FROM PARTIAL MODEL CATEGORIES TO ∞-CATEGORIES

C. BARWICK AND D. M. KAN

Abstract. In this article we study the problem of extracting an ∞-category from a relative category. We introduce *partial model categories*, which are relative categories that satisfy mild versions of the axioms of a model category. Since these axioms involve only the weak equivalences, they are general enough to include the vast majority of the relative categories one encounters in practice. We show that the simplicial nerve of a partial model category is "essentially" a complete Segal space, generalizing a result of Charles Rezk. To prove this, we must introduce a significant generalization of a Quillen's Theorem B. We show also that, conversely, any complete Segal space is dimensionwise equivalent to the simplicial nerve of a partial model category.

Contents

0. Introduction

Many examples of homotopy theories arise most naturally as *relative categories* — that is, as categories $\boldsymbol{C}$ equipped with subcategories $\boldsymbol{W} \subset \boldsymbol{C}$ of weak equivalences [BK1]. Examples include:

(i) the category of commutative differential graded algebras over a field, equipped with the subcategory of quasi-isomorphisms,

(ii) the category of perfect complexes of quasicoherent sheaves on a variety, equipped with the subcategory of local quasi-isomorphisms,

(iii) the category of C^*-algebras, equipped with the subcategory of homotopy equivalences,

(iv) the category of topological spaces and proper maps, equipped with the subcategory of proper homotopy equivalences.

When one is presented with a homotopy theory exhibited in this way, it is natural to seek to employ the flexible techniques of higher category theory as presented in [HTT] and elsewhere. But how does one extract a suitable ∞-category from a relative category $(\boldsymbol{C}, \boldsymbol{W})$? To obtain a quasicategory, current technology demands that one choose between the following rather tortuous procedures.

(i) One may first form a **simplicial localization** $L(\boldsymbol{C}, \boldsymbol{W})$ of $(\boldsymbol{C}, \boldsymbol{W})$ [DK1, DK2, DK3], a simplicial category whose spaces of morphisms can be identified with nerves of categories of zigzags of arrows in $\boldsymbol{C}$ in which the backwards maps lie in $\boldsymbol{W}$. One then replaces these mapping spaces with fibrant ones and performs the **nerve construction** of [HTT, Df. 1.1.5.5] or [DS].

(ii) Alternately, one may form the ordinary nerve of $\boldsymbol{C}$ and regard this simplicial set as **marked** by the edges that correspond to morphisms of $\boldsymbol{W}$. Then one performs a fibrant replacement for the cartesian model structure on the category of marked simplicial sets.

The first of these works relatively well if $(\boldsymbol{C}, \boldsymbol{W})$ is a simplicial model category, but for examples like those listed above, in which no natural simplicial structure is available, neither of these processes is particularly tractable for computations. For example, it seems difficult to describe invariants such as the *moduli space of strings of arrows of length* p of the resulting quasicategory X — which may be modeled as the maximal Kan complex contained in the quasicategory $\operatorname{Fun}(\Delta^p, X)$ — in terms of the relative category $(\boldsymbol{C}, \boldsymbol{W})$.

The first major theorem of this article (2.2) asserts that for virtually any relative category $(\boldsymbol{C}, \boldsymbol{W})$ one meets in practice, there is a simpler and more direct way of extracting an ∞-category X. In effect, we show that the moduli space of strings of arrows of length p in X is equivalent to the nerve $N_p(\boldsymbol{C}, \boldsymbol{W})$ of the category whose objects are of strings of arrows of length p in $\boldsymbol{C}$, and whose morphisms are objectwise weak equivalences. In the framework developed by Charles Rezk [R], the resulting simplicial space $N_*(\boldsymbol{C}, \boldsymbol{W})$ correctly models the ∞-category associated with $(\boldsymbol{C}, \boldsymbol{W})$.

Charles Rezk proved this result for simplicial model categories [R, 8.3]; our generalization extends his result to examples like those of the first paragraph. Perhaps surprisingly, the conditions we demand on the relative category $(\boldsymbol{C}, \boldsymbol{W})$ for our result are quite mild; they are enjoyed by any relative category that can be embedded as a **homotopically full subcategory** of a model category $\boldsymbol{M}$ — that is, a full subcategory of $\boldsymbol{M}$ such that any weak equivalence whose source or target lies in $\boldsymbol{C}$ is also a weak equivalence of $\boldsymbol{C}$. We therefore call these relative categories **partial model categories**. These include all of the examples from the first paragraph.

The proof of this result makes use of a generalization of Quillen's Theorem B (5.9) that is of interest in its own right. In [DKS, §6], Theorem B was generalized to give more robust but less simple characterizations of the homotopy fibres of the nerve of a functor. We extend this result yet further to describe more general homotopy pullbacks.

Our second major theorem (3.1) is a converse to Rezk's result. It asserts, in effect, that any ∞-category is equivalent to one associated with a partial model category. More precisely, given a relative category $(\boldsymbol{C}, \boldsymbol{W})$, we construct a **relative Yoneda embedding**

$$y\colon (\boldsymbol{C}, \boldsymbol{W}) \longrightarrow \boldsymbol{S}^{\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{W}^{\mathrm{op}}}$$

into a relative category of relative functors from $(\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{W}^{\mathrm{op}})$ to the model category of simplicial sets. and note that y induces an equivalence of homotopy theories between $(\boldsymbol{C}, \boldsymbol{W})$ and a homotopically full relative subcategory of the model category $\boldsymbol{S}^{\boldsymbol{C}^{\mathrm{op}}}$ of diagrams of simplicial sets.

Organization of the paper. In §1 we introduce partial model categories and discuss the immediate consequences of their definition. In §2 and §3 we then state the main results of this paper — our generalization of Rezk's result (2.2) and its converse (3.1).

In §3, we also give a proof of (3.1), modulo a partial modelization lemma (3.3), which we will deal with in §7.

Our proof of (2.2), which will be given in §6, relies upon our generalization of Quillen's Theorem B (5.9). In order to formulate this result, we discuss in §4 various *Grothendieck constructions* and give a precise formulation of what we call *Quillen's lemma*. Then (§5), we recall the properties B_n and C_n and prove the *Theorems* B_n *for homotopy fibres* and for *homotopy pullbacks*.

1. Partial model categories

It was noted in [DK2, DK3] that in a simplicial model category $\boldsymbol{M}$, if X is cofibrant and Y is fibrant, then the function complex $\boldsymbol{M}_*(X, Y)$ has the same homotopy type as the mapping space $L(\boldsymbol{M}, \boldsymbol{W})(X, Y)$ of the simplicial localization $L(\boldsymbol{M}, \boldsymbol{W})$ where $\boldsymbol{W} \subset \boldsymbol{M}$ denotes its category of weak equivalences. A key step in the proof of this result was the observation that if a relative category $(\boldsymbol{C}, \boldsymbol{U})$ with the two out of three property admits a 3-*arrow calculus* — that is, if there exists subcategories $\boldsymbol{U}_c$ and $\boldsymbol{U}_f \subset \boldsymbol{U}$ that enjoy some of the properties of the categories of the trivial cofibrations and trivial fibrations in a model category —, then for every pair of objects $X, Y \in \boldsymbol{C}$, the homotopy type of $L(\boldsymbol{C}, \boldsymbol{U})(X, Y)$ admits a rather simple description in terms of 3-arrow zigzags

$$X \longleftarrow \cdot \longrightarrow \cdot \longleftarrow Y$$

in which the outside maps are weak equivalences.

On the other hand, it was noted in [DHKS] that if a relative category $(\boldsymbol{C}, \boldsymbol{U})$ has the *two out of six property* — a strengthening of the more usual two out of three property —, then one can make sense of homotopy limit and colimit functors that enjoy many of the expected properties. Better still, if $(\boldsymbol{C}, \boldsymbol{U})$ also admits a 3-arrow calculus, then the relative category $(\boldsymbol{C}, \boldsymbol{U})$ is *saturated* in the sense that a map in $\boldsymbol{C}$ is a weak equivalence iff its image in $\mathrm{Ho}(\boldsymbol{C}, \boldsymbol{U})$ is an isomorphism. This allows one to formulate very simple conditions that ensure the existence of such homotopy limit and colimit functors.

This suggests that the notion of a relative category that has the two out of six property and admits a 3-arrow calculus is a particularly useful one that deserves further investigation. This leads us to the following definition.

1.1. **Partial model categories.** A **partial model category** will be a pair $(\boldsymbol{C}, \boldsymbol{W})$ consisting of a category $\boldsymbol{C}$ and a subcategory $\boldsymbol{W} \subset \boldsymbol{C}$ (the maps of which will be called **weak equivalences**) which, roughly speaking, satisfies those parts of the model category axioms (as for instance reformulated in [DHKS, 9.1]) which involve only the weak equivalences. More precisely we require that

A. $(\boldsymbol{C}, \boldsymbol{W})$ be a **relative category** — that is, that $\boldsymbol{W}$ contains all the objects of $\boldsymbol{C}$ (and hence also their identity maps),

B. $(\boldsymbol{C},\boldsymbol{W})$ has the **two out of six property** — that is, if r, s and t are maps in $\boldsymbol{C}$ such that the *two* compositions sr and ts exists and are in $\boldsymbol{W}$, then the *four* maps r, s, t and tsr are also in $\boldsymbol{W}$ (which together with A) readily implies that $(\boldsymbol{C},\boldsymbol{W})$ *has the two out of three property and that* $\boldsymbol{W}$ *contains all the isomorphisms.*

C. $(\boldsymbol{C},\boldsymbol{W})$ admits a 3**-arrow calculus**, i.e. there exists subcategories $\boldsymbol{U},\boldsymbol{V}\subset\boldsymbol{W}$ which have the property that
   (i) for every map $u\in\boldsymbol{U}$, its *pushouts* in $\boldsymbol{C}$ exist and are again in $\boldsymbol{U}$,
   (ii) for every map $v\in\boldsymbol{V}$, it' *pullbacks* in $\boldsymbol{C}$ exist and are again in $\boldsymbol{V}$, and
   (iii) the maps $w\in\boldsymbol{W}$ admit a *functorial factorization* $w=vu$ with $u\in\boldsymbol{U}$ and $v\in\boldsymbol{V}$ (which implies that $\boldsymbol{U}$ and $\boldsymbol{V}$ contain all the objects).

It should be noted that conditions (i) and (ii) are *stronger* than the ones that were used in [DK2] and [DHKS]. However we prefer them as they are cleaner and easier to work with and are likely to be usually automatically satisfied.

1.2. **Examples of partial model categories.**

(i) Any model category is a partial model category. In particular the relative category $\boldsymbol{S}$ of simplicial sets is a partial model category.
(ii) If $(\boldsymbol{C},\boldsymbol{W})$ is a partial model category, then so is any **homotopically full relative subcategory** — that is any relative subcategory of the form $(\boldsymbol{C}',\boldsymbol{C}'\cap\boldsymbol{W})$ where $\boldsymbol{C}'$ is a full subcategory of $\boldsymbol{C}$ with the property that, for every object $C'\in\boldsymbol{C}'$, any object of $\boldsymbol{C}$ that is weakly equivalent to C' lies in $\boldsymbol{C}'$.
(iii) If $(\boldsymbol{C},\boldsymbol{W})$ is a partial model category, then for every relative category $(\boldsymbol{A},\boldsymbol{X})$, the relative category $(\boldsymbol{C},\boldsymbol{W})^{\boldsymbol{A},\boldsymbol{X}}$ of relative functors $(\boldsymbol{A},\boldsymbol{X})\to(\boldsymbol{C},\boldsymbol{W})$ is a partial model category as well.
(iv) If $(\boldsymbol{C},)$ is a partial model category, then so is $(\boldsymbol{W},\boldsymbol{W})$.

1.3. **Saturation** [DHKS, 36.4]**.** *Every partial model category* $(\boldsymbol{C},\boldsymbol{W})$ *is* ***saturated*** *in the sense that a map of* $\boldsymbol{C}$ *is in* $\boldsymbol{W}$ *iff it goes to an isomorphism in the* ***homotopy category*** $\mathrm{Ho}(\boldsymbol{C},\boldsymbol{W})$*, i.e. the category obtained from* $\boldsymbol{C}$ *by "formally inverting" the weak equivalences.*

2. A generalization of a result of Rezk

We now aim to state a strengthening of a result of Rezk on simplicial model categories [R, 8.3], which applies for partial model categories. The proof will appear in §6. First, we recall Rezk's model for ∞-categories.

2.1. **Rezk's complete Segal model structure.** In [R],

A. Rezk constructed a "homotopy theory of homotopy theories" model structure on the category s$\boldsymbol{S}$ of simplicial spaces (i.e. bisimplicial sets) by means of an appropriate left Bousfield localization of the Reedy model structure, the fibrant objects of which he referred to as **complete Segal spaces**, and

B. described a **Rezk** (or **simplicial**) **nerve functor** N from the category **RelCat** of relative categories (1.1) and relative functors between them to s$\boldsymbol{S}$ which sends a relative category $(\boldsymbol{C},\boldsymbol{W})$ to the simplicial space which in dimension $k \geq 0$ has as its n-simplices ($n \geq 0$) the commutative squares of the form

$$\begin{array}{ccccccc}
\cdot & \xrightarrow{c_1} & \cdot & \cdots & \cdot & \xrightarrow{c_k} & \cdot \\
{\scriptstyle w_1}\downarrow & & & & & & \downarrow \\
\cdot & & & & & & \cdot \\
\vdots & & & & & & \vdots \\
\cdot & & & & & & \cdot \\
{\scriptstyle w_n}\downarrow & & & & & & \downarrow \\
\cdot & \longrightarrow & \cdot & \cdots & \cdot & \longrightarrow & \cdot
\end{array}$$

in which the vertical maps are in $\boldsymbol{W}$.

He then noted that [R, 8.3]

* *if $\boldsymbol{M}$ is a simplicial model category, then any Reedy fibrant replacement of the simplicial space $N\boldsymbol{M}$ is a complete Segal space.*

2.2. **Theorem.** *If $(\boldsymbol{C}, \boldsymbol{W})$ is a partial model category (1.1), then any Reedy fibrant replacement of $N(\boldsymbol{C}, \boldsymbol{W})$ is a complete Segal space.*

3. A converse of Rezk's result

3.1. **Theorem.** *Every complete Segal space is Reedy equivalent to the simplicial nerve of a partial model category and in fact of a homotopically full relative subcategory of a category of diagrams of simplicial sets.*

The key to this is a *partial modelization lemma* which we will state in 3.3 but prove in §7 below. Its formulation requires the following.

3.2. **A relative Yoneda embedding.** Let L^H denote the hammock localization of [DK2]. Given a relative category $(\boldsymbol{C}, \boldsymbol{U})$, its **relative Yoneda embedding** will be the relative functor between relative categories

$$y = y_{\boldsymbol{C},\boldsymbol{U}} \colon (\boldsymbol{C}, \boldsymbol{U}) \longrightarrow \boldsymbol{S}^{\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{U}^{\mathrm{op}}}$$

which sends each object $A \in \boldsymbol{C}$ to the relative functor $yA \colon (\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{U}^{\mathrm{op}}) \to \boldsymbol{S}$ which sends each object $B \in \boldsymbol{C}^{\mathrm{op}}$ to the simplicial set $L^H(\boldsymbol{C}, \boldsymbol{U})(B, A)$.

We denote by Ey the **essential image** of the relative Yoneda embedding — that is, Ey is the smallest homotopically full relative subcategory (1.2(ii)) of $S^{\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{U}^{\mathrm{op}}}$ that contains all of the relative functors of the form yA.

3.3. **The partial modelization.** *Given a relative category $(\boldsymbol{C}, \boldsymbol{U})$,*

(i) *the essential image Ey of the relative Yoneda embedding (3.2) is a partial model category and in fact a homotopically full relative subcategory of a category of diagrams of simplicial sets, and*

(ii) *the embedding $e \colon (\boldsymbol{C}, \boldsymbol{U}) \to Ey$ is a* ***DK-equivalence*** *— that is, its simplicial localization is a weak equivalence of simplicial categories [Be], or, equivalently, [BK1, 1.8] its simplicial nerve is a Rezk (i.e. complete Segal) equivalence of simplicial spaces.*

Using this we now can give

3.4. **A proof of 3.1.** First recall from [BK1, 5.3 and 4.4] the existence of

(i) an adjunction $K_\xi\colon \mathbf{RelCat} \leftrightarrow \mathrm{s}\boldsymbol{S} : N_\xi$ of which the unit $\eta\colon 1 \to N_\xi K_\xi$ is a natural Reedy equivalence, and

(ii) a natural Reedy equivalence $\pi^*\colon N \to N_\xi$ (2.1b)

and from [R, 7.2] that

(iii) every Reedy equivalence in $\mathrm{s}\boldsymbol{S}$ is a Rezk equivalence (3.3(ii)) and every Rezk equivalence between two complete Segal spaces is a Reedy equivalence.

Given a complete Segal space X one then can consider the zigzag

$$X \xrightarrow{\ \eta\ } N_\xi K_\xi X \xleftarrow{\ \pi^*\ } NK_\xi X \xrightarrow{\ e\ } NEy_{K_\xi, X}$$

in which, in view of (i) and (ii) above and 3.3(ii) respectively, the first two maps are Reedy equivalences, while the third is a Rezk equivalence, and note that it follows from (iii) above and 2.2 that every Reedy fibrant replacement of the partial model category $Ey_{K_\xi X}$ (3.3(i)) is Reedy equivalent to X.

4. Grothendieck constructions and Quillen's lemma

To prove our generalization of Rezk's theorem (2.2), we will in the next section prove a generalization of Quillen's Theorem B. In preparation for the formulation and the proofs of this generalization, we first briefly discuss Grothendieck constructions and formulate, in terms of Grothendieck constructions, a categorical version of the lemma that Quillen used in his proof of Theorem B.

4.1. **Terminology.** We will work in the category **Cat** of small categories with the Thomason model structure [T2] in which a map is a *weak equivalence* iff its *nerve* is a weak equivalence of simplicial sets and in which *homotopy fibres* and *homotopy pullbacks* have a similar meaning.

4.2. **Grothendieck constructions.** Given a small category $\boldsymbol{D}$ and a functor $F\colon \boldsymbol{D} \to \mathbf{Cat}$ (4.1), the **Grothendieck construction** on F is the category $\boldsymbol{Gr}\, F$ which has

(i) as *objects* the pairs (D, A) consisting of objects

$$D \in \boldsymbol{D} \qquad \text{and} \qquad A \in FD\ ,$$

(ii) as *maps* $(D_1, A_1) \to (D_2, A_2)$ the pairs (d, a) of maps

$$d\colon D_1 \to D_2 \in \boldsymbol{D} \qquad \text{and} \qquad a\colon (Fd)A_1 \to A_2 \in FD_2$$

and

(iii) in which the *composition* is given by the formula

$$(d', a')(d, a) = \big(d'd, a'\big((Fd)a\big)\big)\ .$$

Moreover

(iv) $\boldsymbol{Gr}\, F$ comes with a *projection functor* $\pi\colon \boldsymbol{Gr}\, F \to \boldsymbol{D}$ which sends an object (D, A) (resp. a map (d, a)) in $\boldsymbol{Gr}\, F$ to the object D (resp. the map d) in $\boldsymbol{D}$.

The usefulness of Grothendieck constructions is due to the following property, which was noticed by Bob Thomason [T1, 1.2]:

4.3. **Proposition.**

(i) *The Grothendieck construction is a homotopy colimit construction on the category* **Cat**,

and hence

(ii) *it is homotopy invariant in the sense that every natural weak equivalence* (4.1) *between two functors* $F_1, F_2 \colon \boldsymbol{D} \to \mathbf{Cat}$ *induces a weak equivalence* $\boldsymbol{Gr}\, F_1 \to \boldsymbol{Gr}\, F_2$.

Next we note that Quillen's key observation in the lemma that he used to prove Theorem B was that certain functors $\boldsymbol{D} \to \mathbf{Cat}$ had what we will call

4.4. **Property** Q**.** Given a small category $\boldsymbol{D}$, a functor $F \colon \boldsymbol{D} \to \mathbf{Cat}$ will be said to have **property** Q if it sends all maps of $\boldsymbol{D}$ to weak equivalences in **Cat**.

A categorical version of the lemma that Quillen used in the proof of Theorem B (a proof of which can be found in [GJ, IV, 5.7]) then becomes in view of 4.3(i) above

4.5. **Quillen's lemma.** *If, given a small category* $\boldsymbol{D}$*, a functor* $F \colon \boldsymbol{D} \to \mathbf{Cat}$ *has property* Q*, then, for every object* $D \in \boldsymbol{D}$*, the fibre*

$$\pi^{-1} D = FD$$

of π (4.2(iv)) *over* $\boldsymbol{D}$ *is a homotopy fibre.*

5. A Quillen Theorem B_n for homotopy pullbacks

In [Q, §1] Quillen proved his Theorem B which gave a simple description of the homotopy fibres of a functor $f \colon \boldsymbol{X} \to \boldsymbol{Y}$ if f had a certain *property* B_1. This was generalized in [DKS, §6] where it was shown that increasingly weaker *properties* B_n ($n > 1$) allowed for increasingly less simple descriptions of these homotopy fibres. Moreover it was noted that a sufficient condition for a functor $f \colon \boldsymbol{X} \to \boldsymbol{Y}$ to have property B_n ($n > 1$) was that the category $\boldsymbol{Y}$ has a certain *property* C_n.

The main result of this section (5.9) asserts that for a zigzag $f \colon \boldsymbol{X} \to \boldsymbol{Y} \leftarrow \boldsymbol{Z} : g$ in which f has property B_n (and in particular if $\boldsymbol{Y}$ has property C_n), its homotopy pullback admits a description rather similar to the ones that appear in [DKS, §6]. Moreover, the pullback $\boldsymbol{X} \times_{\boldsymbol{Y}} \boldsymbol{Z}$ of this zigzag comes with a monomorphism into this homotopy pullback and hence is itself a homotopy pullback if the monomorphism is a weak equivalence.

The homotopy fibre results of [DKS, §6]. were obtained by an induction on n which at each stage used Quillen's Theorem B. To prove our homotopy pullback results, it turns out to be convenient to go one step further back to the lemma that Quillen used to prove his Theorem B:

- If $F \colon \boldsymbol{D} \to \mathbf{Cat}$ is a $\boldsymbol{D}$-diagram of categories and *weak equivalences* between them, $\boldsymbol{Gr}\, F$ its Grothendieck construction and $\pi \colon \boldsymbol{Gr}\, F \to \boldsymbol{D}$ the associated projection functor, then, *for every object* $D \in \boldsymbol{D}$*, the fibre*

  $$\pi^{-1} D = FD$$

  of π *over* D *is also a homotopy fibre.*

Using this result we first give a different non-inductive proof of the results of [DKS, §6] and then note that this proof almost effortlessly extends to a proof of our homotopy pullback result (5.9).

5.1. **Two Grothendieck constructions associated with a functor $\boldsymbol{X} \to \boldsymbol{Y}$.** Given an integer $n \geq 1$ and a functor $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ between small categories, we denote by $(f\boldsymbol{X} \downarrow_n \boldsymbol{Y})$ the category of which

(i) an object consists of a pair of objects

$$X \in \boldsymbol{X} \qquad \text{and} \qquad Y \in \boldsymbol{Y}$$

together with an alternating zigzag

$$fX = Y_n \quad \cdots \quad Y_2 \longleftarrow Y_1 \longrightarrow gZ \qquad \text{in } \boldsymbol{Y}$$

and of which

(ii) a *map* consists of a pair of maps

$$x\colon X \to X' \in \boldsymbol{X} \qquad \text{and} \qquad y\colon Y \to Y' \in \boldsymbol{Y}$$

together with a commutative diagram

$$\begin{array}{ccccccc} fX = Y_n & \cdots & Y_2 & \longleftarrow & Y_1 & \longrightarrow & Y \\ \downarrow{\scriptstyle fx} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle y} \\ fX' = Y'_n & \cdots & Y'_2 & \longleftarrow & Y'_1 & \longrightarrow & Y' \end{array} \qquad \text{in } \boldsymbol{Y}$$

(iii) This category comes with a monomorphism

$$h\colon X \longrightarrow (f\boldsymbol{X} \downarrow_n \boldsymbol{Y})$$

which sends each object $X \in \boldsymbol{X}$ to the zigzag of identity maps which starts at fX.

Furthermore

(iv) let, for every object $Y \in \boldsymbol{Y}$

$$(f\boldsymbol{X} \downarrow_n Y) \subset (f\boldsymbol{X} \downarrow_n \boldsymbol{Y})$$

denote the subcategory consisting of the objects which end at Y and the maps which end at $!_Y$

and similarly

(v) let, for every object $X \in \boldsymbol{X}$

$$(fX \downarrow_n \boldsymbol{Y}) \subset (f\boldsymbol{X} \downarrow_n \boldsymbol{Y})$$

denote the subcategory consisting of the objects which start at fX and the maps which start at 1_{fX}.

The naturality of $(f\boldsymbol{X} \downarrow_n Y)$ and $(fX \downarrow_n \boldsymbol{Y})$ in respectively Y and X then readily implies

5.2. **Proposition.** *For every integer $n \geq 1$ and functor $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ between small categories* (4.2)

(i) $(f\boldsymbol{X} \downarrow_n \boldsymbol{Y}) = \boldsymbol{Gr}\big((f\boldsymbol{X} \downarrow_n -)\colon \boldsymbol{Y} \to \mathbf{Cat}\big)$

and

(ii) $$(f\boldsymbol{X} \downarrow_n \boldsymbol{Y}) = \begin{cases} \boldsymbol{Gr}\big((f- \downarrow_n \boldsymbol{Y})\colon \boldsymbol{X} \to \mathbf{Cat}\big) \\ \text{or} \\ \boldsymbol{Gr}\big((f- \downarrow_n \boldsymbol{Y})\colon \boldsymbol{X}^{\mathrm{op}} \to \mathbf{Cat}\big) \end{cases}$$

depending on whether n is even or odd.

5.3. **A Grothendieck construction associated with a zigzag $\boldsymbol{X} \to \boldsymbol{Y} \leftarrow \boldsymbol{Z}$.** Given an integer $n \geq 1$ and a zigzag $f\colon \boldsymbol{X} \to \boldsymbol{Y} \leftarrow \boldsymbol{Z} :g$ between small categories, we denote by $(f\boldsymbol{X} \downarrow_n g\boldsymbol{Z})$ the category of which

(i) an *object* consists of a pair of objects

$$X \in \boldsymbol{X} \qquad \text{and} \qquad Z \in \boldsymbol{Z} \ ,$$

together with an alternating zigzag

$$fX = Y_n \quad \cdots \quad Y_2 \longleftarrow Y_1 \longrightarrow gZ \qquad \text{in } \boldsymbol{Y}$$

and of which

(ii) a map consists of a pair of maps

$$x\colon X \to X' \in \boldsymbol{X} \qquad \text{and} \qquad z\colon Z \to Z' \in \boldsymbol{Z} \ ,$$

together with a commutative diagram

$$\begin{array}{ccccccc} fX = Y_n & \cdots & Y_2 & \longleftarrow & Y_1 & \longrightarrow & gZ \\ {\scriptstyle fx}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle gz} \\ fX' = Y'_n & \cdots & Y'_2 & \longleftarrow & Y'_1 & \longrightarrow & gZ' \end{array} \qquad \text{in } \boldsymbol{Y} \ .$$

(iii) This category comes with a monomorphism

$$K\colon (\boldsymbol{X} \times_{\boldsymbol{Y}} \boldsymbol{Z}) \longrightarrow (f\boldsymbol{X} \downarrow_n g\boldsymbol{Z})$$

which sends each object $(X, Z) \in \boldsymbol{X} \times_{\boldsymbol{Y}} \boldsymbol{Z}$ to a zigzag of identity maps starting at fX and ending at gZ.

Furthermore

(iv) we denote, for every object $Z \in \boldsymbol{Z}$, by

$$(f\boldsymbol{X} \downarrow_n gZ) \subset (f\boldsymbol{X} \downarrow_n g\boldsymbol{Z})$$

the subcategory consisting of the objects which end at gZ and the maps which end at 1_{gZ}.

The naturality of $(f\boldsymbol{X} \downarrow_n gZ)$ with respect to Z then readily implies

5.4. **Proposition.** For every integer $n \geq 1$ and zigzag $f : \boldsymbol{X} \to \boldsymbol{Y} \leftarrow \boldsymbol{Z} : g$ between small categories (4.2)

$$(f\boldsymbol{X} \downarrow_n g\boldsymbol{Z}) = \boldsymbol{Gr}\big((f\boldsymbol{X} \downarrow_n g-)\colon \boldsymbol{Z} \to \mathbf{Cat}\big) \ .$$

We now proceed to the main results.

5.5. **Property B_n.** Given an integer $n \geq 1$, a functor $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ between small categories is said to have **property** B_n if the functor (5.2(i))

$$(f\boldsymbol{X} \downarrow_n -)\colon \boldsymbol{Y} \longrightarrow \mathbf{Cat}$$

has property Q (4.4).

5.6. **Theorem B_n.** *If a functor $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ between small categories has property B_n ($n \geq 1$), then, for every object $Y \in \boldsymbol{Y}$, the category $(f\boldsymbol{X} \downarrow_n Y)$ (5.1(iv)) is a homotopy fibre of f over Y.*

Proof. Given an object $Y \in \boldsymbol{Y}$, it follows from 4.5 and 5.2(i) that

(i) $(f\boldsymbol{X} \downarrow_n Y)$ is the fibre as well as a homotopy fibre over Y of the projection functor

$$\pi\colon \boldsymbol{Gr}(f\boldsymbol{X} \downarrow_n -) = (f\boldsymbol{X} \downarrow_n \boldsymbol{Y}) \longrightarrow \boldsymbol{Y}\ .$$

That it is also a homotopy fibre over Y of the functor $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ therefore is a consequence of

(ii) the commutativity of the diagram

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\ h\ } & (f\boldsymbol{X} \downarrow_n \boldsymbol{Y}) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \boldsymbol{Y} & \end{array}$$

in which h is as in 5.1(iii)

and the readily verifiable fact that

(iii) h is a weak equivalence. □

5.7. **Property C_n.** Let $\boldsymbol{O}$ denote the category consisting of a single object and its identity map and let n be an integer ≥ 1. Then a small category $\boldsymbol{Y}$ is said to have **property** C_n if

(i) every functor $e\colon \boldsymbol{O} \to \boldsymbol{Y}$ has property B_n, i.e.

(ii) every functor $e\colon \boldsymbol{O} \to \boldsymbol{Y}$ gives rise to a functor $(e\boldsymbol{O} \downarrow_n -)\colon \boldsymbol{Y} \to \mathbf{Cat}$ which has property Q (4.4).

The usefulness of this notion is due to the fact that, in view of 4.3(ii), 5.2(ii) and 5.7(ii), one has

5.8. **Theorem C_n.** *If $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a functor between small categories and $\boldsymbol{Y}$ has property C_n ($n \geq 1$), then f has property B_n.*

5.9. **Theorem B_n for homotopy pullbacks.** *Let n be an integer ≥ 1 and let $f\colon \boldsymbol{X} \to \boldsymbol{Y} \leftarrow \boldsymbol{Z} : g$ be a zigzag between small categories. If f has property B_n (5.5) (and in particular if $\boldsymbol{Y}$ has property C_n (5.7)), then*

(i) *the category $(f\boldsymbol{X} \downarrow_n g\boldsymbol{Z})$ (5.3) is a homotopy pullback of this zigzag.*

Moreover if in addition the monomorphism (5.3(iii))

$$k\colon (\boldsymbol{X} \times_{\boldsymbol{Y}} \boldsymbol{Z}) \longrightarrow (f\boldsymbol{X} \downarrow_n g\boldsymbol{Z})$$

is a weak equivalence, then

(ii) *the pullback $(\boldsymbol{X} \times_{\boldsymbol{Y}} \boldsymbol{Z})$ of this zigzag is also a homotopy pullback.*

Proof. As the functor $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ has property B_n, i.e.

- the functor $(f\boldsymbol{X} \downarrow_n -)\colon \boldsymbol{Y} \to \mathbf{Cat}$ has property Q (4.4)

it readily follows that

- the functor $(f\boldsymbol{X} \downarrow_n g-)\colon \boldsymbol{Z} \to \mathbf{Cat}$ (5.4) also has property Q.

Consequently (4.5 and 5.4)

(i) for every object $Z \in \boldsymbol{Z}$, $(f\boldsymbol{X} \downarrow_n gZ)$ is the fibre as well as the homotopy fibre over Z of the projection functor

$$\boldsymbol{Gr}\big(f\boldsymbol{X} \downarrow_n g-\big) = (f\boldsymbol{X} \downarrow_n g\boldsymbol{Z}) \longrightarrow \boldsymbol{Z}$$

Now consider the commutative square

$$\begin{array}{ccc} (f\boldsymbol{X} \downarrow_n \boldsymbol{Y}) & \xleftarrow{g'} & (f\boldsymbol{X} \downarrow_n g\boldsymbol{Z}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \boldsymbol{Y} & \xleftarrow{\quad g \quad} & \boldsymbol{Z} \end{array}$$

in which g' is induced by g.

Then clearly

(ii) this square is a pullback square and hence, for every object $Z \in \boldsymbol{Z}$, g' maps the fibre over Z isomorphically onto the fibre over $gZ \in \boldsymbol{Y}$.

Therefore, in view of (i) and 5.6(i)

(iii) this pullback square is a homotopy pullback square.

With other words $(f\boldsymbol{X} \downarrow_n g\boldsymbol{Z})$ is a homotopy pullback of the zigzag

$$\pi\colon (f\boldsymbol{X} \downarrow_n \boldsymbol{Y}) \longrightarrow \boldsymbol{Y} \longleftarrow \boldsymbol{Z} : g$$

That it is also a homotopy pullback of the zigzag

$$f\colon \boldsymbol{X} \longrightarrow \boldsymbol{Y} \longleftarrow \boldsymbol{Z} : g$$

now follows from 5.6(ii) and 5.6(iii). □

6. A proof of our generalization of Rezk's theorem (2.2)

The proof consists of two parts, a *Segal* part and a *completion* part.

6.1. The Segal part. To deal with the Segal part

(i) let for every integer $k \geq 0$, $\boldsymbol{A}_k$ denote the category which has as its objects the sequences

$$\cdot \xrightarrow{a_1} \cdot \quad \cdots \quad \cdot \xrightarrow{a_k} \cdot \qquad \text{in } \boldsymbol{C}$$

and as its maps the commutative diagrams of the form

$$\begin{array}{ccccccc} \cdot & \xrightarrow{a_1} & \cdot & \cdots & \cdot & \xrightarrow{a_k} & \cdot \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdot & \xrightarrow{a'_1} & \cdot & \cdots & \cdot & \xrightarrow{a'_k} & \cdot \end{array} \qquad \text{in } \boldsymbol{C}$$

in which the vertical maps are in $\boldsymbol{W}$.

Then we have to show that, for every integer $k \geq 2$, the pullback square

$$\begin{array}{ccc} \boldsymbol{A}_k & \longrightarrow & \boldsymbol{A}_{k-1} \\ \downarrow & & \downarrow \\ \boldsymbol{A}_1 & \longrightarrow & \boldsymbol{A}_0 = \boldsymbol{W} \end{array}$$

is a *homotopy pullback square.*

To do this

(ii) for every integer $k \geq 2$, denote by $\boldsymbol{B}_k$ the category which has as its objects the zigzags

$$\cdot \xrightarrow{b_1} \cdot \xrightarrow{x} \cdot \xleftarrow{w} \cdot \xrightarrow{y} \cdot \xrightarrow{b_2} \cdot \quad \cdots \quad \cdot \xrightarrow{b_k} \cdot \qquad \text{in } \boldsymbol{C}$$

in which x, y and w are in $\boldsymbol{W}$ and as its maps the commutative diagrams of the form

$$\begin{array}{ccccccccccccccc} \cdot & \xrightarrow{b_1} & \cdot & \xrightarrow{x} & \cdot & \xleftarrow{w} & \cdot & \xrightarrow{y} & \cdot & \xrightarrow{b_2} & \cdot & \cdots & \cdot & \xrightarrow{b_k} & \cdot \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdot & \xrightarrow{b'_1} & \cdot & \xrightarrow{x'} & \cdot & \xleftarrow{w'} & \cdot & \xrightarrow{y'} & \cdot & \xrightarrow{b'_2} & \cdot & \cdots & \cdot & \xrightarrow{b'_k} & \cdot \end{array} \qquad \text{in } \boldsymbol{C}$$

in which the vertical maps are in $\boldsymbol{W}$, and

(iii) for every integer $k \geq 2$, denote by

$$h_k \colon \boldsymbol{A}_k \longrightarrow \boldsymbol{B}_k$$

the monomorphism which between the first two maps inserts three identity maps, and denote by

$$\boldsymbol{A}'_k \subset \boldsymbol{B}_k$$

the image of $\boldsymbol{A}_k$ under h_k.

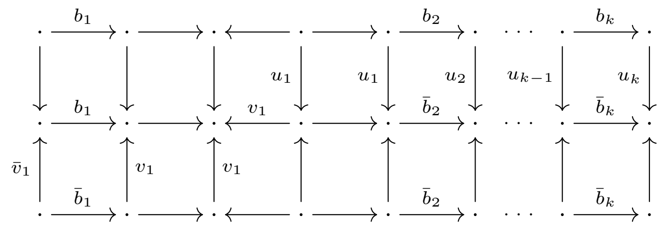
In view of the Quillen Theorem B_3 for homotopy pullbacks (5.9) and the fact that, in view of 1.2(iv), $\boldsymbol{A}_0 = \boldsymbol{W}$ has property C_3, it then suffices to show that, for every integer $k \geq 2$, the inclusion $i \colon \boldsymbol{A}'_k \to \boldsymbol{B}_k$ is a *homotopy equivalence*, i.e. that there exists a retraction $r \colon \boldsymbol{B}_k \to \boldsymbol{A}'_k$ such that the compositions ir and ri are naturally weakly equivalent to the identity functor of $\boldsymbol{B}_k$ and $\boldsymbol{A}'_k$ respectively.

Such a retraction, together with a zigzag of natural weak equivalences connecting the functors ir and $1_{\boldsymbol{B}_k}$ can be obtained by means of the following (natural) commutative diagram in $\boldsymbol{C}$

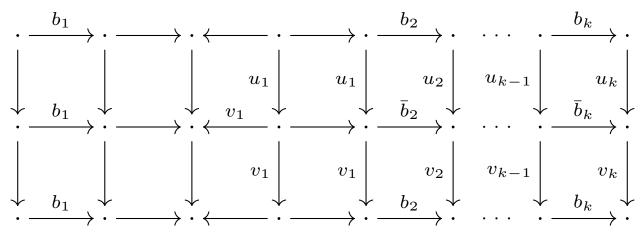
$$\begin{array}{ccccccccccccccc} \cdot & \xrightarrow{b_1} & \cdot & \xrightarrow{x} & \cdot & \xleftarrow{w} & \cdot & \xrightarrow{y} & \cdot & \xrightarrow{b_2} & \cdot & \cdots & \cdot & \xrightarrow{b_k} & \cdot \\ \downarrow & & \downarrow{\scriptstyle x} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdot & \xrightarrow{xb_1} & \cdot & \longrightarrow & \cdot & \xleftarrow{w} & \cdot & \xrightarrow{y} & \cdot & \xrightarrow{b_2} & \cdot & \cdots & \cdot & \xrightarrow{b_k} & \cdot \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle y} & & \uparrow & & \uparrow & & \uparrow \\ \cdot & \xrightarrow{xb_1} & \cdot & \longrightarrow & \cdot & \xleftarrow{w} & \cdot & \longrightarrow & \cdot & \xrightarrow{b_2y} & \cdot & \cdots & \cdot & \xrightarrow{b_k} & \cdot \\ \downarrow & & \downarrow & & \downarrow & & {\scriptstyle u_1}\downarrow & & {\scriptstyle u_1}\downarrow & & {\scriptstyle u_2}\downarrow & & {\scriptstyle u_{k-1}}\downarrow & & \downarrow{\scriptstyle u_k} \\ \cdot & \xrightarrow{xb_1} & \cdot & \longrightarrow & \cdot & \xleftarrow{v_1} & \cdot & \longrightarrow & \cdot & \xrightarrow{\overline{b_2y}} & \cdot & \cdots & \cdot & \xrightarrow{\overline{b_k}} & \cdot \\ {\scriptstyle \overline{v_1}}\uparrow & & \uparrow{\scriptstyle v_1} & & \uparrow{\scriptstyle v_1} & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \cdot & \xrightarrow{\overline{xb_1}} & \cdot & \longrightarrow & \cdot & \longleftarrow & \cdot & \longrightarrow & \cdot & \xrightarrow{\overline{b_2y}} & \cdot & \cdots & \cdot & \xrightarrow{\overline{b_k}} & \cdot \end{array}$$

in which all the unmarked arrows are identity maps, $w = v_1 u_1$ with $u_1 \in \boldsymbol{U}$ and $v_1 \in \boldsymbol{V}$ (1.1c) and the squares involving two u's are pushout squares and those involving two v's are pullback squares.

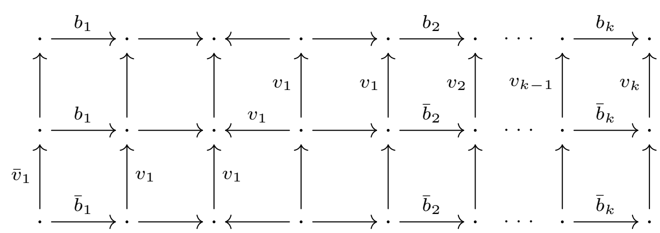
On $\boldsymbol{A}'_k$ this zigzag reduces to the zigzag



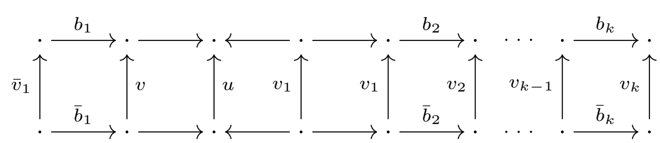
which does not completely lie inside $\boldsymbol{A}'_k$. To remedy this, i.e. to get a natural weak equivalence connecting the top with the bottom inside $\boldsymbol{A}'_k$ we note the existence of the zigzag



in which the bottom row is obtained from the top row by pushing out along $v_1 u_1$ which is an identity map. Combining the bottom halves of the last two diagrams we now get two composable natural weak equivalences



of which the composition



yields the desired natural weak equivalence between ri and $1_{\boldsymbol{A}'}$.

6.2. **The completeness part.** It thus remains to deal with the completeness part of the proof. However this is essentially the same as Rezk's proof of [R, 8.3] in view of the fact that the partial model category $(\boldsymbol{C}, \boldsymbol{W})$ is saturated (1.3).

7. A proof of the partial modelization lemma (3.3)

In preparation for the proof of lemma 3.3 (in 7.4 below) we first

- discuss in 7.1 relative *simplicial* categories and in particular relative *partly simplicial* ones in which the weak equivalences form an ordinary category, and

- review in 7.2 and 7.3 the notions of *fully faithfulness* and *essential surjectivity* and of *essential image* in the categories of categories, simplicial categories, relative categories and relative simplicial categories.

7.1. **Relative (partly) simplicial categories.** Let $\boldsymbol{S}$-**Cat** denote the category of **simplicial categories**, i.e. categories enriched over simplicial sets, and let **Rel SCat** denote the resulting category of **relative simplicial categories**, i.e. pairs consisting of a simplicial category and a sub-simplicial category (of which the maps are called **weak equivalences**) that contains all the objects. Then it turns out that, for our purposes here, it is convenient to work in the somewhat simpler full subcategory

$$\mathbf{Rel}\,\mathrm{P}\,\mathbf{SCat} \subset \mathbf{Rel}\,\mathbf{SCat}$$

spanned by what we will call the **relative partly simplicial categories**, i.e. the objects of which *the weak equivalences form an ordinary category.*

A *simplicial* model category then can be considered as

- an object of **RelCat** consisting of the underlying *model category* and its weak equivalences

or as

- an object of **Rel** P **SCat** consisting of the larger *simplicially enriched model category* and those same weak equivalences.

Moreover in the remainder of this paper **we will consider the category $\boldsymbol{S}$ of simplicial sets only as an object of Rel** P **SCat**.

An object $\boldsymbol{L} \in \boldsymbol{S}$-**Cat** thus gives rise to

- an object $(\boldsymbol{S}^{\boldsymbol{L}}, \sim) \in \mathbf{RelCat}$ in which $\boldsymbol{S}^{\boldsymbol{L}}$ denotes the (model) category which has as objects the simplicial functors $\boldsymbol{L} \to \boldsymbol{S}$ and as maps the natural transformations between them and $\sim$ denotes the subcategory of the natural weak equivalences, and
- an object $(\boldsymbol{S}^{\boldsymbol{L}}_{*}, \sim) \in \mathbf{Rel}\,\mathbf{SCat}$ in which $\boldsymbol{S}^{\boldsymbol{L}}_{*}$ denotes the simplicial (model) category of the simplicial functors $\boldsymbol{L} \to \boldsymbol{S}$ ([DK4, 1.3(v)] and [GJ, IX, 1.4]) and $\sim$ is as above.

We end with noting that similarly an object $(\boldsymbol{L}, \boldsymbol{Z}) \in \mathbf{Rel}\,\mathrm{P}\,\mathbf{SCat}$ gives rise to

- an object $(\boldsymbol{S}^{\boldsymbol{L},\boldsymbol{Z}}, \sim) \in \mathbf{RelCat}$ which is the subobject of $(\boldsymbol{S}^{\boldsymbol{L}}, \sim)$ spanned by the relative simplicial functors $(\boldsymbol{L}, \boldsymbol{Z}) \to \boldsymbol{S}$

and that

(i) *if $\boldsymbol{Z}$ is* **neglectible** *in $\boldsymbol{L}$ in the sense that every map in $\boldsymbol{Z}$ goes to an isomorphism in* $\mathrm{Ho}\,\boldsymbol{L}$*, then* $(\boldsymbol{S}^{\boldsymbol{L},\boldsymbol{Z}}, \sim) = (\boldsymbol{S}^{\boldsymbol{L}}, \sim)$*, and*

(ii) *for every object* $(\boldsymbol{C}, \boldsymbol{U}) \in \mathbf{RelCat}$*, the object* $(\boldsymbol{S}^{\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}}}, \sim) \in \mathbf{RelCat}$ *is exactly the same as the object* $\boldsymbol{S}^{\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}}}$ *mentioned in 3.2.*

7.2. **Fully faithfulness and essential surjectivity.** We will denote by L^H not only the functor $\mathbf{RelCat} \to \boldsymbol{S}$-**Cat** which sends each object to its hammock localization [DK2, 2.1], but also the functor $\mathbf{Rel}\,\mathbf{SCat} \to \boldsymbol{S}$-**Cat** which sends each object to the diagonal of the bisimplicial category obtained from it by dimensionwise application of the hammock localization [DK2, 2.5].

Then we recall the following.

A functor $f\colon \boldsymbol{G} \to \boldsymbol{H}$ between categories (respectively, simplicial categories) is called **fully faithful** if, for every two objects $G_1, G_2 \in \boldsymbol{G}$, it induces an isomorphism (resp. weak equivalence) $\boldsymbol{G}(G_1, G_2) \to \boldsymbol{H}(fG_1, fG_2)$, and similarly a relative functor $f\colon (\boldsymbol{C}, \boldsymbol{U}) \to (\boldsymbol{D}, \boldsymbol{V})$ between relative categories (resp. relative simplicial categories) is called **fully faithful** if, for every two objects $C_1, C_2 \in \boldsymbol{C}$, it induces a weak equivalence

$$L^H(\boldsymbol{C}, \boldsymbol{U})(C_1, C_2) \longrightarrow L^H(\boldsymbol{D}, \boldsymbol{V})(fC_1, fC_2) \in \boldsymbol{S}$$

which implies that

(i) *if f and g are (relative) functors such that gf is defined and g is fully faithful, then gf is fully faithful iff f is so.*

A functor $f\colon \boldsymbol{G} \to \boldsymbol{H}$ between categories (respectively, simplicial categories) is called **essentially surjective** if every object in $\boldsymbol{H}$ is isomorphic in $\boldsymbol{H}$ (resp. $\operatorname{Ho}\boldsymbol{H}$) to an object in the image of f (resp. $\operatorname{Ho} f$), and similarly a relative functor $f\colon (\boldsymbol{C}, \boldsymbol{U}) \to (\boldsymbol{D}, \boldsymbol{V})$ between relative categories (resp. relative simplicial categories) is called **essentially surjective** if the induced functor

$$L^H f\colon L^H(\boldsymbol{C}, \boldsymbol{U}) \longrightarrow L^H(\boldsymbol{D}, \boldsymbol{V})$$

is so, which implies that

(ii) *if f and g are (relative) functors such that gf is defined and f is essentially surjective, then gf is essentially surjective iff g is so.*

Then

(iii) *a map in* **Cat** *is an equivalence of categories iff it is fully faithful and essentially surjective, and*

(iv) *a map in* **RelCat**, $\boldsymbol{S}$-**Cat** *or* **Rel SCat** *is a DK-equivalence iff it is fully faithful and essentially surjective.*

7.3. **Essential images.** The **essential image** Ef of a functor $f\colon \boldsymbol{G} \to \boldsymbol{H}$ between categories (respectively, simplicial categories) is the full subcategory (resp. full simplicial subcategory) of $\boldsymbol{H}$ spanned by the objects which are isomorphic in $\boldsymbol{H}$ (resp. $\operatorname{Ho}\boldsymbol{H}$) to objects in the image of f (resp. $\operatorname{Ho} f$) and similarly the **essential image** Ef of a relative functor $f\colon (\boldsymbol{C}, \boldsymbol{U}) \to (\boldsymbol{D}, \boldsymbol{V})$ between relative categories (resp. relative simplicial categories) is defined by the pullback diagram

$$\begin{array}{ccc} Ef & \longrightarrow & (\boldsymbol{D}, \boldsymbol{V}) \\ \downarrow & & \downarrow \\ EL^H f & \longrightarrow & L^H(\boldsymbol{D}, \boldsymbol{V}) \end{array}$$

which implies that

(i) *the resulting maps*

$$\boldsymbol{G} \longrightarrow Ef \qquad \text{and} \qquad (\boldsymbol{C}, \boldsymbol{U}) \longrightarrow Ef$$

and

$$Ef \longrightarrow \boldsymbol{H} \qquad \text{and} \qquad Ef \longrightarrow (\boldsymbol{D}, \boldsymbol{V})$$

are respectively essentially surjective and fully faithful.

We end with noting that

(ii) *the essential image defined in* **??** *is a special case of the ones defined above.*

Now we are finally ready for

7.4. **A proof of the relative modelization lemma** (3.3)**.** It follows from 7.3(i) and (ii) that the map $e\colon (\boldsymbol{C},\boldsymbol{U}) \to Ef$ is essentially surjective and it thus (7.2(iv)) remains to prove that it is also fully faithful. To do this it suffices, in view of 7.3(ii) and 7.2(i), to show that, in the notation of 7.1(ii),

(i) the Yoneda embedding $y\colon (\boldsymbol{C},\boldsymbol{U}) \to (\boldsymbol{S}^{\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}}}, \sim)$ is fully faithful.

For this we note that y admits a factorization (7.1)

$$(\boldsymbol{C},\boldsymbol{U}) \xrightarrow{y'} (\boldsymbol{S}^{L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}})}, \sim) \overset{7.1(i)}{=} (\boldsymbol{S}^{L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}}),\boldsymbol{U}^{\mathrm{op}}}, \sim) \xrightarrow{(c^{\mathrm{op}})^*} (\boldsymbol{S}^{\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}}}, \sim)$$

in which y' sends each object $A \in \boldsymbol{C}$ to the simplicial functor $L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}}) \to \boldsymbol{S}$ which sends each object $B \in L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}})$ to $L^H(B,A) \in \boldsymbol{S}$, and $c\colon (\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}}) \to L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}}),\boldsymbol{U}^{\mathrm{op}}$ is the obvious inclusion [DK2, 3.1]. The latter is a DK-equivalence [BK1, 3.2] and hence [DK4, 2.2] so is the map $(c^{\mathrm{op}})^*$. Hence, in view of 7.2(iv) and 7.2(i), the condition (i) above is equivalent to condition

(ii) the map $y'\colon (\boldsymbol{C},\boldsymbol{U}) \to (\boldsymbol{S}^{L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}})}, \sim)$ is fully faithful.

To prove this we embed this map in the commutative diagram

$$\begin{array}{ccc} (\boldsymbol{C},\boldsymbol{U}) & \xrightarrow{\quad y' \quad} & (\boldsymbol{S}^{L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}})}, \sim) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \text{incl.}} \\ \big(L^H(\boldsymbol{C},\boldsymbol{U}),\boldsymbol{U}\big) & \xrightarrow{\quad r' \quad} & (\boldsymbol{S}_*^{L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}})}, \sim) \end{array}$$

in which the map in the right is as in 7.1 and r' is induced by the *simplicial Yoneda embedding* of [DK4, 1.3(vi)]

$$r\colon L^H(\boldsymbol{C},\boldsymbol{U}) \longrightarrow \boldsymbol{S}_*^{L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}})} \in \boldsymbol{S}\text{-}\mathbf{Cat}$$

which sends each object $A \in L^H(\boldsymbol{C},\boldsymbol{U})$ to the simplicial functor $L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}}) \to \boldsymbol{S}$ which sends each object $B \in L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}})$ to $L^H(B,A) \in \boldsymbol{S}$. The map on the left is (see above) a DK-equivalence and so is, in view of [DK3, 4.8] the map on the right and hence, to prove (ii), it suffices to show that the bottom map is fully faithful.

For this we embed this map in the following diagram

$$\begin{array}{ccc} L^H\big(L^H(\boldsymbol{C},\boldsymbol{U}),\boldsymbol{U}\big) & \xrightarrow{\quad L^H r' \quad} & L^H(\boldsymbol{S}_*^{L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}})}, \sim) \\ \uparrow & & \uparrow \\ L^H(\boldsymbol{C},\boldsymbol{U}) & \xrightarrow{\quad r \quad} & \boldsymbol{S}_*^{L^H(\boldsymbol{C}^{\mathrm{op}},\boldsymbol{U}^{\mathrm{op}})} \end{array}$$

in which the vertical maps are the obvious inclusions [DK2, 3.1]. As both categories of weak equivalences are neglectible 7.1(i), it follows from [DK1, 6.4] that both vertical maps are DK-equivalences. Moreover it was noted in [DK4, 1.3(vi)] that the map r is fully faithful (and in fact so in the strong sense that the required weak equivalences are actually isomorphisms). All this implies that $L^H r'$ is fully faithful and so is therefore the map r' itself.

References

[BK1] C. Barwick and D. M. Kan, *Relative categories: another model for the homotopy theory of homotopy theories*, Indag. Math. (N.S.) **23** (2012), no. 1-2, 42–68.

[BK1] ———, *A characterization of simplicial localization functors and a discussion of DK-equivalences*, Indag. Math. (N.S.) **23** (2012), no. 1-2, 69–79.

[BK4] C. Barwick and D. M Kan, *A Quillen Theorem B_n for homotopy pullbacks of (∞, k)-categories*, To appear.

[Be] J. E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 2043–2058.

[DHKS] W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, and J. H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, Providence, RI, 2004.

[DK1] W. G. Dwyer and D. M. Kan, *Simplicial localizations of categories*, J. Pure Appl. Algebra **17** (1980), no. 3, 267–284.

[DK2] ———, *Calculating simplicial localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35.

[DK3] ———, *Function complexes in homotopical algebra*, Topology **19** (1980), no. 4, 427–440.

[DK4] ———, *Equivalences between homotopy theories of diagrams*, Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), Ann. of Math. Stud., vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 180–205.

[DKS] W. G. Dwyer, D. M. Kan, and J. H. Smith, *Homotopy commutative diagrams and their realizations*, J. Pure Appl. Algebra **57** (1989), no. 1, 5–24.

[DS] D. Dugger and D. Spivak, *Mapping spaces in quasi-categories*, Algebr. Geom. Topol. **11** (2011), no. 1, 263–325.

[GJ] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999.

[HTT] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, 2009.

[Q] D. Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.

[R] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007 (electronic).

[T1] R. W. Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Cambridge Philos. Soc. **85** (1979), no. 1, 91–109.

[T2] ———, *Cat as a closed model category*, Cahiers Topologie Géom. Différentielle **21** (1980), no. 3, 305–324.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139

E-mail address: clarkbar@math.mit.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139