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On the Tate spectrum of tmf at the prime 2

Scott M. Bailey
Nicolas Ricka

Computations involving the root invariant prompted Mahowald and Shick to develop the slogan: "the root invariant of v_n-periodic homotopy is v_n-torsion." While neither a proof, nor a precise statement, of this slogan appears in the literature, numerous authors have offered computational evidence in support of its fundamental idea. The root invariant is closely related to Mahowald's inverse limit description of the Tate spectrum, and computations have shown the Tate spectrum of v_n-periodic cohomology theories to be v_n-torsion. The purpose of this paper is to split the Tate spectrum of tmf as a wedge of suspensions of kO, providing yet another example in support of the slogan to the existing literature.

55P42; 55T15

1 Introduction

Let λ denote the canonical line bundle over $\mathbb{R}P^\infty = B(\mathbb{Z}/2\mathbb{Z})$. For $\ell \in \mathbb{Z}$, define P_ℓ to be the Thom spectrum of $\ell\lambda$. Induced maps on the level of Thom spectra give a naturally defined inverse system of projective spaces

$$\cdots \to P_{n-1} \xrightarrow{\jmath_{n-1}} P_n \xrightarrow{\jmath_n} P_{n+1} \to \dots \tag{1}$$

W.H. Lin [8] demonstrated that the homotopy limit of (1) has the homotopy type of a desuspended 2-complete sphere, i.e.,

$$\varprojlim P_n \simeq \hat{S}^{-1} \tag{2}$$

Suppose X is a finite complex and consider a cohomotopy class $\alpha \in [X, S^0]_j$. The equivalence (2) guarantees the existence of a largest $\ell \in \mathbb{Z}$ such that the composite $\Sigma^{j-1}X \xrightarrow{\alpha} S^{-1} \to P_\ell$ is nontrivial. In particular, this induces a map $R(\alpha) : \Sigma^{j-1}X \to S^\ell$, the homotopy class of which is called the root invariant of α. Computations inside the EHP sequence led Mahowald and Ravenel [13] to conjecture that the root invariant of a v_n-periodic element is v_{n+1}-periodic. This prompted Mahowald and Shick [14] to discuss the related slogan: "The root invariant of v_n-periodic homotopy is v_n-torsion." They show, for a finite complex X having a v_n-self map, that

$$\varprojlim \left([X, P_\ell](v_n^{-1})\right) = 0 \tag{3}$$

In particular, Mahowald and Shick point out that if $\alpha \in [X, S^0]$ is v_n-periodic then its root invariant, at least when considered as an element of $[X, P_\ell]$, is v_n-torsion.

Let X be a spectrum. Mahowald's description of the Tate spectrum of X is the homotopy inverse limit

$$t(X) = \Sigma \varprojlim (P_\ell \wedge X) \tag{4}$$

If X is a finite spectrum, then (4) implies the Tate spectrum functor corresponds to completion at the prime 2. This is certainly not the case for all spectra since homotopy limits do not commute with the smash product.

While neither a proof, nor even a precise statement of the phenomenon suggested by the slogan has appeared in the literature, many authors have demonstrated the Tate spectrum functor sends v_n-periodic cohomology theories to v_{n-1}-periodic theories:

1984: Davis and Mahowald [5], for $p = 2$, show $t(\mathrm{kO}) \simeq \bigvee_{j\in\mathbb{Z}} \Sigma^{4j}\widehat{H\mathbb{Z}}$;

1986: Davis, Johnson, Klippenstein, Mahowald, and Wegmann [4] demonstrate that, if p any prime and $q = 2(p-1)$, then there are equivalences of p-complete spectra $t(BP\langle 2\rangle) \simeq \prod_{j\in\mathbb{Z}} \Sigma^{qj}\widehat{BP}\langle 1\rangle$, and conjecture a similar splitting of $t(BP\langle n\rangle)$;

1998: Ando, Morava, and Sadofsky [1] prove the existence of a ring isomorphism $(tE(n)_*)^\wedge_{I_{n-1}} \cong E(n-1)_*((x))^\wedge_{I_{n-1}}$ where $I_{n-1} = (p, v_1, \ldots, v_{n-2})$ and construct a map of spectra $\bigvee_{j\in\mathbb{Z}} \Sigma^{2j}E(n-1) \to tE(n)^\wedge_{I_{n-1}}$ which, after completion at I_{n-1} (or equivalently after localization with respect to the $(n-1)$st Morava K-theory) induces the isomorphism of homotopy groups.

The purpose of this paper is to provide yet another example to the literature. Let tmf denote the connective ring spectrum of topological modular forms (see [2, 6, 10]) at the prime 2. The main theorem is:

Theorem 1.1 *There is a weak equivalence of spectra*

$$t(\mathrm{tmf}) \simeq \prod_{i\in\mathbb{Z}} \Sigma^{8i}\mathrm{kO}, \tag{5}$$

In the context of the above machinery, computations involving the homotopy of $t(\mathrm{tmf})$ greatly benefit from Mahowald's inverse limit description of the Tate spectrum. However, the Tate spectrum functor conserves other properties, such as ring structure, of the spectrum. This fact, however, is not immediately clear from the inverse limit point of view. On the other hand, such a structure is clear when placed in the framework established by Greenlees and May [7]. In their notation: let G be a compact Lie group, EG a free contractible G-space and $\tilde{E}G$ the cofiber of the map $EG_+ \to S^0$. If k_G is a G-spectrum, then $\mathbf{t}(k_G) = F(EG_+, k_G) \wedge \tilde{E}G$, where $F(EG_+, k_G)$ is the function G-spectrum of maps $EG_+ \to k_G$, is the Tate spectrum of k_G. Since EG_+ is equipped with a coproduct, if k_G is a ring spectrum then $F(EG_+, k_G)$ is also a ring spectrum. Combining this with the product on $\tilde{E}G$, $\mathbf{t}(k_G)$ is also a ring. Lewis-May fixed points give a lax monoidal functor, so $\mathbf{t}(k_G)^G$ also has a ring structure.

The link to Mahowald's inverse limit description is as follows [7]: If G is cyclic of order 2 and k_G is the equivariant G-spectrum associated to a non-equivariant spectrum k, then there is a homotopy equivalence $\mathbf{t}(k_G)^G \simeq \Sigma \varprojlim (P_\ell \wedge k) = t(k)$.

With the above correspondence in mind, we can restate Theorem 1.1 as

Theorem 1.2 *There is a weak equivalence of ring spectra*

$$t(\mathrm{tmf}) \simeq \mathrm{kO}[x^{\pm 1}] \tag{6}$$

where x is degree 8.

2 Some particular $\mathcal{A}(2)$-modules

Let $\mathcal{A}$ denote the mod-2 Steenrod algebra generated by the squaring operations $\{Sq^{2^i}\}_{i \geq 0}$. Let M be an $\mathcal{A}$-module, and consider the $\mathcal{A}$-modules $\mathcal{A}/\!/\mathcal{A}(n) \otimes M$ via the diagonal action and $\mathcal{A} \otimes_{\mathcal{A}(n)} M$ via left action on $\mathcal{A}$. There is an isomorphism

$$\Phi : \mathcal{A}/\!/\mathcal{A}(n) \otimes M \to \mathcal{A} \otimes_{\mathcal{A}(n)} M \tag{7}$$

defined by $\Phi(a \otimes m) = \sum a' \otimes a''m$ where $\psi(a) = \sum a' \otimes a''$ is the coproduct on $\mathcal{A}$. This isomorphism induces a change-of-rings isomorphism on the level of Ext-groups

$$\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathcal{A}/\!/\mathcal{A}(n) \otimes M, N) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{A}(n)} M, N) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(n)}(M, N) \tag{8}$$

which is often invoked to simplify computations within the Adams sepctral sequence E_2-term. For instance, since $H^*(\mathrm{tmf}) \cong \mathcal{A}/\!/\mathcal{A}(2)$ [15, 12], to compute the homotopy groups of $P_i \wedge \mathrm{tmf}$ it suffices to understand the left $\mathcal{A}(2)$-module structure of H^*P_i.

The following two propositions are results of Lin, Davis, Mahowald, and Adams [9].

Proposition 2.1 *As a $\mathbb{F}_2$-vector space, $H^*P_i = x^i\mathbb{F}_2[x]$, where x is in degree 1. The action of $\mathcal{A}$ on H^*P_i is determined by*

$$Sq^j x^k = \binom{k}{j} x^{j+k} \tag{9}$$

Denote by $\mathbb{F}_2[x^{\pm 1}]$ the colimit of these $\mathcal{A}$-modules. Note that this is certainly not the cohomology of the limit given by (2). A consequence is the following interpretation of the $\mathcal{A}(2)$-module structure of some quotients of $\mathbb{F}_2[x^{\pm 1}]$:

Proposition 2.2 *Let F_ℓ be the sub-$\mathcal{A}(2)$-module of $\mathbb{F}_2[x^{\pm 1}]$ generated by the classes in degree less than ℓ. Then there is an isomorphism of $\mathcal{A}(2)$-modules*

$$\mathbb{F}_2[x^{\pm 1}]/F_\ell \cong \bigoplus_{j \geq \frac{\ell+1}{8}} \Sigma^{8j-1}\mathcal{A}(2)/\!/\mathcal{A}(1) \tag{10}$$

Definition 2.3 Let L_0 denote the spectrum $(S^1 \cup_2 e^2 \cup_\eta e^4 \cup_\nu e^8)_+$. By construction, $H^*L_0 = \mathbb{F}_2\{1, x, Sq^1x, Sq^2Sq^1x, Sq^4Sq^2Sq^1x\}$, with the action of $\mathcal{A}$ indicated by the names of the elements.

Proposition 2.4 *There is a filtration of $\mathcal{A}(2)$-modules of H^*P_{-1} with associated graded $H^*L_0 \oplus \bigoplus_{j\geq 0} \Sigma^{8j-1}\mathcal{A}(2)/\!/\mathcal{A}(1)$.*

Proof Note that there is an isomorphism of $\mathcal{A}(2)$-modules $\mathbb{F}_2[x^{\pm 1}] \cong \big(\mathbb{F}_2[x^{\pm 1}]/F_{-1}\big) \oplus F_{-1}$ so that $H^*P_{-1} \cong \big(\mathbb{F}_2[x^{\pm 1}]/F_{-1}\big) \oplus (F_{-1} \cap H^*P_{-1})$. By construction $H^*L_0 \cong F_{-1} \cap H^*P_{-1}$, hence Proposition 2.2 yields the result. ◻

Proposition 2.5 *There is a* tmf *-module map $\iota : \mathrm{tmf} \wedge P_{-1} \to \mathrm{tmf} \wedge L_0$ realizing the inclusion $\bar\iota : H^*L_0 \to H^*P_{-1}$ of $\mathcal{A}(2)$-modules. Explicitly,*

$$H^*\iota : \mathcal{A}/\!/\mathcal{A}(2) \otimes H^*L_0 \to \mathcal{A}/\!/\mathcal{A}(2) \otimes H^*P_{-1}$$

is $\mathcal{A}/\!/\mathcal{A}(2) \otimes \bar\iota$.

Proof By adjunction, it suffices to show that the corresponding map of spectra $P_{-1} \to \mathrm{tmf} \wedge L_0$ survives the Adams spectral sequence. By the change-of-rings isomorphism (8) the Adams E_2-page computing the desired homotopy classes is

$$E_2 \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(H^*L_0, x^{-1}\mathbb{F}_2[x]) \tag{11}$$

One concludes the argument by observing that the relevant extension group vanishes whenever $t-s=-1$. Indeed, there is a spectral sequence

$$E_1 \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(H^*L_0, H^*L_0) \oplus \bigoplus_{j\geq 0} \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(H^*L_0, \Sigma^{8j-1}\mathcal{A}(2)/\!/\mathcal{A}(1)) \tag{12}$$

which abuts to the E_2-term (11). It then suffices to check there is nothing in degree $t-s=-1$ in (12).

The first summand, $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(H^*L_0, H^*L_0)$, requires a direct computation. The relevant Ext chart is displayed in Figure 1 clearly has no classes in stem $t-s=-1$.

The observant reader will note that, as an $\mathcal{A}(2)$-module, the vector space dual of $\mathcal{A}(2)/\!/\mathcal{A}(1)$ is $\Sigma^{-17}\mathcal{A}(2)/\!/\mathcal{A}(1)$. In particular, by adjunction and change-of-rings, the second summand is isomorphic to $\bigoplus_{j\geq 0}\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\Sigma^{-8(j+2)}H^*L_0, \mathbb{F}_2)$. This is a straightforward computation in $\mathcal{A}(1)$-modules, since $H^*L_0 = \mathbb{F}_2 \oplus \Sigma QM \oplus \Sigma^8\mathbb{F}_2$ where QM is the question mark complex, yielding the E_2-term of $\mathrm{kO} \vee \mathrm{kO}\langle 1\rangle \vee \Sigma^8\mathrm{kO}$ which are all 8-periodic. Figure 2 displays $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*L_0, \mathbb{F}_2)$, and shows there cannot be a class in stem $t-s=-1$ for degree reasons. □

Definition 2.6 Choose a tmf-module map π satisfying the hypothesis of Proposition 2.5. Let $t(\mathrm{tmf})_{-1}$ be the fiber of π.

3 The Tate spectrum of tmf

The proof of Theorem 1.1 is decomposed into a series of lemmas. The argument presented here is similar to that given for the splitting of $t(\mathrm{kO})$ [5]. In what follows, we denote $\Sigma t(\mathrm{tmf})_{-1}$ by $\overline{t(\mathrm{tmf})}$. We will show that $\overline{t(\mathrm{tmf})}$ splits as a wedge of copies of kO. This is done in two steps: first, we show that the cohomology of $\overline{t(\mathrm{tmf})}$ splits as a module over $\mathcal{A}$ in Lemma 3.2; second, we compute the E_2-page of the Adams spectral sequence converging to $[\overline{t(\mathrm{tmf})}, \mathrm{kO}]$ to show that there are classes in $[\overline{t(\mathrm{tmf})}, \Sigma^{8j}\mathrm{kO}]$ for all $j \in \mathbb{Z}$ that realize the splitting (this is done in Lemma 3.4). The main result we need is the following computation of Mahowald [11]:

Lemma 3.1 *The stable $\mathcal{A}(1)$-module $\mathcal{A}/\!/\mathcal{A}(1)$ splits (in the stable category of $\mathcal{A}(1)$-modules) as*

$$\bigoplus_{\ell\geq 0}\Sigma^{12\ell+\sigma(\ell)}\Omega^{4\ell-\sigma(\ell)}\mathbb{F}_2\oplus\bigoplus_{\ell\geq 0}\Sigma^{12\ell+\sigma(\ell)+4}\Omega^{4\ell-\sigma(\ell)}\Lambda(Sq^2).$$

Note that, in particular, this gives that

$$\begin{aligned}\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\mathbb{F}_2,\mathcal{A}/\!/\mathcal{A}(1))\cong&\bigoplus_{\ell\geq 0}\mathrm{Ext}^{s+4\ell-\sigma(\ell),t+12\ell-\sigma(\ell)}_{\mathcal{A}(1)}(\mathbb{F}_2,\mathbb{F}_2)\oplus\\&\bigoplus_{\ell\geq 0}\mathrm{Ext}^{s+4\ell-\sigma(\ell),t+12\ell-\sigma(\ell)+4}_{E(1)}(\mathbb{F}_2,\mathbb{F}_2)\end{aligned}$$

for $s>0$, and these groups are zero as soon as $t-s\equiv 3 \mod 4$.

Lemma 3.2 *There is an isomorphism of $\mathcal{A}$-modules $H^*\overline{t(\mathrm{tmf})}\cong\mathcal{A}/\!/\mathcal{A}(1)[x^8]$ where x^8 is in degree 8.*

Proof By Proposition 2.4, Proposition 2.5, and Definition 2.6 there is a filtration of $H^*\overline{t(\mathrm{tmf})}$ whose associated graded is $\bigoplus_{j\geq 0}\Sigma^{8j-1}\mathcal{A}(2)/\!/\mathcal{A}(1)$. To conclude, we show inductively that there are no non-trivial extensions

$$\mathrm{Ext}^{1,0}_{\mathcal{A}}\left(\bigoplus_{j=0}^{n}\Sigma^{8j}\mathcal{A}/\!/\mathcal{A}(1),\mathcal{A}/\!/\mathcal{A}(1)\right)=\mathrm{Ext}^{1,t+8}_{\mathcal{A}(1)}(\mathbb{F}_2,\mathcal{A}/\!/\mathcal{A}(1)[x^8]/(x^n))$$

By Lemma 3.1, this Ext vanishes whenever $t-s\equiv -1 \mod 4$. The degrees in which we are looking for non-trivial extensions are of the form $(1,8j)$. The result follows. □

Note that, in particular, the Adams spectral sequence computing $\pi_*(\overline{t(\mathrm{tmf})})$ gives homotopy classes $x^{8j}\in\pi_{8j}(\overline{t(\mathrm{tmf})})$. Indeed, the E_2-page of this spectral sequence is given by

$$E_2=\mathrm{Ext}_{\mathcal{A}}(H^*\overline{t(\mathrm{tmf})},\mathbb{F}_2)\cong\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2[x^8],\mathbb{F}_2)\tag{13}$$

and Lemma 3.1 ensures that the elements of x^{8j} survive the Adams spectral sequence for degree reasons.

Lemma 3.3 *We can arrange the cofiber sequences*

$$\Sigma^{-8k}t(\mathrm{tmf})_{-1}\to\mathrm{tmf}\wedge P_{-1-8k}\to\Sigma^{-8k}\mathrm{tmf}\wedge L_0\tag{14}$$

in a commutative diagram

$$
\begin{tikzcd}
\Sigma^{-8k}\overline{t(\mathrm{tmf})} \arrow[r] \arrow[d] & \Sigma\mathrm{tmf}\wedge P_{-1-8k} \arrow[r] \arrow[d] & \Sigma^{-8k+1}\mathrm{tmf}\wedge L_0 \arrow[d] \\
\Sigma^{-8k+8}\overline{t(\mathrm{tmf})} \arrow[r] & \Sigma\mathrm{tmf}\wedge P_{7-8k} \arrow[r] & \Sigma^{-8k+9}\mathrm{tmf}\wedge L_0
\end{tikzcd}
\tag{15}
$$

Moreover, $t(\mathrm{tmf})$ *is the limit of both the leftmost and middle terms.*

Proof The existence of the left commutative square in (15) comes from the identification $\Sigma^8\mathrm{tmf}\wedge P_n \simeq \mathrm{tmf}\wedge P_{n+8}$ for all $n\in\mathbb{Z}$ [3]. This gives the asserted compatibility between the cofiber sequences.

We now take the inverse limit on the three terms. By definition, the limit of the middle term is $\Sigma^{-1}t(\mathrm{tmf})$. We need to show that the map $\varprojlim \Sigma^{-1}\mathrm{tmf}\wedge P_{-1-8k}\to t(\mathrm{tmf})$ is a weak equivalence. Note that the composite of the right-most vertical maps in (15) is zero, since it belongs to $[L_0,\mathrm{tmf}\wedge L_0]_{-16}=0$, by adjunction. Thus the limit is contractible. $\square$

In light of Lemma 3.3, we reduce our analysis of $t(\mathrm{tmf})$ to $\overline{t(\mathrm{tmf})}$. This, in turn, reduces to a simple Adams spectral sequence computation.

Lemma 3.4 *There is a weak equivalence* $\overline{t(\mathrm{tmf})}\simeq \mathrm{kO}[x^8]$. *Moreover, the maps* $\Sigma^{-8k}\overline{t(\mathrm{tmf})}\to\Sigma^{-8k+8}\overline{t(\mathrm{tmf})}$ *coincide with multiplication by* x^8.

Proof We argue as follows: we build a map $\phi_j:\overline{t(\mathrm{tmf})}\to\Sigma^{8j}\mathrm{kO}$ for all $j\geq 0$, which realizes the injection $\phi_j^*:\Sigma^{8j}\mathcal{A}/\!/\mathcal{A}(1)\to\mathcal{A}/\!/\mathcal{A}(1)[x^8]$. Then the coproduct of the ϕ_j will be the desired weak equivalence.

First, fix $j\geq 0$ and build the map ϕ_j. To this end, we compute the Adams spectral sequence converging to $[\overline{t(\mathrm{tmf})},\Sigma^{8j}\mathrm{kO}]$. Its E_2-page is $\mathrm{Ext}_{\mathcal{A}}^{s,t+8j}(\mathcal{A}/\!/\mathcal{A}(1),\mathcal{A}/\!/\mathcal{A}(1))[x^{\pm 8}]$. In particular, Lemma 3.1 guarantees an element in $(s,t+8j)=(0,8j)$ cannot be the source of a differential for degree reasons. This precisely means that any $\mathcal{A}$-module map $\Sigma^{8j}H^*\mathrm{kO}\to H^*\overline{t(\mathrm{tmf})}$ comes from a map $\overline{t(\mathrm{tmf})}\to\Sigma^{8j}\mathrm{kO}$. Choose one map corresponding to the inclusion of $\Sigma^{8j}\mathcal{A}/\!/\mathcal{A}(1)$ into $\mathcal{A}/\!/\mathcal{A}(1)[x^{\pm 8}]$ and call it ϕ_j.

Let $\phi:\overline{t(\mathrm{tmf})}\to\mathrm{kO}[x^8]$ be the wedge of the above ϕ_j for $j\geq 0$. By construction, it induces an isomorphism in cohomology between connective spectra, and thus it is a (2-local) equivalence.

The assertion about the maps $\Sigma^{-8k}\overline{t(\mathrm{tmf})}\to\Sigma^{-8k+8}\overline{t(\mathrm{tmf})}$ follows from its compatibility with the decomposition $\mathrm{Ext}_{\mathcal{A}}(\mathcal{A}/\!/\mathcal{A}(1)\otimes\mathcal{A}/\!/\mathcal{A}(1),\mathbb{F}_2)[x^8]$. $\square$

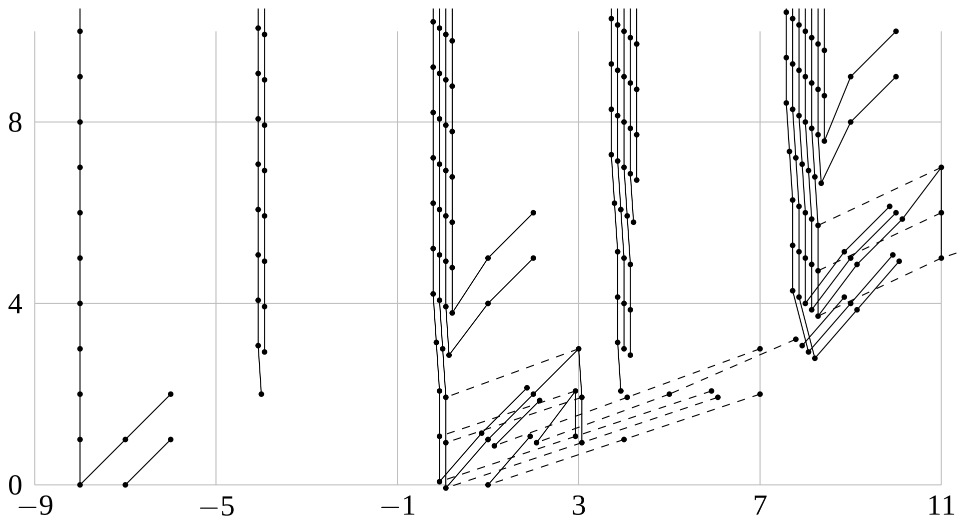

Figure 1: $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(H^*L_0, H^*L_0)$

Proof of Theorem 1.1 By Lemma 3.3, $t(\mathrm{tmf}) \simeq \varprojlim \Sigma^{-8k}\overline{t(\mathrm{tmf})}$ which, by Lemma 3.4, is weakly equivalent to $\varprojlim \Sigma^{-8k}\mathrm{kO}[x^8] \simeq \mathrm{kO}[x^{\pm 8}]$. □

References

[1] **M Ando**, **J Morava**, **H Sadofsky**, *Completions of* $\mathbf{Z}/(p)$*-Tate cohomology of periodic spectra*, Geom. Topol. 2 (1998) 145–174 (electronic)

[2] **M Behrens**, *Notes on the construction of tmf*, manuscript available at `http://math.mit.edu/~mbehrens/papers/index.html` (2007)

[3] **R R Bruner**, **D M Davis**, **M Mahowald**, *Nonimmersions of real projective spaces implied by tmf*, from: "Recent progress in homotopy theory (Baltimore, MD, 2000)", Contemp. Math. 293, Amer. Math. Soc., Providence, RI (2002) 45–68

[4] **D M Davis**, **D C Johnson**, **J Klippenstein**, **M Mahowald**, **S Wegmann**, *The spectrum* $(P \wedge \mathrm{BP}\langle 2\rangle)_{-\infty}$, Trans. Amer. Math. Soc. 296 (1986) 95–110

[5] **D M Davis**, **M Mahowald**, *The spectrum* $(P \wedge bo)_{-\infty}$, Math. Proc. Cambridge Philos. Soc. 96 (1984) 85–93

[6] **P G Goerss**, *Topological modular Forms [after Hopkins, Miller, and Lurie]*Article to accompany Bourbaki presentation of March 14, 2009, manuscript available at `http://www.math.northwestern.edu/~pgoerss`

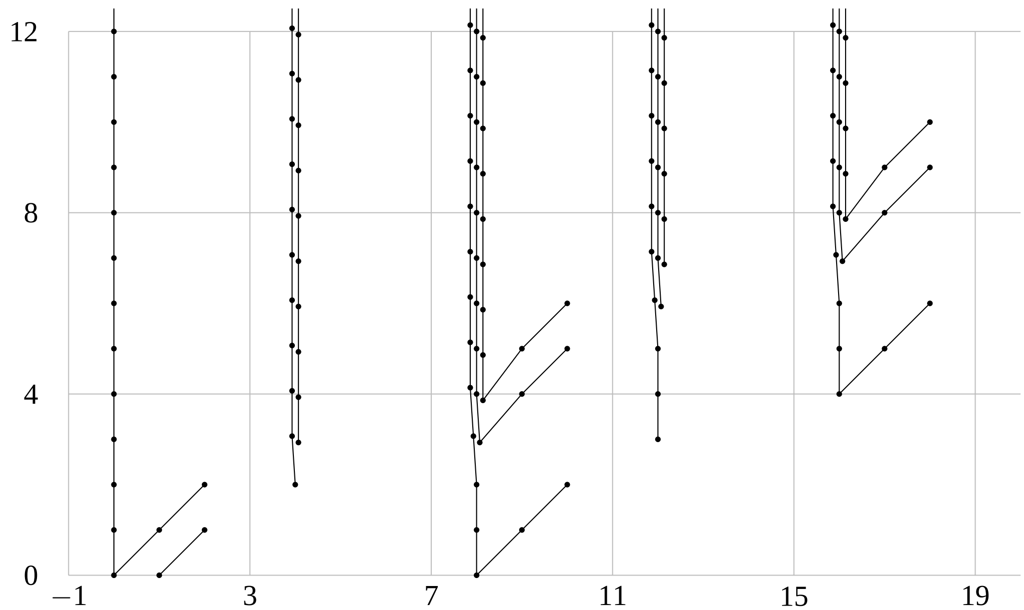

Figure 2: $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*L_0, \mathbb{F}_2)$

[7] **J Greenlees**, **J May**, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 113 (1995) viii+178

[8] **W H Lin**, *On conjectures of Mahowald, Segal and Sullivan*, Math. Proc. Cambridge Philos. Soc. 87 (1980) 449–458

[9] **W H Lin**, **D M Davis**, **M E Mahowald**, **J F Adams**, *Calculation of Lin's Ext groups*, Math. Proc. Cambridge Philos. Soc. 87 (1980) 459–469

[10] **J Lurie**, *Survey article on elliptic cohomology*, manuscript 2007, available at `http://www-math.mit.edu/~lurie/`

[11] **M Mahowald**, *bo-resolutions*, Pacific J. Math. 92 (1981) 365–383

[12] **M Mahowald**, **M Hopkins**, *From elliptic curves to homotopy theory*, manuscript available at `http://hopf.math.purdue.edu/` (1998)

[13] **M E Mahowald**, **D C Ravenel**, *Toward a global understanding of the homotopy groups of spheres*, from: "The Lefschetz centennial conference, Part II (Mexico City, 1984)", Contemp. Math. 58, Amer. Math. Soc., Providence, RI (1987) 57–74

[14] **M Mahowald**, **P Shick**, *Root invariants and periodicity in stable homotopy theory*, Bull. London Math. Soc. 20 (1988) 262–266

[15] **A Mathew**, *The homology of* tmf, manuscript available at `http://arxiv.org/abs/1305.6100` (2013)

Clayton State University, Department of Mathematics, 2000 Clayton State Blvd., Morrow, GA 30260

Wayne State University, Department of Mathematics, 656 W. Kirby, Detroit, MI 48202

ScottBailey@clayton.edu, nicolas.ricka@wayne.edu

http://faculty.clayton.edu/sbailey21,
https://sites.google.com/site/nicoricka/