})$ denote the new linear category with $\mathrm{Obj}(T_n(\mathcal{A})) = \mathrm{Obj}(\mathcal{A})^n$ and $\mathrm{Hom}_{T_n(\mathcal{A})}((A_1, \ldots, A_n), (B_1, \ldots, B_n))$ the set of all $n \times n$ "matrices" of the form $(\alpha_{i,j})$ where $\alpha_{i,j} \in \mathrm{Hom}_{\mathcal{A}}(A_j, B_i)$, $\alpha_{i,j} = 0$ if $j < i$ and composition is defined by

$$(\alpha \circ \beta)_{i,j} = \Sigma_{k=1}^{n} \alpha_{i,k} \circ \beta_{k,j}$$

(i.e., matrix multiplication). Thus, $T_n(\ )$ is an endo-functor of the category small linear categories.

COROLLARY 1.6. *For any small linear category $\mathcal{A}$ with a zero object, the functor from $T_n(\mathcal{A})$ to $\mathcal{A}^n$ given by the identity on objects induces a homotopy equivalence $F.(T_n(\mathcal{A})) \xrightarrow{\simeq} F.(\mathcal{A}^n)$.*

*Proof.* Let $G$ denote the functor from $\mathcal{A}$ to $T_{n-1}(\mathcal{A})$ defined by sending an object $A$ to $(A, 0, \ldots, 0)$ (where 0 is a zero object). Then $T_n(\mathcal{A})$ is naturally isomorphic to $\mathcal{A}_G T_{n-1}(\mathcal{A})$ and the result follows by Proposition 1.4 and induction. $\square$

## 2. Incorporating the $S$ construction

We recall some definitions and facts about Waldhausen's $S$ construction from [22]. Let $\mathcal{C}$ be an exact category (with a chosen zero object 0). Let $[n]$ denote the ordered set $(0 < 1 < \ldots < n)$ (which we think of as a category), let $\mathrm{Ar}[n]$ denote the category of arrows in $[n]$, and let $(j/i)$ denote the arrow from $i$ to $j$ for $i \leq j$. We call a sequence of the form $(j/i) \to (k/i) \to (k/j)$ in $\mathrm{Ar}[n]$ exact. We define $S_n\mathcal{C}$ to be the set of *exact* functors $\bar{C}\colon \mathrm{Ar}[n] \to \mathcal{C}$, which means:

(a) For all $i$, $\bar{C}(i/i) = 0$;
(b) For all $i \leq j \leq k$, the complex $0 \to \bar{C}(j/i) \to \bar{C}(k/i) \to \bar{C}(k/j) \to 0$ is an exact sequence in $\mathcal{C}$.

We think of an element of $S_n\mathcal{C}$ as a sequence of admissible monomorphisms $0 = C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_n$ plus choices $\bar{C}(j/i)$ for all quotients $C_j/C_i$. We can form a simplicial set $[n] \to S_n\mathcal{C}$ (called $s.\mathcal{C}$ in Section 1.4 of [22]) where the face and degeneracy maps amount to forgetting or duplicating a $C_i$, except that for $\mathrm{d}_0$ (which would forget $0 = C_0$) we must also factor out by $C_1$. We define the algebraic K-theory of $\mathcal{C}$ to be

$$\mathrm{K}(\mathcal{C}) = \Omega|S.\mathcal{C}|.$$

Each $S_n\mathcal{C}$ can also be considered as a category with the morphisms the natural transformations of functors. We can further consider this as an exact category by declaring a sequence $\bar{C}'' \to \bar{C} \to \bar{C}'$ to be exact if the associated sequences for all $i \leq j$ are exact as sequences of $\mathcal{C}$. With these conventions, we can consider $S.\mathcal{C}$ not only as a simplicial set but as a simplicial category or even as a simplicial exact category.

*Remark* 2.1. Suppose we have a sub-category $\mathcal{T}$, of $\mathcal{C}$, with the same set of objects, whose morphisms are always isomorphisms. This determines a subcategory $tS_n\mathcal{C}$ of each $S_n\mathcal{C}$, again having all the objects (a morphism is a natural transformation consisting of morphisms from $\mathcal{T}$). Let the bi-simplicial set $N.tS.\mathcal{C}$ be the nerve of the resulting simplicial category. We note that $N_0tS.\mathcal{C}$ is just $S.\mathcal{C}$ and that by 1.4.1 of [22] the natural map of bi-simplicial sets (given by degeneracies) $S.\mathcal{C} \longrightarrow N.tS.\mathcal{C}$ is a homotopy equivalence. This applies in particular if we choose $\mathcal{T}$ to be the sub-category of all isomorphisms which we will denote $N.iS.\mathcal{C}$.

We now note some very general facts related to "additivity" before returning to our specific examples. We let $X.$ denote a simplicial functor from the category of small exact categories to based sets such that

(i) $X.(*) = *$;
(ii) $X.(\mathcal{C} \times \mathcal{D}) \xrightarrow{\simeq} X.(\mathcal{C}) \times X.(\mathcal{D})$.

PROPOSITION 2.2. *The composite functor $X.S.$ satisfies additivity. That is, for all exact categories $\mathcal{C}$,*

$$\begin{array}{ccc} X.S.S_2(\mathcal{C}) & \xrightarrow{\simeq} & X.S.(\mathcal{C}) \times X.S.(\mathcal{C}) \\ & & \\ A \rightarrowtail C \twoheadrightarrow B & \mapsto & A \times B. \end{array}$$

*Proof.* The proof of additivity found in [15] uses only natural homotopies by exact functors between exact categories. Thus, we see that for each $n \in \mathbf{N}$ we have $X_nS.S_2\mathcal{C} \xrightarrow{\simeq} X_nS.(\mathcal{C} \times \mathcal{C})$. By the realization lemma (see for example Lemma 5.1 of [20]) and (ii) we are done. □

LEMMA 2.3 (after 1.5.5 of [22]). *Suppose that $X.\mathcal{D}$ is connected or a simplicial abelian group for every exact category $\mathcal{D}$. Let $\mathcal{C}$ be an exact category such that the natural projections $X.S_n\mathcal{C} \longrightarrow X.\mathcal{C}^n$ are homotopy equivalences for all $n$. Then the natural map from $X.\mathcal{C}$ to $\Omega X.S.\mathcal{C}$ is a homotopy equivalence.*

*Proof.* Let $PS.\mathcal{C}$ denote the simplicial path space of $S.\mathcal{C}$ and consider the sequence $X.\mathcal{C} \longrightarrow X.PS.\mathcal{C} \longrightarrow X.S.\mathcal{C}$. For each fixed $n$, the associated sequence $X.\mathcal{C} \longrightarrow X.PS_n\mathcal{C} \longrightarrow X.S_n\mathcal{C}$ is homotopy equivalent to the trivial fibration sequence $X.\mathcal{C} \longrightarrow (X.\mathcal{C})^{n+1} \longrightarrow (X.\mathcal{C})^n$. We can conclude by B.4 of [5] that the original sequence was a fibration sequence up to homotopy. Since $PS.\mathcal{C}$ simplicially contracts to the trivial category (by exact functors), we see by the realization lemma and (i) that $X.PS.\mathcal{C}$ is also contractible and hence the result. □

*Definition* 2.4. For $\mathcal{E}$ an exact category, we define the *topological Hochschild homology of $\mathcal{E}$* by

$$\mathrm{THH}_*(\mathcal{E}) = \pi_{*+1}(F.S.\mathcal{E}).$$

We note that by Theorem 1.2, Corollary 1.6 and Lemma 2.3 this definition agrees with the usual definition for a discrete ring if one uses the exact category of finitely generated projective right modules over that ring. That is, $S_n\mathcal{P}$ is naturally equivalent to $T_n\mathcal{P}$ and the composition $T_n\mathcal{P} \longrightarrow S_n\mathcal{P} \longrightarrow \mathcal{P}^n$ with the map of Lemma 2.3 is the map used in Corollary 1.6.

LEMMA 2.5. *Suppose $X.$ is a functor to simplicial abelian groups (instead of just pointed sets). Then the two maps $X.S.\mathrm{d}_0 + X.S.\mathrm{d}_2$ and $X.S.\mathrm{d}_1$ from $X.S.S_2$ to $X.S.$ are homotopic.*

*Proof.* Let $\phi_0$ and $\phi_1$ denote the exact functors from $\mathcal{C}$ to $S_2\mathcal{C}$ defined by sending an object $C$ to $C = C \twoheadrightarrow 0$ and $0 \rightarrowtail C = C$ respectively. Let $\phi$ denote the simplicial map $X.S.\phi_0 + X.S.\phi_1$ from $X.S.\mathcal{C} \oplus X.S.\mathcal{C}$ to $X.S.S_2\mathcal{C}$.

By proposition 2.2, $\phi$ is a homotopy equivalence with inverse $(X.S.\mathrm{d}_0, X.S.\mathrm{d}_2)$ and since $X.S.\mathrm{d}_1 \circ \phi$ is simply degree-wise addition, we are done. □

THEOREM 2.6. *For any exact category $\mathcal{C}$, the natural map $F_0S.\mathcal{C} \longrightarrow F.S.\mathcal{C}$ is a homotopy equivalence.*

*Proof.* More generally, we show that for all $n \in \mathbf{N}$, $F_nS.\mathcal{C}$ is simplicially homotopic to $F_0S.\mathcal{C}$ which implies the result by the realization lemma. Let $c$ be the natural transformation from $F_n$ to $F_0$ defined by sending $(\alpha_0; \alpha_1, \dots, \alpha_n)$ to $(\alpha_0\alpha_1 \cdots \alpha_n)$. Let dgn denote the natural transformation from $F_0$ to $F_n$ obtained from lifting by degeneracies. Since $c \circ \mathrm{dgn} = \mathrm{id}_{F_0}$ we need to show that $C = \mathrm{dgn} \circ c$ is homotopic to the identity when we include $S.$ in the picture. In other words, we want to show that the simplicial self-map $C$ of $F_nS.\mathcal{C}$ defined by sending $(\alpha_0; \alpha_1, \dots, \alpha_n)$ to $(\alpha_0\alpha_1 \cdots \alpha_n; \mathrm{id}_{C_0}, \dots, \mathrm{id}_{C_0})$ is a homotopy equivalence.

To prove this we are going to use the fact that by Proposition 2.2, $F_nS.$ satisfies additivity (since $F_n(\mathcal{C} \times \mathcal{D}) \cong F_n\mathcal{C} \times F_n\mathcal{D}$). We define three natural transformations from $F_n$ to $F_nS_2$ which then assemble to give simplicial maps from $F_nS.\mathcal{C}$ to $F_nS.S_2\mathcal{C}$. We will use these to prove that $C$ is a homotopy equivalence. We define the natural transformations $T_{\mathrm{id}}$, $T_{-c}$ and $T_t$ as follows.

Let $\vec{\alpha} = (C_0 \xleftarrow{\alpha_0} C_1 \xleftarrow{\alpha_1} \cdots \longleftarrow C_n \xleftarrow{\alpha_n} C_0)$ denote an element of $F_n\mathcal{C}$ and let $\alpha_{i\cdots j}$ denote the composite $\alpha_i\alpha_{i+1}\cdots\alpha_j$. Then

$$T_{\mathrm{id}}(\vec{\alpha}) = \left[\begin{array}{ccccccccccc}
0 & \longleftarrow & C_0 & = & C_0 & = & \cdots & = & C_0 & \longleftarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle 1,0} & & \downarrow{\scriptstyle 1,0} & & & & \downarrow{\scriptstyle 1,0} & & \downarrow \\
C_0 & \xleftarrow{0+\alpha_0} & C_0 \oplus C_1 & \xleftarrow{1\oplus\alpha_1} & C_0 \oplus C_2 & \xleftarrow{1\oplus\alpha_2} & \cdots & \xleftarrow{1\oplus\alpha_{n-1}} & C_0 \oplus C_n & \xleftarrow{1,\alpha_n} & C_0 \\
\| & & \downarrow{\scriptstyle \pi_{C_1}} & & \downarrow{\scriptstyle \pi_{C_2}} & & & & \downarrow{\scriptstyle \pi_{C_n}} & & \| \\
C_0 & \xleftarrow{\alpha_0} & C_1 & \xleftarrow{\alpha_1} & C_2 & \xleftarrow{\alpha_2} & \cdots & \xleftarrow{\alpha_{n-1}} & C_n & \xleftarrow{\alpha_n} & C_0
\end{array}\right],$$

$$T_{-c}(\vec{\alpha}) = \left[\begin{array}{ccccccccccc}
0 & \longleftarrow & C_1 & \xleftarrow{\alpha_1} & C_2 & \xleftarrow{\alpha_2} & \cdots & \xleftarrow{\alpha_{n-1}} & C_n & \longleftarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle 0,1} & & \downarrow{\scriptstyle 0,1} & & & & \downarrow{\scriptstyle 0,1} & & \downarrow \\
C_0 & \xleftarrow{-\alpha_{0\cdots n}+0} & C_0 \oplus C_1 & \xleftarrow{1\oplus\alpha_1} & C_0 \oplus C_2 & \xleftarrow{1\oplus\alpha_2} & \cdots & \xleftarrow{1\oplus\alpha_{n-1}} & C_0 \oplus C_n & \xleftarrow{1,\alpha_n} & C_0 \\
\| & & \downarrow{\scriptstyle \pi_{C_0}} & & \downarrow{\scriptstyle \pi_{C_0}} & & & & \downarrow{\scriptstyle \pi_{C_0}} & & \| \\
C_0 & \xleftarrow{-\alpha_{0\cdots n}} & C_0 & = & C_0 & = & \cdots & = & C_0 & = & C_0
\end{array}\right],$$

$$T_t(\vec{\alpha}) = \left[\begin{array}{ccccccccccc}
0 & \xleftarrow{0} & C_0 & = & C_0 & = & \cdots & = & C_0 & = & C_0 \\
\| & & \downarrow{\scriptstyle 0,\alpha_{1\cdots n}} & & \downarrow{\scriptstyle 1,\alpha_{2\cdots n}} & & & & \downarrow{\scriptstyle 0,\alpha_n} & & \downarrow \\
C_0 & \xleftarrow{-\alpha_{0\cdots n}+\alpha_0} & C_0 \oplus C_1 & \xleftarrow{1\oplus\alpha_1} & C_0 \oplus C_2 & \xleftarrow{1\oplus\alpha_2} & \cdots & \xleftarrow{1\oplus\alpha_{n-1}} & C_0 \oplus C_n & \xleftarrow{1,\alpha_n} & C_0 \\
\downarrow & & \downarrow{\scriptstyle \alpha_{1\cdots n}-1} & & \downarrow{\scriptstyle \alpha_{2\cdots n}-1} & & & & \downarrow{\scriptstyle \alpha_n-1} & & \downarrow \\
0 & \leftarrow & C_0 & = & C_0 & = & \cdots & = & C_0 & = & C_0
\end{array}\right].$$

Now we note the following relations:

$$\mathrm{d}_0 T_{\mathrm{id}} = \mathrm{id} \quad \mathrm{d}_0 T_{-c} = -C \quad \mathrm{d}_0 T_t = 0$$

$$\mathrm{d}_1 T_t = \mathrm{d}_1 T_{\mathrm{id}} + \mathrm{d}_1 T_{-c} \quad \mathrm{d}_2 T_{\mathrm{id}} = d_2 T_{-c} = d_2 T_t = 0.$$

By Lemma 2.5 we obtain

$$\begin{aligned} \mathrm{id} - C &= \mathrm{d}_0 T_{\mathrm{id}} + \mathrm{d}_0 T_{-c} \\ &= (\mathrm{d}_0 T_{\mathrm{id}} + \mathrm{d}_2 T_{\mathrm{id}}) + (\mathrm{d}_0 T_{-c} + \mathrm{d}_2 T_{-c}) \\ &\simeq \mathrm{d}_1 T_{\mathrm{id}} + \mathrm{d}_1 T_{-c} \\ &= \mathrm{d}_1 T_t \\ &\simeq \mathrm{d}_0 T_t + \mathrm{d}_2 T_t = 0. \end{aligned}$$

## 3. THH($R$) as a Goodwillie derivative

We are now going to show how one can interpret $\mathrm{THH}(R)$ as the underlying space of a *derivative* in the sense of T. Goodwillie in [9]. We let $R$ denote a ring, $\mathcal{P}$ its exact category of finitely generated projective right $R$-modules and $\mathcal{M}$ its exact category of right modules.

*Definition* 3.1. For $M$ an $R$-bimodule, we define $\mathrm{K}(R;M)$ to be:

$$\mathrm{K}(R;M) = \Omega \left| \coprod_{\bar{C} \in S.\mathcal{P}} \mathrm{Hom}_{S.\mathcal{M}}(\bar{C}, \bar{C} \otimes_R M) \right|$$

We note that $\mathrm{K}(R;0) \cong \mathrm{K}(R)$, $\mathrm{K}(R;R) \cong \mathrm{K}(\mathrm{End}(\mathcal{P}))$ and that $\mathrm{K}(R;\ )$ is a functor of $R$-bimodules. We also note that $\mathrm{K}(R;M)$ is the usual algebraic K-theory for the exact category with objects the pairs $(P, \alpha)$ consisting of $P \in \mathcal{P}$ and $\alpha$ an $R$-module homomorphism from $P$ to $P \otimes_R M$ with morphisms

$$\mathrm{Hom}((P,\alpha),(Q,\beta)) = \{f \in \mathrm{Hom}_R(P,Q) | \beta \circ f = (f \otimes \mathrm{id}_M) \circ \alpha\}.$$

A sequence $(P'', \alpha'') \to (P, \alpha) \to (P'\alpha')$ is exact if and only if $P'' \to P \to P'$ is an exact sequence in $\mathcal{P}$.

We extend $\mathrm{K}(R;\ )$ to simplicial $R$-bimodules degree-wise. That is, for $M.$ a simplicial $R$-bimodule, $\mathrm{K}(R;M.)$ is the realization of the simplicial space $[n] \to \mathrm{K}(R;M_n)$. Since $\coprod_{\bar{C} \in S.\mathcal{P}} \mathrm{Hom}_{S.\mathcal{M}}(\bar{C}, \bar{C} \otimes_R M)$ is connected, $\mathrm{K}(R;M.)$ can equivalently be defined as the loop space of the realization of the associated bi-simplicial set. For $X$ a pointed space (= pointed finite simplicial set), we define $\mathrm{K}(R;M;X)$ to be $\mathrm{K}(R;M[X]/M[*])$. By the realization lemma (1.5 of [20]), $\mathrm{K}(R;M;\ )$ is a *homotopy functor* taking homotopy equivalent spaces to homotopy equivalent spaces.

*Definition* 3.2. For $M$ an $R$-bimodule, we let

$$\mathrm{THH}(R;M) = \Omega \left| \bigoplus_{\bar{C} \in S.\mathcal{P}} \mathrm{Hom}_{S.\mathcal{M}}(\bar{C}, \bar{C} \otimes_R M) \right|$$

By a direct transliteration of the methods of Sections 1 and 2, we see that $\mathrm{THH}(R;M)$ is naturally homotopy equivalent to the realization of the simplicial abelian group

$$F_n(\mathcal{P};M) = \bigoplus_{\vec{P} \in N_n(\mathcal{P})} \mathrm{Hom}_{\mathcal{P}}(P_1, P_0 \oplus_R M), \qquad \vec{P} = P_1 \leftarrow \cdots \leftarrow P_n \leftarrow P_0$$

(with structure maps like $F.(\mathcal{P})$ in 1.1). Thus, $\mathrm{THH}(R;M)$ is naturally weak homotopy equivalent with the space $\mathrm{THH}(R;M)$ as defined in [17].

We extend this definition to simplicial bimodules degree-wise and define $\mathrm{THH}(R;M;X)$ for a pointed space $X$ by $\mathrm{THH}(R;M[X]/M[*])$. We note that $\mathrm{THH}(R;M;*) = 0$, $\mathrm{THH}(R;R;S^0) \cong \mathrm{THH}(R)$ and that $\mathrm{THH}(R;M;\ )$ is *excisive*: a generalized homology theory.

*Definition* 3.3. We define $\tilde{\mathrm{K}}(R;M;X)$ to be the (homotopy) fiber of the natural retraction from $\mathrm{K}(R;M;X)$ to $\mathrm{K}(R;0;*) = \mathrm{K}(R)$. Since $\mathrm{K}(R;M;X)$ is an infinite loop space and the map in question is a map of infinite loop spaces, we see that $\mathrm{K}(R;M;X)$ is weakly homotopic to $\mathrm{K}(R) \times \tilde{\mathrm{K}}(R;M;X)$. The *Goodwillie differential* of the homotopy functor $\mathrm{K}(R;M;\ )$ at $X$ is

$$\mathrm{DK}(R;M;X) = \lim_{n \to \infty} \Omega^n \tilde{\mathrm{K}}(R;M;S^n \wedge X).$$

The *derivative* of $\mathrm{K}(R;M;\ )$ is a spectrum naturally associated to its differential and in our case it is simply the spectrum $\tilde{\mathrm{K}}(R;M;S^n)$ with the obvious structure maps.

We note that there is a natural transformation from $\mathrm{K}(R;M;\ )$ to $\mathrm{THH}(R;M;\ )$ induced by the natural map from the coproduct of the underlying set of groups to the direct sum of the groups. Since $\mathrm{THH}(R;M;\ )$ is excisive, the natural map $\mathrm{THH}(R;M;X) \longrightarrow \Omega\,\mathrm{THH}(R;M;\Sigma X)$ is an equivalence for all $X$ and we have a natural diagram

$$\begin{array}{ccc} \tilde{\mathrm{K}}(R;M;X) & \longrightarrow & \mathrm{THH}(R;M;X) \\ \downarrow & & \downarrow \simeq \\ \mathrm{DK}(R;M;X) & \longrightarrow & \mathrm{DTHH}(R;M;X) \end{array}$$

THEOREM 3.4. *For $M.$ an $m$-connected simplicial $R$-bimodule ($\pi_*(|M.|) = 0$ for $* \leq m$), the map $\tilde{\mathrm{K}}(R;M) \longrightarrow \mathrm{THH}\,(R;M)$ is $2m$-connected and thus $\mathrm{DK}\,(R;M;\ )$ is naturally weak homotopy equivalent to $\mathrm{DTHH}\,(R;M;\ )$.*

*Proof*. We are indebted to T. Goodwillie for pointing out an error in our earlier version of this proof. For each $p$, we let $S^{(p)}$ denote the diagonal of the iteration of the $S$ construction $p$-times. We define

$$\mathrm{fib}^{(p)} = \text{fiber}: \left| \coprod_{\bar{C}\in S_{\cdot}^{(p)}\mathcal{P}} \mathrm{Hom}_{S_{\cdot}^{(p)}\mathcal{M}}(\bar{C}, \bar{C}\otimes_R M)\right| \longrightarrow |S_{\cdot}^{(p)}\mathcal{P}|,$$

$$\mathrm{cof}^{(p)} = \text{cofiber}: |S_{\cdot}^{(p)}\mathcal{P}| \xrightarrow{0} \left| \coprod_{\bar{C}\in S_{\cdot}^{(p)}\mathcal{P}} \mathrm{Hom}_{S_{\cdot}^{(p)}\mathcal{M}}(\bar{C}, \bar{C}\otimes_R M)\right|,$$

$$\mathrm{THH}^{(p)} = \left| \bigoplus_{\bar{C}\in S_{\cdot}^{(p)}\mathcal{P}} \mathrm{Hom}_{S_{\cdot}^{(p)}\mathcal{M}}(\bar{C}, \bar{C}\otimes_R M)\right|.$$

We now note that the natural map from $\tilde{\mathrm{K}}(R;M)$ to $\mathrm{THH}(R;M)$ can be put into the following commutative diagram

$$\begin{array}{ccccc} \tilde{\mathrm{K}}(R;M) & & \longrightarrow & & \mathrm{THH}(R;M) \\ \downarrow \simeq & & & & \downarrow \simeq \\ \Omega^p \,\mathrm{fib}^{(p)} & \xrightarrow{\alpha_p} & \Omega^p \,\mathrm{cof}^{(p)} & \xrightarrow{\beta_p} & \Omega^p \,\mathrm{THH}^{(p)}. \end{array}$$

The vertical maps are homotopy equivalences by additivity and Lemma 2.3. We claim that the map $\alpha_p$ is at least $p-3$ connected and the map $\beta_p$ is $2m$ connected. Assuming our claims, we are finished by choosing $p \geq 2m+3$.

To see that the maps $\alpha_p$ are at least $p-3$ connected, we consider the commuting diagram

$$\begin{array}{ccc} \left|\coprod_{\bar{C}\in S_{\cdot}^{(p)}\mathcal{P}} \mathrm{Hom}_{S_{\cdot}^{(p)}\mathcal{M}}(\bar{C}, \bar{C}\otimes_R M)\right| & \longrightarrow & |S_{\cdot}^{(p)}\mathcal{P}| \\ \downarrow & & \downarrow \\ \left|\bigvee_{\bar{C}\in S_{\cdot}^{(p)}\mathcal{P}} \mathrm{Hom}_{S_{\cdot}^{(p)}\mathcal{M}}(\bar{C}, \bar{C}\otimes_R M)\right| & \longrightarrow & *. \end{array}$$

This diagram satisfies the Blakers-Massey theorem (see for example [24], 7.4) with all the spaces at least $p-1$-connected. Consequently the map of the homotopy fibers is at least $2p-3$-connected. Thus, $\alpha_p$ is at least $p-3$-connected.

Now we show that the maps $\beta_p$ are at least $2m$-connected for all $p$. First we note that for each fixed $n$, the natural map

$$(1) \qquad \bigvee_{\bar{C}\in S_n^{(p)}\mathcal{P}} \mathrm{Hom}_{S_n^{(p)}\mathcal{M}}(\bar{C}, \bar{C}\otimes_R M) \longrightarrow \bigoplus_{\bar{C}\in S_n^{(p)}\mathcal{P}} \mathrm{Hom}_{S_n^{(p)}\mathcal{M}}(\bar{C}, \bar{C}\otimes_R M)$$

is $2m+1$-connected. This is because the map from the wedge of a finite number of $m$-connected spaces to their product is $2m+1$-connected (essentially by the

Blakers-Massey theorem again), and to extend this to arbitrary index systems we take the inductive limit over all finite subsets. Letting $n$ vary, we see that the (homotopy) fiber of the natural map in (1) is equivalent to a bi-simplicial space $Z_{..}$ such that $Z_{q,\cdot}$ is contractible if $0 \leq q \leq p-1$ and $2m+1$-connected if $q \geq p$. By standard spectral sequence arguments it follows that the natural map is $2m+p$-connected and hence that $\beta_p$ is $2m$-connected. □

## 4. The relation between $\mathrm{K}(R;M)$ and $\mathrm{K}(R\oplus M)$

In order to compare stable K-theory to THH it is going to be convenient for us to examine first the relationship between $\mathrm{K}(R;M)$ and $\mathrm{K}(R\oplus M)$ (where $R\oplus M$ denotes the augmented ring with underlying group $R\oplus M$ and multiplication $(r,m)(r',m') = (rr', rm' + mr'))$ for any $R$-bimodule $M$. We will let $\tilde{\mathrm{K}}(R\oplus M)$ denote the fiber of the natural map from $\mathrm{K}(R\oplus M)$ to $\mathrm{K}(R)$ induced by the ring homomorphism sending $(r,m)$ to $r$. This is a map of infinite loop spaces with a section, so $\mathrm{K}(R\oplus M)$ is weakly homotopic to $\mathrm{K}(R)\times\tilde{\mathrm{K}}(R\oplus M)$.

The rest of this section is devoted to proving the following theorem. We let $B.M$ denote the bar construction naturally considered as a simplicial $R$-bimodule; in particular $\mathrm{K}(R;B.M)\cong \mathrm{K}(R;M;S^1)$.

THEOREM 4.1. *For any $R$-bimodule $M$, there exists a natural weak homotopy equivalence $\Psi(R,M)$*

$$\mathrm{K}(R;B.M) \xrightarrow{\simeq} \mathrm{K}(R\oplus M)$$

*which factors to give a homotopy equivalence $\tilde{\Psi}(R,M)$ from $\tilde{\mathrm{K}}(R,B.M)$ to $\tilde{\mathrm{K}}(R\oplus M)$.*

*Construction of* $\Psi(R,M)$.

Recall that $B_nM = M^{\bigoplus n}$ and $\mathrm{Hom}_{S.\mathcal{M}}(\bar{C},\bar{C}\otimes_R(M^{\bigoplus n}))$ is naturally isomorphic to $\mathrm{Hom}_{S.\mathcal{M}}(\bar{C},\bar{C}\otimes_R M)^{\bigoplus n}$. We construct a natural map of bi-simplicial sets $\Psi_{*,\cdot}$ from $\coprod_{\bar{C}\in S.\mathcal{P}}\mathrm{Hom}_{S.\mathcal{M}}(\bar{C},\bar{C}\otimes_R M)^*$ to $N_*iS.\mathcal{P}_{R\oplus M}$ (notation is from Remark 2.1). Using the ring map from $R$ to $R\oplus M$ (sending $r$ to $(r,0)$) we see that there is an exact functor from $\mathcal{P}$ to $\mathcal{P}_{R\oplus M}$. This functor is naturally isomorphic to the exact functor $G$ which sends $P$ to $P\oplus(P\otimes_R M)$ with $(p,(p'\otimes m'))(r,m) = (pr, p'\otimes m'r + p\otimes m)$. Given an $R$-module homomorphism $\alpha$ from $P$ to $P\otimes_R M$, we define the isomorphism $(\mathrm{id},\alpha)$ of $G(P)$ (as an $R\oplus M$-module) by $(\mathrm{id},\alpha)(p,p'\otimes m) = (p,\alpha(p)+p'\otimes m)$. We note that $(\mathrm{id},\alpha)\circ(\mathrm{id},\beta) = (\mathrm{id},\alpha+\beta)$. The map $\Psi_{p,q}$ is defined by sending

$$(\bar{C};\alpha_1,\ldots,\alpha_p)\in S_q\mathcal{P}\times\mathrm{Hom}_{S_q\mathcal{M}}(\bar{C},\bar{C}\otimes_R M)^p$$

to

$$(G(\bar{C}) \overset{(\mathrm{id},\alpha_1)}{\longleftarrow} G(\bar{C}) \overset{(\mathrm{id},\alpha_2)}{\longleftarrow} \cdots \overset{(\mathrm{id},\alpha_p)}{\longleftarrow} G(\bar{C})) \in N_p i S_q \mathcal{P}_{R\oplus M}.$$

There is a commuting diagram

$$\begin{array}{ccc} \mathrm{K}(R; B.M) & \xrightarrow{\Psi} & \mathrm{K}(R \oplus M) \\ \downarrow & & \downarrow \\ \mathrm{K}(R;0) & \xrightarrow{=} & \mathrm{K}(R); \end{array}$$

thus, $\Psi$ naturally produces a map $\tilde{\Psi}$ from $\tilde{\mathrm{K}}(R; B.M)$ to $\tilde{\mathrm{K}}(R \oplus M)$ and $\Psi$ is a weak homotopy equivalence if and only if $\tilde{\Psi}$ is one also.

To show that $\Psi$ is a homotopy equivalence it is convenient to restrict our attention to free modules. That is, to use the subcategories of finitely generated free modules (denoted $\mathcal{F}$) instead of all of the projectives. We note that the map $\Psi$ is still well-defined if we restrict to $\mathcal{F}$.

*Reduction.* It suffices to show that $\Psi$ is an equivalence on the bi-simplicial subsets defined using free modules.

*Proof.* Let the decoration "$f$" mean the appropriate construction using the exact category of free modules as opposed to projectives. By cofinality (see for example [10], 1.1) we know that the natural map $\mathrm{K}_f(R; M) \to \mathrm{K}(R; M)$ (induced by the exact inclusion functor) is an isomorphism on all the homotopy groups except possibly 0. Each class $[(P, \alpha)]$ of $\pi_0\mathrm{K}(R; M)$ is equivalent to $[(P \oplus P', \alpha \oplus 0)] - [(P', 0)]$ and in choosing $P'$ so that $P \oplus P'$ is free we see that $\pi_0\widetilde{\mathrm{K}_f}(R; M) \xrightarrow{\cong} \pi_0\tilde{\mathrm{K}}(R; M)$ and hence the natural map $\widetilde{\mathrm{K}_f}(R; M) \to \tilde{\mathrm{K}}(R; M)$ is a weak homotopy equivalence. By the realization lemma, we conclude that this is true for simplicial $R$-bimodules as well.

Again by cofinality, we know that the natural map from $\mathrm{K}_f(R \oplus M)$ to $\mathrm{K}(R \oplus M)$ is an isomorphism on all homotopy groups except possibly zero. Since $\pi_0\mathrm{K}_f(R) \xrightarrow{\cong} \pi_0\mathrm{K}_f(R \oplus M)$ and $\pi_0\mathrm{K}(R) \xrightarrow{\cong} \pi_0\mathrm{K}(R \oplus M)$ (use Nakayama's lemma; (see for example [1], IX.1.3)) we see that the natural map $\widetilde{\mathrm{K}_f}(R \oplus M) \to \tilde{\mathrm{K}}(R \oplus M)$ is a weak homotopy equivalence.

By these remarks, if $\widetilde{\Psi_f}(R, M)$ is a weak homotopy equivalence, then $\tilde{\Psi}(R, M)$ will be one also which will imply that $\Psi(R, M)$ is one too. □

*Proof* (of Theorem 4.1). The following is a simplification by the referee of our original argument. By the above reduction, we can assume that we are working with the respective categories of free modules. We pass to equivalent exact categories with one object for each nonnegative integer. We can consider $\mathcal{F}_R$ as a subcategory of $\mathcal{F}_{R\oplus M}$ with all of the objects (one for each natural number), but having only the morphisms $(\beta, 0)$. Note, however, that for $q > 1$ the subcategory $S_q\mathcal{F}_R$ of $S_q\mathcal{F}_{R\oplus M}$ does *not* have all the objects.

Let $U$ be the exact functor from $\mathcal{F}_{R\oplus M}$ to $\mathcal{F}_R$ which takes every object to itself and takes $(\beta, \alpha)$ to $(\beta, 0)$. It is a retraction $(UU = U)$ with image $\mathcal{F}_R$. Let $\mathcal{T}$ be the class of isomorphisms of the form $(1, \alpha)$. These are precisely the morphisms which $U$ takes to the identity maps.

The bi-simplicial map $\Psi_{.,.}$ which defines $\Psi$ is an injection into $N.tS.\mathcal{F}_{R\oplus M}$. By Remark 2.1 and the realization lemma it will be enough if we show that for each $q$ the inclusion of the image of $\Psi_{.,q}$ into $N.tS_q\mathcal{F}_{R\oplus M}$ is a homotopy equivalence. The image of $\Psi_{.,q}$ is the nerve of a full subcategory of $tS_q\mathcal{F}_{R\oplus M}$, namely that which has the same objects as $S_q\mathcal{F}_R$. We complete the proof by showing that every object $\bar{C}$ of $S_q\mathcal{F}_{R\oplus M}$ is $t$-isomorphic to an object of $S_q\mathcal{F}_R$, namely $S_q(U)(\bar{C})$.

The fact that $\bar{C}$ is isomorphic to $S_q(U)(\bar{C})$ follows from the fact that in $\mathcal{F}_{R\oplus M}$ every short exact sequence splits. (Every filtered object $\bar{C}$ is a split object, and so its isomorphism class is determined by the isomorphism classes of its subquotients $\bar{C}(i+1/i), 1 \leq i \leq q$). If $\eta$ is an isomorphism from $\bar{C}$ to $S_q(U)(\bar{C})$, then putting $\hat{\eta} = S_q(U)(\eta)$, we see that $\hat{\eta}^{-1} \circ \eta$ is a $t$-isomorphism because $S_q(U)(\hat{\eta}^{-1} \circ \eta) = S_q(U)(\hat{\eta})^{-1} \circ S_q(U)(\eta) = \hat{\eta}^{-1} \circ \hat{\eta} = 1$. □

## 5. Stable K-theory and THH

For $R.$ a simplicial ring, we let $\mathrm{K}(R.)$ denote its algebraic K-theory as defined by F. Waldhausen in [21], Section 6. For $M.$ a simplicial $R.$-bimodule, we let $R. \oplus M.$ denote the new simplicial ring which is $R_n \oplus M_n$ in dimension $n$. We let $\tilde{\mathrm{K}}(R. \oplus M.) = \mathrm{fiber}(\mathrm{K}(R. \oplus M.) \longrightarrow \mathrm{K}(R.))$. Since this is a map of infinite loop spaces with a retract, $\mathrm{K}(R. \oplus M.)$ is weakly homotopic to $\mathrm{K}(R.) \times \tilde{\mathrm{K}}(R. \oplus M.)$. In general, $\mathrm{K}(R.)$ is *not* equivalent to the realization of the simplicial space $[n] \to \mathrm{K}(R_n)$. We recall, however, a fact noted by T. Goodwillie.

PROPOSITION 5.1 (Lemma I.2.2 of [8]). *For any simplicial ring $R.$ and $R.$-bimodule $M.$, $\tilde{\mathrm{K}}(R. \oplus M.)$ is naturally weak homotopy equivalent to the realization of the simplicial space $[n] \to \tilde{\mathrm{K}}(R_n \oplus M_n)$.*

There is a gap in the proof of this statement in [8]. The diagram on page 359 of [8] is not shown to be homotopy-cartesian, but this can now be deduced from 2.7 of [6].

For $A.$ a simplicial abelian group and $X$ a pointed space (= pointed finite simplicial set) we let $\tilde{A}.[X]$ denote the new simplicial abelian group $[n] \to A_n[X_n]/A_n[*]$.

*Definition* 5.2. C. Kassel defined the *stable* K-*theory* of the simplicial ring $R.$ with coefficients in the simplicial $R.$-bimodule $M.$ (see [12], 2.3) by

$$\mathrm{K}^s(R.,M.) = \lim_{n\to\infty} \Omega^n \tilde{\mathrm{K}}(R. \oplus \tilde{M}.[S^{n-1}]).$$

We recall that for any pointed space $X$, this is an equivalent definition (up to weak equivalence) with $\mathrm{K}^s(R,X)$ defined in [21] Section 6 when we let $M.$ be $\tilde{R}.[X]$ (see page 388 of [21]).

THEOREM 5.3. *For any simplicial ring $R.$ and $R.$-bimodule $M.$, there is a natural weak equivalence between $\mathrm{K}^s(R.,M.)$ and $\mathrm{THH}\,(R.;M.)$ (where $\mathrm{THH}\,(R.;M.)$ is the realization of the simplicial space $[n] \to \mathrm{THH}\,(R_n;M_n)$).*

*Proof*. We note that by Proposition 5.1 above, it suffices to prove the case for $R$ and $M$ discrete. By Theorem 4.1 we have a natural homotopy equivalence $\Psi(R,M)$ from $\tilde{\mathrm{K}}(R;B.M)$ to $\tilde{\mathrm{K}}(R\oplus M)$. If we use the model of $S^1$ obtained by using the category of based finite sets (see for example [19]), then there is a natural isomorphism of simplicial abelian groups $\tilde{M}[S^1\wedge X] \cong \widetilde{B.M}[X]$ for any pointed space $X$. We define $\Psi(R,M,X)$ to be the natural map of simplicial spaces given in degree $n$ by $\Psi(R,\tilde{M}[X_n])$. Using the fact that we can define $\mathrm{K}(R\oplus\tilde{M}[X])$ degree-wise (again by Proposition 5.1) and by the realization lemma, $\Psi(R,M,X)$ produces a natural homotopy equivalence from $\tilde{\mathrm{K}}(R;M;S^1\wedge X)$ to $\tilde{\mathrm{K}}(R\oplus\tilde{M}[X])$ for all pointed spaces $X$. Thus, we have constructed a map of limit systems (starting with $q=1$)

$$\Omega^q\Psi(R,M,S^q)\colon\ \Omega^q\tilde{\mathrm{K}}(R;M;S^q) \xrightarrow{\simeq} \Omega^q\tilde{\mathrm{K}}(R\oplus\tilde{M}[S^{q-1}]).$$

Hence, $\mathrm{K}^s(R,M)$ is naturally weak homotopy equivalent to $\mathrm{DK}(R;M;S^0)$ (see Definition 3.3) and by Proposition 3.4 this is naturally weak homotopy equivalent to $\mathrm{THH}(R;M;S^0) = \mathrm{THH}(R;M)$. □

*Remark* 5.4 (the Dennis trace map). We recall that there is a natural transformation from $\mathrm{K}(R)$ to $\mathrm{K}^s(R,R)$. This map is produced by a natural map from $\mathrm{K}(R)$ to $\Omega\tilde{\mathrm{K}}(R\oplus R)$ which one obtains by taking a product with the element "$e$" of $\pi_1\tilde{\mathrm{K}}(\mathbf{Z}\oplus\mathbf{Z})$ determined by the class of $\mathbf{Z}\oplus\mathbf{Z} \xrightarrow{e} \mathbf{Z}\oplus\mathbf{Z}$ in $N_1 iS_1\mathcal{P}_{\mathbf{Z}\oplus\mathbf{Z}}$ with $e = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (see for example 4.2 of [12]).

The zero and identity endomorphisms produce natural exact functors from $\mathcal{P}_R$ to $\mathrm{End}(\mathcal{P}_R)$ and hence a natural map "$1-0$" (using the $H$-space structure) from $\mathrm{K}(R)$ to $\mathrm{K}(\mathrm{End}(R)) = \mathrm{K}(R;R)$ which then maps to $\tilde{\mathrm{K}}(R;R)$. This produces a natural map from $\mathrm{K}(R)$ to $\mathrm{THH}(R;R)$ which agrees with the usual map to stable K-theory via our natural isomorphism by the commuting diagram:

$$\begin{array}{ccc} \mathrm{K}(R) & \xrightarrow{1-0} & \tilde{\mathrm{K}}(R;R) \\ \Big\downarrow{\scriptstyle \times e} & & \Big\downarrow \\ \Omega\tilde{\mathrm{K}}(R\oplus R) & \xleftarrow{\Omega\Psi(R,R)} & \Omega\tilde{\mathrm{K}}(R;B.R) \end{array}$$

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