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Abstract

Let S be a Noetherian scheme of finite dimension and denote by $\rho \in [\mathbb{1}, \mathbb{G}_m]_{\mathbf{SH}(S)}$ the (additive inverse of the) morphism corresponding to $-1 \in \mathcal{O}^\times(S)$. Here $\mathbf{SH}(S)$ denotes the motivic stable homotopy category. We show that the category obtained by inverting ρ in $\mathbf{SH}(S)$ is canonically equivalent to the (simplicial) local stable homotopy category of the site $S_{\text{rét}}$, by which we mean the *small* real étale site of S, comprised of étale schemes over S with the real étale topology. One immediate application is that $\mathbf{SH}(\mathbb{R})[\rho^{-1}]$ is equivalent to the classical stable homotopy category. In particular this computes all the stable homotopy sheaves of the ρ-local sphere (over $\mathbb{R}$). As further applications we show that $D_{\mathbb{A}^1}(k, \mathbb{Z}[1/2])^- \simeq \mathbf{DM}_W(k)[1/2]$ (improving a result of Ananyevskiy–Levine–Panin), reprove Röndigs' result that $\underline{\pi}_i(\mathbb{1}[1/\eta, 1/2]) = 0$ for $i = 1, 2$ and establish some new rigidity results.

Contents

1. Introduction

For a scheme S we denote by $\mathbf{SH}(S)$ the motivic stable homotopy category [MV99, Ayo07]. We recall that this is a triangulated category which is the homotopy category of a stable model category that (roughly) is obtained from the homotopy theory of (smooth, pointed) schemes by making the 'Riemann sphere' $\mathbb{P}^1_S$ into an invertible object.

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If $\alpha : k \hookrightarrow \mathbb{C}$ is an embedding of a field k into the complex numbers, then we obtain a complex realisation functor $R_{\alpha,\mathbb{C}} : \mathbf{SH}(k) \to \mathbf{SH}$ (where now $\mathbf{SH}$ denotes the classical stable homotopy category) connecting the world of motivic stable homotopy theory to classical stable homotopy theory [MV99, § 3.3.2]. This functor is induced from the functor which sends a smooth scheme S over k to its topological space of complex points $S(\mathbb{C})$ (this depends on α). Similarly if $\beta : k \hookrightarrow \mathbb{R}$ is an embedding into the real numbers, then there is a real realisation functor $R_{\beta,\mathbb{R}} : \mathbf{SH}(k) \to \mathbf{SH}$ induced from $S \mapsto S(\mathbb{R})$ [MV99, § 3.3.3] [HO16, Proposition 4.8].

These functors serve as a good source of inspiration and a convenient test of conjectures in stable motivic homotopy theory. For example, in order for a morphism $f : E \to F$ to be an equivalence it is necessary that $R_{\alpha,\mathbb{C}}(f)$ and $R_{\beta,\mathbb{R}}(f)$ are equivalences, for all such embeddings α, β. On the other hand, this criterion is clearly not sufficient; there are fields without any real or complex embeddings!

It is thus a very natural question to ask how far these functors are from being an equivalence, or what their 'kernel' is. The aim of this article is to give some kind of complete answer to this question in the case of real realisation. We begin with the simplest formulation of our result. Write $R_\mathbb{R}$ for the (unique) real realisation functor for the field $k = \mathbb{R}$. The first clue comes from the observation that $R_\mathbb{R}(\mathbb{G}_m) = \mathbb{R}\backslash 0 \simeq \{\pm 1\} = S^0$. That is to say $R_\mathbb{R}$ identifies $\mathbb{G}_m$ and S^0. We can even do better. Write $\rho' : S^0 \to \mathbb{G}_m$ for the map of pointed motivic spaces corresponding to $-1 \in \mathbb{R}^\times$. Then one may check easily that $R_\mathbb{R}(\rho')$ is an equivalence between $S^0 \simeq R_\mathbb{R}(S^0)$ and $R_\mathbb{R}(\mathbb{G}_m)$.

We prove that $\mathbf{SH}(\mathbb{R})[\rho'^{-1}] \simeq \mathbf{SH}$ via real realisation. That is to say $R_\mathbb{R}$ is in some sense the universal functor turning ρ' into an equivalence. More precisely, the functor $R_\mathbb{R} : \mathbf{SH}(\mathbb{R}) \to \mathbf{SH}$ has a right adjoint R^* (e.g. by Neeman's version of Brown representability) and we show that R^* is fully faithful with image consisting of the ρ'-stable motivic spectra, i.e. those $E \in \mathbf{SH}(\mathbb{R})$ such that $E(X \wedge \mathbb{G}_m) \xrightarrow{\rho^*} E(X)$ is an equivalence for all $X \in Sm(\mathbb{R})$.

Of course, our description of $\mathbf{SH}(\mathbb{R})[\rho'^{-1}]$ is just an explicit description of a certain Bousfield localisation of $\mathbf{SH}(\mathbb{R})$. Moreover the element ρ' exists not only over $\mathbb{R}$ but already over $\mathbb{Z}$, so we are lead to study more generally the category $\mathbf{SH}(S)[\rho'^{-1}]$, for more or less arbitrary base schemes S. Actually, for some formulas it is nicer to consider $\rho := -\rho' \in [S, \Sigma^\infty \mathbb{G}_m]$ and we shall write this from now on. Of course $\mathbf{SH}(S)[\rho'^{-1}] = \mathbf{SH}(S)[\rho^{-1}]$. In this generality we can no longer expect that $\mathbf{SH}(S)[\rho^{-1}] \simeq \mathbf{SH}$. Indeed as we have said before in general there is no real realisation! As a first attempt, one might guess that if X is a scheme over $\mathbb{R}$, then $\mathbf{SH}(S)[\rho^{-1}] \simeq \mathbf{SH}(S(\mathbb{R}))$, where the right-hand side denotes some form of parametrised homotopy theory [MS06]. This cannot be quite true unless S is proper, because the category $\mathbf{SH}(S(\mathbb{R}))$ will then not be compactly generated. The way out is to use semi-algebraic topology. For this we have to recall that if S is a scheme, then there exists a topological space $R(S)$ [Sch94, (0.4.2)]. Its points are pairs (x, α) with $x \in S$ and α an ordering of the residue field $k(x)$. This is given a topology incorporating all of these orderings. Write $\mathrm{Shv}(RS)$ for the category of sheaves on this topological space.

Now, given any topos $\mathcal{X}$, there is a naturally associated stable homotopy category $\mathrm{SH}(\mathcal{X})$. If $\mathcal{X} \simeq \mathrm{Set}$ then $\mathrm{SH}(\mathcal{X})$ is just the ordinary stable homotopy category. In general, if $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$ where $\mathcal{C}$ is a Grothendieck site, then $\mathrm{SH}(X)$ is the local homotopy category of presheaves of spectra on $\mathcal{C}$.

With this preparation out of the way, we can state our main result as follows.

Theorem (See Theorem 35). *Let S be a Noetherian scheme of finite dimension. Then there is a canonical equivalence of categories*

$$\mathbf{SH}(S)[\rho^{-1}] \simeq \mathrm{SH}(\mathrm{Shv}(RS)).$$

A more detailed formulation is given later in this introduction. For now let us mention one application. We go back to $S = \mathrm{Spec}(\mathbb{R})$. In this case Proposition 36 in § 10 assures us that the equivalence from the above theorem does indeed come from real realisation. But given $E \in \mathbf{SH}(\mathbb{R})$, its ρ-localisation can be calculated quite explicitly (see Lemma 15). From this one concludes that $\pi_i(R_{\mathbb{R}}E) = \mathrm{colim}_n\, \underline{\pi}_i(E)_n(\mathbb{R})$, where the colimit is along multiplication by ρ in the second grading of the bigraded homotopy sheaves of E. (Recall that $\underline{\pi}_i(E)_n(\mathbb{R}) = [\mathbb{1}[i], E \wedge \mathbb{G}_m^{\wedge n}]$ so ρ indeed induces $\rho : \underline{\pi}_i(E)_n(\mathbb{R}) \to \underline{\pi}_i(E)_{n+1}(\mathbb{R})$.)

This may seem slightly esoteric, but actually $\mathbf{SH}(S)[\rho^{-1}, 2^{-1}] = \mathbf{SH}(S)[\eta^{-1}, 2^{-1}]$ and so our computations apply, after inverting two, to the more conventional η-localisation as well. As a corollary, we obtain the following.

Theorem. *The motivic stable 2-local, η-local stems over $\mathbb{R}$ agree with the classical stable 2-local stems:*

$$\underline{\pi}_i(\mathbb{1}_{\eta,2})_j(\mathbb{R}) = \pi_i^s \otimes_{\mathbb{Z}} \mathbb{Z}[1/2].$$

Some more applications will be described later in this introduction.

Overview of the proof. The proof uses a different description of the category $\mathrm{Shv}(RS)$. Namely, there is a topology on all schemes called the *real étale topology* and abbreviated rét-topology [Sch94, (1.2)]. (The covers are families of étale morphisms which induce a jointly surjective family on the associated real spaces $R(\bullet)$.) We write $Sm(S)_{\text{rét}}$ for the site of all smooth schemes over S with this topology, and $S_{\text{rét}}$ for the site of all étale schemes over S with this topology. Then $\mathrm{Shv}(S_{\text{rét}}) \simeq \mathrm{Shv}(RS)$ [Sch94, Theorem (1.3)].

Write $\mathbf{SH}(S)$ for the motivic stable homotopy category, $\mathbf{SH}(S)[\rho^{-1}]$ for the ρ-local motivic stable homotopy category, $\mathbf{SH}(S)^{\text{rét}}$ for the rét-local motivic stable homotopy category (i.e. the category obtained from the site $Sm(S)_{\text{rét}}$ by precisely the same construction as is used to build $\mathbf{SH}(S)$ from $Sm(S)_{\text{Nis}}$), and $\mathbf{SH}^{S^1}(S)$ for the motivic S^1-stable homotopy category. We trust that $\mathbf{SH}^{S^1}(S)^{\text{rét}}$, $\mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ and so on have evident meanings. Write $\mathrm{SH}(S_{\text{rét}})$ for the rét-local stable homotopy category on the small real étale site. This is just the homotopy category of the category of presheaves of spectra on $S_{\text{rét}}$ with the local model structure. Similarly $\mathrm{SH}(Sm(S)_{\text{rét}})$ means the rét-local presheaves of spectra on $Sm(S)$. Then for example $\mathbf{SH}^{S^1}(S)^{\text{rét}}$ is the $\mathbb{A}^1$-localisation of $\mathrm{SH}(Sm(S)_{\text{rét}})$.

The canonical functor $e : \mathrm{SH}(S_{\text{rét}}) \to \mathrm{SH}(Sm(S)_{\text{rét}})$ (extending a (pre)sheaf on the small site to the large site) is fully faithful by general results (see Corollary 6). It is moreover t-exact: for $E \in \mathrm{SH}(S_{\text{rét}})$ we have $\underline{\pi}_i(eE) = e\underline{\pi}_i(E)$. Here $\underline{\pi}_*$ denotes the homotopy sheaves.

If F is a sheaf on the small real étale site of a scheme Y, then $H^p(Y \times \mathbb{A}^1, F) = H^p(Y, F)$ and $H^p(Y_+ \wedge \mathbb{G}_m, F) = H^p(Y, F)$. If Y is of finite type over $\mathbb{R}$ and F is locally constant, then this follows by comparison of real étale cohomology with Betti cohomology of the real points [Del91, Theorem II.5.7]. For the general case, see Theorem 8.

Now the category $\mathbf{SH}^{S^1}(S)^{\text{rét}}[\rho^{-1}]$ is obtained from $\mathrm{SH}(Sm(S)_{\text{rét}})$ by $(\mathbb{A}^1, \rho)$-localisation. It follows from t-exactness of e, the descent spectral sequence, and the above result about rét-cohomology that the composite $\mathrm{SH}(S_{\text{rét}}) \to \mathrm{SH}(Sm(S)_{\text{rét}}) \to \mathbf{SH}^{S^1}(S)^{\text{rét}}[\rho^{-1}]$ is still fully faithful.

The category $\mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ is obtained from $\mathbf{SH}^{S^1}(S)^{\text{rét}}[\rho^{-1}]$ by $\otimes$-inverting $\mathbb{G}_m$. However in the latter category we have $\mathbb{G}_m \simeq \mathbb{1}$ (via $\rho!$), so $\mathbb{G}_m$ is already invertible, and inverting it has no effect: $\mathbf{SH}^{S^1}(S)^{\text{rét}}[\rho^{-1}] \simeq \mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$. We have thus shown that

$$\mathrm{SH}(S_{\text{rét}}) \to \mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$$

is fully faithful.

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The next step is to show that it is essentially surjective. This follows from the proper base change theorem by a clever argument of Cisinski–Déglise. Of course this first requires that we know that $\mathrm{SH}(S_{\text{rét}})$ and $\mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ satisfy proper base change. For $\mathrm{SH}(S_{\text{rét}})$ this is a consequence of the proper base change theorem in real étale cohomology established by Scheiderer, see Theorem 9. For $\mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ this would follow from the axiomatic six functors formalism of Voevodsky/Ayoub/Cisinski–Déglise, see § 5. It is in fact not very hard to show directly that $\mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ satisfies the six functors formalism. Instead we shall show (without assuming the six functors formalism) that $\mathbf{SH}(S)^{\text{rét}}[\rho^{-1}] \simeq \mathbf{SH}(S)[\rho^{-1}]$, and that this latter category satisfies the six functors formalism.

The next step is thus to show that the localisation functor $\mathbf{SH}(S)[\rho^{-1}] \to \mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ is an equivalence. It clearly has dense image, so it suffices to show that it is fully faithful. Using the fact that $\mathbf{SH}(S)[\rho^{-1}]$ satisfies continuity and gluing (which follows quite easily from the same statement for $\mathbf{SH}(S)$), we may reduce to the case where S is the spectrum of a field k. The case where $\mathrm{char}(k) > 0$ is easily dealt with (note that such fields are never orderable), so we may assume that k has characteristic zero and so in particular is *perfect*.

The ρ-localisation can be described rather explicitly. For $E \in \mathbf{SH}(k)$, consider the directed system

$$E \xrightarrow{\rho} E \wedge \mathbb{G}_m \xrightarrow{\rho} E \wedge \mathbb{G}_m \wedge \mathbb{G}_m \xrightarrow{\rho} \cdots .$$

Then $\mathrm{hocolim}_n\, E \wedge \mathbb{G}_m^{\wedge n}$ is a model for the ρ-localisation $E[\rho^{-1}]$ of E (see Lemma 15). It follows that its homotopy sheaves are given by

$$\underline{\pi}_i(E[\rho^{-1}]) = \underline{\pi}_i(E)_*[\rho^{-1}] =: \mathrm{colim}_n\, \underline{\pi}_i(E)_n.$$

Here the colimit is along multiplication by ρ. (Let us remark here that the homotopy sheaves in $\mathbf{SH}(k)$ are *bigraded*, and so, technically, are those in $\mathbf{SH}(k)[\rho^{-1}]$. However inverting ρ means that up to canonical isomorphism, the homotopy sheaf is independent of the second index, so we suppress it.) It then follows from the descent spectral sequence that in order to prove that the functor $\mathbf{SH}(k)[\rho^{-1}] \to \mathbf{SH}(k)^{\text{rét}}[\rho^{-1}]$ is an equivalence, it is enough to prove that if F_* is a homotopy module (element in the heart of $\mathbf{SH}(k)$) such that $\rho : F_n \to F_{n+1}$ is an isomorphism for all n (we call such a homotopy module ρ-*stable*), then $H^n_{\text{rét}}(X, F_*) = H^n_{\text{Nis}}(X, F_*)$ for all X smooth over k. In particular, we need to show that F_* is a sheaf in the real étale topology. This is actually sufficient, because Nisnevich, Zariski and real étale cohomology of real étale sheaves all agree [Sch94, Proposition 19.2.1].

This ties in with work of Jacobson and Scheiderer. Recall that $\underline{\pi}_0(\mathbb{1})_* = \underline{K}^{MW}_*$, i.e. the zeroth stable motivic homotopy sheaf is unramified Milnor–Witt K-theory. A theorem of Jacobson [Jac17] together with work of Morel implies that $\underline{K}^{MW}_*[\rho^{-1}] = \mathrm{colim}_n\, \underline{I}^n = a_{\text{rét}}\underline{\mathbb{Z}}$; here $\underline{I}$ is the sheaf of fundamental ideals. Finally if F_* is a general ρ-stable homotopy module, we use properties of transfers for homotopy modules together with the structure of F_* as a module over $\underline{K}^{MW}_*[\rho^{-1}] = a_{\text{rét}}\underline{\mathbb{Z}}$ to show that F_* is a sheaf in the real étale topology. This concludes the overview of the proof.

Throughout the article we actually establish all our results for both the stable motivic homotopy category $\mathbf{SH}(S)$ and the stable $\mathbb{A}^1$-derived category $D_{\mathbb{A}^1}(S)$. The proofs in the latter case are essentially always the same as in the former, so we do not tend to give them. (In fact in some cases proofs just for the latter category would be simpler.)

Overview of the article. In § 2 we recall some results from local homotopy theory, including the existence and basic properties of the homotopy t-structure, a general compact generation criterion and a fully faithfulness result.

In § 3 we recall the real étale topology and establish some supplements.

In § 4 we recall some results about motivic stable homotopy categories and transfers for finite étale morphisms. In particular we establish the base change and projection formulas for these.

In § 5 we recall the formalism of pre-motivic and motivic categories and how it can be used to establish that a category satisfies the six functors formalism.

In § 6 we carefully prove some basic facts about monoidal Bousfield localisation.

We judge these five sections as preliminary and the results as not very original. The 'real work' is contained in the next three sections. In § 7 we review Jacobson's theorem on the colimit of the powers of the sheaf of fundamental ideals and use it together with our results on transfers to prove that ρ-stable homotopy modules are sheaves in the real étale topology.

Section 8 contains various preliminary observations and reductions.

Finally in § 9 we carry out the proof as outlined above.

The remaining three sections contain some applications. In § 10 we show that our functor $\mathbf{SH}(\mathbb{R}) \to \mathbf{SH}(\mathbb{R})[\rho^{-1}] \simeq \mathrm{SH}(\mathrm{Spec}(\mathbb{R})_{\text{rét}}) \simeq \mathbf{SH}$ coincides with the real realisation functor. It follows that the ρ-inverted stable homotopy sheaves of $E \in \mathbf{SH}(\mathbb{R})$ are just the stable homotopy groups of its real realisation.

In § 11 we collect some consequences for the η-inverted sphere. We use that $\mathbb{1}[1/2, 1/\rho] \simeq \mathbb{1}[1/2, 1/\eta]$. Since the classical stable stems $\pi_i^s = \mathbb{Z}/2$ for $i = 1, 2$ are 2-torsion, it follows that $\underline{\pi}_i(\mathbb{1}[1/2, 1/\eta])(\mathbb{R}) = 0$ for $i = 1, 2$. Since the ρ-local homotopy sheaves are unramified sheaves in the real étale topology, this (more or less) implies that $\underline{\pi}_i(\mathbb{1}[1/2, 1/\eta]) = 0$ for $i = 1, 2$. This reproves a result of Röndigs [Rön16].

A different but related question is to determine rational motivic stable homotopy theory. By a recent result of Ananyevskiy *et al.* [ALP17] we have $\mathbf{SH}(k)^-_{\mathbb{Q}} \simeq \mathbf{DM}_W(k, \mathbb{Q})$, where the right-hand side denotes a category of rational Witt-motives. Our results show easily that $\mathbf{DM}_W(k, \mathbb{Z}[1/2]) \simeq D_{\mathbb{A}^1}(k, \mathbb{Z}[1/2])^- \simeq D(\mathrm{Spec}(k)_{\text{rét}}, \mathbb{Z}[1/2])$ and more generally that $D_{\mathbb{A}^1}(k, \mathbb{Z})[1/\rho] \simeq D(\mathrm{Spec}(k)_{\text{rét}})$. By the same proof as in classical rational stable homotopy theory we have $\mathbf{SH}(k)^-_{\mathbb{Q}} \simeq D_{\mathbb{A}^1}(k, \mathbb{Q})^-$, and so we consider our results as one version of an integral strengthening of the result of Ananyevskiy–Levine–Panin.

In § 12 we collect some applications to the rigidity problem. A sheaf F on $Sm(k)$ is called *rigid* if for every essentially smooth, Henselian local scheme X with closed point x we have $F(X) = F(x)$. For example, sheaves with transfers in the sense of Voevodsky which are of torsion prime to the characteristic of the perfect base field are rigid (see [SV96, Theorem 4.4]). Our results imply that the homotopy sheaves of any $E \in \mathbf{SH}(k)[\rho^{-1}]$ are real étale sheaves extended from the small real étale site of k. One might already call this a rigidity result, but it is also not hard to see (and we show) that all such sheaves are rigid in the above sense. As an application, we show that the motivic stable homotopy sheaves $\underline{\pi}_i(\mathbb{1})_0[1/e]$ are all rigid, where e is the exponential characteristic. This ties up a loose end of the author's PhD thesis.

Notation. If S is a scheme, we denote the motivic stable homotopy category by $\mathbf{SH}(S)$. We denote the S^1-stable motivic homotopy category (i.e. where $\mathbb{G}_m$ has not been inverted yet) by $\mathbf{SH}^{S^1}(S)$. If $\mathcal{X}$ is a topos or site, we denote by $\mathrm{SH}(\mathcal{X})$ the associated stable homotopy category, see § 2. In particular $\mathrm{SH}(S_{\text{rét}}), \mathrm{SH}(Sm(S)_{\text{rét}})$ and $\mathbf{SH}(S)^{\text{rét}}$ should be carefully distinguished: the first is the stable homotopy category of the small rét-site on S, the second is the stable homotopy category of the site of all smooth schemes, with the rét-topology, and the latter is the rét-localisation of the motivic stable homotopy category. This last category is $\mathbb{A}^1$-local and $\mathbb{G}_m$-stable, whereas the second category is neither, and these notions do not even make sense for the first category.

The classical stable homotopy category will still be denoted by **SH**.

We denote the unit of a monoidal category $\mathcal{C}$ by $\mathbb{1}_{\mathcal{C}}$ or just by $\mathbb{1}$, if $\mathcal{C}$ is clear from the context. Thus if $\mathcal{C}$ is a stable homotopy category of some sort, then $\mathbb{1}$ is the sphere spectrum.

2. Recollections on local homotopy theory

If $(\mathcal{C}, \tau)$ is a Grothendieck site, we can consider the associated category $\mathrm{Shv}(\mathcal{C}_\tau)$ of sheaves (a topos), the category $\mathrm{sPre}(\mathcal{C})$ of *simplicial presheaves* on $\mathcal{C}$, as well as the categories $\mathcal{SH}(\mathcal{C})$ of presheaves of spectra and $C(\mathcal{C})$ of presheaves of complexes of abelian groups on $\mathcal{C}$. The latter three categories carry various local model structures, in particular the injective and the projective one [Jar15]. We denote the homotopy category of $\mathcal{SH}(\mathcal{C}_\tau)$ by $\mathrm{SH}(\mathcal{C}_\tau)$ and the homotopy category of $C(\mathcal{C}_\tau)$ by $D(\mathcal{C}_\tau)$.

It is also possible to model $\mathrm{SH}(\mathcal{C}_\tau)$ and so on by *sheaves*. For this, let $\mathrm{sShv}(\mathcal{C}_\tau)$ denote the category of sheaves of simplicial sets, and similarly let $\mathcal{SH}^{\mathrm{s}}(\mathcal{C}_\tau)$ be the category of sheaves of spectra, and let $C^{\mathrm{s}}(\mathcal{C}_\tau)$ be the category of sheaves of chain complexes. (Here we mean sheaves in the 1-categorical sense, so this category is equivalent to the category of chain complexes of sheaves of abelian groups, and similarly for the spectra.) These also afford local model structures, and $\mathrm{Ho}(\mathrm{sShv}(\mathcal{C}_\tau)) \simeq \mathrm{Ho}(\mathrm{sPre}(\mathcal{C}_\tau))$, and so on.

Given a functor $f^* : \mathcal{C} \to \mathcal{D}$, there is an induced restriction functor $f_* : \mathrm{Pre}(\mathcal{D}) \to \mathrm{Pre}(\mathcal{C})$, where $\mathrm{Pre}(\mathcal{C})$ denotes the category of presheaves (of sets) on $\mathcal{C}$ (and similarly for $\mathcal{D}$). The functor f_* has a left adjoint $f^* : \mathrm{Pre}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{D})$. It is in fact the left Kan extension of $f^* : \mathcal{C} \to \mathcal{D}$.

If $\mathcal{C}, \mathcal{D}$ are sites the functor f^* is called continuous if $f_* : \mathrm{Pre}(\mathcal{D}) \to \mathrm{Pre}(\mathcal{C})$ preserves sheaves. In this case the induced functor $f_* : \mathrm{Shv}(\mathcal{D}) \to \mathrm{Shv}(\mathcal{C})$ has a left adjoint still denoted $f^* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$. If this induced functor is left exact (commutes with finite limits) then f is called a *geometric morphism*.

More generally, an adjunction $f^* : \mathrm{Shv}(\mathcal{C}) \leftrightarrows \mathrm{Shv}(\mathcal{D}) : f_*$ (where $f^* \vdash f_*$ does not necessarily come from a functor $f^* : \mathcal{C} \to \mathcal{D}$) is called a geometric morphism if f^* preserves finite limits.

If $f : \mathcal{C} \to \mathcal{D}$ is any functor, then there are induced adjunctions $f^* : \mathrm{sPre}(\mathcal{C}) \leftrightarrows \mathrm{sPre}(\mathcal{D}) : f_*$, and similarly for spectra and chain complexes. Similarly if $f^* : \mathrm{Shv}(\mathcal{C}) \leftrightarrows \mathrm{Shv}(\mathcal{D}) : f_*$ is any adjunction, then there are induced adjunctions $f^* : \mathrm{sShv}(\mathcal{C}) \leftrightarrows \mathrm{sShv}(\mathcal{D}) : f_*$, and so on. If $f^* \vdash f_*$ is a geometric morphism in either of the above senses, then the induced adjunctions on presheaves (sheaves) of simplicial sets, spectra, and chain complexes are Quillen adjunctions in the local model structure [Jar15, § 5.3] [CD09, Theorem 1.18].

The above discussion allows us to prove the following useful result.

Lemma 1. *Let $f^* : \mathrm{Shv}(\mathcal{C}) \leftrightarrows \mathrm{Shv}(\mathcal{D}) : f_*$ be a geometric morphism such that f^* is fully faithful and f_* preserves colimits. Then the induced functors*

$$Lf^* : \mathrm{SH}(\mathcal{C}) \to \mathrm{SH}(\mathcal{D})$$

and

$$Lf^* : D(\mathcal{C}) \to D(\mathcal{D})$$

are fully faithful.

The same result also holds for $Lf^* : \mathrm{Ho}(\mathrm{sPre}(\mathcal{C})) \to \mathrm{Ho}(\mathrm{sPre}(\mathcal{D}))$, with the same proof.

Proof. We give the proof for the derived categories, it is the same for spectra.

Since f_* preserves colimits it affords a right adjoint $f^!$. Then $f_* \vdash f^!$ is a geometric morphism in the opposite direction (note that f_* preserves finite limits, and in fact all limits, since it is a right adjoint) and consequently f_* is *bi*-Quillen. It follows that $f_* : C^s(\mathcal{D}) \to C^s(\mathcal{C})$ preserves weak equivalences, and consequently coincides (up to weak equivalence) with its derived functor.

Now to show that Lf^* is fully faithful we need to show that $Rf_*Lf^* \simeq \mathrm{id}$. But $Rf_* \simeq f_*$ since f_* is bi-Quillen. Let $E \in C^s(\mathcal{C})$ be cofibrant. Then $Lf^*E \simeq f^*E$ and consequently $Rf_*Lf^*E \simeq f_*f^*E$. Since f^* is fully faithful we have $f_*f^*E \cong E$. This concludes the proof. □

We will also make use of t-structures. We shall use *homological* notation for t-structures [Lur16, Definition 1.2.1.1]. Briefly, a t-structure on a triangulated category $\mathcal{C}$ consists of two (strictly full) subcategories $\mathcal{C}_{\geqslant 0}$ and $\mathcal{C}_{\leqslant 0}$, satisfying various axioms. We put $\mathcal{C}_{\geqslant n} = \mathcal{C}_{\geqslant 0}[n]$ and $\mathcal{C}_{\leqslant n} = \mathcal{C}_{\leqslant 0}[n]$. One then has $\mathcal{C}_{\geqslant n+1} \subset \mathcal{C}_{\geqslant n}$ and $\mathcal{C}_{\leqslant n} \subset \mathcal{C}_{\leqslant n+1}$ and $[\mathcal{C}_{\geqslant n+1}, \mathcal{C}_{\leqslant n}] = 0$. In fact $E \in \mathcal{C}_{\geqslant n+1}$ if and only if for all $F \in \mathcal{C}_{\leqslant n}$ we have $[E, F] = 0$, and vice versa. The inclusion $\mathcal{C}_{\geqslant n} \hookrightarrow \mathcal{C}$ has a right adjoint which we denote $E \mapsto E_{\geqslant n}$, and the inclusion $\mathcal{C}_{\leqslant n} \hookrightarrow \mathcal{C}$ has a left adjoint which we denote $E \mapsto E_{\leqslant n}$. The adjunctions furnish map $E_{\geqslant n+1} \to E \to E_{\leqslant n}$ and this extends to a distinguished triangle in a unique and functorial way. The intersection $\mathcal{C}^\heartsuit := \mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}$ called the *heart*. It is an abelian category. We put $\pi_0^{\mathcal{C}}(E) = (E_{\leqslant 0})_{\geqslant 0} \simeq (E_{\geqslant 0})_{\leqslant 0} \in \mathcal{C}^\heartsuit$ and $\pi_i^{\mathcal{C}}(E) = \pi_0^{\mathcal{C}}(E[i])$. Then $\pi_*^{\mathcal{C}}$ is a homological functor on $\mathcal{C}$. The t-structure is called *non-degenerate* if $\pi_i^{\mathcal{C}}(E) = 0$ implies that $E \simeq 0$.

By a t-category we mean a triangulated category with a fixed t-structure.

Suppose that $(\mathcal{C}, \tau)$ is a site. Let for $E \in \mathcal{SH}(\mathcal{C}_\tau)$ and $i \in \mathbb{Z}$ the sheaf $\underline{\pi}_i(E) \in \mathrm{Shv}(\mathcal{C}_\tau)$ be defined as the sheaf associated with the presheaf $\mathcal{C} \ni X \mapsto \pi_i(E(X))$. Here we view E as a presheaf of spectra. By definition, local weak equivalences of spectra induce isomorphisms on $\underline{\pi}_i$, so $\underline{\pi}_i(E)$ is well defined for $E \in \mathrm{SH}(\mathcal{C}_\tau)$. This is a sheaf of *abelian groups*. Put

$$\mathrm{SH}(\mathcal{C}_\tau)_{\geqslant 0} = \{E \in \mathrm{SH}(\mathcal{C}_\tau) : \underline{\pi}_i(E) = 0 \text{ for } i < 0\},$$
$$\mathrm{SH}(\mathcal{C}_\tau)_{\leqslant 0} = \{E \in \mathrm{SH}(\mathcal{C}_\tau) : \underline{\pi}_i(E) = 0 \text{ for } i > 0\}.$$

We define similarly for $E \in D(\mathcal{C}_\tau)$ the sheaf $\underline{h}_i(E)$, and then the subcategories $D(\mathcal{C}_\tau)_{\geqslant 0}$, $D(\mathcal{C}_\tau)_{\leqslant 0}$.

LEMMA 2. *If $(\mathcal{C}, \tau)$ is a Grothendieck site, then the above construction provides $\mathrm{SH}(\mathcal{C}_\tau)$ with a non-degenerate t-structure. The functor $\underline{\pi}_0 : \mathrm{SH}(\mathcal{C}_\tau)^\heartsuit \to \mathrm{Shv}(\mathcal{C}_\tau)$ is an equivalence of categories. Moreover let $F \in \mathrm{Shv}(\mathcal{C}_\tau) \simeq \mathrm{SH}(\mathcal{C})^\heartsuit$. Then for $X \in \mathcal{C}$ there is a natural isomorphism $[\Sigma^\infty X_+, F[n]] = H^n_\tau(X, F)$.*

Similar statements hold for $D(\mathcal{C}_\tau)$ in place of $\mathrm{SH}(\mathcal{C}_\tau)$.

Proof. For derived categories, this result is classical. For $\mathrm{SH}(\mathcal{C}_\tau)$, the result is also fairly well known, but the author does not know an explicit reference, so we sketch a proof.

Note that there is a Quillen adjunction (in the local model structures)

$$\Sigma^\infty : \mathrm{sPre}(\mathcal{C}_\tau)_* \leftrightarrows \mathcal{SH}(\mathcal{C}_\tau) : \Omega^\infty.$$

By direct computation using the above adjunction, we find that $\underline{\pi}_i(\Omega^\infty E) = \underline{\pi}_i(E)$, for $E \in \mathrm{SH}(\mathcal{C}_\tau)$ and $i \geqslant 0$.

By [Lur16, Proposition 1.4.3.4 and Remark 1.4.3.5] the category $\mathrm{SH}(\mathcal{C}_\tau)$ admits a t-structure,[1] where $E \in \mathrm{SH}(\mathcal{C}_\tau)_{\leqslant 0}$ if and only if $\Omega^\infty(E) \simeq *$, and the subcategory $\mathrm{SH}(\mathcal{C}_\tau)_{\geqslant 0}$ is generated under

[1] The author would like to thank Saul Glasman for pointing out this reference.

homotopy colimits and extensions by $\Sigma^\infty \mathcal{C}_+$. We first need to show that this is the t-structure we want, i.e. that the positive and negative parts are determined by vanishing of homotopy sheaves. Since $\underline{\pi}_i(\Omega^\infty E) = \underline{\pi}_i(E)$, this is correct for the negative part. I claim that if $E \in \mathrm{SH}(\mathcal{C}_\tau)_{\geqslant 0}$, then $\underline{\pi}_i(E) = 0$ for $i < 0$. If $X \in \mathrm{sPre}(\mathcal{C}_\tau)_*$, then $\underline{\pi}_i(\Sigma^\infty X) = 0$ for $i < 0$ by direct computation. It thus remains to show that the subcategory of $E \in \mathrm{SH}(\mathcal{C}_\tau)$ with $\underline{\pi}_i(E) = 0$ for $i < 0$ is closed under homotopy colimits and extensions. For extensions this is clear. Homotopy colimits are generated by pushouts and filtered colimits [Lur09, Propositions 4.4.2.6 and 4.4.2.7], so we need only deal with cones and filtered colimits. For cones this is again clear, and for filtered colimits it holds because homotopy groups of spectra commute with filtered colimits, and hence the same is true for homotopy sheaves (see the proof of Corollary 3 for more details on this). This proves the claim. Conversely, let $E \in \mathrm{SH}(\mathcal{C}_\tau)$ with $\underline{\pi}_i(E) = 0$ for $i < 0$. Consider the decomposition $E_{\geqslant 0} \to E \to E_{<0}$. Then $\underline{\pi}_i(E_{\geqslant 0}) = 0$ for $i < 0$, so $0 = \underline{\pi}_i(E) = \underline{\pi}_i(E_{<0})$ for $i < 0$. It follows that $E_{<0} \simeq 0$ and so $E \simeq E_{\geqslant 0} \in \mathrm{SH}(E)_{\geqslant 0}$.

The t-structure is non-degenerate because it is defined in terms of homotopy sheaves, and homotopy sheaves detect weak equivalences by definition.

We have an adjunction

$$M : \mathrm{SH}(\mathcal{C}_\tau) \leftrightarrows D(\mathcal{C}_\tau) : U.$$

By construction U is t-exact and thus M is right t-exact. Consider the induced adjunction

$$M^\heartsuit : \mathrm{SH}(\mathcal{C}_\tau)^\heartsuit \leftrightarrows D(\mathcal{C}_\tau)^\heartsuit : U.$$

By direct computation using the classical Hurewicz isomorphism (and the above adjunction), $\underline{\pi}_0(UME) = \underline{\pi}_0(E)$ if $E \in \mathrm{SH}(\mathcal{C}_\tau)_{\geqslant 0}$. It follows that $UM^\heartsuit \simeq \mathrm{id}$. Since U is faithful by definition, from this we deduce that $M^\heartsuit U \simeq \mathrm{id}$ as well. Thus $\mathrm{SH}(\mathcal{C}_\tau)^\heartsuit \simeq D(\mathcal{C}_\tau)^\heartsuit \simeq \mathrm{Shv}(\mathcal{C}_\tau)$, the latter equivalence being classical. Finally if $X \in \mathcal{C}$ and $F \in \mathrm{Shv}(\mathcal{C}_\tau)$ then $[\Sigma^\infty X_+, F[n]] = [\Sigma^\infty X_+, UF[n]] = H^n_\tau(X, F)$, the first equality by definition and the second by adjunction and the same result in $D(\mathcal{C}_\tau)$. □

COROLLARY 3. *Let $(\mathcal{C}, \tau)$ be a Grothendieck site.*

(1) *Let $X \in \mathcal{C}$. If τ-cohomology on X commutes with filtered colimits of sheaves and the τ-cohomological dimension of X is finite, then $\Sigma^\infty X_+ \in \mathrm{SH}(\mathcal{C}_\tau)$ is a compact object.*

(2) *For any collection $E_i \in \mathrm{SH}(\mathcal{C})$ and $j \in \mathbb{Z}$ we have $\underline{\pi}_j(\bigoplus_i E_i) = \bigoplus_i \underline{\pi}_j(E_i)$.*

Similarly for $D(\mathcal{C}_\tau)$.

Proof. Let us show that (1) reduces to (2). For $E \in \mathrm{SH}(\mathcal{C}_\tau)$ there is a conditionally convergent spectral sequence

$$H^p_\tau(X, \underline{\pi}_{-q}E) \Rightarrow [X, E[p+q]].$$

Under our assumptions on the cohomological dimension of X, it converges strongly to the right-hand side. Under the assumption of commutation of cohomology with filtered colimits, by spectral sequence comparison, it thus suffices to show that for $E_i \in \mathrm{SH}(\mathcal{C}_\tau)$ we have $\underline{\pi}_n(\bigoplus_i E_i) = \bigoplus_i \underline{\pi}_n E_i$.

Now we prove (2). For $E \in \mathcal{SH}(\mathcal{C})$ write $\underline{\pi}^p_j(E)(X) = \pi_j(E(X))$; this defines a presheaf of abelian groups on $\mathcal{C}$. By definition $\underline{\pi}_j(E) = a_\tau \underline{\pi}^p_j(E)$. Let $\{E_i\}_i \in \mathcal{SH}(\mathcal{C})$. Then $\underline{\pi}^p_j(\bigoplus_i E_i) = \bigoplus_i \underline{\pi}^p_j(E_i)$, since homotopy groups of spectra commute with filtered colimits. We may assume that all the E_i are cofibrant, so their presheaf direct sum coincides with the derived direct sum. In this case it remains to show that

$$a_\tau \bigoplus_i \underline{\pi}^p_j(E_i) \cong \bigoplus_i a_\tau \underline{\pi}^p_j(E_i).$$

(Note that here we write $\bigoplus_i$ for both direct sums of presheaves and direct sums of sheaves, depending on whether the terms on the right are presheaves or sheaves.) But this holds for any collection of presheaves on any site (both sides satisfy the same universal property).

The proof for D is the same. □

We can enhance the functoriality of the SH construction as follows. Recall that a triangulated functor $F : \mathcal{C} \to \mathcal{D}$ between t-categories is called right (respectively left) t-exact if $F(\mathcal{C}_{\geqslant 0}) \subset \mathcal{D}_{\geqslant 0}$ (respectively $F(\mathcal{C}_{\leqslant 0}) \subset \mathcal{D}_{\leqslant 0}$). The functor is called t-exact if it is both left and right t-exact.

Lemma 4. *Let $f^* : \mathrm{Shv}(\mathcal{C}) \leftrightarrows \mathrm{Shv}(\mathcal{D}) : f_*$ be a geometric morphism, where $\mathrm{Shv}(\mathcal{D})$ has enough points. Then in the adjunction*

$$Lf^* : \mathrm{SH}(\mathcal{C}) \leftrightarrows \mathrm{SH}(\mathcal{D}) : Rf_*$$

the left adjoint Lf^* is t-exact, the right adjoint Rf_* is left t-exact, and the induced functors

$$(Lf^*)^{\heartsuit} : \mathrm{SH}(\mathcal{C})^{\heartsuit} \leftrightarrows \mathrm{SH}(\mathcal{D})^{\heartsuit} : (Rf_*)^{\heartsuit}$$

coincide (under the identification from Lemma 2) with $f^* \vdash f_*$.

Similar statements hold for D in place of SH.

The author contends that the assumption that $\mathcal{D}$ has enough points is not really necessary. See also [Lur09, Remark 6.5.1.4].

Proof. Certainly Rf_* is left t-exact if Lf^* is t-exact by adjunction, and $(Rf_*)^{\heartsuit}$ is right adjoint to $(Lf^*)^{\heartsuit}$, so it suffices to prove the claims for Lf^*.

Since $\mathcal{D}$ has enough points, it is then enough to assume that $\mathrm{Shv}(\mathcal{D}) = \mathrm{Set}$. (Indeed let $p : \mathrm{Set} \to \mathrm{Shv}(\mathcal{D})$ be a point; we will have

$$p^*\underline{\pi}_i(Lf^*E) = \pi_i(Lp^*Lf^*E) = p^*f^*\underline{\pi}_iE$$

for all $E \in \mathrm{SH}(\mathcal{C})$ by applying the reduced case to p and fp which are points of $\mathcal{D}$ and $\mathcal{C}$, respectively. Since $\mathcal{D}$ has enough points it follows that $\underline{\pi}_i(Lf^*E) = f^*\underline{\pi}_i(E)$, as was to be shown.)

Let $p^* : \mathrm{Shv}(\mathcal{C}) \leftrightarrows \mathrm{Set} : p_*$ be a point of $\mathcal{C}$. Then p^* corresponds to a pro-object in $\mathcal{C}$, which is to say that there is a filtered family $X_\alpha \in \mathcal{C}$ such that for $F \in \mathrm{Shv}(\mathcal{C})$ we have $p^*(F) = \mathrm{colim}_\alpha F(X_\alpha)$ [GK15, Proposition 1.4 and Remark 1.5].

It follows that for $E \in \mathcal{SH}^{\mathrm{s}}(\mathcal{C})$ we have

$$\pi_i(p^*E) = \pi_i(\mathrm{colim}_\alpha E(X_\alpha)) \cong \mathrm{colim}_\alpha \pi_i(E(X_\alpha)) = p^*\underline{\pi}_i(E),$$

where the isomorphism in the middle holds because homotopy groups commute with filtered colimits of spectra. In particular p^* preserves weak equivalences and so $p^* \simeq Lp^*$. Thus the previous equation is precisely what we intended to prove. □

3. Recollections on real étale cohomology

If X is a scheme, let $R(X)$ be the set of pairs (x, p) where $x \in X$ and p is an ordering of the residue field $k(x)$. For a ring A we put $\mathrm{Sper}(A) = R(\mathrm{Spec}(A))$. A family of morphisms $\{\alpha_i : X_i \to X\}_{i \in I}$ is called a *real étale covering* if each α is étale and $R(X) = \bigcup_i \alpha(R(X_i))$. (Note that for $(x, p) \in X_i$ the extension $k(x)/k(\alpha(x))$ defines by restriction an ordering of $k(\alpha(x))$.) The real étale coverings

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define a topology on all schemes [Sch94, (1.1)] called the *real étale topology*. We often abbreviate this name to 'rét-topology'.

For a scheme X, we let $X_{\text{rét}}$ denote the small real étale site on X and $Sm(X)_{\text{rét}}$ the site of smooth (separated, finite type) schemes over X with the real étale topology. If $f : X \to Y$ is any morphism of schemes, we get the usual base change functors $f^* : Y_{\text{rét}} \to X_{\text{rét}}$ and $f^* : Sm(Y) \to Sm(X)$. Also the natural inclusion $e : X_{\text{rét}} \to Sm(X)$ induces an adjunction $e^p : \text{Pre}(X_{\text{rét}}) \leftrightarrows \text{Pre}(Sm(X)) : r = e_*$.

Lemma 5. *If X is a scheme, the above adjunction induces a geometric morphism $e : \text{Shv}(X_{\text{rét}}) \leftrightarrows \text{Shv}(Sm(X)_{\text{rét}}) : r$ where e is fully faithful and r preserves colimits.*

Proof. The functor r is restriction and e is left Kan extension. Since e preserves covers, r preserves sheaves. Moreover r commutes with taking the associated sheaf, because every cover of $Y \in X_{\text{rét}}$ in $Sm(X)$ comes from a cover in $X_{\text{rét}}$ (because étale morphisms are stable under composition). It follows that r commutes with colimits. Since $e : X_{\text{rét}} \to Sm(X)_{\text{rét}}$ preserves pullbacks (and $X_{\text{rét}}$ has pullbacks!), the adjunction is a geometric morphism [Sta17, Tag 00X6]. In order to see that e is fully faithful, i.e. $F \to reF$ an isomorphism for every $F \in \text{Shv}(X_{\text{rét}})$, we note that for the presheaf adjunction $e^p : \text{Pre}(X_{\text{rét}}) \leftrightarrows \text{Pre}(Sm(k)) : r$ we have $re^pF = F$. Indeed this holds for F representable by definition, every sheaf is a colimit of representables, and e^p and f both commute with taking colimits. Finally note that for a sheaf F we have $eF = a_{\text{rét}}e^pF$ and thus $reF = ra_{\text{rét}}e^pF = a_{\text{rét}}re^pF = a_{\text{rét}}F = F$, where we have used again that r commutes with taking the associated sheaf. □

Corollary 6. *If X is a scheme, the induced derived functor $Le : \text{SH}(X_{\text{rét}}) \to \text{SH}(Sm(X)_{\text{rét}})$ is t-exact and fully faithful. Similarly for D in place of* SH.

Proof. The functor is fully faithful by Lemmas 5 and 1. It is t-exact by Lemma 4. □

Lemma 7. *If $f : X \to Y$ is a morphism of schemes, then the induced functor $f^* : Y_{\text{rét}} \to X_{\text{rét}}$ is the left adjoint of a geometric morphism of sites. Moreover the derived functor*

$$Lf^* : \text{SH}(Y_{\text{rét}}) \to \text{SH}(X_{\text{rét}})$$

is t-exact, and similarly for $Lf^* : D(Y_{\text{rét}}) \to D(X_{\text{rét}})$.

Proof. The 'moreover' part follows from Lemma 4.

Since $f^* : Y_{\text{rét}} \to X_{\text{rét}}$ preserves covers $f_* : \text{Pre}(X_{\text{rét}}) \to \text{Pre}(Y_{\text{rét}})$ preserves sheaves and the morphism is continuous. It is a geometric morphism of sites because f^* preserves pullbacks [Sta17, Tag 00X6]. □

If X is a scheme, there is the natural map $X \to X \times \mathbb{A}^1$ corresponding to the point $0 \in \mathbb{A}^1$. Similarly there is the natural map $X \coprod X \to X \times (\mathbb{A}^1 \backslash 0)$ corresponding to the points $\pm 1 \in \mathbb{A}^1 \backslash 0$.

Theorem 8. *Let X be a scheme and $F \in \text{Shv}(X_{\text{rét}})$. Then for any $p \geqslant 0$ the natural maps $X \to X \times \mathbb{A}^1$ and $X \coprod X \to X \times (\mathbb{A}^1 \backslash 0)$ induce isomorphisms*

$$H^p_{\text{rét}}(X \times \mathbb{A}^1, F) \to H^p_{\text{rét}}(X, F)$$
$$H^p_{\text{rét}}(X \times (\mathbb{A}^1 \backslash 0), F) \to H^p_{\text{rét}}(X, F) \oplus H^p_{\text{rét}}(X, F).$$

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Proof. The first statement is homotopy invariance, see [Sch94, Example 16.7.2].

For the second statement, we follow closely that proof. Let $f\colon X\coprod X\to X\times(\mathbb{A}^1\backslash 0)$ be the canonical map. It suffices to show that $R^nf_*F=0$ for $n>0$ and $R^0f_*F=F$, where we identify F with its pullback to $X\coprod X$ and $X\times(\mathbb{A}^1\backslash 0)$ for notational convenience. All of these statements are local on X, so we may assume that X is affine.

Then one may assume that F is constructible (since rét-cohomology commutes with filtered colimits of sheaves, and all sheaves on a spectral space are filtered colimits of constructible sheaves; see again [Sch94, Example 16.7.2]). Next, writing $X=\mathrm{Spec}(A)$ as the inverse limit of the filtering system $\mathrm{Spec}(A')$, with $A'\subset A$ finitely generated over $\mathbb{Z}$, and using Proposition (A.9) of [Sch94, Example 16.7.2], we may assume that X is of finite type over $\mathbb{Z}$.

But $\mathrm{Sper}(\mathbb{Z})=\mathrm{Sper}(\mathbb{Q})=\mathrm{Sper}(\mathbb{R})$, whence $H^p_{\text{rét}}(X,F)=H^p_{\text{rét}}(X\times_{\mathbb{Z}}\mathbb{R},F)$, so we may assume that X is of finite type over $\mathbb{R}$.

We may further assume that $F=M_Z$ is the constant sheaf on a closed, constructible subset of X (Proposition (A.6) of [Sch94, Example 16.7.2]).

It is thus enough to prove the analog of our result for an affine semi-algebraic space X over $\mathbb{R}$ and $F=M$ a constant sheaf. But then $H^*_{\text{rét}}(X,M)=H^*_{\text{sing}}(X(\mathbb{R}),M)$ [Del91, Theorem II.5.7] and so on, so this is obvious. □

Theorem 9 (Proper base change). *Consider a cartesian square of schemes*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

with f proper and Y finite-dimensional Noetherian. Then for any $E\in\mathrm{SH}(X_{\text{rét}})$ (respectively $E\in D(X_{\text{rét}})$) the canonical map

$$g^*Rf_*(E)\to Rf'_*g'^*(E)$$

is a weak equivalence.

Proof. We prove the claim for SH, the proof we give will work just as well for D. We proceed in several steps.

Step 0. If g is étale, then the claim follows from the observation that $f^*g_\#=g'_\#f'^*$.

Step 1. If $f\colon X\to Y$ is any morphism and $E\in\mathrm{SH}(X_{\text{rét}})$, then there is a conditionally convergent spectral sequence

$$E_2^{pq}=R^pf_*\underline{\pi}_{-q}E\Rightarrow\underline{\pi}_{-p-q}(Rf_*E).$$

For this, let $E\in\mathrm{Spt}(X_{\text{rét}})$ also denote a fibrant model. Then $Rf_*E\simeq f_*E$ and for $U\in Y_{\text{rét}}$ we have $f_*(E)(U)=E(f^*U)$. Since E is fibrant there is a conditionally convergent descent spectral sequence

$$H^p(f^*U,\underline{\pi}_{-q}(E))\Rightarrow\pi_{-p-q}(E(f^*U)).$$

By varying U, this yields a presheaf of spectral sequences on $Y_{\text{rét}}$. Equivalently, this is a spectral sequence of presheaves. Taking the associated sheaf on both sides we obtain a conditionally convergent spectral sequence

$$a_{\text{rét}}H^p_{\text{rét}}(f^*\bullet,\underline{\pi}_{-q}(E))\Rightarrow\underline{\pi}_{-p-q}(f_*E).$$

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It remains to see that $a_{\text{rét}} H^p_{\text{rét}}(f^* \bullet, F) = R^p f_* F$, for any sheaf F on $X_{\text{rét}}$. For this we view $F \in D(X_{\text{rét}})^{\heartsuit}$. Then by definition $R^p f_* F = \underline{\pi}_{-p} R f_* F$. Repeating the above argument with $D(X_{\text{rét}})$ in place of $\mathrm{SH}(X_{\text{rét}})$, we obtain a conditionally convergent spectral sequence

$$a_{\text{rét}} H^p_{\text{rét}}(f^* \bullet, \underline{\pi}_{-q} F) \Rightarrow R^{p+q} f_* F.$$

Since $\underline{\pi}_{-q} F = 0$ for $q \neq 0$ this spectral sequence converges strongly, yielding the desired identification.

Step 2. If f is proper and of relative dimension at most n, then for $F \in \mathrm{Shv}(X_{\text{rét}})$ and $p > n$ we have $R^p f_* F = 0$.

Indeed in this situation, by the proper base change theorem in real étale cohomology [Sch94, Theorem 16.2], for any real closed point $y \to Y$ we get $(R^p f_* F)_y = H^p_{\text{rét}}(X_y, F|_{X_y})$. Since real closed fields are the stalks of the rét-topology, in order for a sheaf $G \in \mathrm{Shv}(Y_{\text{rét}})$ to be zero it is necessary and sufficient that $G_y = 0$ for all such y. But real étale cohomological dimension is bounded by Krull dimension [Sch94, Theorem 7.6], so we find that $R^p f_* F = 0$ for $p > n$, as claimed.

Conclusion of proof. Since isomorphism in $\mathrm{SH}(Y'_{\text{rét}})$ is local on Y', it is an easy consequence of step 0 that we may assume that Y' is quasi-compact (e.g. affine). Then f' is of bounded relative dimension (being of finite type).

Now let $E \in \mathrm{SH}(X_{\text{rét}})$. By t-exactness of g^* and g'^* we get from step 1 conditionally convergent spectral sequences

$$g^* R^p f_* \underline{\pi}_{-q} E \Rightarrow \underline{\pi}_{-p-q}(g^* R f_* E)$$

and

$$R^p f'_* g'^* \underline{\pi}_{-q} E \Rightarrow \underline{\pi}_{-p-q}(R f'_* g'^* E).$$

The exchange transformation $g^* R f_*(E) \to R f'_* g'^*(E)$ induces a morphism of spectral sequences (i.e. respecting the differentials and filtrations). By proper base change for sheaves, we have $g^* R^p f_* \cong R^p f'_* g'^*$. Thus the two spectral sequences are isomorphic. By step 2 the second one converges strongly, and hence so does the first. Thus the result follows from spectral sequence comparison. □

Remark. The only place in the above proof where we have used the assumption on Y is in step 1, namely in the construction of the conditionally convergent spectral sequence

$$R^p f_* \underline{\pi}_{-q} E \Rightarrow \underline{\pi}_{-p-q}(R f_* E).$$

The author does not know how to construct such a spectral sequence in general. He nonetheless contends that the proper base change theorem should be true without assumptions on Y, but perhaps a different proof is needed.

Remark. In the above proof we deduce proper base change for spectra and unbounded complexes from proper base change for bounded complexes. Since we are dealing with hypercomplete toposes, this is not tautological; see for example [Lur09, Counterexample 6.5.4.2 and Remark 6.5.4.3]. The crucial property which seems to make the proof work is encapsulated in step 2 and might be phrased as ‘a proper morphism is locally of finite relative rét-cohomological dimension’. The same is true in étale (instead of real étale) cohomology and this seems to be what the proof of proper base change for unbounded étale complexes [CD13, Theorem 1.2.1] ultimately rests on, in the guise of [CD13, Lemma 1.1.7]. This fails for a general proper morphism of topological spaces (consider for example an infinite product of compact positive dimensional spaces mapping to the point).

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4. Recollections on motivic homotopy theory

We denote the stable motivic homotopy category over a base scheme X [Ayo07] by $\mathbf{SH}(X)$, and the stable $\mathbb{A}^1$-derived category over X [CD12, § 5.3] by $D_{\mathbb{A}^1}(X)$. We write $\mathbb{1}_X \in \mathbf{SH}(X)$ for the monoidal unit. If the context is clear we may just write $\mathbb{1}$.

Let $f : Y \to X$ be a finite étale morphism of schemes. Then in the category $\mathbf{SH}(X)$ we have an induced morphism $f : f_\# \mathbb{1}_Y \to \mathbb{1}_X$ and consequently $D(f) : D(\mathbb{1}_X) \to D(f_\# \mathbb{1}_Y)$. Here $DE := \underline{\mathrm{Hom}}(E, \mathbb{1})$. Now in fact whenever $f : Y \to X$ is smooth proper then $D(f_\# \mathbb{1}_Y) \simeq f_* \mathbb{1}_Y$ [CD12, Proposition 2.4.31] and if f is étale then $f_*(\mathbb{1}_Y) \simeq f_\#(\mathbb{1}_Y)$ [CD12, Example 2.4.3(2), Definition 2.4.24 and Proposition 2.4.31]. Let us write $\alpha_{X,Y} : f_\# \mathbb{1}_Y \to D(f_\# \mathbb{1}_Y)$ for this canonical isomorphism. We can then form the commutative diagram

$$\begin{tikzcd} D(f_\# \mathbb{1}_Y) & f_\# \mathbb{1}_Y \arrow[l, "\alpha_{X,Y}"'] \\ D(\mathbb{1}_X) \arrow[u, "D(f)"] & \mathbb{1}_X \arrow[l, "\alpha_{X,X}"] \arrow[u, "\mathrm{tr}_f"'] \end{tikzcd}$$

where tr_f is defined so that the diagram commutes. This is the *duality transfer* of f as defined in [RØ08, § 2.3].

Now suppose that k is a perfect field. Recall that then $\mathbf{SH}(k)$ has a t-structure. To define it, for $E \in \mathbf{SH}(k)$ denote by $\underline{\pi}_i(E)_j$ the Nisnevich sheaf associated with the presheaf $X \mapsto [\Sigma^\infty X_+[i], E \wedge \mathbb{G}_m^{\wedge j}]$. Then $E \in \mathbf{SH}(k)_{\geqslant 0}$ if and only if $\underline{\pi}_i(E)_j = 0$ for all $i < 0$ and all $j \in \mathbb{Z}$. This indeed defines a t-structure [Mor03, § 5.2], and then its heart can be described explicitly: it is equivalent to the category of *homotopy modules* [Mor03, Theorem 5.2.6].

Let $F_* \in \mathbf{SH}(k)$ is a homotopy module, which we identify with an element in the heart of the homotopy t-structure. Given a finite étale morphism $f : Y \to X$ of essentially k-smooth schemes, write $s : X \to \mathrm{Spec}(k)$ for the structure map. We then define $\mathrm{tr}_f : F_n(Y) \to F_n(X)$ as

$$\mathrm{tr}_f(F) := \mathrm{tr}_f^* : [f_\# \mathbb{1}_Y, s^* F \wedge \mathbb{G}_m^{\wedge n}] \to [\mathbb{1}_X, s^* F \wedge \mathbb{G}_m^{\wedge n}].$$

This transfer has the usual properties, of which we recall two.

Proposition 10 (Base change). *Let k be a perfect field, $g : V \to X$ be a morphism of essentially k-smooth schemes and $f : Y \to X$ finite étale. Consider the following cartesian square.*

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "f"] \\ V \arrow[r, "g"'] & X \end{tikzcd}$$

Then for any homotopy module F_*, we have $g^* \mathrm{tr}_f = \mathrm{tr}_p q^* : F_*(Y) \to F_*(V)$.

Proof. Note that $p : W \to V$ is finite étale, so this makes sense. By continuity (of F), we may assume that X and V are smooth (and hence so are Y and W). Write $s : X \to \mathrm{Spec}(k)$ for the structure map.

If $t : A \to B$ is any map in $\mathbf{SH}(X)$, then the canonical diagram

$$\begin{tikzcd} F_*(B) = [B, s^* F] \arrow[r, "\circ t"] \arrow[d, "g^*"'] & {[A, s^* F] = F_*(A)} \arrow[d, "g^*"] \\ F_*(g^* B) = [g^* B, g^* s^* F] \arrow[r, "\circ g(t)"] & {[g^* A, g^* s^* F] = F_*(g^* A)} \end{tikzcd}$$

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commutes, since g^* is a functor. Applying this to $\mathrm{tr}_f : \mathbb{1}_X \to f_\# \mathbb{1}_Y$ it is enough to prove that $g^*(\mathrm{tr}_f) = \mathrm{tr}_p$ under the canonical identifications.

Let $f_+ : f_\# \mathbb{1}_Y \simeq \Sigma^\infty_X Y_+ \to \Sigma^\infty_X X_+ = \mathbb{1}_X$ be the canonical map (so that $\mathrm{tr}_f = D(f_+)$ via $\alpha_{X,Y}$), and similarly for p_+. Then $g^*(f_+) \simeq p_+$ and consequently $g^*(D(f_+)) \simeq D(p_+)$. It thus remains to show that $\alpha_{\bullet,\bullet}$ is natural, i.e. that $g^* \alpha_{X,Y} = \alpha_{V,W} : \Sigma^\infty_V W_+ \to D(\Sigma^\infty_V W_+)$.

For this we use the notation of [CD12, Example 2.4.3(2), Definition 2.4.24 and Proposition 2.4.31]. The isomorphism $\alpha_{X,Y} : f_\# \mathbb{1} \to D(f_\# \mathbb{1})$ is factored into the isomorphisms $D(f_\# \mathbb{1}) \to f_* \mathbb{1}$, the Thom transformation $f_\# \Omega_f \mathbb{1} \to f_* \mathbb{1}$ [CD12, Definition 2.4.21] and $\Omega_f \mathbb{1} \to \mathbb{1}$. All of these are natural in the required sense. □

Lemma 11 (Commutation of transfer with external product). *Let $f : X' \to X$ and $g : Y' \to Y$ be finite étale. Then*

$$\begin{aligned} s_{X\times Y\#}(\mathrm{tr}_{f\times g}) &= s_{X\#}(\mathrm{tr}_f) \wedge s_{Y\#}(\mathrm{tr}_g) : \Sigma^\infty (X' \times Y')_+ \simeq \Sigma^\infty X'_+ \wedge \Sigma^\infty Y'_+ \\ &\to \Sigma^\infty (X \times Y)_+ \simeq \Sigma^\infty X_+ \wedge \Sigma^\infty Y_+. \end{aligned}$$

Here we write $s_X : X \to \mathrm{Spec}(k)$ for the canonical map, and similarly for $Y, X \times Y$.

Proof. Write $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ for the projections. I claim that the following diagram commutes up to natural isomorphism.

$$\begin{tikzcd} \mathbf{SH}(X) \times \mathbf{SH}(Y) \arrow[r, "p_X^* \wedge p_Y^*"] \arrow[d, "s_{X\#} \wedge s_{Y\#}"'] & \mathbf{SH}(X \times Y) \arrow[d, "s_{X\times Y\#}"'] \\ \mathbf{SH}(k) \arrow[r, equal] & \mathbf{SH}(k) \end{tikzcd}$$

To prove the claim, first note that there is, for $T \in \mathbf{SH}(X), U \in \mathbf{SH}(Y)$, a natural map $s_{X\times Y\#}(p_X^* T \wedge p_Y^* U) \to s_{X\#} T \wedge s_{y\#} U$, which can be obtained by adjunctions, using that the pullback functors are monoidal, and that $s_{X\times Y} = s_X \circ p_X$ (and similarly for Y). Then to prove that the comparison map is an isomorphism it suffices to consider $T = \Sigma^\infty X', U = \Sigma^\infty Y'$ for $X' \to X$ smooth any $Y' \to Y$ smooth (note that all our functors are left adjoints and so commute with arbitrary sums, and objects of the forms T, U are generators). But then the claim boils down to

$$X' \times_k Y' \cong (X' \times Y) \times_{X\times Y} (X \times Y')$$

which is clear.

To prove the lemma, we now specialise to $f : X' \to X$ and $g : Y' \to Y$ finite étale. Then

$$\mathrm{tr}_{f\times g} = s_{X\times Y\#}(D\Sigma^\infty_{X\times Y}(f \times g)_+).$$

Note that

$$\Sigma^\infty_{X\times Y}(f \times g)_+ = p_X^* \Sigma^\infty_X f_+ \wedge p_Y^* \Sigma^\infty_Y g_+.$$

Since p_X^*, p_Y^* are monoidal we compute

$$\mathrm{tr}_{f\times g} = s_{X\times Y\#} p_X^* D\Sigma^\infty_X f_+ \wedge p_Y^* D\Sigma^\infty_Y g_+ = s_{X\#} D\Sigma^\infty_X f_+ \wedge s_{Y\#} D\Sigma^\infty_Y g_+,$$

where in the last equality we have used the claim. Since $s_{X\#} D\Sigma^\infty_X f_+ = \mathrm{tr}_f$ by definition (and similarly for Y), this is what we wanted to prove. □

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Recall also the homotopy module $\underline{K}_*^{MW} = \underline{\pi}_0(\mathbb{1})_*$ of Milnor–Witt K-theory [Mor12, ch. 3]. Every homotopy module F_* is a module over $\underline{K}_*^{MW}$ in the sense that there are natural pairings $\underline{K}_*^{MW}(X) \otimes F_*(X) \to F_{*+*}(X)$.

Corollary 12 (Projection formula). *Let k be a perfect field, $f : Y \to X$ a finite étale morphism of essentially k-smooth schemes, and F_* a homotopy module. Then for $a \in \underline{K}_*^{MW}(Y)$ and $b \in F_*(X)$ we have $\mathrm{tr}_f(af^*b) = \mathrm{tr}_f(a)b$. Similarly for $a \in \underline{K}_*^{MW}(X)$ and $b \in F_*(Y)$ we have $\mathrm{tr}_f(f^*(a)b) = a\mathrm{tr}_f(b)$.*

Proof. The usual proof works, see for example [CF17, Proof of Corollary 3.5]. We review it. We only show the first statement, the second is similar. Consider the cartesian square

$$\begin{tikzcd} Y \arrow[r, "(\mathrm{id} \times f)\delta_Y"] \arrow[d, "f"'] & Y \times X \arrow[d, "f \times \mathrm{id}"] \\ X \arrow[r, "\delta_X"'] & X \times X \end{tikzcd}$$

where $\delta_X : X \to X \times X$ is the diagonal and similarly for Y. We have the map $\beta : \Sigma^\infty Y_+ \wedge \Sigma^\infty X_+ \to \underline{K}_*^{MW} \wedge F \to F$, where $\underline{K}_*^{MW} \wedge F \to F$ is the module structure and the first map is the tensor product of $\Sigma^\infty Y_+ \to \underline{K}_*^{MW}$ (corresponding to a) and $\Sigma^\infty X_+ \to F$ (corresponding to b). This defines an element $\beta \in F(Y \times X)$. We have $\mathrm{tr}_f((\mathrm{id} \times f)\delta_Y)^*\beta = \mathrm{tr}_f(af^*b)$ and $\delta_X^* \mathrm{tr}_{f \times \mathrm{id}_Y} \beta = \mathrm{tr}_f(a)b$ (the latter since $\mathrm{tr}_{\mathrm{id}} = \mathrm{id}$ and $\mathrm{tr}_{f \times g}(x \otimes y) = \mathrm{tr}_f(x) \otimes \mathrm{tr}_g(y)$ by Lemma 11). These two elements are equal by the base change formula, i.e. Proposition 10. $\square$

5. Recollections on pre-motivic categories

The six functors formalism [CD12, § A.5] is a very strong, and very general, duality theory. As such it is no surprise that proving that any theory satisfies it requires some work. Fortunately it is now possible to reduce this to establishing a few axioms.

Let $\mathcal{S}$ be a base category of schemes. Recall that a *pre-motivic category* $\mathcal{M}$ over $\mathcal{S}$ consists of [CD13, Definition A.1.1] a pseudofunctor $\mathcal{M}$ on $\mathcal{S}$, taking values in triangulated, closed symmetric monoidal categories. Often these categories will be obtained as the homotopy categories of a pseudofunctor taking values in suitable Quillen model categories and left Quillen functors. For $f : X \to Y \in \mathcal{S}$, the functor $\mathcal{M}(f) : \mathcal{M}(Y) \to \mathcal{M}(X)$ is denoted f^*. For any f, the functor f^* has a triangulated right adjoint f_* (which is not required to be monoidal). If f is smooth, then f^* has a triangulated left adjoint $f_\#$ (also not required to be monoidal). Moreover, $\mathcal{M}$ needs to satisfy smooth base change and the smooth projection formula, in the following sense.

Let

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r, "p"] & S \end{tikzcd}$$

be a cartesian square in $\mathcal{S}$, with p smooth. Then smooth base change means that the natural transformation $q_\# g^* \to f^* p_\#$ is required to be a natural isomorphism.

Finally, let $f : Y \to X$ be a smooth morphism in $\mathcal{S}$. Then the smooth projection formula means that, for $E \in \mathcal{M}(X)$ and $F \in \mathcal{M}(Y)$ we have $f_\#(F \otimes f^*E) \simeq f_\#(F) \otimes E$, via the canonical map.

Here are some further properties a pre-motivic category can satisfy. We say $\mathcal{M}$ satisfies the *homotopy property* if for every $X \in \mathcal{S}$ the natural map $p_\# \mathbb{1} \to \mathbb{1} \in \mathcal{M}(X)$ is an isomorphism, where $p : \mathbb{A}^1 \times X \to X$ is the canonical map.

Let now $q : \mathbb{P}^1 \times X \to X$ be the canonical map. We say that $\mathcal{M}$ satisfies the *stability property* if the cone of the canonical map $q_\# \mathbb{1} \to \mathbb{1} \in \mathcal{M}(X)$ is a $\otimes$-invertible object. In this case we write $\mathbb{1}(1) = fib(q_\# \mathbb{1} \to \mathbb{1})[-2]$ and then as usual $E(n) = E \otimes \mathbb{1}(1)^{\otimes n}$ for $n \in \mathbb{Z}, E \in \mathcal{M}(X)$.

Finally, let $X \in \mathcal{S}$, $j : U \to X \in \mathcal{S}$ an open immersion, and $i : Z \to \mathcal{S}$ a complementary closed immersion. Then for $E \in \mathcal{M}(U)$ there are the adjunction maps

$$j_\# j^* E \to E \to i_* i^* E.$$

We say that $\mathcal{M}$ satisfies the *localisation property* if these maps are always part of a distinguished triangle.

One then has the following fundamental result. It was discovered by Voevodsky, first worked out in detail by Ayoub, and then formalised by Cisinski–Déglise.

Theorem 13 (Ayoub, Cisinski–Déglise). *Let $\mathcal{S}$ be the category of Noetherian schemes of finite dimension and $\mathcal{M}$ a pre-motivic category which satisfies the homotopy property, the stability property, and the localisation property. Then if $\mathcal{M}(X)$ is a well-generated triangulated category for every X, $\mathcal{M}$ satisfies the full six functors formalism.*

Proof. This is proved for 'adequate categories of schemes' in [CD12, Theorem 2.4.50], of which Noetherian finite dimensional schemes are an example. □

One further property we will make use of is *continuity*. This can be formulated as follows. Let $\{S_\alpha\}_{\alpha \in A}$ be an inverse system in $\mathcal{S}$, where all the transition morphisms are affine and the limit $S := \lim_\alpha S_\alpha$ exists in $\mathcal{S}$. Write $p_\alpha : S \to S_\alpha$ for the canonical projection. Let $E \in \mathcal{M}(S_{\alpha_0})$ for some $\alpha_0 \in A$ and write for $\alpha > \alpha_0$, $E_\alpha = (S_\alpha \to S_{\alpha_0})^* E$. We say that $\mathcal{M}$ satisfies the continuity property if for every affine inverse system S_α as above, every E and every $i \in \mathbb{Z}$ the canonical map

$$\mathrm{colim}_{\alpha > \alpha_0} [\mathbb{1}(i), E_\alpha]_{\mathcal{M}(S_\alpha)} \to [\mathbb{1}(i), p_{\alpha_0}^* E]_{\mathcal{M}(S)}$$

is an isomorphism.

We in particular use the following consequence of continuity and localisation.

Corollary 14. *Suppose that $\mathcal{M}$ be a pre-motivic category over $\mathcal{S}$ (where $\mathcal{S}$ contains all Henselizations of its schemes), coming from a pseudofunctor valued in model categories. Assume that $\mathcal{M}$ satisfies continuity and localisation.*

Let $E \in \mathcal{M}(X)$, where X is Noetherian of finite dimension. Then $E \simeq 0$ if and only if for every morphism $f : \mathrm{Spec}(k) \to X$ with k a field we have $f^* E \simeq 0$.

Proof. By localisation, we may assume that X is reduced (see for example [CD12, Proposition 2.3.6(1)]). By [CD12, Proposition 4.3.9] (this result requires $\mathcal{M}$ to come from a model category) we may assume that X is (Henselian) local with closed point x and open complement U. By localisation, it suffices to show that $E|_x \simeq 0$ and $E|_U \simeq 0$. The former holds by assumption, and the latter by induction on the dimension. This concludes the proof. □

Example. The pseudofunctors $X \mapsto \mathbf{SH}(X)$ and $X \mapsto D_{\mathbb{A}^1}(X)$ satisfy the six functors formalism and continuity (for the base category of Noetherian finite dimensional schemes) [Ayo07, CD12].

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6. Recollections on monoidal bousfield localisation

Let $\mathcal{M}$ be a monoidal model category and $\alpha : Y' \to Y \in \mathcal{M}$ a morphism. We wish to 'monoidally invert α', by which we mean passing to a model category $L^{\otimes}_{\alpha}\mathcal{M}$ obtained by localising $\mathcal{M}$ and such that for every $T \in L^{\otimes}_{\alpha}\mathcal{M}$ the induced map $\alpha_T : T \otimes^L Y' \to T \otimes^L Y$ is a weak equivalence. We will also write $L^{\otimes}_{\alpha}\mathcal{M} =: \mathcal{M}[\alpha^{-1}]$ and even $\mathrm{Ho}(\mathcal{M}[\alpha^{-1}]) =: \mathrm{Ho}(\mathcal{M})[\alpha^{-1}]$.

The monoidal α-localisation exists very generally. Suppose that Y' and Y are cofibrant, and that $\mathcal{M}$ admits a set of cofibrant homotopy generators G (for example $\mathcal{M}$ combinatorial [Bar10, Corollary 4.33]). Let $H_\alpha = \{Y' \otimes T \xrightarrow{\alpha\otimes \mathrm{id}} Y \otimes T \mid T \in G\}$. When no confusion can arise, we will denote H_α just by H. Then the Bousfield localisation $L_H\mathcal{M}$, if it exists (for example if $\mathcal{M}$ is left proper and combinatorial) is $\mathcal{M}[\alpha^{-1}]$. We will call H_α-local objects α-local. As a further sanity check, the model category $L_H\mathcal{M}$ is still monoidal as follows from [Bar10, Proposition 4.47].

The situation simplifies somewhat if Y' and Y are invertible and $\mathcal{M}$ is stable. Then we may as well assume that $Y' = \mathbb{1}$. Given $T \in \mathcal{M}$ cofibrant we can consider the directed system

$$T \cong T \otimes \mathbb{1} \xrightarrow{\mathrm{id}\otimes\alpha} T \otimes Y \cong T \otimes Y \otimes \mathbb{1} \to T \otimes Y^{\otimes 2} \to \cdots$$

and its homotopy colimit $T[\alpha^{-1}] := \mathrm{hocolim}_n\, T \otimes X^{\otimes n}$. More generally, if T is not cofibrant, we can either first cofibrantly replace it, or use the derived tensor product. Either way, we denote the result still by $T[\alpha^{-1}]$. The main point of this section is to show that under suitable conditions, $T[\alpha^{-1}]$ is the α-localisation of T.

Clearly this is only a reasonable expectation under some compact generation assumption. More generally, one would expect a transfinite iteration of α. Since all our applications will be in compactly generated situations, we refrain from giving the more general argument.

Recall that by a set of compact homotopy generators G for $\mathcal{M}$ we mean a set of (usually cofibrant) objects $G \subset Ob(\mathcal{M})$ such that $\mathcal{M}$ is generated by the objects in G under homotopy colimits, and such that for any directed system $X_1 \to X_2 \to \cdots \in \mathcal{M}$ and $T \in G$, the canonical map $\mathrm{hocolim}_i \operatorname{Map}^d(T, X_i) \to \operatorname{Map}^d(T, \mathrm{hocolim}_i X_i)$ is an equivalence.

Lemma 15. *Let $\alpha : \mathbb{1} \to Y$ be a map between objects in a symmetric monoidal, stable model category such that Y is invertible (in the homotopy category). Assume that $\mathcal{M}$ has a set of compact homotopy generators G, and that $\mathcal{M}[\alpha^{-1}]$ exists.*

Then for each $U \in \mathcal{M}$ the object $U[\alpha^{-1}]$ is α-local and α-locally weakly equivalent to U. In other words, $U \mapsto U[\alpha^{-1}]$ is an α-localisation functor.

Also G defines a set of compact homotopy generators for $\mathcal{M}[\alpha^{-1}]$.

Proof. We first show that the images of G in $\mathrm{Ho}(\mathcal{M}[\alpha^{-1}])$ are compact homotopy generators. Generation is clear, and for homotopy compactness it is enough to show that a filtered homotopy colimit of α-local objects is α-local. But this follows from homotopy compactness of $T \otimes Y^{\otimes n}$ (for $T \in G$ and $n \in \{0, 1\}$) and definition of α-locality.

In a model category $\mathcal{N}$ with compact homotopy generators, if $T_1 \to T_2 \to \cdots$ is a directed system of weak equivalences then $\mathrm{hocolim}_i\, T_i$ is weakly equivalent to T_1. (This follows from the same result in the category of simplicial sets.) Thus $U[\alpha^{-1}]$ is α-locally weakly equivalent to U.

It remains to see that $U[\alpha^{-1}]$ is α-local. This follows from the next two lemmas. ☐

In the above lemma, we have defined an object X to be α-local if for all $T \in \mathcal{M}$ the induced map $\alpha^* : \operatorname{Map}^d(T \otimes^L Y, X) \to \operatorname{Map}^d(T, X)$ is an equivalence, because this is the way Bousfield localisation works. Another intuitively appealing property would be for the canonical map $X \to X \otimes Y$ to be an equivalence. As the next lemma shows, these two notions agree in our case.

Lemma 16. *Let $\mathcal{M}$ be a symmetric monoidal model category and $\alpha : \mathbb{1} \to Y$ a morphism with Y invertible.*

Call an object $X \in \mathcal{M}$ α'-local if $X \to X \otimes^L Y$ is a weak equivalence. Then X is α-local if and only if X is α'-local, if and only if X is $\alpha \otimes \alpha$-local.

Proof. We shall show that (1) X is α-local if and only if it is $\alpha \otimes \alpha$-local, (2) X is α'-local if and only if it is $(\alpha \otimes \alpha)'$-local, (3) X is α'-local if it is α-local and (4) X is $\alpha \otimes \alpha$-local if it is $(\alpha \otimes \alpha)'$-local.

All tensor products and mapping spaces will be derived in this proof.

(1) Consider the string of maps

$$\mathrm{Map}(T \otimes Y^{\otimes 3}, X) \to \mathrm{Map}(T \otimes Y^{\otimes 2}, X) \to \mathrm{Map}(T \otimes Y, X) \to \mathrm{Map}(T, X).$$

If X is $\alpha \otimes \alpha$-local, then the composite of any two consecutive maps is an equivalence, and hence all maps are equivalences by 2-out-of-6. Consequently X is α-local. The converse is clear.

(2) Consider the string of maps

$$X \to X \otimes Y \to X \otimes Y^{\otimes 2} \to X \otimes Y^{\otimes 3}.$$

If X is $(\alpha \otimes \alpha)'$-local then so is $Z \otimes X$ for any Z, since (derived) tensor product preserves weak equivalences. It follows that $X \otimes Y$ is $(\alpha \otimes \alpha)'$-local, and hence the composite of any two consecutive maps is an equivalence. Again by 2-out-of-6 this implies that X is α'-local. The converse is clear.

(3) An object X is α-local if (and only if) for all $T \in \mathcal{M}$ the map $\mathrm{Map}(T \otimes Y, X) \to \mathrm{Map}(T, X)$ is a weak equivalence (of simplicial sets). In particular $T \to T \otimes Y$ is an α-local weak equivalence for all T. It also follows that $X \otimes Y$ is α-local if X is (here we use invertibility of Y). Since $X \to X \otimes Y$ is an α-local weak equivalence, it is a weak equivalence if X (and hence $X \otimes Y$) is α-local. Thus X is α'-local if it is α-local.

(4) For any simplicial set K we have $[K, \mathrm{Map}(T, X)] = [K \otimes T, X]$ (using a framing if the model category is not simplicial). It follows that X is α-local if and only if for all $T \in \mathcal{M}$ the map $\alpha^* : [T \otimes Y, X] \to [T, X]$ is an isomorphism. In particular, this property can be checked entirely in the homotopy category of $\mathcal{M}$, in which we will work from now on.

Suppose, for now, that X is α'-local. (We will find that our strategy does not work, but it will work for $\alpha \otimes \alpha$, and this is all that is left to prove.) We can choose an inverse equivalence $\beta : X \otimes Y \to X$. We consider the map $\overline{\beta} : [T, X] \to [T \otimes Y, X]$ sending $f : T \to X$ to $T \otimes Y \xrightarrow{f \otimes \mathrm{id}} X \otimes Y \xrightarrow{\beta} X$. We would like to say that $\overline{\beta}$ is inverse to α^*. Given $f : T \to X$ we get the following commutative diagram.

$$\begin{tikzcd} T \otimes Y \arrow[r, "f \otimes \mathrm{id}"] & X \otimes Y \\ T \arrow[u, "\alpha"] \arrow[r, "f"] & X \arrow[u, "\alpha"'] \end{tikzcd}$$

Consequently $\alpha_* \alpha^* \overline{\beta} = \alpha_* : [T, X] \to [T, X \otimes Y]$ and thus $\alpha^* \overline{\beta} = \mathrm{id}$ (note that α_* means composition with $X \to X \otimes Y$, which is an isomorphism).

The problem is with showing that $\overline{\beta} \alpha^* = \mathrm{id}$. For this we fix $f : T \otimes Y \to X$ and consider the following diagram.

$$\begin{tikzcd} T \otimes Y \otimes Y \arrow[r, "f \otimes \mathrm{id}"] & X \otimes Y \\ T \otimes \mathbb{1} \otimes Y \arrow[u, "\mathrm{id} \otimes \alpha \otimes \mathrm{id}"] \arrow[r, "f"] & X \arrow[u, "\alpha"'] \end{tikzcd}$$

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If it commutes for all such f, then $\overline{\beta}\alpha^* = \mathrm{id}$. But this is not clear; the two paths differ by a switch of Y.

However, in any symmetric monoidal category, the switch isomorphism on the *square* of an invertible object is the identity [Dug14, Propositions 4.20 and 4.21]. Consequently our argument works for $\alpha \otimes \alpha$, and this is what we set out to prove. $\square$

Remark. The assumption that Y is invertible is necessary in general for the above result. For example, if $\mathcal{M}$ is a *cartesian* symmetric monoidal model category, then there cannot be any α'-local objects unless $* = \mathbb{1} \to Y$ is already an equivalence.

Lemma 17. *Notations and assumptions as in Lemma 15.*

For any (cofibrant) $X \in \mathcal{M}$, the object $X[\alpha^{-1}]$ is α-local.

Proof. By the previous lemma, it suffices to show that $X[\alpha^{-1}]$ is $(\alpha\otimes\alpha)'$-local. Clearly $X[\alpha^{-1}] \simeq X[(\alpha\otimes\alpha)^{-1}]$, i.e. we may assume without loss of generality that Y is a square, and so its switch isomorphism (in the homotopy category) is the identity.

Since tensor product commutes with colimits (in each variable) we have $X[1/f] \simeq X \otimes^L \mathbb{1}[1/f]$, and we can simplify notation by assuming without loss of generality that $X = \mathbb{1}$.

What we need to prove is that the following diagram induces an equivalence on homotopy colimits.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "f_1"] \arrow[d, "h_1"'] & G \arrow[r, "f_2"] \arrow[d, "h_2"'] & G\otimes G \arrow[r, "f_3"] \arrow[d, "h_3"'] & G\otimes G\otimes G \arrow[r] \arrow[d, "h_4"'] & \cdots \\
G \arrow[r, "f_2'"] & G\otimes G \arrow[r, "f_3'"] & G\otimes G\otimes G \arrow[r, "f_4'"] & G\otimes G\otimes G \arrow[r] & \cdots
\end{tikzcd}$$

Because of the domains and codomains, it is tempting to guess that $f_i \simeq h_i \simeq f_i'$. Here we write $f \simeq g$ to mean that the maps become equal in the homotopy category. We claim that this guess is correct. Then if T is any homotopy compact object, applying $[T, \bullet]$ to our diagram we get a diagram of abelian groups which we need to show induces an isomorphism on colimits. Homotopic maps become equal when applying $[T, \bullet]$, and then the desired result follows from an easy diagram chase. By compact generation and stability, this will conclude the proof.

It remains to prove the claim. For this we may work entirely in the homotopy category, which we will do from now on. It is easy to see that indeed $f_i = h_i$. For general Y, it would not be true that $f_i' = f_i$; one may check that the maps differ by appropriate switches of Y. However, we have assumed that the switch on Y *is* the identity, so indeed $f_i = f_i'$ as well. $\square$

Remark. The stability assumption was used in the above proof in the following form: if $A \to B$ is any morphism in $\mathcal{M}$ and $[T, A] \to [T, B]$ is an isomorphism for all homotopy compact T, then $A \to B$ is a weak equivalence. This fails for example in the homotopy category of spaces.

The stability assumption is in fact necessary for the above result. The author learned the following counterexample from Marc Hoyois: let $\mathcal{M}$ be the model category of small, stable ∞-categories, $Y = \mathbb{1}$ the category of finite spectra and $\alpha = 2$, i.e. the functor which sends a finite spectrum s to $s \oplus s$. Then $\mathcal{C} \in \mathcal{M}$ is α'-local only if it is trivial. Indeed for $c \in \mathcal{C}$ the map $[c, c] \to [c \oplus c, c \oplus c]$ needs to be an isomorphism, which forces $c \simeq 0$. But one may show that $\mathbb{1}[1/\alpha]$ is not the zero category, and so is not α'-local (let alone α-local).

See [Hoy17, Theorem 3.8] for a criterion that can be applied in unstable situations.

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7. The theorem of Jacobson and ρ-stable homotopy modules

Throughout this section, k is a field of characteristic zero. Recall that the real étale topology is finer than the Nisnevich topology; in particular every real étale sheaf is a Nisnevich sheaf.

THEOREM 18 (Jacobson [Jac17, Theorem 8.5]). *There is a canonical isomorphism (in* $\mathrm{Shv}_{\mathrm{Nis}}(Sm(k))$*)*

$$\operatorname{colim}_n \underline{I}^n \to a_{\text{rét}}\underline{\mathbb{Z}},$$

where the transition maps $\underline{I}^n \to \underline{I}^{n+1}$ *are given by multiplication with* $2 = \langle 1, 1\rangle \in \underline{I}$.

Here $\underline{I}$ denotes the sheaf of fundamental ideals on $Sm(k)_{\mathrm{Nis}}$, i.e. the sheaf associated with the presheaf $X \mapsto I(X)$, where $I(X)$ is the fundamental ideal of the Witt ring of X [Kne77]. We similarly write $\underline{W}$ for the sheaf of Witt rings, etc.

Let us recall the construction of the isomorphism in Jacobson's theorem. If $\phi \in W(K)$, where K is a field, and p is an ordering of K, then there is the signature $\sigma_p(\phi) \in \mathbb{Z}$. If $\phi \in W(X)$, define $\sigma(\phi) : R(X) \to \mathbb{Z}$ as follows. For $(x, p) \in R(X)$ put $\sigma(\phi)(x, p) = \sigma_p(\phi|_x)$. Then one shows that $\sigma(\phi)$ is a *continuous* function from $R(X)$ to $\mathbb{Z}$, i.e. an element of $H^0_{\text{rét}}(X, \mathbb{Z})$.

Next if $\phi \in I(k)$ then $\sigma_p(\phi) \in 2\mathbb{Z}$. Consequently if $\phi \in I(X)$ also $\sigma(\phi) \in 2H^0_{\text{rét}}(X, \mathbb{Z})$. We may thus define $\tilde{\sigma}(\phi) = \sigma(\phi)/2$ and in this way we obtain $\tilde{\sigma} : I(X) \to H^0_{\text{rét}}(X, \mathbb{Z})$. Similarly we get $\tilde{\sigma} : I^n(X) \to H^0_{\text{rét}}(X, \mathbb{Z})$ with $\tilde{\sigma}(\phi) = \sigma(\phi)/2^n$ for $\phi \in I^n(X)$. For each n there is a commutative diagram as follows.

$$\begin{array}{ccc} I^n(X) & \xrightarrow{\tilde{\sigma}} & H^0_{\text{rét}}(X, \mathbb{Z}) \\ \downarrow 2 & & \| \\ I^{n+1}(X) & \xrightarrow{\tilde{\sigma}} & H^0_{\text{rét}}(X, \mathbb{Z}) \end{array}$$

Consequently there is an induced map $\tilde{\sigma} : \operatorname{colim}_n I^n(X) \to H^0_{\text{rét}}(X, \mathbb{Z})$. The claim is that this is an isomorphism after sheafifying, i.e. for X local.

COROLLARY 19. *Let* $\underline{K}^{MW}_n$ *denote the nth unramified Milnor–Witt K-theory sheaf. Then there is a canonical isomorphism* $\operatorname{colim}_n \underline{K}^{MW}_n \to a_{\text{rét}}\mathbb{Z}$*. Here the colimit is along multiplication with* $\rho := -[-1] \in K^{MW}_1(k)$.

Proof. Recall the element $h \in K^{MW}_0(k)$ with the following properties: $\underline{K}^{MW}_n/h = \underline{I}^n$ [Mor04, Théorème 2.1] and for $a \in K^{MW}_1(k)$ we have $a^2 h = 0$ [Mor12, Corollary 3.8] (this relation is the analogue of the fact that in a graded commutative ring R_* with $a \in R_1$ we have $a^2 = -a^2$ by graded commutativity, so $2a^2 = 0$). Consequently $\rho^2 h = 0$ and so $\operatorname{colim}_n \underline{K}^{MW}_n \to \operatorname{colim}_n \underline{I}^n$ is an isomorphism. It remains to note that the image of ρ in $K^{MW}_1/h(k) \cong I(k)$ is given by $-(\langle -1\rangle - 1) = 2 \in I(k) \subset W(k)$, so the induced transition maps in the colimit are precisely those used in Jacobson's theorem. □

Note that the sheaves $\underline{I}^n$ form a homotopy module, namely the homotopy module of Witt K-theory [Mor12, Examples 3.33 and 3.33] [Mor04, Theoreme 2.1]; see also [GSZ16]. Consequently they have transfers for finite separable field extensions. The sheaf $a_{\text{rét}}\underline{\mathbb{Z}}$ also has transfers for finite (separable) field extensions. Indeed if l/k is finite then $\mathrm{Sper}(l) \to \mathrm{Sper}(k)$ is a finite-sheeted local homeomorphism [Sch85, 3.5.6 Remark(ii)] and hence we transfer by 'taking sums over the values at the preimages'.

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Lemma 20. *The isomorphism* $\mathrm{colim}_n \underline{I}^n \to a_{\mathrm{r\acute{e}t}}\underline{\mathbb{Z}}$ *is compatible with transfers on fields.*

Proof. It suffices to prove that for a field k, the total signature $W(k) \to H^0_{\mathrm{r\acute{e}t}}(k, \mathbb{Z})$ is compatible with transfer. Let l/k be a finite extension and R/k a real closure. There is a commutative diagram

$$\begin{tikzcd} W(l) \arrow[r] \arrow[d, "\mathrm{tr}"'] & W(R \otimes_k l) \arrow[d, "\mathrm{tr}"'] \\ W(k) \arrow[r] & W(R) \end{tikzcd}$$

by the base change formula, i.e. Proposition 10. Note that $\mathrm{Sper}(R \otimes_k l)$ is the fibre of $\mathrm{Sper}(l) \to \mathrm{Sper}(k)$ over the ordering corresponding to the inclusion $k \subset R$. Consequently we also have the following commutative diagram.

$$\begin{tikzcd} H^0_{\mathrm{r\acute{e}t}}(l, \mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{tr}"'] & H^0_{\mathrm{r\acute{e}t}}(R \otimes_k l, \mathbb{Z}) \arrow[d, "\mathrm{tr}"'] \\ H^0_{\mathrm{r\acute{e}t}}(k, \mathbb{Z}) \arrow[r] & H^0_{\mathrm{r\acute{e}t}}(R, \mathbb{Z}) \end{tikzcd}$$

Since the signature maps are determined by pulling back to a real closure, this means that we may assume that k is real closed. (Since both sides we are trying to prove equal are additive, we may still assume that l is a field.) But then either $l = k$ or $l = k[\sqrt{-1}]$. In the former case the transfer on both sides is the identity, and in the latter it is zero. $\square$

We will make good use of the following observation.

Corollary 21. *Let* $l_1, \ldots, l_r/k$ *be finite extensions such that* $\coprod_i \mathrm{Spec}(l_i) \to \mathrm{Spec}(k)$ *is a rét-cover. Then* $\mathrm{tr} : \bigoplus_i H^0_{\mathrm{r\acute{e}t}}(l_i, \mathbb{Z}) \to H^0_{\mathrm{r\acute{e}t}}(k, \mathbb{Z})$ *is surjective.*

Proof. The map $\coprod_i \mathrm{Sper}(l_i) \to \mathrm{Sper}(k)$ is a surjective local homeomorphism of compact, Hausdorff, totally disconnected spaces [Sch85, Theorem 3.5.1 and Remarks 3.5.6]. The result thus follows from the next lemma. $\square$

Lemma 22. *Let* $\phi : X \to Y$ *be a surjective local homeomorphism of compact, Hausdorff, totally disconnected spaces. Then* ϕ *has finite fibers, and the 'summing over preimages' transfer* $H^0(X, \mathbb{Z}) \to H^0(Y, \mathbb{Z})$ *is surjective.*

Proof. The claim that ϕ has finite fibers is well known. We include a proof for convenience of the reader: since ϕ is a local homeomorphism the fibers are discrete, since Y is Hausdorff they are closed, and since X is compact they are compact. Now observe that a compact discrete space is finite.

We now prove the surjectivity of the transfer. First we make the following claim: if X is a compact, Hausdorff, totally disconnected space, then given $x \in U \subset X$ with U open, there exists $x \in V \subset U$ such that V is clopen in X. Indeed for $y \neq x$ let U_y be a clopen neighbourhood of y disjoint from x. Then $\bigcup_{y \in X \setminus U} U_y$ is an open cover of the compact (since closed) complement $X \setminus U$. Let $U_1, \ldots, U_n$ be a finite subcover. Then $V = X \setminus \bigcup_i U_i$ works.

Now consider the morphism $\phi : X \to Y$. For $y \in Y$ choose a clopen neighbourhood U_y of $y \in Y$ such that there exists a clopen set $V_y \subset X$ with $\phi(V_y) = U_y$ and $\phi : V_y \to U_y$ a homeomorphism. We will say in this situation that ϕ *splits strongly over* U_y. We note that such

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V_y, U_y exist: since ϕ is a local homeomorphism, there exists $V'_y \subset X$ such that $U'_y := \phi(V_y)$ is an open neighbourhood of y and $\phi : V'_y \to U'_y$ is a homeomorphism. By the claim, we may assume that V'_y is clopen. Now choose a clopen neighbourhood $U_y \subset U'_y$, using the claim again. Then $V_y := \phi^{-1}(U_y) \cap V'_y$ is clopen in X and maps homeomorphically to U_y.

We obtain in this way an open cover $\{U_y\}_{y \in Y}$ of Y. Since Y is compact, we can choose a finite subcover $U_1, \ldots, U_n$. Using that all the U_i are clopen we can refine further until we have found a disjoint clopen cover (replace U_i by $U_i \setminus (U_1 \cup U_2 \cup \cdots \cup U_{i-1})$) over which ϕ splits strongly. (Note that if ϕ splits strongly over a clopen $U \subset Y$, then it also splits strongly over any clopen $U' \subset U$.)

Since $H^0(Y, \mathbb{Z})$ is the set of continuous functions from Y to $\mathbb{Z}$, it suffices to prove that the indicator function $\chi_{U_i} : Y \to \mathbb{Z}$ of the clopen subset U_i is in the image of transfer (because $1 = \sum_i \chi_{U_i}$, and the transfer is additive). But ϕ is strongly split over U_i by construction, so there exists some clopen subset $U \subset X$ such that $\phi : U \to U_i$ is a homeomorphism. Then $\chi_U \in H^0(X, \mathbb{Z})$ and this is taken by transfer to χ_{U_i}, as follows from the explicit description of transfer in terms of 'summing over preimages'. $\square$

We will want to show that certain presheaves are sheaves in the rét-topology. We find it easiest to first develop a criterion for this. We start with the following result, which is surely well known.

Lemma 23. *Let τ be a topology on a category $\mathcal{C}$ and F a presheaf on $\mathcal{C}$ which is τ-separated. Let $X \in \mathcal{C}$ and $U_\bullet, V_\bullet \to X$ be τ-coverings. Suppose that $V_\bullet$ refines $U_\bullet$, i.e. we are given a morphism $f : V_\bullet \to U_\bullet$ over X. Then if F satisfies the sheaf condition with respect to $V_\bullet$, it also satisfies the sheaf condition with respect to $U_\bullet$.*

Proof. The proof can be extracted from the proof of [Sta17, Tag 00VX]. We repeat the argument for convenience. For simplicity, suppose that $U_\bullet$ and $V_\bullet$ use the same indexing set I, and that the refinement is of the form $V_i \to U_i$. We are given $s_i \in F(U_i)$ for each i, such that $s_i|_{U_i \times_X U_j} = s_j|_{U_i \times_X U_j}$, and we need to show that there is a (necessarily unique) $s \in F(X)$ with $s|_{U_i} = s_i$.

Let $t_i = f^* s_i$. Then $t_\bullet$ is a compatible family for the covering $V_\bullet$, and hence there is $s \in F(X)$ with $s|_{V_i} = t_i$ for all i. We need to show that also $s|_{U_i} = s_i$. For this, fix $i_0 \in I$ and consider the coverings $U'_\bullet, V'_\bullet \to U_{i_0}$ obtained by base change. Then $V'_\bullet$ refines $U'_\bullet$. We find that $s_{i_0}|_{U_{i_0} \times_X U_i} = s_i|_{U_{i_0} \times_X U_i}$ by assumption, and hence $f^*(s_{i_0}|_{U_{i_0} \times_X U_i}) = f^*(s_i|_{U_{i_0} \times_X U_i}) = t_i|_{U_{i_0} \times_X V_i} = s|_{U_{i_0} \times_X V_i}$ by construction. But now because F is separated in the τ-topology and $V'_\bullet \to U_{i_0}$ is a cover we conclude that $s_{i_0} = s|_{U_{i_0}}$, as needed. $\square$

Corollary 24. *Let F be a sheaf on $Sm(k)_{\mathrm{Nis}}$. Then F is a sheaf in the rét-topology if and only if F satisfies the sheaf condition for every rét-cover $f : U \to X$, where X is (essentially) smooth, Henselian local and f is finite étale.*

Proof. For this proof, we call a morphism with the properties of f a frét-cover.

The condition is clearly necessary; we show the converse.

$(*)$ We first claim that every rét-cover $U_\bullet \to X$ with X smooth Henselian local can be refined by a frét-cover. We can certainly refine $U_\bullet$ by an affine cover, so assume that each U_i is affine. Then by [Sta17, Tag 04GJ] each U_i splits as $U'_i \coprod U''_i$ with $U'_i \to X$ finite étale and $U''_i \to X$ not hitting the closed point m of X (note that $U_i \to X$ is everywhere quasi-finite). I claim that $U'_\bullet$ is also a rét-cover. Indeed étale morphisms induce open maps on real spectra [Sch94, Proposition 1.8] and $U'_\bullet$ covers $R(m) \subset R(X)$ by construction. But the only open subset of $R(X)$ containing $R(m)$ is all of $R(X)$, by [ABR12, Propositions II.2.1 and II.2.4]. Finally the real spectrum of any ring is quasi-compact [ABR12, II.1.5] whence we can always refine by a finite subcover, and then taking the disjoint union we refine by a singleton cover.

If $X \in Sm(k)$, we write $F_X := F|_{X_{\mathrm{Nis}}}$ for the restriction to the small site. Write $\underline{\mathrm{Hom}}$ for the internal mapping presheaf functor in this category. Recall that $\underline{\mathrm{Hom}}(V, F_X)(V') = F(V \times_X V')$; in particular this functor preserves sheaves.

Let $U_\bullet \to X$ be a rét-cover. To show that $F(X) \to F(U_\bullet) \rightrightarrows F(U_\bullet \times_X U_\bullet)$ is an equaliser diagram (respectively the first map is injective), it is sufficient to show that $F_X \to \underline{\mathrm{Hom}}(U_\bullet, F_X) \rightrightarrows \underline{\mathrm{Hom}}(U_\bullet \times_X U_\bullet, F_X)$ is an equaliser diagram of sheaves (respectively the first map is an injection of sheaves), since limits of sheaves are computed in presheaves. But finite limits (respectively injectivity) are detected on stalks, whence in both situations we may assume that X is Henselian local. $(**)$

Now we show that F is rét-separated. Let $U_\bullet \to X$ be a rét-cover. By the above, to show that $F(X) \to F(U_\bullet)$ is injective we may assume that X is Henselian local. Then $U_\bullet \to X$ is refined by a frét-cover $V \to X$, by $(*)$. But then $F(X) \to F(U_\bullet) \to F(V)$ is injective since F satisfies the sheaf condition for $V \to X$ by assumption, so $F(X) \to F(U_\bullet)$ is injective and F is rét-separated.

Finally let $U_\bullet \to X$ be any rét-cover. We wish to show that F satisfies the sheaf condition for this cover. By $(**)$, we may assume that X is Henselian local. Then $U_\bullet \to X$ is refined by a frét-cover $V \to X$ and F satisfies the sheaf condition with respect to $V \to X$ by assumption, so it satisfies the sheaf condition with respect to $U_\bullet \to X$ by Lemma 23. □

Remark. Using [ABR12, Corollary II.1.15], the claim $(*)$ can be extended as follows: Every rét-cover $U_\bullet \to X$ with X arbitrary is refined by a cover $V'_\bullet \to V_\bullet \to X$, where $V_\bullet \to X$ is a Nisnevich cover and each $V'_i \to V_i$ is a frét-cover.

THEOREM 25. *Let F_* be a homotopy module such that $\rho : F_n \to F_{n+1}$ is an isomorphism for all n. Then F_* consists of rét-sheaves.*

Proof. We apply Corollary 24. Hence let $\phi : U \to X$ be a rét-cover with ϕ finite étale and X essentially smooth, Henselian local. We need to show that F satisfies the sheaf condition with respect to this cover. Note that U is then a finite disjoint union of essentially smooth, Henselian local schemes, by [Sta17, Tag 04GH (1)].

We now use the transfer $\mathrm{tr} : F_*(U) \to F_*(X)$ from § 4. Any homotopy module is a module over K_*^{MW} and satisfies the projection formula with respect to this module structure. It follows from Corollary 19 and our assumption that ρ acts invertibly on F_* that F_* is a module over $a_{\mathrm{r\acute{e}t}}\mathbb{Z}$, and satisfies the projection formula with respect to that module structure.

We know that for a Henselian local ring A with residue field κ, we have $H^0_{\mathrm{r\acute{e}t}}(A, \mathbb{Z}) = H^0_{\mathrm{r\acute{e}t}}(\kappa, \mathbb{Z})$. This follows from [ABR12, Propositions II.2.2 and II.2.4] (the author learned this argument from [KSW16, proof of Lemma 6.4]). Consequently by Corollary 21, Proposition 10 and stability of rét-covers under base change, there exists $a \in H^0_{\mathrm{r\acute{e}t}}(U, \mathbb{Z})$ such that $\mathrm{tr}(a) = 1$.

Now suppose given $b \in F_*(X)$ such that $b|_U = 0$. Then $b = 1b = \mathrm{tr}(a)b = \mathrm{tr}(a \cdot b|_U) = 0$ by the projection formula (i.e. Corollary 12). Consequently $F_*(X) \to F_*(U)$ is injective.

Write $p_1, p_2 : U \times_X U \to U$ for the two projections and suppose given $b \in F_*(U)$ such that $p_1^* b = p_2^* b$. We have to show that there is $c \in F_*(X)$ such that $b = c|_U$. I claim that $c := \mathrm{tr}(ab)$ works. Indeed we have $\mathrm{tr}(ab)|_U = \phi^*(\mathrm{tr}_\phi(ab)) = \mathrm{tr}_{p_2}(p_1^*(ab))$ by Proposition 10. Now $p_1^* b = p_2^* b$ by assumption, and so $\mathrm{tr}_{p_2}(p_1^*(a)p_1^*(b)) = \mathrm{tr}_{p_2}(p_1^*(a)p_2^*(b)) = \mathrm{tr}_{p_2}(p_1^* a)b$ by the projection formula again. Finally $\mathrm{tr}_{p_2}(p_1^* a) = \phi^* \mathrm{tr}_\phi(a) = \phi^* 1 = 1$ by base change again, so we are done. □

8. Preliminary observations

We are now almost ready to prove our main theorems. This section collects some preliminary observations and reductions.

Lemma 15 from § 6 applies in particular to $\mathbf{SH}(S)$ and $D_{\mathbb{A}^1}(S)$ for a Noetherian base scheme S. We will be particularly interested in the case $Y = \mathbb{G}_m$ and $\alpha = \rho : S \to \mathbb{G}_m$ the additive inverse of the morphism corresponding to -1. What the lemma says is that the ρ-localisation can be computed as the obvious colimit.

We write $\mathbf{SH}(S)^{\text{rét}}$ for the real étale localisation of $\mathbf{SH}(S)$ and $D_{\mathbb{A}^1}(S)^{\text{rét}}$ for the real étale localisation of $D_{\mathbb{A}^1}(S)$. There is possibly a slight confusion as to what this means, since it could mean the localisation at desuspensions of real étale (hyper-) covers, or the category obtained by the same procedure as $\mathbf{SH}(S)$ but replacing the Nisnevich topology by the real étale one from the start. This does not actually make a difference:

Lemma 26. *Let $\mathcal{M}$ be a monoidal model category, $T \in \mathcal{M}$ cofibrant and H a set of maps. There is an isomorphism of Quillen model categories*

$$\operatorname{Spt}(L_H\mathcal{M}, T) = L_{H'} \operatorname{Spt}(\mathcal{M}, T),$$

provided that all the localisations exist (e.g. $\mathcal{M}$ left proper and combinatorial). Here $H' = \bigcup_{i\in\mathbb{Z}} \Sigma^{\infty+i} H$ and $\operatorname{Spt}(\mathcal{N}, U)$ denotes the model category of (non-symmetric) U-spectra in $\mathcal{N}$ with the local model structure.

Proof. We follow [Hov01]. Recall that $\operatorname{Spt}(\mathcal{N}, U)$ denotes the category of sequences $(X_1, X_2, X_3, \dots)$ together with bonding maps $X_i \otimes U \to X_{i+1}$, and morphisms the compatible sequences of morphisms. This is firstly provided with a global model structure $\operatorname{Spt}(\mathcal{N}, U)_{\text{gl}}$ in which a map $(X_\bullet) \to (Y_\bullet)$ is a fibration or weak equivalence if and only if $X_i \to Y_i$ is for all i. This is also called a levelwise fibration or weak equivalence. The local model structure is then obtained by localisation at a set of maps which is not important to us, because it only depends on a choice of set of generators of $\mathcal{M}$, and for $L_H\mathcal{M}$ we can just choose the same generators.

Since in any model category $L_{H_1}L_{H_2}\mathcal{N} = L_{H_1\cup H_2}\mathcal{N}$, it is enough to show that $L_{H'} \operatorname{Spt}(\mathcal{M}, T)_{\text{gl}} = \operatorname{Spt}(L_H\mathcal{M}, T)_{\text{gl}}$. Note that an acyclic fibration in $\operatorname{Spt}(L_H\mathcal{M}, T)_{\text{gl}}$ is the same as a levelwise acyclic H-local fibration in $\mathcal{M}$, i.e. a levelwise acyclic fibration. Consequently the cofibrations in $\operatorname{Spt}(L_H\mathcal{M}, T)_{\text{gl}}$ are the same as in $\operatorname{Spt}(\mathcal{M}, T)_{\text{gl}}$, whereas the former has more weak equivalences. Thus the former is a Bousfield localisation of the latter and hence it is enough to show that $L_{H'} \operatorname{Spt}(\mathcal{M}, T)_{\text{gl}}$ and $\operatorname{Spt}(L_H\mathcal{M}, T)_{\text{gl}}$ have the same fibrant objects. An object of $\operatorname{Spt}(L_H\mathcal{M}, T)_{\text{gl}}$ is fibrant if and only if it is levelwise H-locally fibrant. An object E of $L_{H'} \operatorname{Spt}(\mathcal{M}, T)_{\text{gl}}$ is fibrant if and only if it is levelwise fibrant and H'-local, which means that for each $\alpha : X \to Y \in H$ and every $n \in \mathbb{Z}$ the map

$$\operatorname{Map}^d(\Sigma^{\infty+n}\alpha, E) : \operatorname{Map}^d(\Sigma^{\infty+n}Y, E) \to \operatorname{Map}^d(\Sigma^{\infty+n}Y, E)$$

is a weak equivalence. By adjunction, this is the same as $\operatorname{Map}^d(\alpha, E_n)$ being an equivalence, i.e. all E_n being H-local. This concludes the proof. $\square$

Write $\mathbf{SH}^{S^1}(S)$ for the S^1-stable homotopy category (i.e. obtained from motivic spaces by just inverting S^1, but not $\mathbb{G}_m$).

Lemma 27. *There are canonical Quillen equivalences $\mathbf{SH}^{S^1}(S)[\rho^{-1}] \simeq \mathbf{SH}(S)[\rho^{-1}]$ and similarly for the real étale topology.*

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Proof. By Lemma 26 we know that $\mathrm{Spt}(\mathcal{SH}^{S^1}(S)[\rho^{-1}], \mathbb{G}_m) = \mathrm{Spt}(\mathcal{SH}^{S^1}(S), \mathbb{G}_m)[\rho^{-1}] \simeq \mathcal{SH}(S)[\rho^{-1}]$. But the map $\rho : S \to \mathbb{G}_m$ is invertible in $\mathbf{SH}^{S^1}(S)[\rho^{-1}]$ and thus $\mathrm{Spt}(\mathcal{SH}^{S^1}(S)[\rho^{-1}], \mathbb{G}_m) \simeq \mathcal{SH}^{S^1}(S)[\rho^{-1}]$, i.e. inverting an invertible object has no effect [Hov01, Theorem 5.1]. □

We also observe the following.

Proposition 28. *The pseudofunctor $X \mapsto \mathbf{SH}(X)[\rho^{-1}]$ satisfies the full six functors formalism (on Noetherian schemes of finite dimension), compact generation, and continuity.*

Proof. If $i : Z \to X$ is a closed immersion then the functor $i_* : \mathbf{SH}(Z) \to \mathbf{SH}(X)$ commutes with filtered homotopy colimits (being right adjoint to a functor preserving compact objects) and satisfies $i_*(X \otimes \mathbb{G}_m) \simeq i_*(X) \otimes \mathbb{G}_m$ [CD12, A.5.1(6) and (3)]. It follows from the explicit description of ρ-localisation in Lemma 15 that i_* commutes with $L : \mathbf{SH}(X) \to \mathbf{SH}(X)[\rho^{-1}]$. Thus $\mathbf{SH}(X)[\rho^{-1}]$ satisfies localisation, by [CD12, Proposition 2.3.19]. Since $\mathbf{SH}(X)[\rho^{-1}]$ clearly satisfies the homotopy and stability properties, it satisfies the six functors formalism by Theorem 13.

Since $\mathbf{SH}(X)$ is compactly generated so is $\mathbf{SH}(X)[\rho^{-1}]$, by the last sentence of Lemma 15.

For any morphism $f : X \to Y$ the functor $f^* : \mathbf{SH}(Y) \to \mathbf{SH}(X)$ commutes with (filtered) homotopy colimits (being a left adjoint), and consequently it commutes with ρ-localisation, as above. Thus continuity for $\mathbf{SH}(X)[\rho^{-1}]$ follows from continuity for $\mathbf{SH}(X)$. □

For completeness, we include the following rather formal observation. It is not used in the remainder of this text (except that it is restated as part of Theorem 35).

Proposition 29. *The canonical functor $\mathbf{SH}(S)^{\text{rét}} \to \mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ is an equivalence. In other words, ρ is a weak equivalence in $\mathbf{SH}(S)^{\text{rét}}$.*

Proof. I claim that in $\mathbf{SH}(S)^{\text{rét}}$ there is a splitting $\mathbb{G}_m \simeq \mathbb{1} \vee \Delta$ such that the composite $\mathbb{1} \xrightarrow{\rho} \mathbb{G}_m \simeq \mathbb{1} \vee \Delta \to \mathbb{1}$ is the identity. It will follow from Lemma 30 below that $\Delta \simeq 0$, proving this lemma.

Call $a \in \mathcal{O}^\times(X)$ *totally positive* if for every real closed field r and morphism $\alpha : \mathrm{Spec}(r) \to X$ we have $\alpha^*(a) > 0$. Note that in particular any square of a unit is totally positive.

This defines a sub-presheaf $G_+ \subset R_{\mathbb{A}^1\backslash 0}$ of the presheaf represented by $\mathbb{A}^1\backslash 0$. Define G_- analogously using totally negative units. I claim that $a_{\text{rét}} R_{\mathbb{A}^1\backslash 0} = a_{\text{rét}} G_+ \coprod a_{\text{rét}} G_{-1}$. We may prove this on stalks, which are Henselian rings with real closed residue fields [Sch94, (3.7.3)]. If A is such a ring and $a \in A^\times$, then the reduction $\bar{a} \in A/m$ is a unit and so either positive or negative. It follows that either $\bar{a}$ or $-\bar{a}$ is a square, whence either a or $-a$ is a square (A being Henselian of characteristic zero). Consequently a is either totally positive or totally negative, proving the claim.

We may thus define a map $a_{\text{rét}}\mathbb{G}_m \to a_{\text{rét}} S^0 = a_{\text{rét}}(* \coprod *)$ by mapping $a_{\text{rét}} G_+$ to the base point and $a_{\text{rét}} G_-$ to the other point. Since -1 is totally negative this yields an unstable splitting $a_{\text{rét}} S^0 \to a_{\text{rét}} \mathbb{G}_m \to a_{\text{rét}} S^0$ of the required form. The stable splitting follows. □

Lemma 30. *Let $\mathcal{C}$ be an additive symmetric monoidal category in which $\otimes$ distributes over $\oplus$. If $G \in \mathcal{C}$ is an invertible object, such that $G \cong \mathbb{1} \oplus \Delta$, then $\Delta \cong 0$.*

Proof. The object G is rigid (being invertible) and hence Δ is rigid (being a summand of G). We have $\mathbb{1} \cong D(G) \otimes G \cong (D\mathbb{1} \oplus D\Delta) \otimes (\mathbb{1} \oplus \Delta) \cong \mathbb{1} \oplus \Delta \oplus D(\Delta) \oplus D(\Delta) \otimes \Delta$. Thus in order to prove the claim we may assume that $G = \mathbb{1}$. Now the splitting $\mathbb{1} \cong \mathbb{1} \oplus \Delta$ corresponds to morphisms $\mathbb{1} \xrightarrow{e} \mathbb{1} \oplus \Delta \xrightarrow{f} \mathbb{1}$ with $fe = \mathrm{id}$ and $ef \in \mathrm{End}(\mathbb{1} \oplus \Delta)$ the projection. Fixing an isomorphism $\mathbb{1} \xrightarrow{z} \mathbb{1} \oplus \Delta$ we get corresponding elements $z^{-1}e, fz \in [\mathbb{1}, \mathbb{1}]$. We have $\mathrm{id} = fe = f(zz^{-1})e = (fz)(z^{-1}e)$. But $\mathrm{End}(\mathbb{1})$ is commutative [Bal10, sentence before Proposition 2.2] so $\mathrm{id} = (z^{-1}e)(fz)$ and consequently $ef = zz^{-1} = \mathrm{id}$ and $\Delta = 0$. □

9. Main theorems

Proposition 31. *Let k be a field of characteristic zero. The functor $L : \mathbf{SH}(k)[\rho^{-1}] \to \mathbf{SH}(k)^{\text{rét}}[\rho^{-1}]$ is an equivalence.*

Proof. It is enough to show that all objects in $\mathbf{SH}(k)[\rho^{-1}]$ are rét-local. Let $U_\bullet \to X$ be a rét-hypercover, and let $\hat{X}$ be its homotopy colimit (in $\mathbf{SH}(k)$). We need to show that if $E \in \mathbf{SH}(k)[\rho^{-1}]$, then $[\hat{X}, E] = [X, E]$. We have conditionally convergent descent spectral sequences

$$H^p_{\mathrm{Nis}}(X, \underline{\pi}_{-q}(E)_{-i}) \Rightarrow [\Sigma^\infty X_+ \wedge \mathbb{G}_m^{\wedge i}, E[p+q]] \tag{1}$$
$$[\hat{X}, \underline{\pi}_{-q}(E)_{-i}[p]] \Rightarrow [\hat{X} \wedge \mathbb{G}_m^{\wedge i}, E[p+q]]. \tag{2}$$

Here we display the E_2-pages on the left-hand side. We moreover have the conditionally convergent homotopy colimit spectral sequence

$$[U_*, \underline{\pi}_{-q}(E)_{-i}[p]] \Rightarrow [\hat{X}, \underline{\pi}_{-q}(E)_{-i}[p + *]]. \tag{3}$$

Here the left-hand side is the E_1-page. We have $[U_n, \underline{\pi}_{-q}(E)_{-i}[p]] = H^p_{\mathrm{Nis}}(U_n, \underline{\pi}_{-q}(E)_{-i}) = H^p_{\text{rét}}(U_n, \underline{\pi}_{-q}(E)_{-i})$; indeed since E is ρ-local each $\underline{\pi}_{-q}(E)_{-i}$ is a rét-sheaf, by Theorem 25, and for any rét-sheaf F we have $H^p_{\text{rét}}(U_n, F) = H^p_{\mathrm{Nis}}(U_n, F)$ [Sch94, Proposition 19.2.1]. It follows that spectral sequence (3) converges strongly (because the dimension of X is finite) and identifies with the descent spectral sequence in rét-cohomology for the cover $U_\bullet$. In particular, it converges to $H^{p+*}_{\text{rét}}(X, \underline{\pi}_{-q}(E)_{-i})$. Thus we find that $[\hat{X}, \underline{\pi}_{-q}(E)_{-i}[p]] = H^p_{\text{rét}}(X, \underline{\pi}_{-q}(E)_{-i})$. Using [Sch94, Proposition 19.2.1] again, we conclude that the evident map from spectral sequence (1) to spectral sequence (2) induces an isomorphism on the E_2-pages, and moreover both converge strongly (again for cohomological dimension reasons). Thus the induced map on targets is an isomorphism, which is what we wanted to show. □

Corollary 32. *The proposition holds for all fields.*

Proof. We claim that if k has positive characteristic, then ρ is nilpotent in $\mathbf{SH}(k)$. By base change, it suffices to prove this when $k = \mathbb{F}_p$. That is to say, we wish to show that ρ is nilpotent in $K^{MW}_*(\mathbb{F}_p)$, or equivalently that $\mathrm{colim}_n K^{MW}_n(\mathbb{F}_p) = 0$. By the same argument as in the proof of Corollary 19 we know that $\mathrm{colim}_n K^{MW}_n(\mathbb{F}_p) = \mathrm{colim}_n I^n(\mathbb{F}_p)$. Thus our claim follows from nilpotence of the fundamental ideal of $\mathbb{F}_p$, which is well known [MH73, III(5.9)]. □

Corollary 33. *Let S be a Noetherian scheme of finite dimension. The functor $L : \mathbf{SH}(S)[\rho^{-1}] \to \mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ is an equivalence.*

In particular $\mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ satisfies the full six functors formalism.

Our initial proof of this statement contained a mistake; a correction and vast simplification has kindly been communicated by Denis-Charles Cisinski.

Proof. It suffices to prove that all objects of $\mathbf{SH}(S)[\rho^{-1}]$ are rét-local. Thus let $X \in Sm(S)$ and $U_\bullet \to X$ a rét-hypercover. We need to show that

$$\alpha : \operatorname{hocolim}_\Delta \Sigma^\infty U_\bullet \to \Sigma^\infty X$$

is an equivalence in $\mathbf{SH}(S)[\rho^{-1}]$. (See also Lemma 26.) Since $\mathbf{SH}(S)[\rho^{-1}]$ satisfies the six functors formalism by Proposition 28, it follows from Corollary 14 that suffices to show that if $f : \operatorname{Spec}(k) \to S$ is a morphism (with k a field), then $f^*\alpha$ is an equivalence. But f^* is a left adjoint so commutes with homotopy colimits (and Σ^∞), so $f^*\alpha$ is isomorphic to the map

$$\operatorname{hocolim}_\Delta \Sigma^\infty f^* U_\bullet \to \Sigma^\infty f^* X$$

in $\mathbf{SH}(k)[\rho^{-1}]$. Since rét-covers are stable by pullback, this is an equivalence by Corollary 32. □

Proposition 34. *Let S be a Noetherian scheme of finite dimension. Then the canonical functor $\mathrm{SH}(S_{\text{rét}}) \to \mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ is an equivalence.*

Proof. The functor $\mathrm{SH}(S_{\text{rét}}) \to \mathrm{SH}(Sm(S)_{\text{rét}})$ is fully faithful and t-exact by Corollary 6. The image of $\mathrm{SH}(S_{\text{rét}})$ in $\mathrm{SH}(Sm(S)_{\text{rét}})$ consists of $\mathbb{A}^1$-local and ρ-local objects, by the descent spectral sequence and Theorem 8 (and Corollary 6, which implies that the homotopy sheaves of LeE are the extensions of the homotopy sheaves of E). Consequently $\mathrm{SH}(S_{\text{rét}}) \to \mathbf{SH}^{S^1}(S)^{\text{rét}}[\rho^{-1}]$ is fully faithful. But $\mathbf{SH}^{S^1}(S)^{\text{rét}}[\rho^{-1}] \to \mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ is an equivalence by Lemma 27. We have thus established that the functor is fully faithful. We need to show it is essentially surjective.

The category $\mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ is generated by objects of the form $p_*(\mathbb{1})$ where $p : T \to S$ is projective [CD12, Proposition 4.2.13]. Since the functor $e : \mathrm{SH}(S_{\text{rét}}) \to \mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ has a right adjoint it commutes with arbitrary sums, and hence it identifies $\mathrm{SH}(S_{\text{rét}})$ with a localising subcategory of $\mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$. It thus suffices to show that e commutes with p_*, where $p : T \to S$ is a projective morphism. This is exactly the same as the proof of [CD13, Proposition 4.4.3]. It boils down to the proper base change theorem holding both in $\mathbf{SH}(S)^{\text{rét}}[\rho^{-1}]$ (where it follows from the six functors formalism which we have already established by showing that $\mathbf{SH}(S)^{\text{rét}}[\rho^{-1}] \simeq \mathbf{SH}(S)[\rho^{-1}]$) and in $\mathrm{SH}(S_{\text{rét}})$; the latter is Theorem 9. □

Remark. If S is the spectrum of a field, the above proof can be simplified greatly, by arguing as in [Bac16, § 5]. See in particular Lemma 21, Corollary 26 and Proposition 28 of [Bac16, § 5]. This way we no longer need to use the proper base change theorems, and thus also do not need to know that $\mathbf{SH}(X)^{\text{rét}}[\rho^{-1}]$ satisfies the six functors formalism.

One may also extract from [Bac16, § 5] a proof of Proposition 31 not relying on Theorem 25. Thus if the base is a field, §§ 3, 5, and 7 can be dispensed with.

In summary, we have thus established the following result.

Theorem 35. *Let S be a Noetherian scheme of finite dimension. In the following two diagrams, all functors are the canonical ones, and are equivalences of categories.*

$$\begin{tikzcd}
\mathbf{SH}(S_{\text{rét}}) \arrow[r, "a"] \arrow[d] & \mathbf{SH}^{S^1}(S)^{\text{rét}}[\rho^{-1}] \arrow[d, "b"'] & \mathbf{SH}^{S^1}(S)[\rho^{-1}] \arrow[l] \arrow[d, "b'"'] \\
\mathbf{SH}(S)^{\text{rét}} \arrow[r, "c"] & \mathbf{SH}(S)^{\text{rét}}[\rho^{-1}] & \mathbf{SH}(S)[\rho^{-1}] \arrow[l, "d"']
\end{tikzcd}$$

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$$\begin{tikzcd}
D_{\mathbb{A}^1}(S_{\text{rét}}) \arrow[r] \arrow[d] & D^{S^1}_{\mathbb{A}^1}(S)^{\text{rét}}[\rho^{-1}] \arrow[d] & D^{S^1}_{\mathbb{A}^1}(S)[\rho^{-1}] \arrow[l] \arrow[d] \\
D_{\mathbb{A}^1}(S)^{\text{rét}} \arrow[r] & D_{\mathbb{A}^1}(S)^{\text{rét}}[\rho^{-1}] & D_{\mathbb{A}^1}(S)[\rho^{-1}] \arrow[l]
\end{tikzcd}$$

In particular all these categories satisfy the full six functors formalism, and continuity.

Proof. The functor d is an equivalence by Corollary 33, b and b' are equivalences by Corollary 27, c is an equivalence by Proposition 29 and ba is an equivalence by Proposition 34. It follows that a is an equivalence, and so are the two unlabelled functors.

By Proposition 28, $\mathbf{SH}(\bullet)[\rho^{-1}]$ satisfies the full six functors formalism, and hence so do all the other pseudofunctors, being equivalent.

We have provided the proofs for $\mathbf{SH}$, the ones for $D_{\mathbb{A}^1}$ are exactly the same. □

10. Real realisation

In this section we work over the field $\mathbb{R}$ of real numbers. We then have a composite

$$R_1 : \mathbf{SH}(\mathbb{R}) \xrightarrow{L_\rho} \mathbf{SH}(\mathbb{R})^{\text{rét}}[\rho^{-1}] \simeq \mathbf{SH}^{S^1}(\mathbb{R})^{\text{rét}}[\rho^{-1}] \xrightarrow{r} \mathbf{SH}^{\text{s}}.$$

Here by $\mathbf{SH}^{\text{s}}$ we mean the model of the stable homotopy category $\mathbf{SH}$ built from simplicial sets. Of course $\mathbf{SH}^{\text{s}} \simeq \mathbf{SH}$ canonically (and this may be an equality depending, on our favourite model of $\mathbf{SH}$). Also r denotes the functor induced by the right adjoint of $e : \text{Pre}(\mathbb{R}_{\text{rét}}) \to \text{Pre}(Sm(\mathbb{R}))$ from § 3.

Following Heller and Ormsby [HO16, § 4.4], there is also the real realisation functor $LR_2 : \mathbf{SH}(\mathbb{R}) \to \mathbf{SH}^{\text{t}}$. Here $\mathbf{SH}^{\text{t}}$ is the model of $\mathbf{SH}$ built from topological spaces. The functor LR_2 is defined by starting with the functor $R_2 : Sm(\mathbb{R}) \to \text{Top}, X \mapsto X(\mathbb{R})$ assigning a smooth scheme over $\mathbb{R}$ its set of real points with the strong topology. We then get a functor $R_2 : \text{sPre}(Sm(\mathbb{R}))_* \to \text{Top}$ by left Kan extension, i.e. demanding that $R_2(\Delta^n_+ \wedge X_+) = \Delta^n_+ \wedge X(\mathbb{R})_+$ and that R_2 preserves colimits. Using the projective model structure on $\text{sPre}(Sm(\mathbb{R}))_*$ this functor is left Quillen and then one promotes it to $LR_2 : \mathbf{SH}(\mathbb{R}) \to \mathbf{SH}^{\text{t}}$ in the usual way.

Fortunately the two potential real realisation functors are the same. To state this result, recall that there is an adjunction

$$|\bullet| : \text{sSet} \leftrightarrows \text{Top} : \text{Sing}^{\text{t}},$$

and then by passing to homotopy categories of spectra one obtains the adjoint equivalence

$$L|\bullet| : \mathbf{SH}^{\text{s}} \leftrightarrows \mathbf{SH}^{\text{t}} : R\,\text{Sing}^{\text{t}}.$$

Proposition 36. *The two functors $L|R_1(\bullet)|, LR_2(\bullet) : \mathbf{SH}(\mathbb{R}) \to \mathbf{SH}^{\text{t}}$ are canonically isomorphic.*

Proof. The functor R_2 takes multiplication by ρ into a weak equivalence. Consequently it remains left Quillen in the ρ-local model structure and hence LR_2 canonically factors through the localisation $\mathbf{SH}(\mathbb{R}) \to \mathbf{SH}(\mathbb{R})[\rho^{-1}]$. Since $\mathbf{SH}(\mathbb{R})[\rho^{-1}] \simeq \mathbf{SH}^{S^1}(\mathbb{R})^{\text{rét}}[\rho^{-1}]$ the obvious functor $R_2' : \text{Spt}(Sm(\mathbb{R})) \to \text{Spt}^t$ is left Quillen in the $(\rho, \text{rét}, \mathbb{A}^1)$-local model structure. (Here we have used twice the following well-known observation: if $L : \mathcal{M} \leftrightarrows \mathcal{N} : R$ is a Quillen adjunction and

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H is a set of maps between cofibrant objects in $\mathcal{M}$ which is taken by L into weak equivalences, then $L : L_H\mathcal{M} \leftrightarrows \mathcal{N} : R$ is also a Quillen adjunction. This follows from [Hir09, Propositions 8.5.4 and 3.3.16].)

We now have the following diagram (which we do not know to be commutative so far).

$$\begin{tikzcd} \mathbf{SH}^{S^1}(\mathbb{R})^{\text{rét}}[\rho^{-1}] \arrow[r, "R_2'"] \arrow[d, "r"'] & \mathbf{SH}^{\text{t}} \arrow[d, equal] \\ \mathbf{SH}^{\text{s}} \arrow[r, "|\bullet|"] & \mathbf{SH}^{\text{t}} \end{tikzcd}$$

Here all the functors are derived; we omit the 'L' and 'R'. The functor r is an equivalence with inverse e by Theorem 35. Thus for $E \in \mathbf{SH}^{S^1}(\mathbb{R})^{\text{rét}}[\rho^{-1}]$ we have a canonical isomorphism $R_2'E \simeq R_2'erE$ and so to prove the proposition it suffices to exhibit a canonical isomorphism of functors $R_2'e \simeq |\bullet|$.

But this isomorphism exists on the level of underived functors, and then passes to the homotopy categories. Indeed if $E \in \mathrm{Spt}^s$ then $R_2'eE$ and $|E|$ are both computed by applying functors (of the same names) levelwise to E, so we may just as well show that for $E \in \mathrm{sSet}_*$ we have $R_2'eE \cong |E|$. But now R_2', e and $|\bullet|$ all preserve colimits, so we may just deal with $E = \Delta^n_+$. But then $R_2'eE\Delta^n_+ = |\Delta^n_+|$ holds basically by definition. $\square$

A similar result can be obtained for the $\mathbb{A}^1$-derived category. We have $r : D_{\mathbb{A}^1}(\mathbb{R})[\rho^{-1}] \to D(\mathrm{Spec}(\mathbb{R})_{\text{rét}}) \simeq D(Ab)$. There is also $R_2 : D_{\mathbb{A}^1}(\mathbb{R}) \to D(Ab)$ which is obtained by (derived) left Kan extension from the functor $Sm(\mathbb{R}) \to C(Ab)$ which sends a smooth scheme X the singular complex of its real points $C_*(X(\mathbb{R}))$. Then there is the following commutative diagram.

$$\begin{tikzcd} \mathbf{SH}(\mathbb{R}) \arrow[r, "R_2"] \arrow[d, "C_*"'] & \mathbf{SH} \arrow[d, "C_*"'] \\ D_{\mathbb{A}^1}(\mathbb{R}) \arrow[r, "R_2"] & D(Ab) \end{tikzcd}$$

Proposition 37. *The functors $rL_\rho, R_2 : D_{\mathbb{A}^1}(\mathbb{R}) \to D(Ab)$ are canonically isomorphic.*

Proof. As before R_2 factors through L_ρ as R_2' and we may show that $r, R_2' : D_{\mathbb{A}^1}(\mathbb{R})[\rho^{-1}] \to D(Ab)$ are canonically isomorphic. The functor r is an equivalence with inverse e, so it is enough to show that $R_2'e : D(Ab) \to D_{\mathbb{A}^1}(\mathbb{R})[\rho^{-1}] \simeq D^{S^1}_{\mathbb{A}^1}(\mathbb{R})[\rho^{-1}] \to D(Ab)$ is canonically isomorphic to the identity. This is the same argument as before. $\square$

Let us make explicit the following consequence.

Corollary 38. *Let $E \in \mathbf{SH}(\mathbb{R})$. Then*

$$\underline{\pi}_i(E)(\mathbb{R})[\rho^{-1}] = \pi_i(RE)$$

and

$$\underline{h}_i^{\mathbb{A}^1}(E)(\mathbb{R})[\rho^{-1}] = H_i(RE).$$

Here $R : \mathbf{SH}(\mathbb{R}) \to \mathbf{SH}$ denotes any one of the (canonically isomorphic) real realisation functors we have considered and $\underline{h}_i^{\mathbb{A}^1}(E) := \underline{h}_i(FE)$ where $F : \mathbf{SH}(\mathbb{R}) \to D_{\mathbb{A}^1}(\mathbb{R})$ is the canonical functor.

Proof. Combine Lemma 15 (saying that $L_\rho E = E[\rho^{-1}] = \mathrm{hocolim}_n\, E \wedge \mathbb{G}_m^{\wedge n}$) with compactness of the units of $\mathbf{SH}, D_{\mathbb{A}^1}$ and the above two propositions. $\square$

11. Application 1: the η-inverted sphere

From now on, k will denote a perfect field. Since essentially all our results concern the ρ-inverted situation, they are really only interesting if k has characteristic zero, so this is not a big restriction.

Recall that the motivic Hopf map $\eta : \mathbb{A}^2\backslash 0 \to \mathbb{P}^1$ defines an element of the same name in motivic stable homotopy theory $\eta : \Sigma^\infty \mathbb{G}_m \to \mathbb{1}$. Here we use that $\Sigma^\infty(\mathbb{A}^2\backslash 0) \simeq \Sigma^\infty \mathbb{G}_m \wedge \Sigma^\infty \mathbb{P}^1$. The element $\eta \in \underline{\pi}_0(\mathbb{1})_{-1}$ is non-nilpotent, and so inverting it is very natural. The category $\mathbf{SH}(k)[\eta^{-1}]$ can be constructed very similarly to $\mathbf{SH}(k)[\rho^{-1}]$. In particular the localisation functor $L : \mathbf{SH}(k) \to \mathbf{SH}(k)[\eta^{-1}]$ is just the evident colimit, see Lemma 15. It is typically denoted $E \mapsto E_\eta$ or $E \mapsto E[1/\eta]$. One may similarly invert other endomorphisms of the sphere spectrum. If $0 \neq n \in \mathbb{Z}$ then there is a corresponding automorphism of $\mathbb{1}$, and we denote the localisation by $E \mapsto E[1/n]$.

At least after inverting 2, inverting η is essentially the same as inverting ρ.

Lemma 39. *The endomorphism ring $K_*^{MW}(k)[1/2] = [\mathbb{1}[1/2], \mathbb{1}[1/2] \wedge \mathbb{G}_m^{\wedge *}]$ splits canonically into two summands $K_*^{MW}(k)[1/2] = K^+ \oplus K^-$. In fact $K^- = K_*^{MW}(k)[1/2, 1/\eta]$ and K^+ is characterised by the fact that $\eta K^+ = 0$.*

In K^- we have the equality $\eta\rho = 2$, whereas in K^+ we have $\rho^2 = 0$. In particular

$$K_*^{MW}(k)[1/2, 1/\eta] = K^- = K_*^{MW}(k)[1/2, 1/\rho].$$

Proof. This is well known, see for example [Mor12, § 3.1]. We summarise: for $a \in k^\times$ let $\langle a \rangle = 1 + \eta[a] \in K_0^{MW}(k)$. Put $\epsilon = -\langle -1 \rangle$. Then $\epsilon^2 = 1$ and so after inverting 2, $K_*^{MW}(k)$ splits into the eigenspaces for ϵ. One puts $h = 1 - \epsilon$ and then has $\eta h = 0$. On K^+ we have $\epsilon = -1$, so $h = 2$ and consequently $\eta = 0$ (since 2 is invertible).

By definition $\rho = -[-1]$ and consequently $\eta\rho = 1 + \epsilon$. Thus on K^- where $\epsilon = 1$ we find $\eta\rho = 2$ as claimed, and in particular η is invertible on K^-.

Finally $\rho^2 h = 0$ in $K_*^{MW}(k)$ and thus $2\rho^2 = 0$ in K^+. (This is just another expression of the fact that $K^+ \cong K^M(k)[1/2]$ is graded-commutative and ρ has degree 1, so $\rho^2 = -\rho^2$.) But since 2 is invertible in K^+ we find $\rho^2 = 0$ (in K^+). This concludes the proof. $\square$

Röndigs has studied the homotopy sheaves $\underline{\pi}_1(\mathbb{1}_\eta)$ and $\underline{\pi}_2(\mathbb{1}_\eta)$ and proved that they vanish [Rön16]. (Note that $\underline{\pi}_i(E_\eta)_*$ is independent of $*$, because multiplication by η is an isomorphism, so we shall suppress the second index.) He argues that $\underline{\pi}_i(\mathbb{1})_* \to \underline{\pi}_i(\mathbb{1}[1/2])_*$ is injective for $i = 1, 2$ (see his Lemma 8.2) and consequently an important part of his work is in showing that $\underline{\pi}_i(\mathbb{1}[1/\eta, 1/2]) = 0$ for $i = 1, 2$. We can deduce this as an easy corollary from our work.

Proposition 40. *Let k be a perfect field. Then $\underline{\pi}_i(\mathbb{1}[1/\eta, 1/2]) = 0$ for $i = 1, 2$.*

Proof. By Lemma 39 we know that $\mathbf{SH}(k)[1/2, 1/\eta] = \mathbf{SH}(k)[1/2, 1/\rho]$. By Theorem 35, we have $\mathbf{SH}(k)[1/2, 1/\rho] = \mathrm{SH}(\mathrm{Spec}(k)_{\text{rét}})[1/2]$. In particular this category is trivial unless k has characteristic zero, which we shall assume from now on.

The sheaves $\underline{\pi}_i(\mathbb{1}[1/2, 1/\rho])$ are unramified [Mor05, Lemma 6.4.4], so it suffices to show that $\underline{\pi}_i(\mathbb{1}[1/2, 1/\rho])(K) = 0$ for $i = 1, 2$ and every K (of characteristic zero). Since k was also arbitrary, we may just as well show the result for $k = K$, simplifying notation. We are dealing with rét-sheaves by Theorem 25, and so if

$$\mathrm{Spec}(l_1) \coprod \mathrm{Spec}(l_2) \coprod \cdots \coprod \mathrm{Spec}(l_n) \to \mathrm{Spec}(k)$$

is a rét-cover, the canonical map

$$\underline{\pi}_i(\mathbb{1}[1/2, 1/\rho])(k) \to \prod_m \underline{\pi}_i(\mathbb{1}[1/2, 1/\rho])(l_m)$$

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is injective. Consequently we may assume that k is real closed. But then $\mathrm{SH}(\mathrm{Spec}(k)_{\text{rét}}) = \mathbf{SH}$ is just the ordinary stable homotopy category, so it suffices to show: $\pi_i^s[1/2] = 0$ for $i = 1, 2$, where π_i^s are the classical stable homotopy groups. But $\pi_1^s = \mathbb{Z}/2 = \pi_2^s$ is well known, so we are done. $\square$

In classical algebraic topology, it is well known that rational stable homotopy theory is the same as rational homology theory: $\mathbf{SH}_{\mathbb{Q}} \simeq D(\mathbb{Q})$. In motivic stable homotopy theory, the situation is not so simple. As is well known (and follows for example from Lemma 39) there is a splitting $\mathbf{SH}(k)_{\mathbb{Q}} = \mathbf{SH}(k)_{\mathbb{Q}}^+ \times \mathbf{SH}(k)_{\mathbb{Q}}^-$. The $+$ part has been identified with rational motivic homology theory by Cisinski–Déglise [CD12, § 16]: $\mathbf{SH}(k)_{\mathbb{Q}}^+ \simeq \mathbf{DM}(k, \mathbb{Q})$.

The $-$ part has been only identified recently with an appropriate category of rational Witt motives by Ananyevskiy *et al.* [ALP17]: $\mathbf{SH}(k)_{\mathbb{Q}}^- = \mathbf{DM}_W(k, \mathbb{Q})$. Here the category $\mathbf{DM}_W(k, \mathbb{Q})$ may be conveniently defined as the homotopy category of modules over the (strict ring spectrum model of the) homotopy module of rational Witt theory. That is to say there is the homotopy module $\underline{W}_{\mathbb{Q}}$ with $(\underline{W}_{\mathbb{Q}})_i = \underline{W} \otimes_{\mathbb{Z}} \mathbb{Q}$ for all i. This is the same as $K^- \otimes_{\mathbb{Z}[1/2]} \mathbb{Q}$, or equivalently $\underline{\pi}_0(\mathbb{1}_{\eta,\mathbb{Q}})$. Then corresponding to this homotopy module there is a strict ring spectrum, the Eilenberg–MacLane spectrum $EM\underline{W}_{\mathbb{Q}}$. Finally we may form the model category $EM\underline{W}_{\mathbb{Q}}$**-Mod** and its homotopy category $\mathrm{Ho}(EM\underline{W}_{\mathbb{Q}}\text{-}\mathbf{Mod}) =: \mathbf{DM}_W(k, \mathbb{Q})$.

More generally, one may define $\mathbf{DM}_W(k, \mathbb{Z}[1/2])$ by replacing $\underline{W}_{\mathbb{Q}} = \underline{\pi}_0(\mathbb{1}_{\mathbb{Q}}[1/\eta])$ in the above construction with $\underline{W}[1/2] = \underline{\pi}_0(\mathbb{1}[1/\eta, 1/2])$.

The theorem of Ananyevskiy–Levine–Panin essentially boils down to the computation that $\underline{\pi}_i(\mathbb{1}_{\eta,\mathbb{Q}}) = 0$ for $i > 0$. We can deduce this and more from our general theory. We write $H_{\mathbb{A}^1}\mathbb{Z}$ for the image of the tensor unit in $D_{\mathbb{A}^1}(k)$ under the 'forgetful' functor $D_{\mathbb{A}^1}(k) \to \mathbf{SH}(k)$.

Proposition 41. *We have $\underline{\pi}_i(H_{\mathbb{A}^1}\mathbb{Z}[1/\rho]) = 0$ for $i > 0$, and consequently $\underline{\pi}_i(H_{\mathbb{A}^1}\mathbb{Z}[1/2, 1/\eta]) = 0$ for $i > 0$. Similarly $\underline{\pi}_i(\mathbb{1}_{\mathbb{Q}}[1/\rho]) = \underline{\pi}_i(\mathbb{1}_{\mathbb{Q}}[1/\eta]) = 0$ for $i > 0$.*

Thus we have the equivalences

$$D_{\mathbb{A}^1}(k, \mathbb{Z}[1/2])^- \simeq \mathbf{DM}_W(k, \mathbb{Z}[1/2]) \simeq D(\mathrm{Spec}(k)_{\text{rét}}, \mathbb{Z}[1/2])$$
$$\mathbf{SH}(k)_{\mathbb{Q}}^- \simeq \mathbf{DM}_W(k, \mathbb{Q}) \simeq D(\mathrm{Spec}(k)_{\text{rét}}, \mathbb{Q}).$$

Proof. As in the proof of Proposition 40 we have

$$D_{\mathbb{A}^1}(k)[1/2]^- := D_{\mathbb{A}^1}(k)[1/2, 1/\eta] = D_{\mathbb{A}^1}(k)[1/2, 1/\rho].$$

By Theorem 35, this is the same as $D(\mathrm{Spec}(k)_{\text{rét}})[1/2] = D(\mathrm{Spec}(k)_{\text{rét}}, \mathbb{Z}[1/2])$.

Similarly

$$\mathbf{SH}(k)_{\mathbb{Q}}^- := \mathbf{SH}(k)_{\mathbb{Q}}[1/\eta] = \mathbf{SH}(k)_{\mathbb{Q}}[1/\rho] = \mathrm{SH}(\mathrm{Spec}(k)_{\text{rét}})_{\mathbb{Q}},$$

and the latter category is equivalent to $D(\mathrm{Spec}(k)_{\text{rét}})_{\mathbb{Q}}$ by classical stable rational homotopy theory.

From this we can read off $\underline{\pi}_*(H_{\mathbb{A}^1}\mathbb{Z}[1/2, 1/\eta])$ and so on. The main point is that $\underline{\pi}_n(H_{\mathbb{A}^1}\mathbb{Z}[1/2, 1/\eta]) = 0$ for $n > 0$. It suffices to check this on fields, so we may as well check it for k (k being arbitrary), and we have $\underline{\pi}_n(H_{\mathbb{A}^1}\mathbb{Z}[1/2, 1/\eta])(k) = [\mathbb{1}[n], \mathbb{1}]_{D(\mathrm{Spec}(k)_{\text{rét}}, \mathbb{Z}[1/2])} = H^{-n}_{\text{rét}}(k, \mathbb{Z}[1/2]) = 0$.

It remains to show that $D_{\mathbb{A}^1}(k, \mathbb{Z}[1/2])^- \simeq \mathbf{DM}_W(k, \mathbb{Z}[1/2])$. Our computation of homotopy sheaves implies that $H_{\mathbb{A}^1}\mathbb{Z}[1/\rho, 1/2] \to EM\underline{W}[1/2]$ is a weak equivalence. The result follows. $\square$

Let us also make explicit the following observation.

Corollary 42. *Let k be a real closed field or $\mathbb{Q}$. Then $\underline{\pi}_*(\mathbb{1}[1/\rho])(k) = \pi_*^s$ and in particular $\underline{\pi}_*(\mathbb{1}[1/\eta, 1/2])(k) = \pi_*^s \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$. Here π_*^s denotes the classical stable homotopy groups.*

Proof. This follows immediately from $\mathrm{Shv}(\mathrm{Spec}(k)_{\text{rét}}) = \mathrm{Set}$, Lemma 39 and Theorem 35. $\square$

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12. Application 2: some rigidity results

In this section we establish some rigidity results. We work with ρ-stable sheaves. These sheaves are h-torsion (because $\rho^2 h = 0$), explaining to some extent why we do not need the usual torsion assumptions.

There are various notions of rigidity for sheaves. We shall call a presheaf F on $Sm(k)$ *rigid* if for every essentially smooth, Henselian local scheme X with residue field x, the natural map $F(X) \to F(x)$ is an isomorphism. This notion goes back to perhaps Gillet and Thomason [GT84] and Gabber [Gab92].

Lemma 43. *Let $F \in \mathrm{Shv}(\mathrm{Spec}(k)_{\mathrm{rét}})$. Then $eF \in \mathrm{Shv}(Sm(k)_{\mathrm{rét}})$ is rigid.*

Proof. Extension e and pullback are both left Kan extensions. From this it is easy to show that they commute, and so we find that $(eF)|_{X_{\mathrm{rét}}} = f^*F \in \mathrm{Shv}(X_{\mathrm{rét}})$, where X is (essentially) smooth over k with structural morphism f.

If $\mathrm{char}(k) > 0$ then $\mathrm{Spec}(k)_{\mathrm{rét}}$ and $Sm(k)_{\mathrm{rét}}$ are both the trivial site, so we may assume that k is of characteristic zero and consequently perfect. In this case, for an essentially k-smooth Henselian local scheme X with closed point $i\colon x \to X$, there exists a retraction $s\colon X \to x$. (Write $k(x)/k$ as $k(T_1, \ldots, T_n)[U]/P$ with $P \in k(T_1, \ldots, T_n)[U]$ separable; this is possible because $k(x)/k$ is separable, k being perfect. Lift the elements T_i to $\mathcal{O}_X$ arbitrarily and then use Hensel's lemma to produce a root of P in $\mathcal{O}_X$.)

It is thus enough to prove: if $F \in \mathrm{Shv}(x_{\mathrm{rét}})$ then $H^0(x, F) = H^0(X, s^*F)$. It follows from [Sch94, Discussion after Proposition 19.2.1] and [ABR12, Propositions II.2.2 and II.2.4] that for any $G \in \mathrm{Shv}(X_{\mathrm{rét}})$ we have $H^0(X, G) = H^0(x, i^*G)$. Consequently $H^0(X, s^*F) = H^0(x, i^*s^*F) = H^0(x, F)$, because $si = \mathrm{id}$ by construction. □

Corollary 44. *If $E \in \mathbf{SH}(k)[\rho^{-1}]$ then all the homotopy sheaves $\underline{\pi}_i(E)$ are rigid.*

Proof. By Theorem 35 and Corollary 6 we know that all the homotopy sheaves of E are of the form eF, with $F \in \mathrm{Shv}(\mathrm{Spec}(k)_{\mathrm{rét}})$. Thus the claim follows immediately from Lemma 43. □

Corollary 45. *Let k be a perfect field of finite virtual 2-étale cohomological dimension and exponential characteristic $e \neq 2$. Then the homotopy sheaves $\underline{\pi}_i(\mathbb{1})_0[1/e]$ are rigid.*

Proof. We will first assume that $e = 1$, and explain the necessary changes in positive characteristic at the end.

For $i = 0$ we have $\underline{\pi}_0(\mathbb{1})_0 = \underline{GW}$ and this sheaf is known to be rigid [Gil17, Theorem 2.4]. We consider the arithmetic square [RSØ16, Lemma 3.9] as follows.

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d] & \mathbb{1}[1/2] \arrow[d] \\ \mathbb{1}^\wedge_2 \arrow[r] & \mathbb{1}^\wedge_2[1/2] \end{tikzcd}$$

Since rigid sheaves are stable under extension, kernel and cokernel, the five lemma implies that it is enough to show that $\underline{\pi}_*(\mathbb{1}[1/2])_0, \underline{\pi}_*(\mathbb{1}^\wedge_2)_0$ and $\underline{\pi}_*(\mathbb{1}^\wedge_2[1/2])_0$ are rigid. Since rigid sheaves are stable by colimit, the case of $\mathbb{1}^\wedge_2[1/2]$ follows from $\mathbb{1}^\wedge_2$.

By [HKO11, Theorem 1] and [Rön16, proof of Theorem 8.1], we know that $\mathbb{1}^\wedge_2$ is the target of the convergent motivic Adams spectra sequence. The homotopy sheaves at the E_1 page are all sheaves with transfers in the sense of Voevodsky and torsion prime to the characteristic, and hence rigid, for example by [HY07, paragraph after Lemma 1.6]. Since rigid sheaves are stable by extension etc., it follows that the E_∞ page is rigid, and finally so are the homotopy sheaves of $\mathbb{1}_2^\wedge$.

By motivic Serre finiteness [ALP17, Theorem 6] (beware that their indexing convention for motivic homotopy groups differs from ours!), $\underline{\pi}_i(\mathbb{1}[1/2])_0$ is torsion for $i > 0$. By design, it is of odd torsion prime to the exponential characteristic. Consequently all of the l-torsion subsheaves of $\underline{\pi}_i(\mathbb{1}[1/2]^+)_0$ are rigid by the same argument as before, and so is the colimit $\underline{\pi}_i(\mathbb{1}[1/2]^+)_0$.

It remains to deal with $\underline{\pi}_i(\mathbb{1}[1/2]^-)_0$. But this is just the same as $\underline{\pi}_i(\mathbb{1}[1/2, 1/\rho])_0$ and so is rigid by Corollary 44.

This concludes the proof if $e = 1$. If $e > 2$ the same proof works. The only problem might be that we have torsion prime to the characteristic, but we excluded this possibility by inverting e. □

Remark. We appeal to [ALP17] in order to know that $\underline{\pi}_i(\mathbb{1})_0 \otimes \mathbb{Q} = 0$ for $i > 0$. This can also be deduced from Proposition 41, using that $\mathbf{SH}(k)^+_{\mathbb{Q}} = \mathbf{DM}(k, \mathbb{Q})$.

There is another (older) notion of rigidity first considered by Suslin [Sus83]. This corresponds to (1) in the next result. It is a slightly silly property in our situation, but (2) is a replacement in spirit. It is related to important results in semialgebraic topology due to Coste-Roy, Delfs [Del91, see in particular Corollary II.6.2] and Scheiderer [Sch94].

Proposition 46. *Let $E \in \mathbf{SH}(k)[\rho^{-1}]$ and $i \in \mathbb{Z}$.*

(1) *If $\bar{L}/\bar{K}$ is an extension of algebraically closed fields over k, then*

$$\underline{\pi}_i(E)(\bar{K}) = \underline{\pi}_i(E)(\bar{L}) = 0.$$

(2) *If $L^{\mathrm{r}}/K^{\mathrm{r}}$ is an extension of real closed fields over k, then also*

$$\underline{\pi}_i(E)(K^{\mathrm{r}}) = \underline{\pi}_i(E)(L^{\mathrm{r}}).$$

Proof. As before, by Theorem 35 and Corollary 6 we know that all the homotopy sheaves of E are of the form eF, with $F \in \mathrm{Shv}(\mathrm{Spec}(k)_{\mathrm{rét}})$. For such sheaves we have $eF(\bar{K}) = 0 = eF(\bar{L})$, so (1) holds. Since pullback $\mathrm{Spec}(K^{\mathrm{r}})_{\mathrm{rét}} \to \mathrm{Spec}(L^{\mathrm{r}})_{\mathrm{rét}}$ induces an isomorphism of sites, (2) also follows immediately. (See also the first paragraph of the proof of Lemma 43.) □

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