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A CLASSIFICATION THEOREM FOR DIAGRAMS OF SIMPLICIAL SETS†

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§1. INTRODUCTION

1.1 Summary

THE AIM of this paper is to prove a rather general *classification theorem for diagrams of simplicial sets*, which encompasses the classification results for *Postnikov conjugates* of [15] and [3] and those for *simplicial fibrations* of [1] and [4]. This theorem will be applied in [10] to analyze the category of topological spaces on which a topological group G acts. It leads to a classification of these G-spaces with respect to weak equivariant homotopy equivalences, i.e. with respect to equivariant maps which restrict to weak homotopy equivalences on the fixed point sets of a given collection of subgroups of G.

1.2 Motivation

To motivate our result we start with recalling the essence of the two classification results mentioned above:

I. CLASSIFICATION OF POSTNIKOV CONJUGATES ([3, 15]). *Given a fibrant simplicial set X, there is a simplicial set L, which is constructed as a homotopy inverse limit and has the following properties*:

(i) *There is a natural* 1–1 *correspondence between the components of L and the homotopy types of the Postnikov conjugates of X (i.e. the fibrant simplicial sets Y such that, for every integer $n \geqslant 0$, the* nth *Postnikov approximations P_nX and P_nY have the same homotopy type).*

(ii) *For every Postnikov conjugate Y of X, the corresponding (see* (i)) *component of L has the homotopy type of the classifying complex of the simplicial monoid of self homotopy eqiuvalences of Y.*

II. CLASSIFICATION OF SIMPLICIAL FIBRATIONS ([1, 4]). *Given a simplicial set B and a fibrant simplicial set X, there is a simplicial set L, which is constructed as a function complex and has the following properties*:

(i) *There is a natural* 1–1 *correspondence between the components of L and the homotopy equivalence classes of fibrations with base B and all fibres homotopically equivalent to X.*

(ii) *For every fibration p with base B and all fibres homotopically equivalent to X, the corresponding (see* (i)) *component of L has the homotopy type of the classifying complex of the simplicial monoid of self homotopy equivalences of p.*

The similarity as well as the dissimilarity of these two results suggests that they are special cases of one much more general classification theorem. The nature of this more general result becomes clear, once one realizes that *a function complex is a special case of a homotopy inverse limit* and that *a fibration can be considered as a diagram of simplicial sets indexed by the simplices of the base.*

1.3 The main result

Our classification theorem consists essentially of two parts, a more or less *formal* first part and a second part which is more *computational.*

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Let $\mathbf{S}$ denote the category of simplicial sets, let $\mathbf{D}$ be an arbitrary but fixed small category and let $\mathbf{S}^{\mathbf{D}}$ denote the *category of* $\mathbf{D}$*-diagrams* of simplicial sets (i.e. functors $\mathbf{D}\to\mathbf{S}$ and natural transformations between them). Call a map $f\colon X\to Y\in\mathbf{S}^{\mathbf{D}}$ a *weak equivalence* if, for every object $D\in\mathbf{D}$, the map $fD\colon XD\to YD\in\mathbf{S}$ is a weak (homotopy) equivalence and call two objects $X, Y\in\mathbf{S}^{\mathbf{D}}$ *weakly equivalent* if they can be connected by a finite string of weak equivalences. Finally, for every integer $n\geqslant 0$, denote by $\mathbf{n}$ the category which has the integers $0,\ldots,n$ as objects and which has exactly one map $i\to j$ whenever $i\leqslant j$, and call two diagrams $X, Y\in\mathbf{S}^{\mathbf{D}}$ *conjugate* if, for every integer $n\geqslant 0$ and every functor $J\colon\mathbf{n}\to\mathbf{D}$, the induced $\mathbf{n}$-diagram J^*X and J^*Y are weakly equivalent.

Our main result now is a *classification of the weak equivalence classes of the conjugates with an object* $X\in\mathbf{S}^{\mathbf{D}}$. If we define the *classification complex* cX of X as the nerve of the subcategory $\mathbf{c}X\subset\mathbf{S}^{\mathbf{D}}$ which consists of all weak equivalences between conjugates of X, then our result can be summarized as follows:

FORMAL PART. (i) *There is an obvious* 1–1 *correspondence between the components of* cX *and the weak equivalence classes of the conjugates of* X.

(ii) *For every conjugate* Y *of* X, *the component of* cX *containing it has* (*in a sense that will be made precise in* §2) *the homotopy type of a classifying complex for the self weak equivalences of* Y.

COMPUTATIONAL PART. (iii) *The classification complex* cX *is weakly* (*homotopy*) *equivalent to* (*see* §3) *a homotopy inverse limit involving the much more accessible classification complexes* cJ^*X *of the induces* $\mathbf{n}$*-diagrams mentioned above.*

The substantive part of this result is part (iii) which asserts, along the lines of [9], that a certain problem in diagram homotopy theory (the classification problem) can be reduced, in an explicit way, to a problem in ordinary homotopy theory (the problem of computing a particular homotopy inverse limit).

1.4 Remark

By combining the arguments used to prove (i), (ii) and (iii) with 7.3 one readily obtains the following somewhat stronger result. Let $\pi\colon\mathbf{S}\to ho\mathbf{S}$ denote the natural functor from $\mathbf{S}$ to its homotopy category $ho\mathbf{S}$ [2, Chap. VIII, §3], let $\bar{X}\colon\mathbf{D}\to ho\mathbf{S}$ be a $\mathbf{D}$-diagram in $ho\mathbf{S}$, let a $\mathbf{D}$-diagram $Y\colon\mathbf{D}\to\mathbf{S}$ be called conjugate to $\bar{X}$ if, for every integer $n\geqslant 0$ and every functor $J\colon\mathbf{n}\to\mathbf{D}$, the induced $\mathbf{n}$-diagram π_*J^*Y and $J^*\bar{X}\colon\mathbf{n}\to ho\mathbf{S}$ are naturally equivalent and let $c\bar{X}$ denote the nerve of the subcategory of $\mathbf{c}\bar{X}\subset\mathbf{S}^{\mathbf{D}}$ which consists of all weak equivalences between conjugates of $\bar{X}$. Then *the above statements* (i), (ii) *and* (iii) *remain valid with* X *replaced by* $\bar{X}$ *and* cX *replaced by* $c\bar{X}$. *In particular the homotopy inverse limit referred to in* (*iii*) *then is non-empty iff* $\bar{X}$ *is realizable up to conjugacy*, i.e. *iff there exists a* $\mathbf{D}$*-diagram* $Y\in\mathbf{S}^{\mathbf{D}}$ *which is conjugate to* $\bar{X}$.

1.5 Organization of the paper

In §2 we make clear what we mean by *the homotopy type of a classifying complex for the self weak equivalences of a diagram of simplicial sets* or more generally *the homotopy type of a classifying complex for the self weak equivalences of an object in a closed simplicial model category*. We then obtain a very general *classification result for closed simplicial model categories* which implies the formal part of our main result.

In §3 we discuss the *computational* part. Its statement involves a homotopy inverse limit taken over the *division* of the category $\mathbf{D}$, i.e. the category $d\mathbf{D}$ which, roughly speaking, has as objects the simplices of the nerve of $\mathbf{D}$ and as maps the simplicial operators between them. One could also (see §4) have used a homotopy inverse limit over the much smaller *subdivision* of the category $\mathbf{D}$, i.e. the category $sd\mathbf{D}$ which, roughly speaking, has as objects the non-degenerate simplices of the nerve of $\mathbf{D}$ and as maps certain equivalence classes of iterated face maps between them. This turns out to be particularly useful when (the nerve of) $\mathbf{D}$ is finite or finite dimensional as in that case the same holds for $sd\mathbf{D}$. Another and also very useful *variation* involves (see §5) a function complex between diagrams indexed by a category which can be even smaller than the subdivision.

In §6 we discuss a special type of functor between small categories which we call *L-cofinal*. We then use this discussion in §7 and §8 to produce a three step proof of the computational part of our main result: first a reduction to so-called direct categories, next a further reduction to finite dimensional categories and finally a proof for finite dimensional categories by induction on the dimension.

In an Appendix (§9) we briefly review some basic and more or less well known results on *homotopy limits*.

1.6 Notation, terminology, etc.

(i) We will freely use the notation and terminology of[9]. In particular *we won't distinguish between a small category and its nerve* and, if $\mathbf{D}$ is a small category, then *we endow the category $\mathbf{S}^{\mathbf{D}}$, of $\mathbf{D}$-diagrams of simplicial sets, with the closed simplicial model category structure of* [9, §2] (in which a map $f: X \to Y \in \mathbf{S}^{\mathbf{D}}$ is a weak equivalence or a fibration whenever, for every object $D \in \mathbf{D}$, the map $fD: XD \to YD \in \mathbf{S}$ is so).

(ii) Many of the simplicial sets used in this paper are not small, but only *homotopically small* in the sense of [8, 2.2]. As explained in [8, §2], this does not really matter, i.e. one can "do homotropy theory" with them as usual.

(iii) As [2, Chap. XI] homotopy inverse limits only have invariant homotopy meaning when applied to fibrant diagrams, we often have to replace a given diagram X by a weakly equivalent fibrant one such as, for instance, $Ex^{\infty}X$, where Ex^{∞} denotes the functor of [11]. To simplify the notation *we will write X^f instead of $Ex^{\infty}X$*. (Of course, instead of Ex^{∞}, we could just as well have used the singular functor on the geometric realization[12]).

§2. CLASSIFICATION COMPLEXES

In this section we

(i) make clear what we mean by *the homotopy type of a classifying complex for the self weak equivalences of a diagram of simplicial sets*, or more generally, *the homotopy type of a classifying complex for the self weak equivalences of an object in a closed simplicial model category*, and

(ii) obtain a *classification result for closed simplicial model categories* which implies the formal part of our main result.

We start with defining

2.1 Classification complexes for closed simplicial model categories

Let $\mathbf{M}$ be a closed simplicial model category, i.e. [13, Chap. II] a category with a notion of function complexes and with three distinguished classes of maps, weak equivalences, fibrations and cofibrations, satisfying certain axioms. Call two objects X, $Y \in \mathbf{M}$ *weakly equivalent* if they can be connected by a finite string of weak equivalences. By a *classification complex* of $\mathbf{M}$ we mean the nerve c of any subcategory $\mathbf{c} \subset \mathbf{M}$ such that

(i) every map in $\mathbf{c}$ is a weak equivalence,

(ii) if $f: X \to Y \in \mathbf{M}$ is a weak equivalence and either X or Y is in $\mathbf{c}$, then f is in $\mathbf{c}$ and

(iii) c is homotopically small (1.6(ii)).

Important special cases are the

2.2 Special classification complexes

Let $X \in \mathbf{M}$ be an object. *The special classification complex* of X then will be the (unique) classification complex scX which is connected and which contains X. Its usefulness is due to the fact that

(i) *scX depends only on the weak equivalence class of X*, and

(ii) *scX has "the homotopy type of a classifying complex for the self weak equivalences of X."*

By this last statement we mean that scX has the property mentioned in the following proposition. (Note that this proposition also guarantees that scX has the smallness property 2.1(ii).)

2.3 PROPOSITION. *Let $\mathbf{M}$ be a closed simplicial model category, let $X \in \mathbf{M}$ be an object which is both fibrant and cofibrant and let* haut$X$ *be its simplicial monoid of self weak eqiuvalences (which has as n-simplices the weak equivalences $X \otimes \Delta[n] \to X \in \mathbf{M}$*[13, Chap. II, §1]). *Then the classifying complex $\bar{W}$*haut$X$ [12, p. 87] *is weakly (homotopy) equivalent to scX.*

Proof. This follows readily from [6, 5.5], [7, 2.2] and [8, 4.6 and 4.8].

2.4 Remark

The above proof of proposition 2.3 actually implies the slightly stronger statement that $\bar{W}$haut$X$ *and scX can be connected by a finite string of weak equivalences which is natural with respect to all simplicial functors $f: \mathbf{M} \to \mathbf{N}$ between closed simplicial model categories which preserve weak equivalences and are such that $fX \in \mathbf{N}$ is also both fibrant and cofibrant.*

An immediate consequence is the classification result mentioned at the beginning of this section:

2.5 COROLLARY. *Let $\mathbf{M}$ be a closed simplicial model category and let c be a classification complex for $\mathbf{M}$. Then*

(i) *there is an obvious* 1–1 *correspondence between the components of c and the weak equivalence classes of the objects of $\mathbf{M}$ which are in c, and*

(ii) *for every object $Y \in \mathbf{M}$ which is in c, the component of c containing it is exactly scY and thus has (in the above sense) the homotopy type of a classifying complex for the self weak equivalences of Y.*

2.6 Examples

(i) Let $\mathbf{D}$ be a small category. Then [9] $\mathbf{S}^{\mathbf{D}}$ is a closed simplicial model category and the above results thus apply. In particular, if $X \in \mathbf{S}^{\mathbf{D}}$, *then the classification complex cX of* 1.3 *is a classification complex in the sense of* 2.1. (The smallness property 2.1(iii) follows from the computational part of our main result (3.4)).

(ii) If n is an integer ≥ 0 and $X \in \mathbf{S}^{\mathbf{n}}$, then $cX = scX$. Moreover, if n' is an integer ≥ 0 and $j: \mathbf{n}' \to \mathbf{n}$ is a functor which is onto, then it is easy to see (by constructing suitable functors and natural transformations) that the induced diagram $j^*X \in \mathbf{S}^{\mathbf{n}'}$ is such that $cX = scX$ is a deformation retract of $cj^*X = scj^*X$.

§3. THE CLASSIFICATION THEOREM

To state our main result we need

3.1 The classification diagram $c_{d\mathbf{D}}X$ of a diagram $X \in \mathbf{S}^{\mathbf{D}}$

Let $d\mathbf{D}$ be the *division of* $\mathbf{D}$, i.e. [9, §5] the category which has as objects the functors $\mathbf{n} \to \mathbf{D}$ $(n \geq 0)$ and as maps $(J_1: \mathbf{n}_2 \to \mathbf{D}) \to (J_2: \mathbf{n}_2 \to \mathbf{D})$ the commutative diagrams

$$\begin{tikzcd} \mathbf{n}_2 \arrow[rr] \arrow[dr, "J_2"'] & & \mathbf{n}_1 \arrow[dl, "J_1"] \\ & \mathbf{D} & \end{tikzcd}$$

and consider, for every object $J: \mathbf{n} \to \mathbf{D} \in d\mathbf{D}$, the classification complex cJ^*X of the induced $\mathbf{n}$-diagram. It is easy to see that these give rise to a *classification diagram* $c_{d\mathbf{D}}X \in \mathbf{S}^{d\mathbf{D}}$ which has the property.

3.2 PROPOSITION. *Let* $X \in \mathbf{S}^{\mathbf{D}}$. *Then*

$$cX = \underleftarrow{\lim}^{d\mathbf{D}} c_{d\mathbf{D}} X$$

In view of this we define

3.3 The homotopy classification complex hoc X **of a diagram** $X \in \mathbf{S}^{\mathbf{D}}$

This is the simplicial set defined by ([2, Chap. XI] and 1.6)

$$\text{hoc}\, X = \text{ho}\underleftarrow{\lim}^{d\mathbf{D}}(c_{d\mathbf{D}}X)^f.$$

Our main result now is the

3.4 Classification theorem

Let $X \in \mathbf{S}^{\mathbf{D}}$. *Then*

(*i*) *there is an obvious* 1–1 *correspondence between the components of* cX *and the weak equivalence classes of the conjugates of* X,

(*ii*) *for every conjugate* Y *of* X, *the component of* cX *containing it is exactly* scY *and thus has* (*in the sense of* §2) *the homotopy type of the classifying complex for the self weak equivalences of* Y, *and*

(*iii*) *the obvious* ([2, Chap. XI], [11]) *map*

$$cX = \underleftarrow{\lim}^{d\mathbf{D}} c_{d\mathbf{D}} X \to \text{ho}\underleftarrow{\lim}^{d\mathbf{D}}(c_{d\mathbf{D}}X)^f = \text{hoc}\, X \in \mathbf{S}$$

is a weak (*homotopy*) *equivalence.*

Parts (i) and (ii) of this theorem are a special case of 2.4 and it thus remains to prove (iii), which will be done in §8.

3.5 A slight generalization

Define two diagrams X, $Y \in \mathbf{S}^{\mathbf{D}}$ to be 0-*conjugate* if, for every object $D \in \mathbf{D}$, the simplicial sets XD and YD are weakly (homotopy) equivalent and define the corresponding notions of 0-*classification complex* $c^\circ X$ and *homotopy* 0-*classification complex* hoc°X of a diagram $X \in \mathbf{S}^{\mathbf{D}}$. It is easy to see that these are disjoint unions of classification complexes and homotopy classification complexes respectively and that therefore *Theorem* 3.4 *remains valid if one replaces conjugate by* 0-*conjugate and substitutes* $c^\circ X$ *and* hoc°X *for* cX *and* hoc X. (The homotopical smallness of $c^\circ X$ is not immediately obvious, even from the computational part of our main result. One needs in addition the observation that, if $\mathbf{D} = \mathbf{n}$ for some integer $n \geq 0$ and $X \in \mathbf{S}^{\mathbf{D}}$, then $c^\circ X$ is homotopically small, or equivalently, has a *set* of components.)

We end by pointing out the

3.6 Accessibility of the homotopy type of cX **if** $X \in \mathbf{S}^{\mathbf{n}}$

This follows readily from 2.3 and 2.6. For instance, if $n = 0$ and $Y \in \mathbf{S}^0 = \mathbf{S}$ is fibrant (and automatically cofibrant) and weakly equivalent to X, then cX has the same homotopy type as $\bar{W}$ haut Y. Similarly, if $n > 0$, $Y \in \mathbf{S}^{\mathbf{n}}$ is fibrant and cofibrant and weakly equivalent to X and $Y' \in \mathbf{S}^{n-1}$ consists of the first $(n-1)$ maps of Y and $g: Y(n-1) \to Y(n)$ denotes the last, then cX has the homotopy type of the "g-component" of the total complex of the quasi-fibration over $(\bar{W}$ haut $Y') \times (\bar{W}$ haut $Y(n)$ which has the function complex $\hom(Y(n-1), Y(n))$ as fibre and which is associated with the natural actions of haut Y' and haut $Y(n)$ on this function complex. In particular if all the maps in X are weak (homotopy) equivalences, then cX has the common homotopy type of all the $\bar{W}$ haut $Y(j)$.

§4. REDUCTION TO THE SUBDIVISION

We now show that the homotopy type of the classification complex cX can also be expressed as a homotopy inverse limit over the *subdivision* of the category $\mathbf{D}$, i.e. [9, §5] the category $sd\mathbf{D}$ obtained from the division $d\mathbf{D}$ by turning all the "degeneracy maps" (i.e. diagrams as in 3.1 in which the top map is onto) into identity maps. This is particularly useful when (the nerve of) $\mathbf{D}$ is finite or finite dimensional as in that case $sd\mathbf{D}$ has the same property.

To do this we note that 3.4(ii), 9.1 and 9.5 immediately imply.

4.1 PROPOSITION. *Given a small category* $\mathbf{D}$ *and a diagram* $X \in \mathbf{S}^{\mathbf{D}}$, *let* $g: d\mathbf{D} \to \mathbf{E}$ *be a left cofinal* (9.3) *functor between small categories and let* $e \in \mathbf{S}^{\mathbf{E}}$ *be a fibrant diagram such that* $c_{d\mathbf{D}}X$ *is weakly equivalent to the pull back diagram* g^*e. *Then the classification complex* cX *is weakly equivalent to the homotopy inverse limit* $\underleftarrow{\operatorname{holim}}^{\mathbf{E}} e$.

Our main result in this section then is

4.2 Reduction to the subdivision

Let $\mathbf{D}$ be a small category, let $X \in \mathbf{S}^{\mathbf{D}}$, let $s: d\mathbf{D} \to sd\mathbf{D}$ be the projection functor [9, §5], let $s_*: \mathbf{S}^{d\mathbf{D}} \to \mathbf{S}^{sd\mathbf{D}}$ be the "*homotopy push down*" functor which (9.8) assigns to every diagram $Y \in \mathbf{S}^{d\mathbf{D}}$ and object $I \in sd\mathbf{D}$, the simplicial set $\underrightarrow{\operatorname{holim}}^{s \downarrow I} j^* Y$ (where j denotes the forgetful functor). Then 2.6(ii), 5.3, 5.8, 6.3, 9.1 and 9.2 imply that

(i) *the classification diagram* $c_{d\mathbf{D}}X$ *is weakly equivalent to the pull back diagram* $s^* s_* c_{d\mathbf{D}}X$, and hence (4.1)

(ii) *the classification complex* cX *is weakly equivalent to* $\underleftarrow{\operatorname{holim}}^{sd\mathbf{D}}(s_* c_{d\mathbf{D}}X)^f$.

This result can sometimes be simplified:

4.3 The retract-free case

Let $\mathbf{D}$ be a small category which is *retract-free*, i.e. no two non-identity maps in $\mathbf{D}$ compose to an identity map (this is, for instance, the case if $\mathbf{D}$ is finite or finite dimensional or more generally, if $\mathbf{D}$ or its opposite $\mathbf{D}^{op}$ are direct in the sense of [9, §4]) and let $X \in \mathbf{S}^{\mathbf{D}}$. Then *the projection* $s: d\mathbf{D} \to sd\mathbf{D}$ *admits an obvious cross section* $t: sd\mathbf{D} \to d\mathbf{D}$ and 2.6(ii), 5.6, 6.3 and 9.3 readily imply that

(i) *the classification diagram* $c_{d\mathbf{D}}X$ *is weakly equivalent to the pull back diagram* $s^*t^*c_{d\mathbf{D}}X$ and hence (4.1)

(ii) *the classification complex* cX *is weakly equivalent to* $\underleftarrow{\operatorname{holim}}^{sd\mathbf{D}}(t^* c_{d\mathbf{D}}X)^f$.

4.4 The direct case

If $\mathbf{D}$ is *direct* [9, §4] and has *left cancellation* (i.e. $fg = fh$ implies $g = h$) and $X \in \mathbf{S}^{\mathbf{D}}$ is both *fibrant and cofibrant*, then so are the induced diagrams $J^*X \in \mathbf{S}^{\mathbf{n}}$. Thus one can (see 2.3) consider the resulting $d\mathbf{D}$-diagram $\bar{W}\mathrm{haut}_{d\mathbf{D}}X$ of the simplicial sets $\bar{W}\mathrm{haut}J^*X$, which is readily seen to be the pull back along $s: d\mathbf{D} \to sd\mathbf{D}$ of an $sd\mathbf{D}$-diagram $\bar{W}\mathrm{haut}_{sd\mathbf{D}}X$ and note that (see 2.4).

(i) *The classification diagram* $c_{d\mathbf{D}}X$ *is weakly equivalent to the pull back diagram* $s^*\bar{W}\mathrm{haut}_{sd\mathbf{D}}X$ and hence (4.1)

(ii) *the classification complex cX is weakly equivalent to*

$$\operatorname{holim}^{d\mathbf{D}}_{\leftarrow}(\bar{W}\mathrm{haut}_{d\mathbf{D}}X)^{f} \quad \text{and} \quad \operatorname{holim}^{sd\mathbf{D}}_{\leftarrow}(\bar{W}\mathrm{haut}_{sd\mathbf{D}}X)^{f}.$$

Dually one has

4.5 The inverse case

If $\mathbf{D}$ is *inverse* (i.e. $\mathbf{D}^{op}$ is direct), then it is not difficult to verify that $\mathbf{S}^{\mathbf{D}}$ also admits a closed simplicial model category structure in which the simplicial structure and the weak equivalences are as in [9, §2] and a map $f: X \to Y \in \mathbf{S}^{\mathbf{D}}$ is a cofibration iff, for every object $D \in \mathbf{D}$, the map $fD: XD \to YD \in \mathbf{S}$ is a cofibration. If $\mathbf{D}$ has *right cancellation* (i.e. $gf = hf$ implies $g = h$) and a diagram $X \in \mathbf{S}^{\mathbf{D}}$ is both *fibrant* and *cofibrant* with respect to this model category structure, then so are the induced diagrams J^*X and hence *the conclusions of* 4.4 *hold.*

4.6 The Postnikov case

Using 4.5 one can recover the first classification result mentioned in 1.2. Let $\mathbf{N}$ be the category which has the non-negative integers as objects and which has exactly one map $n_1 \to n_2$ whenever $n_2 \geqslant n_1$. Let $Y \in \mathbf{S}$ be fibrant and let $X \in \mathbf{S}^{\mathbf{N}}$ be its Postnikov tower, i.e. $X(n) = P_n Y$, the nth Postnikov approximation of Y, for every integer $n \geqslant 0$. Then X is both fibrant and cofibrant with respect to the model category structure of 4.5. Moreover, if $\{\bar{W}\mathrm{haut}P_nY\}$ denotes the resulting $\mathbf{N}$-diagram of the simplicial sets $\bar{W}\mathrm{haut}P_nY$, then the $sd\mathbf{N}$-diagram $\bar{W}\mathrm{haut}_{sd\mathbf{N}}X$ (4.5) is clearly a pull back of the $\mathbf{N}$-diagram $\{\bar{W}\mathrm{haut}P_nY\}$ along the functor $p: sd\mathbf{N} \to \mathbf{N}$ given by [9, §5] $(J: \mathbf{n} \to \mathbf{N}) \to J(0)$. As a result (4.5) *the classification diagram $c_{d\mathbf{N}}X$ is weakly eqiuvalent to the pull back diagram $s^*p^*\{\bar{W}\mathrm{haut}P_nY\}$* and hence (4.1) *the classification complex cX is weakly equivalent to* $\operatorname{holim}^{n}\{\bar{W}\mathrm{haut}P_nY\}^{f}$.

§5. A USEFUL VARIATION

We now consider another *variation* on our main result which enables one to express the homotopy type of the classification complex cX as a function complex between diagrams indexed by a category which can be considerably smaller than the subdivision of $\mathbf{D}$. A similar result holds for the 0-classification complex $c^{\circ}X$ (3.5).

5.1. PROPOSITION. *Given a small category $\mathbf{D}$ and a diagram $X \in \mathbf{S}^{\mathbf{D}}$, let $g: d\mathbf{D} \to E$ be a functor between small categories and let $e \in \mathbf{S}^{\mathbf{E}}$ be a fibrant diagram such that $c_{d\mathbf{D}}X$ is weakly equivalent to the pull back diagram g^*e. Then the classification complex cX is weakly equivalent to the function complex* $\hom^{\mathbf{E}}((g\downarrow -), e)$.

Proof. This follows immediately from 3.4(iii), 9.1 and 9.5.

5.2 Example

Let (see §4) $\mathbf{E} = sd\mathbf{D}$ and $g = s: d\mathbf{D} \to sd\mathbf{D}$ and let $e = s_*c_{d\mathbf{D}}X$. Then (see 4.2) $\hom^{\mathbf{E}}((g\downarrow -), e)$ *is weakly equivalent to* $\operatorname{holim}^{sd\mathbf{D}}_{\leftarrow}(s_*c_{d\mathbf{D}}X)^{f}$.

5.3 Example

Let $\mathbf{D}$ be a *group* (i.e. $\mathbf{D}$ has only one object D and all maps of $\mathbf{D}$ are invertible) and let $X \in \mathbf{S}^{\mathbf{D}}$ be *fibrant*. Then one readily verifies (2.3 and 2.6(ii)) that the diagram $c_{d\mathbf{D}}X$ *is weakly equivalent to the constant $d\mathbf{D}$-diagram with value* $\bar{W}\mathrm{haut}XD$ and hence (5.1) *the classification complex cX is weakly equivalent to the ordinary function complex*. $\hom(\mathbf{D}, (\bar{W}\mathrm{haut}XD)^{f})$. In view of [5, §2] this implies the second classification result mentioned in 1.2 for the case that B has the homotopy type of a $K(\pi, 1)$. The general case can be obtained by a simplicial version of the above argument (see [10, 5.5]).

5.4 Example

Let $\mathbf{D}$ be a small category with *connected* nerve, let $X \in \mathbf{S}^{\mathbf{D}}$ be a *fibrant* diagram such that, for every map $d \in \mathbf{D}$, the induced map $Xd \in \mathbf{S}$ is a weak equivalence and let $D \in \mathbf{D}$ be an object. As in 5.3 one then readily sees that *the diagram* $c_{d\mathbf{D}}X$ *is weakly equivalent to the constant* $d\mathbf{D}$*-diagram with value* $\bar{W}\mathrm{haut}XD$ *and the classification complex* cX *is therefore* (5.1) *weakly equivalent to the ordinary function complex* $\hom(\mathbf{D}, (\bar{W}\mathrm{haut}XD)^f)$.

5.5 Remark

There is of course, an analogue of Proposition 5.1 for the 0-classification complex $c^\circ X$ (3.5). In fact, if (see §3) $c^\circ_{d\mathbf{D}}X$ denotes the $d\mathbf{D}$-diagram consisting of the 0-classification complexes $c^\circ J^*X$, then *Proposition* 5.1 *clearly remains true if one replaces* $c_{d\mathbf{D}}X$ *by* $c^\circ_{d\mathbf{D}}X$ *and* cX *by* $c^\circ X$.

5.6 Example

Given a small category $\mathbf{D}$ and a diagram $X \in \mathbf{S}^{\mathbf{D}}$ let $\pi.\mathbf{D}$ be the category obtained from $\mathbf{D}$ by identifying two maps iff they have the same domain and range, let $\mathbf{E} = d\pi.\mathbf{D}$ and let $g: d\mathbf{D} \to \mathbf{E}$ be the division of the projection. Clearly the functor $c^\circ_{d\mathbf{D}}X: d\mathbf{D} \to \mathbf{S}$ factors through $\mathbf{E}$ and if e denotes the resulting $\mathbf{E}$-diagram, then (5.5) *the* 0*-classification complex* $c^\circ X$ *is weakly equivalent to the function complex* $\hom^{\mathbf{E}}((g\downarrow -), e^f)$. Moreover, if $g^{-1} \in \mathbf{S}^{\mathbf{E}}$ denotes the more or less evident diagram which assigns to every object of $\mathbf{E}$ (the nerve of) its inverse image under g (which is discrete), then it is not difficult to verify that *the obvious map* $(g\downarrow -) \to g^{-1} \in \mathbf{S}^{\mathbf{E}}$ *is a weak eqiuvalence.*

If $\mathbf{D}$ has only one object, then $\pi.\mathbf{D}$ is trivial and hence $\mathbf{E} \approx \Delta^{op}$ and g^{-1} consists of the simplices of the nerve of $\mathbf{D}$ with the usual simplicial operators between them.

§6. L-COFINAL FUNCTORS

In preparation for the proof of Theorem 3.4 we discuss here a special type of functors between small categories, called an *L-cofinal* functor, and derive the following

6.1 Properties of L-cofinal functors

(i) An L-cofinal functor $u: \mathbf{A} \to \mathbf{B}$ induces an equivalence (6.5) between the homotopy theory of $\mathbf{B}$-diagrams of simplicial sets and the homotopy theory of $u^{-1}\mathbf{B}$-diagrams of simplicial sets. (An $u^{-1}\mathbf{B}$-diagram is a functor $X: \mathbf{A} \to \mathbf{S}$ such that $Xa \in \mathbf{S}$ is a weak equivalence whenever $ua \in \mathbf{B}$ is an identity map.) This we prove by means of a *pair of adjoint functors* (6.4)

$$\underset{\sim}{u}_*: \mathbf{S}^{u^{-1}\mathbf{B}} \leftrightarrow \mathbf{S}^{\mathbf{B}}: \underset{\sim}{u}^*$$

where $\mathbf{S}^{u^{-1}\mathbf{B}} \subset \mathbf{S}^{\mathbf{A}}$ denotes the full subcategory generated by the $u^{-1}\mathbf{B}$-diagrams, $\underset{\sim}{u}^*$ is a blown up version of the pull back functor $\mathbf{S}^{\mathbf{B}} \to \mathbf{S}^{u^{-1}\mathbf{B}}$ and $\underset{\sim}{u}_*$ is a "homotopy push down functor".

(ii) L-cofinal functors are very well behaved with respect to homotopy limits. Their definition immediately implies that they are *left cofinal* (6.6), but it turns out that they are also *right cofinal* (6.7) and in addition permit a *mixed push down theorem* (6.8) for homotopy inverse limits.

6.2 L-cofinal functors

A functor $u: \mathbf{A} \to \mathbf{B}$ between two small categories will be called *L-cofinal* if, for every object $B \in \mathbf{B}$,

(i) the inverse image category $u^{-1}B$ is contractible (1.6(i)), and

(ii) its inclusion in the over category $u^{-1}B \to u\downarrow B$ is right cofinal (9.4).

In view of 9.4 and 9.10 this definition readily implies the following

6.3 Key property

If u: $\mathbf{A}\to\mathbf{B}$ is an L-cofinal functor and X: $\mathbf{A}\to\mathbf{S}$ is a $u^{-1}\mathbf{B}$-diagram (6.1), *then, for every object $A\in\mathbf{A}$, the obvious map* [2, Chap. XII]

$$XA \to \underrightarrow{\mathrm{holim}}^{u\downarrow uA} j^* X \in \mathbf{S}$$

(*where j denotes the forgetful functor*) *is a weak equivalence.*

In order to formulate the equivalence of homotopy theories mentioned above (6.1(i)) we need

6.4 A pair of adjoint functors

Let u: $\mathbf{A}\to\mathbf{B}$ be a functor between small categories. A lengthy but straightforward calculation then yields that *the homotopy push down functor $\underset{\to}{u}_*$: $\mathbf{S}^{\mathbf{A}}\to\mathbf{S}^{\mathbf{B}}$ of* 9.8 *has as right adjoint the functor $\underset{\to}{u}^*$: $\mathbf{S}^{\mathbf{B}}\to\mathbf{S}^{\mathbf{A}}$ given by*

$$\underset{\to}{u}^* YA = \hom^{\mathbf{B}}((A\downarrow u\downarrow\mathbf{B}), Y)$$

for every diagram $Y\in\mathbf{S}^{\mathbf{B}}$ and object $A\in\mathbf{A}$. Here $A\downarrow u\downarrow\mathbf{B}\in\mathbf{S}^{\mathbf{B}}$ assigns to every object $B\in\mathbf{B}$ the (nerve of the) "between category" $A\downarrow u\downarrow B$ which has as objects the pairs of maps $(A\to A'\in\mathbf{A}, uA'\to B\in\mathbf{B}$.

Actually $\underset{\to}{u}^*Y$ is a $u^{-1}\mathbf{B}$-diagram and in fact a blown up version of the induced $u^{-1}\mathbf{B}$-diagram u^*Y. To prove this one notes that

(i) *$A\downarrow u\downarrow\mathbf{B}$ is a free* [9, 2.4] *and hence cofibrant* **B**-*diagram,*

(ii) *$\pi_0(A\downarrow u\downarrow\mathbf{B})$ is a strong deformation retract of $A\downarrow u\downarrow\mathbf{B}$,*

(iii) *$\pi_0(A\downarrow u\downarrow\mathbf{B})\approx uA\downarrow\mathbf{B}$ is free on one generator and hence* $\hom^{\mathbf{B}}(\pi_0(A\downarrow u\downarrow B), Y)\approx u^*YA$, and

(iv) by (ii) and (iii) *the projections $A\downarrow u\downarrow\mathbf{B}\to\pi_0(A\downarrow u\downarrow\mathbf{B})$ induce weak equivalences* [9, §1 and §2]

$$u^*YA \approx \hom^{\mathbf{B}}(\pi_0(A\downarrow u\downarrow\mathbf{B}), Y)\to\hom^{\mathbf{B}}((A\downarrow u\downarrow\mathbf{B}, Y)\approx \underset{\to}{u}^*YA.$$

Combined with standard arguments, Proposition 6.3 leads to the following result:

6.5 Equivalence of homotopy theories

Let u: $\mathbf{A}\to\mathbf{B}$ be a functor between small categories. Then one has:

(*i*) *The functor $\underset{\to}{u}^*$ perserves fibrations and weak equivalences.*

(*ii*) *The functor $\underset{\to}{u}_*$ preserves cofibrations and weak equivalences.*

(*iii*) *If u is L-cofinal, then, for every cofibrant object $X\in\mathbf{S}^{u^{-1}\mathbf{B}}$ and every fibrant object $Y\in\mathbf{S}^{\mathbf{B}}$, a map $X\to\underset{\to}{u}^*Y\in\mathbf{S}^{u^{-1}\mathbf{B}}$ is a weak equivalence iff its adjoint $\underset{\to}{u}_*X\to Y\in\mathbf{S}^{\mathbf{B}}$ is so.*

(*iv*) *Consequently, if u is L-cofinal, then the functors $\underset{\to}{u}^*$ and $\underset{\to}{u}_*$ induce equivalences between the homotopy theory of the diagram category $\mathbf{S}^{\mathbf{B}}$ and the homotopy theory of the diagram category $\mathbf{S}^{u^{-1}\mathbf{B}}$* (i.e. [8, 5.4] these functors induce weak equivalences between the appropriate simplicial localizations).

Next we investigate the behavior of L-cofinal functors with respect to homotopy limits. Definition 6.2 immediately implies

6.6 PROPOSITION. *Every L-cofinal functor is left cofinal* (9.3).

Somewhat surprisingly one also has

6.7 PROPOSITION. *Every L-cofinal functor is right cofinal* (9.4).

Proof. Let $B \in \mathbf{B}$ be an arbitrary but fixed object and consider the functor $\hom(B, -)$: $\mathbf{B} \to (\mathbf{sets}) \subset \mathbf{S}$. By inspection, (the nerve of) the over category $B \downarrow u$ is isomorphic to $\underrightarrow{\operatorname{holim}}^{\mathbf{A}} u^* \hom(B, -)$ which, by 9.8, is weakly equivalent to $\underrightarrow{\operatorname{holim}}^{\mathbf{B}} \underset{\rightarrow}{u}_* u^* \hom(B, -)$. On the other hand $B \downarrow \mathbf{B} \approx \underrightarrow{\operatorname{holim}}^{\mathbf{B}} \hom(B, -)$ is contractible and the desired result now follows readily from 6.3.

Also useful and somewhat unexpected is the

6.8 Mixed push down theorem for homotopy inverse limits

Let $u: \mathbf{A} \to \mathbf{B}$ be an L-cofinal functor, let $X \in \mathbf{S}^{u^{-1}\mathbf{B}}$ be fibrant and let $\underset{\rightarrow}{u}_* X \to Y \in \mathbf{S}^{\mathbf{B}}$ be a weak equivalence such that Y is fibrant. Then $\underleftarrow{\operatorname{holim}}^{\mathbf{A}} X$ *is in a natural manner, weakly equivalent to* $\underleftarrow{\operatorname{holim}}^{\mathbf{B}} Y$.

Proof. In view of 6.5, 6.6 and 9.1 there is a natural sequence of weak equivalences

$$\underleftarrow{\operatorname{holim}}^{\mathbf{B}} Y \xrightarrow{\sim} \underleftarrow{\operatorname{holim}}^{\mathbf{A}} u^* Y \xrightarrow{\sim} \underleftarrow{\operatorname{holim}}^{\mathbf{A}} \underset{\cdot}{u}^* Y \xleftarrow{\sim} \underleftarrow{\operatorname{holim}}^{\mathbf{A}} X$$

Some easy examples of L-cofinal functors are the result of the readily verified

6.9 PROPOSITION. *Let $u: \mathbf{A} \to \mathbf{B}$ be a functor between small categories such that for every object $B \in \mathbf{B}$,*

(i) *the inverse image category $u^{-1}B$ is contractible, and*

(ii) *its inclusion in the over category $u^{-1}B \to u \downarrow B$ has a left adjoint.*

Then u is L-cofinal.

6.10 *Examples.* (i) Let $\mathbf{D}$ be a small category and let $\mathbf{C}$ be a small category which is contractible. Then *the projection $\mathbf{D} \underline{\times} \mathbf{C} \to \mathbf{D}$ is L-cofinal.*

(ii) For a small category $\mathbf{D}$, let $\overline{sd}\mathbf{D}$ be the opposite of its subdivision (see §4). Then *the functor $q: \overline{sd}\mathbf{D} \to \mathbf{D}$ given by the formula $(J: \mathbf{n} \to \mathbf{D}) \to J(n)$, is L-cofinal* [9, 5.5 and 5.6].

The remainder of this section will be devoted to proving the following two propositions which provide less obvious examples of L-cofinal functors.

6.11 PROPOSITION. *Let $\mathbf{D}$ be a small category. Then the projection $s: d\mathbf{D} \to sd\mathbf{D}$ is L-cofinal.*

This is a special case

6.12 PROPOSITION. *Let $u: \mathbf{A} \to \mathbf{B}$ be a functor between small categories. Then the composition*

$$v: d\mathbf{A} \xrightarrow{s} sd\mathbf{A} \xrightarrow{(sd)u} sd\mathbf{B}$$

satisfied 6.2(ii). *Moreover, if u is L-cofinal, then so is v.*

6.13 COROLLARY. *Let $u: \mathbf{A} \to \mathbf{B}$ be an L-cofinal functor. Then the induced functor $(sd)u: sd\mathbf{A} \to sd\mathbf{B}$ is both left and right cofinal.*

This follows from 6.11, 6.12 and the observation that, *if a functor y and a composite functor xy are both left (or both right) cofinal, then x is also left (or right) cofinal.*

Proof of 6.12. Let $\bar{y} \in s\,\mathbf{dB}$ and $z \in \mathbf{v}\downarrow\mathbf{Y}$ be objects and let $c: v^{-1}\bar{y} \to v\downarrow\bar{y}$ denote the inclusion functor. We first show that $z\downarrow c$ is contractible. To do this let $y: \mathbf{n} \to \mathbf{B}$ be a "non-degenerate" [9, 5.3] functor such that $sy = \bar{y}$, let

$$\begin{array}{ccc} \mathbf{k} & \xrightarrow{x} & \mathbf{A} \\ f\downarrow & & \downarrow u \\ \mathbf{n} & \xrightarrow{y} & \mathbf{B} \end{array}$$

be a commutative diagram of functors representing z and let, for every integer i with $0 \leqslant i \leqslant n$, V_i denote the set integers j such that $0 \leqslant j \leqslant k$ and

$$ux(\min(fi, j) \to \max(fi, j)) \in \mathbf{B}$$

is an identity map. Then the objects of $z\downarrow c$ are in an obvious 1–1 correspondence with the sequences of integers

$$0 \leqslant a_0^0 \leqslant \cdots \leqslant a_{N_0}^0 < a_0^1 \leqslant \cdots a_{N_1}^1 < \cdots < a_0^n \leqslant \cdots \leqslant a_{N_n}^n \leqslant k$$

such that $a_g^i \in V_i$ for all $0 \leqslant g \leqslant N_i$ and $0 \leqslant i \leqslant n$, and the maps of $z\downarrow c$ correspond to deletions and/or repetitions of integers. Using this presentation it is not difficult to show that the identity functor of $z\downarrow c$ can be connected to a constant functor by a sequences of natural transformations and that $z\downarrow c$ is thus contractible. As moreover $s^{-1}\bar{y} \in d\mathbf{B}$ has y as initial object and hence $s^{-1}\bar{y}$ is contractible, this proves 6.11.

To complete the proof of 6.12, assume that $\bar{y}$ and y are as above. If $n = 0$, then $v^{-1}\bar{y} = \mathrm{d}(u^{-1}y(0))$ and hence (6.11) $v^{-1}\bar{y}$ is contractible. If $n > 0$, let y': $\mathbf{n}-\mathbf{1} \to \mathbf{B}$ be the restriction of y to all of $\mathbf{n}$ but the object n, let $\bar{y}' = sy'$, let $x: \mathbf{k} \to \mathbf{A}$ be an object of $v^{-1}\bar{y}'$ and let r: $v^{-1}\bar{y} \to v^{-1}\bar{y}'$ be the restriction functor. Then it is not difficult to see that $r\downarrow x$ retracts to $r^{-1}x$ and that $r^{-1}x$ is isomorphic to $\mathrm{d}(x'\downarrow c)$, where $c: u^{-1}y(n) \to u\downarrow y(n)$ is the inclusion functor and $x' \in u\downarrow y(n)$ is the object determined by $x(k)$ and $y(n-1 \to n)$. The desired result now follows from the definition of L-cofinality.

6.14 Remark

The above argument actually proves the slightly stronger statement: *In the notation of* 6.12, *let $sd'\mathbf{B} \subset sd\mathbf{B}$ be a full subcategory and let $d'\mathbf{A} = v^{-1}(sd'\mathbf{B}) \subset d\mathbf{A}$. Then the restriction v': $d'\mathbf{A} \to ds'\mathbf{B}$ of v satisfies* 6.2(ii). *Moreover, if u is L-cofinal, then so is v'.*

We end with some obvious comments on

6.15 R-cofinal functors

One calls a functor $u: \mathbf{A} \to \mathbf{B}$ between two small categories *R-cofinal* if its opposite is L-cofinal, i.e. if, for every object $B \in \mathbf{B}$,

(i) the inverse image category $u^{-1}B$ is contractible, and

(ii) its inclusion in the under category $u^{-1}B \to B\downarrow u$ is left cofinal (9.3).

It is clear that all the results of this section can be reformulated to apply to R-cofinal functors. In particular, *an R-cofinal functor u: $\mathbf{A} \to \mathbf{B}$ induces an equivalence between the homotopy theory of $\mathbf{S}^{\mathbf{B}}$ and the homotopy theory of $\mathbf{S}^{u^{-1}\mathbf{B}}$.*

§7. A LEMMA

This section contains a Lemma (7.2) which plays an important role in the proof of part (iii) of Theorem 3.4 (in §8). In order to formulate it we need the notion of

7.1 Initial subcategories

A subcategory $\mathbf{C}$ of a category $\mathbf{D}$ is called *initial* if $d\colon D_1 \to D_2 \in \mathbf{D}$ and $D_2 \in \mathbf{C}$ imply $d \in \mathbf{C}$. One then has:

7.2 Lemma. *Let*

$$\begin{tikzcd} \mathbf{C}' \arrow[r, "f"] \arrow[d, "i'"'] & \mathbf{C} \arrow[d, "i"] \\ \mathbf{D}' \arrow[r, "g"'] & \mathbf{D} \end{tikzcd}$$

be a push out diagram of small categories such that i' and f map $\mathbf{C}'$ isomorphically onto initial subcategories of $\mathbf{D}'$ and $\mathbf{C}$ respectively (and hence i and g map C and $\mathbf{D}'$ isomorphically onto initial subcategories $\mathbf{D}$). Then, for every diagram $X_{\mathbf{D}} \in \mathbf{S}^{\mathbf{D}}$, the induced diagram

$$\begin{tikzcd} cX_{\mathbf{D}} \arrow[r] \arrow[d] & cX_{\mathbf{D}'} \arrow[d] \\ cX_{\mathbf{C}} \arrow[r] & cX_{\mathbf{C}'} \end{tikzcd}$$

(where $X_{\mathbf{C}}$, $X_{\mathbf{C}}$ and $X_{\mathbf{D}}$ denote the induced $\mathbf{C}$-, $\mathbf{C}'$- and $\mathbf{D}'$-diagrams) is, up to homotopy, a fibre square.

Proof. Let $\mathbf{W}$ denote the category with five objects and four non-identity maps $u \to v \leftarrow w \to x \leftarrow y$, let $\mathbf{E}'$ denote the category obtained from $\mathbf{C}' \times \mathbf{W}$ by "replacing $\mathbf{C}' \times u$ by a copy of $\mathbf{D}'$" and let $\mathbf{E}$ be obtained from $\mathbf{E}'$ by "replacing $\mathbf{C}' \times y$ by a copy of $\mathbf{C}$". Then the obvious functors $r'\colon \mathbf{E}' \to \mathbf{D}'$ and $r\colon \mathbf{E} \to \mathbf{D}$ are readily verified to be R-cofinal (6.15) and hence (6.5) it suffices to prove the desired result for the push out diagram

$$\begin{tikzcd} \mathbf{C}' \arrow[r] \arrow[d, "j"'] & \mathbf{C} \arrow[d, "j"] \\ \mathbf{E}' \arrow[r] & \mathbf{E} \end{tikzcd}$$

and an $r^{-1}\mathbf{D}$-diagram $Y_{\mathbf{E}} \in \mathbf{S}^{r^{-1}\mathbf{D}}$. This we will do by showing that, for every $r^{-1}\mathbf{D}$-diagram $Y_{\mathbf{E}} \in \mathbf{S}^{r^{-1}\mathbf{D}}$, the natural "end point restriction" diagram

$$\begin{tikzcd} cY_{\mathbf{E}} \arrow[r] \arrow[d] & cY_{\mathbf{E}'} \arrow[d] \\ cY_{\mathbf{C}} \times cY_{\mathbf{D}'} \arrow[r] & cY_{\mathbf{C}'} \times cY_{\mathbf{D}'} \end{tikzcd}$$

is, up to homotopy, a fibre square. But this last diagram is a pull back diagram and the top map thus induces isomorphisms of the fibres. It therefore remains to show that the inclusion of these fibres in the homotopy fibres are weak equivalences.

To do this consider the underlying functors

$$\mathbf{c}Y_{\mathbf{E}} \xrightarrow{h} \mathbf{c}Y_{\mathbf{C}} \times \mathbf{c}Y_{\mathbf{D}'} \quad \text{and} \quad \mathbf{c}Y_{\mathbf{E}'} \xrightarrow{h'} \mathbf{c}Y_{\mathbf{C}'} \times \mathbf{c}Y_{\mathbf{D}'}$$

and note that, for every object $(P, Q) \in \mathbf{c}Y_{\mathbf{C}} \times \mathbf{c}Y_{\mathbf{D}'}$, (the nerve of) $b^{-1}(P, Q)$ is a deformation retract of $(P, Q)\downarrow b$. Furthermore it readily follows from [6, 5.5] and [7, 2.2 and 6.2] that $b^{-1}(P, Q)$ has the homotopy type of the loops on scP (2.2) and the functor b thus satisfies the conditions of Theorem A of Quillen[14]. A similar argument can be applied to b' and the desired result is now immediate.

7.3 Remark

In order to prove the somewhat stronger result than 3.4 mentioned in 1.4, one needs the following addition to Lemma 7.2: If $Y_{\mathbf{D}'} \in \mathbf{S}^{\mathbf{D}'}$ and $Z_{\mathbf{C}} \in \mathbf{S}^{\mathbf{C}}$ *are such that the induced diagrams* $Y_{C'}$ *and* $Z_{C'} \in \mathbf{S}^{C'}$ *are weakly equivalent, then there exists a diagram* $X_{\mathbf{D}} \in \mathbf{S}^{\mathbf{D}}$ *such that the induced diagrams* $X_{\mathbf{D}'} \in \mathbf{S}^{\mathbf{D}'}$ and $X_{\mathbf{C}} \in \mathbf{S}^{\mathbf{C}}$ *are weakly eqiuvalent to* $Y_{D'}$ *and* $Z_{\mathbf{C}}$ *respectively.*

To prove this one notes that (in the above notation) there is an $r^{-1}\mathbf{D}$-diagram which "restricts at the end points" to $Y_{\mathbf{D}'}$ and $Z_{\mathbf{C}}$ and the desired result now follows readily from 6.5 and the R-cofinality (6.15) of the functor $r: \mathbf{E} \to \mathbf{D}$.

§8. PROOF OF PART (iii) OF THEOREM 3.4

The proof of 3.4(iii) consists of three parts.

I. Reduction to the direct case

Here we show that it suffices to consider the case that $\mathbf{D}$ is a direct[9, §4] category. To do this let (see [9, §5]) $\overline{sd}\mathbf{D}$ be the opposite of the subdivision of $\mathbf{D}$ and $q: \overline{sd}\mathbf{D} \to \mathbf{D}$ the functor given by $(J: \mathbf{n} \to \mathbf{D}) \to J(n)$. The desired result then follows immediately from the fact that [9, §4] $\overline{sd}\mathbf{D}$ is direct and

8.1 Proposition. *Let* $\mathbf{D}$ *be a small category, let* $X \in \mathbf{S}^{\mathbf{D}}$ *and let*

$$\begin{array}{ccc} cX & \longrightarrow & \mathrm{hoc}X \\ \downarrow & & \downarrow \\ cq^*X & \longrightarrow & \mathrm{hoc}q^*X \end{array}$$

be the commutative diagram in which the horizontal maps are as in 3.4 *the vertical maps are induced by* q. *Then both vertical maps are weak equivalences.*

Proof. For the map on the left this follows from 6.5 and 6.10 and for the map on the right from 6.6, 6.10, 6.11 and 9.3.

II. Reduction to the finite dimensional case

For this it suffices to show:

8.2 Proposition. *Let* $\mathbf{D}$ *be a small category which is the union of an increasing sequence* $\mathbf{D}^\circ \subset \cdots \subset \mathbf{D}^n \subset \ldots$ *of initial subcategories such that* 3.4(*iii*) *holds for all* $\mathbf{D}^i (i \geqslant 0)$. *Then* 3.4(*iii*) *also holds for* $\mathbf{D}$.

To prove this we need the

8.3 Proposition. *If* 3.4(iii) *holds for a set* $\{\mathbf{D}^\alpha\}$ *of small categories, then it also holds for their disjoint union* $\coprod_\alpha \mathbf{D}^\alpha$.

Proof. Let $\mathbf{C} = \coprod_\alpha \mathbf{D}^\alpha$. If $X \in \mathbf{S}^{\mathbf{C}}$, then $cX \approx \prod_\alpha cX_\alpha$ and $\mathrm{hoc}X \approx \prod_\alpha \mathrm{hoc}X_\alpha$, where X_α denotes the restriction of X to $\mathbf{D}^\alpha$. The $\mathrm{hoc}X_\alpha$ are fibrant and hence the product $\prod_\alpha \mathrm{hoc}X_\alpha$ has homotopy meaning (i.e. $\pi_i \prod_\alpha \mathrm{hoc}X_\alpha \approx \prod_\alpha \pi_i \mathrm{hoc}X_\alpha$ for every choice of base point and integer $i \geqslant 0$) and the desired result now follows from the following proposition which states that the product $\prod_\alpha cX_\alpha$ also has homotopy meaning (even though the cX_α are not necessarily fibrant).

8.4 PROPOSITION. *For every set of diagrams* $\{X_\alpha \in \mathbf{S}^{\mathbf{D}^\alpha}\}$ *the obvious map* $\prod_\alpha cX_\alpha \to \prod_\alpha (cX_\alpha)^f$ (1.6(iii)) *is a weak equivalence.*

Proof. One may assume that the X_α are both fibrant and cofibrant. Let again $\mathbf{C} = \coprod_\alpha \mathbf{D}^\alpha$ and let $X \in \mathbf{S}^{\mathbf{C}}$ be the diagram which on $\mathbf{D}^\alpha$ restricts to X_α. Then X is also fibrant and cofibrant and

$$\mathrm{haut}\, X \approx \prod_\alpha \mathrm{haut} X_\alpha \quad \text{and} \quad \bar{W}\mathrm{haut} X \approx \prod_\alpha \bar{W}\,\mathrm{haut} X_\alpha$$

As the $\mathrm{haut}X_\alpha$ are fibrant, the product $\prod_\alpha \mathrm{haut}X_\alpha$ has homotopy meaning and as

$$\pi_* \bar{W}\mathrm{haut}X \approx \pi_{*-1}\, \mathrm{haut}X \approx \prod_\alpha \pi_{*-1}\mathrm{haut}X_\alpha \approx \prod_\alpha \pi_* \bar{W}\, \mathrm{haut}X_\alpha$$

so has the product $\prod_\alpha \bar{W}\mathrm{haut}X_\alpha$. The proposition now follows readily form 2.3.

Proof of Proposition 8.2. Let $\mathbf{C}$ be the disjoint union $\mathbf{C} = \coprod_i \mathbf{D}^i$ and let $\mathbf{V}$ be the category with three objects and two non-identity maps $u \to v \leftarrow w$. Then there is a map of push out diagrams

$$\begin{tikzcd} \mathbf{C} \amalg \mathbf{C} \arrow[r] \arrow[d, "g^*"'] & \mathbf{C} \times \mathbf{V} \arrow[d] \\ \mathbf{C} \times \mathbf{V} \arrow[r] & \mathbf{D}^* \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C} \amalg \mathbf{C} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{C} \arrow[d] \\ \mathbf{C} \arrow[r] & \mathbf{D} \end{tikzcd}$$

in which f is the folding functor and g^* (resp. g) sends the first copy of $\mathbf{D}^i$ into $\mathbf{D}^i \times u$ (resp. $\mathbf{D}^i$) and the second copy into $\mathbf{D}^{i+1} \times w$(resp. $\mathbf{D}^{i+1}$). Given a diagram $X_{\mathbf{D}} \in \mathbf{S}^{\mathbf{D}}$, this map induces (in the notation of §7) a map of pull back diagrams

$$\begin{tikzcd} cX_{\mathbf{D}} \arrow[r] \arrow[d] & cX_{\mathbf{C}} \arrow[d] \\ cX_{\mathbf{C}} \arrow[r] & cX_{\mathbf{C}\amalg\mathbf{C}} \end{tikzcd} \longrightarrow \begin{tikzcd} cX_{\mathbf{D}*} \arrow[r] \arrow[d] & cX_{\mathbf{C}\times\mathbf{V}} \arrow[d] \\ cX_{\mathbf{C}\times\mathbf{V}} \arrow[r] & cX_{\mathbf{C}\amalg\mathbf{C}} \end{tikzcd}$$

One readily verifies that the functors $\mathbf{C} \times \mathbf{V} \to \mathbf{C}$ and $\mathbf{D}^* \to \mathbf{D}$ are R-cofinal (6.15) and therefore induce weak equivalences $cX_{\mathbf{C}} \sim cX_{\mathbf{C}\times\mathbf{V}}$ and $cX_{\mathbf{D}} \sim cX_{\mathbf{D}*}$. Moreover (7.2) the diagram on the right is, up to homotopy, a fibre square and so is therefore the diagram on the left. Using 8.3 it is not difficult to show that the inclusion $cX_{\mathbf{D}} \to \mathrm{holim}^i (cX_{\mathbf{D}^i})^f$ is a weak equivalence and the desired result now follows readily from the fact that $\mathrm{hoc}X_D$ is the inverse limit of the tower of fibrations $\{\mathrm{hoc}X_{\mathbf{D}^i}\}$.

III. The finite dimensional case

The 0-dimensional case follows from 8.3 and the following proposition which is an easy consequence of the fact that $\mathbf{dn}$ has an initial object [2, Chap. XI, 4.1(iii)].

8.5 PROPOSITION. *Theorem* 3.4(iii) *holds if* $\mathbf{D} = \mathbf{n}$ *for some* $n > 0$.

The higher dimensional case is proved by induction or n. Let $\dim\mathbf{D} = n > 0$. Then there is, in the notation of [9, §4], a push out diagram of categories

$$\begin{array}{ccc} \mathbf{A}' = \coprod_{I \in (sd\mathbf{D})_n} \overline{sd}(\mathbf{D}^{n-1})\downarrow I & \longrightarrow & \overline{sd}(\mathbf{D}^{n-1}) = \mathbf{A} \\ u' = \text{incl.}\downarrow & & \downarrow \text{incl.} = u \\ \mathbf{B}' = \coprod_{I \in (sd\mathbf{D})_n} \overline{sd}\mathbf{D}\downarrow I & \longrightarrow & \overline{sd}\mathbf{D} = \mathbf{B} \end{array} \tag{8.6}$$

in which each $\overline{sd}\mathbf{D}\downarrow I \approx \overline{sd}\mathbf{n}$ and hence (8.1) it suffices to show that, in the notation of §7, for each diagram $Y_{\mathbf{B}} \in \mathbf{S}^{\mathbf{B}}$, the induced diagrams

$$\begin{array}{ccc} cY_{\mathbf{B}} & \longrightarrow & cY_{\mathbf{B}'} \\ \downarrow & & \downarrow \\ cY_{\mathbf{A}} & \longrightarrow & cY_{\mathbf{A}'} \end{array} \quad \text{and} \quad \begin{array}{ccc} \operatorname{hoc} Y_{\mathbf{B}} & \longrightarrow & \operatorname{hoc} Y_{\mathbf{B}'} \\ \downarrow & & \downarrow \\ \operatorname{hoc} Y_{\mathbf{A}} & \longrightarrow & \operatorname{hoc} Y_{\mathbf{A}'} \end{array} \tag{8.7}$$

are up to homotopy, fibre squares.

To do this for the diagram on the left, let $\mathbf{V}$ be as in the Proof of 8.2 and consider the factorization of diagram 8.6

$$\begin{array}{ccccc} \mathbf{A}' & \longrightarrow & \mathbf{A}'' & \longrightarrow & \mathbf{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{B}' & \longrightarrow & \mathbf{B}'' & \longrightarrow & \mathbf{B} \end{array} \tag{8.8}$$

in which $\mathbf{A}''$ and $\mathbf{B}''$ are obtained from the push out diagrams

$$\begin{array}{ccc} \mathbf{A}' & \longrightarrow & \mathbf{A} \\ \downarrow & \times w & \downarrow \\ \mathbf{A}' \times \mathbf{V} & \longrightarrow & \mathbf{A}'' \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{A}' & \xrightarrow{\times u} & \mathbf{A}'' \\ \downarrow & & \downarrow \\ \mathbf{B}' & \longrightarrow & \mathbf{B}'' \end{array}$$

and the functors $\mathbf{A}'' \to \mathbf{A}$ and $\mathbf{B}'' \to \mathbf{B}$ are induced by the projection $\mathbf{A}' \times \mathbf{V} \to \mathbf{A}'$. The desired result then follows readily from the R-cofinality (6.15) of the functors $\mathbf{A}'' \to \mathbf{A}$ and $\mathbf{B}'' \to \mathbf{B}$ and the fact that the left hand square in 8.8 satisfies the conditions of Lemma 7.2.

To prove that in 8.7 the square on the right is, up to homotopy, a fibre square, let $P = (c_{d\mathbf{A}} Y_{\mathbf{A}})^f$ and $Q = (c_{d\mathbf{B}} Y_{\mathbf{B}})^f$ and note that there is a commutative diagram (9.7)

$$\begin{array}{ccc} \operatorname{hoc} Y_{\mathbf{B}} = \underleftarrow{\operatorname{holim}}^{d\mathbf{B}} Q & \xrightarrow{\sim} & \underleftarrow{\operatorname{holim}}^{d\mathbf{B}} \underleftarrow{id}_* Q \\ \downarrow & & \downarrow \\ \operatorname{hoc} Y_{\mathbf{A}} = \underleftarrow{\operatorname{holim}}^{d\mathbf{A}} P & \xrightarrow{\sim} & \underleftarrow{\operatorname{holim}}^{d\mathbf{B}} \underleftarrow{du}_* P \end{array}$$

in which the horizontal (push down) maps are weak equivalences. Furthermore, let $\mathbf{E} \subset d\mathbf{B}$ be the full (initial)subcategory generated by the objects which are not in $d\mathbf{A}$, let $e\colon \mathbf{E} \to d\mathbf{B}$ be the inclusion functor and let $e^*\colon \mathbf{S}^{d\mathbf{B}} \to \mathbf{S}^{d\mathbf{B}}$ be the functor given by

$$\begin{array}{ll} e^* ZJ = ZJ & \text{for } J \in \mathbf{E} \\ e^* ZJ = * & \text{otherwise} \end{array}$$

for every object $Z \in \mathbf{S}^{d\mathbf{B}}$. Then there is a commutative diagram

$$\begin{array}{ccccc} \underleftarrow{\operatorname{holim}}^{d\mathbf{B}} \underleftarrow{id}_* Q & \longrightarrow & \underleftarrow{\operatorname{holim}}^{d\mathbf{B}} e^* \underleftarrow{id}_* Q & \longrightarrow & \underleftarrow{\operatorname{holim}}^{\mathbf{E}} e^* \underleftarrow{id}_* Q \\ \downarrow & & \downarrow & & \downarrow \\ \underleftarrow{\operatorname{holim}}^{d\mathbf{B}} \underleftarrow{du}_* P & \longrightarrow & \underleftarrow{\operatorname{holim}}^{d\mathbf{B}} e^* \underleftarrow{du}_* P & \longrightarrow & \underleftarrow{\operatorname{holim}}^{\mathbf{E}} e^* \underleftarrow{du}_* P \end{array}$$

in which the indicated maps are clearly isomorphisms and the square on the left is [2, p. 303] a fibre square, because the square of $d\mathbf{B}$-diagrams involved is so for every object $J \in d\mathbf{B}$. The desired result now follows from the fact that the map $\operatorname{hoc} Y_{\mathbf{B}'} \to \operatorname{hoc} Y_{\mathbf{A}'}$ gives rise to similar diagrams in which the last vertical map (between the $\underleftarrow{\mathrm{holim}}^{\mathbf{E}'}$'s) is, in an obvious manner, isomorphic to the above map (between the $\underleftarrow{\mathrm{holim}}^{\mathbf{E}}$'s).

§9. APPENDIX ON HOMOTOPY LIMITS

This is a brief review of some of the basic properties of homotopy limits from [2, Chaps. XI and XII] as well as some similar "well knowns" results.

9.1 Homotopy invariance of homotopy inverse limits [2, p. 304]

Let $\mathbf{D}$ be a small category and let $g\colon X \to Y \in \mathbf{S}^{\mathbf{D}}$ be a weak equivalence between fibrant objects (1.6(*i*)). *Then the induced map $\underleftarrow{\mathrm{holim}}^{\mathbf{D}} g\colon \underleftarrow{\mathrm{holim}}^{\mathbf{D}} X \to \underleftarrow{\mathrm{holim}}^{\mathbf{D}} Y \in \mathbf{S}$ is also a weak equivalence.*

9.2 Homotopy invariance of homotopy direct limits [2, p. 335]

Let $\mathbf{D}$ be a small category and let $g\colon X \in Y \to \mathbf{S}^{\mathbf{D}}$ be a weak equivalence. Then so is the induced map $\underrightarrow{\mathrm{holim}}^{\mathbf{D}} g\colon \underrightarrow{\mathrm{holim}}^{\mathbf{D}} X \to \underrightarrow{\mathrm{holim}}^{\mathbf{D}} Y \in \mathbf{S}$.

9.3 Cofinality theorem for homotopy inverse limits [2, p. 317]

Let $u\colon \mathbf{A} \to \mathbf{B}$ be a functor between small categories which is left cofinal (i.e. the over category $u \downarrow B$ is contractible, for every object $B \in \mathbf{B}$), *and let $Y \in \mathbf{S}^{\mathbf{B}}$ be fibrant. Then the induced map $\underleftarrow{\mathrm{holim}}^{\mathbf{B}} Y \to \underleftarrow{\mathrm{holim}}^{\mathbf{A}} u^* Y \in \mathbf{S}$ is a weak equivalence.*

Using similar double complex arguments one can also prove:

9.4 Cofinality theorem for homotopy direct limits

Let $u\colon \mathbf{A} \to \mathbf{B}$ be a functor between small categories which is right cofinal (i.e. $B \downarrow u$ is contractible for every object $B \in \mathbf{B}$) *and let $Y \in \mathbf{S}^{\mathbf{B}}$. Then the induced map $\underrightarrow{\mathrm{holim}}^{\mathbf{A}} u^* Y \to \underrightarrow{\mathrm{holim}}^{\mathbf{B}} Y \in \mathbf{S}$ is weak equivalence.*

9.5 Reduction theorem for homotopy inverse limits

Let $\mathbf{A} \to \mathbf{B}$ be a functor between small categories and let $Y \in \mathbf{S}^{\mathbf{B}}$ be fibrant. Then the $\mathbf{B}$-diagram $u \downarrow -$ is cofibrant [9, §2] *and the obvious map $\hom^{\mathbf{B}}((u \downarrow -), Y) \to \underleftarrow{\mathrm{holim}}^{\mathbf{A}} u^* Y \in \mathbf{S}$ is a weak equivalence.*

9.6 Reduction theorem for homotopy direct limits

Let $u\colon \mathbf{A} \to \mathbf{B}$ be a functor between small categories. Let $Y \in \mathbf{S}^{\mathbf{B}}$ and let $(-\downarrow u) \times_{\mathbf{B}} Y$ denote the difference kernel as in [2, *p.* 328]. *Then the obvious map $\underrightarrow{\mathrm{holim}}^{\mathbf{A}} u^* Y \to (-\downarrow u) \times_{\mathbf{B}} Y \in \mathbf{S}$ is a weak equivalence.*

9.7 Push down theorem for homotopy inverse limits

Let $u\colon \mathbf{A} \to \mathbf{B}$ be a functor between small categories. Let $X \in \mathbf{S}^{\mathbf{A}}$ be fibrant and let $\underleftarrow{u_*} X \in \mathbf{S}^{\mathbf{B}}$ be the "homotopy push down" given by $\underleftarrow{u_*} XB = \underleftarrow{\mathrm{holim}}^{B \downarrow u} j^* X$ for every object $B \in \mathbf{B}$ (j denotes as usual the forgetful functor). *Then the obvious map $\underleftarrow{\mathrm{holim}}^{\mathbf{A}} X \to \underleftarrow{\mathrm{holim}}^{\mathbf{B}} \underleftarrow{u_*} X \in \mathbf{S}$* is *a weak equivalence.*

9.8 Push down theorem for homotopy direct limits

Let $u\colon \mathbf{A} \to \mathbf{B}$ be a functor between small categories, let $X \in \mathbf{S}^{\mathbf{A}}$ and let $\underrightarrow{u_*} X \in \mathbf{S}^{\mathbf{B}}$ be the "homotopy push down" given by $\underrightarrow{u_*} XB = \underleftarrow{\mathrm{holim}}^{u \downarrow B} j^* X$ for every object $B \in \mathbf{B}$. Then the obvious map $\underleftarrow{\mathrm{holim}}^{\mathbf{B}} \underrightarrow{u_*} X \to \underleftarrow{\mathrm{holim}}^{\mathbf{A}} X \in \mathbf{S}$ is a weak equivalence.

Other useful results are

9.9 PROPOSITION. *Let* $\mathbf{D}$ *be a small category and let* $X \in \mathbf{S}^{\mathbf{D}}$ *be a constant diagram* [2, p. 300]. *Then there are obvious isomorphisms* (1.6(i))

$$\underrightarrow{\mathrm{holim}}^{\mathbf{D}} X \approx \mathbf{D} \times X \quad \textit{and} \quad \mathrm{holim}^{\mathbf{D}} X = \mathrm{hom}(\mathbf{D}, X)$$

This follows immediately from the definitions.

9.10 PROPOSITION. *Let* $\mathbf{D}$ *be a contractible small category and let* $X \in \mathbf{S}^{\mathbf{D}}$ *be such that, for every map* $\mathrm{d} \in \mathbf{D}$, $X\mathrm{d}$ *is a weak equivalence. Then, for every object* $D \in \mathbf{D}$,

(i) *the obvious map* $XD \to \mathrm{holim}^{\mathbf{D}} X$ *is a weak equivalence, and*

(ii) *if* X *is fibrant, then the obvious map* $\underleftarrow{\mathrm{holim}}^{\mathbf{D}} X \to XD$ *is also a weak equivalence.*

Proof. The first part is a cosnequence of the Lemma on p. 90 (98) of [14] and the second part follows readily from the first part using 9.1 and 9.9.

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