

A SHORT PROOF OF TELESCOPIC TATE VANISHING

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ABSTRACT. We give a short proof of a theorem of Kuhn that Tate constructions for finite group actions vanish in telescopically localized stable homotopy theory. In particular, we observe that Kuhn’s theorem is equivalent to the statement that the transfer $BC_{p+} \rightarrow S^0$ admits a section after telescopic localization, which in turn follows from the Kahn-Priddy theorem.

1. INTRODUCTION

Let Sp denote the ∞ -category of spectra.¹ Thanks to the thick subcategory theorem [HS98], the “primes” of Sp (in the sense of [Bal05]) are indexed by the Morava K -theories $K(n)$, for $n \geq 0$ and an implicit prime p . In chromatic homotopy theory, one studies Sp by first studying the Bousfield-localized categories $L_{K(n)}\mathrm{Sp}$, and then attempting to assemble the local knowledge into global knowledge.

One reason this is a viable approach is that the ∞ -categories $L_{K(n)}\mathrm{Sp}$ have some surprisingly simple properties. An important example is the following theorem, which follows from the main results of [GS96, HS96] and has recently been extended in an interesting direction in [HL13].

Theorem 1.1 (Compare Greenlees-Sadofsky [GS96], Hovey-Sadofsky [HS96]). *Let G be a finite group, and X be a $K(n)$ -local spectrum with a G -action, i.e., an object of the ∞ -category $\mathrm{Fun}(BG, L_{K(n)}\mathrm{Sp})$. Then the norm map*

$$X_{hG} \rightarrow X^{hG}$$

in $L_{K(n)}\mathrm{Sp}$ is an equivalence.

When $n = 0$, applying the localization $L_{K(0)}$ is equivalent to working rationally. Then this theorem is easy to prove, because the obvious composition $X^{hG} \rightarrow X \rightarrow X_{hG}$ provides an inverse to the norm map up to multiplication by the order of G . But for $n > 0$, Theorem 1.1 is surprising, since for example X can easily have torsion dividing the order of G . The proof of [GS96, HS96] is based on the calculation of $K(n)^*(BG)$ for G of prime order.

In addition to the Morava K -theory localization functors $L_{K(n)}$, there are also the closely related telescopic localization functors $L_{T(n)}$. There is a natural transformation $L_{T(n)} \rightarrow L_{K(n)}L_{T(n)} = L_{K(n)}$ which is an equivalence for $n = 0$ and $n = 1$ (proved by Miller [Mil81] at odd primes and Mahowald [Mah82] at $p = 2$), but is believed not to be an equivalence in general; this question is the *telescope conjecture*. In contrast to the $L_{K(n)}$, these $L_{T(n)}$ have a more fundamental finitary construction, but for $n > 1$ it is not known how to compute their values explicitly.

¹The use of ∞ -categories is not really necessary here. We use it for convenience in discussing group actions. The reader can replace $\mathrm{Fun}(BG, \mathrm{Sp})$ (of which we only need the homotopy category) with the subcategory of the homotopy category of genuine G -spectra given by the Borel-equivariant or cofree ones. Compare the discussion in [MNN17, Sec. 6.3].

In [Kuh04, Th. 1.5], Kuhn strengthened Theorem 1.1 to apply to the telescopic localization functors. That is, he showed that Theorem 1.1 holds with $L_{T(n)}\mathrm{Sp}$ replacing $L_{K(n)}\mathrm{Sp}$. Part of this had been previously proved by Mahowald-Shick [MS88] at the prime 2.

Without calculational access to the $L_{T(n)}$, Kuhn's proof is necessarily different from that of [GS96]. Instead it is based on the *Bousfield-Kuhn functor*. For $n > 0$, this is a functor $\Phi: \mathcal{S}_* \rightarrow L_{T(n)}\mathrm{Sp}$ from pointed spaces to $T(n)$ -local spectra such that we have a natural equivalence

$$\Phi \circ \Omega^\infty \simeq L_{T(n)}: \mathrm{Sp} \rightarrow L_{T(n)}\mathrm{Sp}.$$

We refer to [Kuh89, Bou01] for the construction of the functor and [Kuh08] for a survey.

Kuhn's proof applies the Bousfield-Kuhn functor to a sequence of generalizations of the Kahn-Priddy splitting. In this note, we use just the Bousfield-Kuhn functor applied to the Kahn-Priddy splitting. The key observation is that Tate vanishing for a finite group G is equivalent to the localized transfer map $BG_+ \rightarrow S^0$ admitting a section, and this can be proved directly when $G = C_p$ using the Kahn-Priddy theorem. Thus, we obtain a simplification of Kuhn's argument [Kuh04, Sec. 3], avoiding the use of results such as the C_p -Segal conjecture.

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2. THE PROOF

Lemma 2.1. *Let R be a multiplicative cohomology theory and K a pointed connected CW-complex with basepoint k . If $r \in R^0(K)$ restricts to a unit in $R^0(\{k\})$, then r itself is a unit.*

Proof. We prove that for every pointed subcomplex $K' \subset K$, the map $R^*(K') \rightarrow R^*(K')$ given by multiplication by r is an isomorphism. This is true for K' a point by hypothesis. If $K' \subset K$ is a finite-dimensional subcomplex, an Atiyah-Hirzebruch spectral sequence argument shows that the kernel of $R^0(K') \rightarrow R^0(\{k\})$ is nilpotent, which forces r to restrict to a unit in $R^0(K')$. Then a five lemma argument with the Milnor sequence implies it for every K' . Taking $K' = K$ we conclude. \square

Throughout, let $L: \mathrm{Sp} \rightarrow \mathrm{Sp}$ be a Bousfield localization and G a finite group.

Definition 2.2. Given an object $X \in \mathrm{Fun}(BG, L\mathrm{Sp})$, the (L -local) *Tate construction* $X^{tG} \in L\mathrm{Sp}$ is the cofiber of the *norm map* $X_{hG} \rightarrow X^{hG}$.

Note that the homotopy orbits X_{hG} are computed in $L\mathrm{Sp}$, so that they are the L -localization of the homotopy orbits in Sp , and this is the L -localization of the usual Tate construction in spectra.

Our basic observation is the following proposition.

Proposition 2.3. *The following conditions are equivalent:*

- (1) *For every G -object X of $L\mathrm{Sp}$, the norm map $X_{hG} \rightarrow X^{hG}$ in $L\mathrm{Sp}$ is an equivalence, i.e., $X^{tG} = 0$.*
- (2) *Condition 1 holds just for $X = LS$ with trivial G -action.*
- (3) *The transfer map*

$$(1) \quad \Sigma_+^\infty BG \rightarrow \Sigma_+^\infty *$$

of spectra admits a section after applying L .

Proof. Clearly $1 \Rightarrow 2$; for the converse, we use that X^{tG} is a module over LS^{tG} (cf. [GM95, Prop. I.3.5]).

We now prove that 2 and 3 are equivalent. We use the basic diagram in $L\text{Sp}$ (compare [GM95, Sec. I.5])

$$(2) \quad \begin{array}{ccccc} (LS)_{hG} & \xrightarrow{N} & (LS)^{hG} & \longrightarrow & (LS)^{tG} \\ & \searrow f & \downarrow r & & \\ & & LS & & \end{array}$$

Here:

- (1) The Tate construction $(LS)^{tG}$ is a ring spectrum and the map from $(LS)^{hG}$ is a multiplicative map.
- (2) The map f is the L -localization of the transfer (1).
- (3) The map r identifies with the map $F(BG_+, LS) \rightarrow LS$ given by the basepoint of BG .
- (4) The horizontal row is a cofiber sequence.

Suppose 2 holds. Then N is an equivalence. Since r has a section, it follows from the diagram that f does as well, as desired.

Finally, suppose 3 holds, i.e., f has a section. To show that $(LS)^{tG} = 0$, the diagram shows that it suffices to see that the induced map $N: \pi_0((LS)_{hG}) \rightarrow \pi_0((LS)^{hG})$ has image containing a unit, which will then map to zero in $\pi_0((LS)^{tG})$. Since f has a section, it follows that there exists $x \in \pi_0((LS)_{hG})$ whose image in $\pi_0(LS)$ under $f = r \circ N$ is equal to 1. It follows that $Nx \in \pi_0((LS)^{hG}) = (LS)^0(BG)$ is a unit in view of Lemma 2.1, which completes the proof. \square

Next we need a reduction to the group C_p .

Lemma 2.4. *Let L be a Bousfield localization of spectra. If the equivalent conditions of Proposition 2.3 are satisfied for every group G of prime order, then they are satisfied for every finite group G .*

Proof. This follows from [Kuh04, Lemmas 2.7 and 2.8]. \square

Theorem 2.5 (Compare Kahn-Priddy [KP78], Segal [Seg74]). *The transfer map $\Sigma_+^\infty BC_p \rightarrow \Sigma_+^\infty *$ admits a section after applying the functor $\Omega^{\infty+1}$.*

Note that the Kahn-Priddy theorem is usually stated for Σ_p replacing C_p . However, the result as stated follows because it reduces to a statement at the prime p (in fact, $(\Sigma_+^\infty B\Sigma_p)[1/p] \simeq S^0[1/p]$) and the transfer exhibits $(\Sigma_+^\infty B\Sigma_p)_{(p)}$ as a summand of $(\Sigma_+^\infty BC_p)_{(p)}$. Note also that the connected parts of the spectra in question are all torsion and split into a product of their q -localizations for primes q .

Putting things together, we thus obtain our main result.

Theorem 2.6. *Suppose L is a Bousfield localization of spectra such that there exists a functor $\Phi: \mathcal{S}_* \rightarrow L\text{Sp}$ such that $\Phi\Omega^\infty \simeq L$. Then the equivalent conditions of Proposition 2.3 are satisfied for every finite group G . In particular, Tate constructions in $L\text{Sp}$ vanish.*

Proof. By Lemma 2.4, it suffices to assume that G has prime order. Applying Φ to the section of Theorem 2.5, we deduce that condition 3 holds for such G , concluding the proof. \square

Using the $K(n)$ -local Bousfield-Kuhn functors of [Kuh89] and their generalization to the telescopic setting in [Bou01], we recover:

Corollary 2.7 (Cf. [Kuh04, Theorem 1.5]). *The equivalent conditions of Proposition 2.3 hold for $E = T(n)$.*

Remark 2.8. We also obtain as a consequence that there can be no analog of the Bousfield-Kuhn functor for E -localization when $E = K(n_1) \vee K(n_2)$ for $n_1 < n_2$. In fact, we have that $L_{K(n_1)}(E_{n_2})^{tC_p} \neq 0$.

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