

THE STABLE PICARD GROUP OF $\mathcal{A}(2)$

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ABSTRACT. Using a form of descent in the stable category of $\mathcal{A}(2)$ -modules, we show that there are no exotic elements in the stable Picard group of $\mathcal{A}(2)$, *i.e.* that the stable Picard group of $\mathcal{A}(2)$ is free on 2 generators.

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Convention. Through out this paper, \mathbb{F} will denote the field with two elements. Every algebraic structure is implicitly over the base field \mathbb{F} , and tensor products are taken over \mathbb{F} . The Hopf algebras under consideration in this paper are connected, cocommutative finite dimensional graded Hopf algebras, unless explicitly specified otherwise.

1. INTRODUCTION

Let A be a Hopf algebra. The *Picard group* of $\mathbf{St}(A)$, denoted by $\mathrm{Pic}(A)$ is the group of stably \otimes -invertible A -modules,

$$\mathrm{Pic}(A) := \{M \in \mathbf{St}(A) : \exists N \text{ such that } M \otimes N = \mathbb{S}\},$$

where \mathbb{S} is the unit of the symmetric monoidal category $(\mathbf{St}(A), \otimes, \mathbb{S})$. When $B \subset A$ be a Hopf subalgebra, the forgetful functor $U : \mathbf{St}(A) \rightarrow \mathbf{St}(B)$ being monoidal, it induces a group homomorphism $\mathrm{Pic}(U) : \mathrm{Pic}(A) \rightarrow \mathrm{Pic}(B)$. Define the *relative Picard group* $\mathrm{Pic}(A, B)$ as the kernel

$$\mathrm{Pic}(A, B) := \ker(\mathrm{Pic}(U)).$$

Some elements are always in the Picard group of a Hopf algebra. Explicitly, there is a morphism of groups (see (3)):

$$\iota : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathrm{Pic}(A).$$

The interesting part of the Picard group consist in the the elements in the cokernel of ι . These are called *exotic elements*. In this paper, we are interested in the determination of the Picard group of $\mathcal{A}(2)$, the Hopf subalgebra of \mathcal{A} generated by Sq^1, Sq^2 and Sq^4 . This problem is very natural, as the continuation of the study presented in [AP76]. Let $\mathcal{A}(1)$ denote the Hopf subalgebra of the Steenrod algebra \mathcal{A} generated by Sq^1 and Sq^2 . Questions regarding Picard groups of hopf algebra started with [AP76], where the determination of the stable Picard group of the Hopf subalgebra $\mathcal{A}(1)$ is used to show the uniqueness of the infinite loop space structure on the classifying space of the infinite orthogonal group (see *loc cit* for the definition of $\mathcal{A}(1)$). The connective real K -theory ko is also related to $\mathcal{A}(1)$ as $H^*(ko, \mathbb{F}) \cong \mathcal{A} // \mathcal{A}(1)$ (see [AP76]). Here, the result is quite surprising: the group homomorphism

$$\iota : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathrm{Pic}(\mathcal{A}(1))$$

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is not surjective, and there is exactly one exotic element, called the Joker (see [AM74, Bru12]). This $\mathcal{A}(1)$ -module is pictured in figure 1.



FIGURE 1. The joker. Each dot represents a copy of \mathbb{F} . Straight lines represent the action of Sq^1 and curved ones represent the action of Sq^2 .

Studying $\text{Pic}(\mathcal{A}(2))$ is of topological interest for two reasons. Firstly, $\mathcal{A}(2)$ shares the same relationship with the spectrum of topological modular forms as the relationship between $\mathcal{A}(1)$ and connective real K -theory. Secondly, the Joker plays a role in the determination of the Picard group of the $K(1)$ -local stable homotopy category [HMS94].

The main result of this paper shows there are no ‘surprises’ for the stable Picard group of $\mathcal{A}(2)$, more precisely:

Theorem. 4.6 *The morphism of groups*

$$\iota : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Pic}(\mathcal{A}(2)),$$

which sends (n, m) to $\mathbb{S}^{n,m}$, is an isomorphism.

The key idea in our approach is that we consider a chain of inclusions of Hopf subalgebras of $\mathcal{A}(2)$ (which we define in section 2),

$$D(2) \subset C(2) \subset \mathcal{A}(2)$$

of Hopf subalgebras of $\mathcal{A}(2)$. Our starting point is to compute $\text{Pic}(D(2))$ by hand and show that (see Proposition 3.1) $\text{Pic}(D(2))$ does not have any exotic elements. The heuristic reason for considering $D(2)$ is that it is very close to being an exterior algebra (see Remark 2.12) and $\text{Pic}(E)$ for an exterior Hopf algebra is known to be free of exotic elements (see Proposition 3.2). Then, by definition of the relative Picard group, we get a three-stage filtration of the stable Picard group of $\mathcal{A}(2)$.

$$\begin{array}{ccccc} \text{Pic}(\mathcal{A}(2)) & \longrightarrow & \text{Pic}(C(2)) & \longrightarrow & \text{Pic}(D(2)) . \\ \uparrow & & \uparrow & & \parallel \\ \text{Pic}(\mathcal{A}(2), C(2)) & & \text{Pic}(C(2), D(2)) & & \text{Pic}(D(2)), \end{array}$$

where the vertical maps are the inclusion of the kernel of the next horizontal map. Next we use methods of homotopical descent along restriction functors as described in [Ric16]. In Corollary 4.5, we give a criteria for general $B \subset A$ under which the relative Picard group $\text{Pic}(A, B)$ is trivial. It follows that $\text{Pic}(C(2), D(2))$ and $\text{Pic}(\mathcal{A}(2), C(2))$ are trivial. Our main result, Theorem 4.6, is a straightforward consequence of the above observations.

In light of Corollary 4.5, it seems that the exotic element in the stable Picard group of $\mathcal{A}(1)$ (namely the Joker) is just a low-dimensional anomaly and authors expect the stable Picard group of $\mathcal{A}(n)$ not to have any exotic elements whenever $n \geq 2$ ¹. Authors hope to address the case when $n > 2$ in future.

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¹Eric Wofsey made similar conclusions in his unpublished work.

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2. SET UP

Let A be a finite dimensional graded Hopf algebra. Let $\mathbf{Mod}(A)$ be the category of bounded below graded A -modules. For a graded A -module M , we denote by M_n the set of elements in degree n . The diagonal $\Delta : A \rightarrow A \otimes A$ gives $\mathbf{Mod}(A)$ the structure of a symmetric monoidal closed category. Explicitly, if M, N are A -modules, the action of A on $M \otimes N$ is defined by

$$\begin{aligned} A \otimes M \otimes N &\xrightarrow{\Delta \otimes M \otimes N} A \otimes A \otimes M \otimes N \\ &\xrightarrow{\cong} A \otimes M \otimes A \otimes N \\ &\rightarrow M \otimes N, \end{aligned}$$

and the unit \mathbb{S} for this monoidal structure is the A -module \mathbb{F} , concentrated in degree zero.

Recall from [Mar83, Theorem 12.5, Proposition 12.8] that the category $\mathbf{Mod}(A)$ has enough injective and projective modules, and that the class of injective modules, projective modules, and free modules coincide. Let $\mathbf{St}(A)$ (see [SS03, Example 2.4.(v)] for an efficient and detailed definition) denote the stable category of graded A -modules. Objects of $\mathbf{St}(A)$ are the objects of $\mathbf{Mod}(A)$, but the stable morphisms between two A -modules M and N is

$$(1) \quad [M, N]^A := \frac{\mathbf{Mod}(A)(M, N)}{\langle \{f : M \rightarrow P \rightarrow N : P \text{ is a projective } A\text{-module}\} \rangle}.$$

In particular, free (respectively injective, projective, since these notion coincide) A -modules becomes equivalent to $\mathbf{0}$ in this category. It turns out (see [Mar83]) that the structure of a closed monoidal category on $\mathbf{Mod}(A)$ passes to $\mathbf{St}(A)$. We will denote (abusively) the monoidal product in $\mathbf{St}(A)$ by \otimes , the unit by \mathbb{S}_A and the hom-bifunctor by $F_A(-, -)$.

Additionally, the process of ‘killing the free modules’ gives $\mathbf{St}(A)$ the structure of a triangulated category (see [HPS97, Section 9.6]). We recall here the definition of the suspension functor, for completeness.

Definition 2.1. Choose a minimal projective resolution

$$\cdots \rightarrow P^{i+1} \rightarrow P^i \rightarrow \cdots \rightarrow P^1 \rightarrow P^0 \rightarrow \mathbb{S}_A$$

of \mathbb{S}_A as an A -module. For $n \geq 0$, let the A -module $\ker(P^{n+1} \rightarrow P^n)$ be denoted by $\mathbb{S}_A^{n,0}$, and the linear dual of $\mathbb{S}_A^{n,0}$ by $\mathbb{S}_A^{-n,0}$. We denote by $\mathbb{S}_A^{n,m}$ the A -module $\mathbb{S}_A^{n,0}[-m]$, where $[-m]$ is the shift in internal grading. Explicitly $(\mathbb{S}_A^{n,m})_k = (\mathbb{S}_A^{n,0})_{k+m}$.

We can now define the suspension functor

$$\Sigma^{n,m} : \mathbf{St}(A) \rightarrow \mathbf{St}(A)$$

which sends $M \mapsto \mathbb{S}_A^{n,m} \otimes M$.

Remark 2.2. Another choice of projective resolution of \mathbb{S}_A would result in another definition of $\mathbb{S}_A^{n,m}$. However, since the resulting objects would be isomorphic to $\mathbb{S}_A^{n,m}$, the choice we made is harmless.

Definition 2.3. We say that an A -module M is reduced if it does not contain any A -free summand.

Let M be an A -module. As observed in [Bru12], one can construct a reduced A -module M^{red} , which is stably isomorphic to M .

To follow the common notations in algebraic topology, we make the convention

$$[M, N]_{s,t}^A := [\Sigma^{s,t} M, N]^A.$$

Remark 2.4. By [HPS97, Section 9.6], the suspension functor which is part of the triangulated structure is $\Sigma^{1,0}$. In particular, for A -modules M and N , and $s \geq 0$, the bigraded extension groups are isomorphic to

$$(2) \quad [M, N]_{-s,t}^A = [\Sigma^{-s,t} M, N]^A \cong \text{Ext}_A^{s,t}(M, N).$$

In particular when $s = 0$, $\text{Ext}_A^{0,t}(M, N) = [M, N]_{0,t}^A$ is the set of stable maps, as described in (1). Let $\text{hom}_A(M, N)$ denote the space of homomorphisms between M and N . When $s \geq 1$, the vector space

$$[M, N]_{s,t}^A \cong \pi_s(\text{hom}_A(\Sigma^{0,t} M, N)).$$

The reader is referred to [Mar83, Proposition 14.1.8] for the explicit comparison.

We now turn to the definition of our main object of interest: the Picard and relative Picard groups.

Definition 2.5. Let $\text{Pic}(A)$ be the group of stably \otimes -invertible A -modules,

$$\text{Pic}(A) := \{M \in \mathbf{St}(A) : \exists N \text{ such that } M \otimes N = \mathbb{S}_A\}.$$

This group is called the *Picard group* of $\mathbf{St}(A)$, and its elements are called *Picard elements*.

Remark 2.6. Note that Picard elements share a lot of properties with the unit object (see [MS14b, Proposition 2.1.3]). In particular, Picard elements have a finite dimensional representative. This justifies our restriction to bounded below modules.

When $B \subset A$ be a Hopf subalgebra, the forgetful functor $U : \mathbf{St}(A) \rightarrow \mathbf{St}(B)$ is monoidal. This induces a group homomorphism $\text{Pic}(U) : \text{Pic}(A) \rightarrow \text{Pic}(B)$.

Definition 2.7. Let

$$\text{Pic}(A, B) := \ker(\text{Pic}(U)).$$

This group is called the *relative Picard group* associated to the inclusion $B \subset A$.

Note that there is always a family of \otimes -invertible modules. Indeed, for all $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$, the A -module $\mathbb{S}_A^{n,m}$ has inverse $\mathbb{S}_A^{-n,-m}$. This defines a group homomorphism

$$(3) \quad \iota : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Pic}(A).$$

Definition 2.8. The elements in the cokernel of ι are called *exotic elements*.

Let $\mathcal{A}(2)$ be the Hopf subalgebra of the modulo 2 Steenrod algebra \mathcal{A} generated by the Steenrod squares Sq^1, Sq^2, Sq^4 . As an algebra, $\mathcal{A}(2)$ has the following presentation

$$\mathcal{A}(2) \cong \frac{\mathbb{F}[Sq^1, Sq^2, Sq^4]}{\left(\begin{array}{c} Sq^1 Sq^1, \\ Sq^2 Sq^2 + Sq^1 Sq^2 Sq^1, \\ Sq^1 Sq^4 + Sq^4 Sq^1 + Sq^2 Sq^1 Sq^2, \\ Sq^4 Sq^4 + Sq^2 Sq^4 Sq^2 + Sq^4 Sq^2 Sq^2 \end{array} \right)}.$$

The coalgebra structure is given by the Cartan formulas. This Hopf algebra is dual to

$$\mathcal{A}(2)^* \cong \frac{\mathbb{F}[\xi_1, \xi_2, \xi_3]}{(\xi_1^8, \xi_2^4, \xi_3^2)},$$

together with the diagonal

$$\begin{aligned} \Delta(\xi_1) &= \xi_1 \otimes 1 + 1 \otimes \xi_1, \\ \Delta(\xi_2) &= \xi_2 \otimes 1 + \xi_1^2 \otimes \xi_1 + 1 \otimes \xi_2, \\ \Delta(\xi_3) &= \xi_3 \otimes 1 + \xi_2^2 \otimes \xi_1 + \xi_1^4 \otimes \xi_2 + 1 \otimes \xi_3. \end{aligned}$$

By Palmieri's work [Pal97, Theorem 1.3], a stable $\mathcal{A}(2)$ -module M is \otimes -invertible if and only if, for all quasi-elementary Hopf subalgebras E of $\mathcal{A}(2)$, the restriction of M to E is. Moreover, all the elementary sub-Hopf algebras of $\mathcal{A}(2)$ are in fact exterior Hopf subalgebras by [Pal01, Section 2.1.1]. Finally *loc cit* recalls the classification the elementary Hopf subalgebras of \mathcal{A} (the result is originally in [Mar83]). We deduce the following result:

Proposition 2.9. *The maximal elementary Hopf subalgebras of $\mathcal{A}(2)$ are*

$$\begin{aligned} E_1 &\cong E(Q_0, Q_1, Q_2), \\ E_2 &\cong E(Q_1, P_2^1, Q_2), \end{aligned}$$

where

$$\begin{aligned} Q_0 &= Sq^1, \\ Q_{i+1} &= Q_i Sq^{2^{i+1}} + Sq^{2^{i+1}} Q_i, \\ P_2^1 &= Sq^2 Sq^4 + Sq^4 Sq^2. \end{aligned}$$

Definition 2.10. Let $D(2)$ be the Hopf subalgebra of $\mathcal{A}(2)$ generated by Q_0, Q_1, Q_2 , and P_2^1 .

Definition 2.11. Let $C(2)$ be the Hopf subalgebra of $\mathcal{A}(2)$ generated by Sq^1, Sq^2, Q_1, P_2^1 and Q_2 .

The dual of these Hopf algebras have an easy presentation:

$$\begin{aligned} D(2)^* &\cong \frac{\mathbb{F}[\xi_1, \xi_2, \xi_3]}{(\xi_1^2, \xi_2^4, \xi_3^2)}, \\ C(2)^* &\cong \frac{\mathbb{F}[\xi_1, \xi_2, \xi_3]}{(\xi_1^4, \xi_2^4, \xi_3^2)}, \end{aligned}$$

as quotient Hopf algebras of $\mathcal{A}(2)^*$.

Remark 2.12. Note that every generator $D(2)$ is either contained in E_1 or in E_2 . However, $D(2)$ is not an exterior algebra as the commutator

$$[Q_0, P_2^1] = Q_0 P_2^1 + P_2^1 Q_0 = Q_2$$

is nonzero. In fact,

$$D(2) = \frac{\mathbb{F}[Q_0, Q_1, P_2^1, Q_2]}{\langle Q_0^9, Q_1^2, Q_2^2, [Q_0, P_2^1] = Q_2, (P_2^1)^2 \rangle}$$

is a presentation of $D(2)$.

3. AN ELEMENTARY CASE: THE PICARD GROUP OF $D(2)$

In this section we compute the stable Picard group of $D(2)$. An immediate consequence of the study of exterior algebras in [AM74] is that the Picard group of exterior Hopf algebras does not contain any exotic element (we give another argument here in Proposition 3.2, which relies on the analysis of [CT05]). The Hopf algebra $D(2)$ being close to be an exterior Hopf algebra, the reader should expect that its Picard group does not contain exotic elements. Proposition 3.1 shows that it is indeed the case.

Proposition 3.1. *The group homomorphism*

$$\iota : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Pic}(D(2)),$$

is an isomorphism.

The proof is direct, and only relies on the analysis of the Picard group of elementary algebras.

Proposition 3.2. *Let $n \geq 2$ and $E = E(x_1, x_2, \dots, x_n)$ be the exterior Hopf algebra generated by the elements x_i . Then, ι is an isomorphism.*

Proof. Let M be a \otimes -invertible module. Equivalently, the morphism

$$\text{hom}(M, M) \rightarrow \mathbb{S}_E$$

is an isomorphism. The latter assertion depends only on the underlying ungraded module over the exterior algebra E . But the exterior algebra is isomorphic to the group algebra of an elementary abelian 2-group. By the classification of \otimes -invertibles modules in the group algebra case, due to Carlson and Thevenaz in [CT05], M is stably isomorphic to $\mathbb{S}_E^{n,m}$. The result follows. \square

Proof of Proposition 3.1. Let $[M]$ be an isomorphism class of stably \otimes -invertible modules. By Definition 2.3 and subsequent construction, we can assume without loss of generality that M is a reduced module (see Definition 2.3). Moreover, by Remark 2.6), M is finite dimensional in this case.

Let M_1 and M_2 and N be the restrictions of M to E_1 , E_2 and $E(Q_1, Q_2)$ respectively. There is a commutative diagram of abelian groups such that

$$\begin{array}{ccc} \text{Pic}(D(2)) & \longrightarrow & \text{Pic}(E_1) & & [M] & \longmapsto & [M_1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Pic}(E_2) & \longrightarrow & \text{Pic}(E(Q_1, Q_2)) & & [M_2] & \longmapsto & [N]. \end{array}$$

By Proposition 3.2, $\text{Pic}(E_1)$, $\text{Pic}(E_2)$ and $\text{Pic}(E(Q_1, Q_2))$ are isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and therefore the left vertical map and the bottom horizontal map in (3) are isomorphisms as shown below:

$$\begin{array}{ccc} \text{Pic}(D(2)) & \longrightarrow & \text{Pic}(E_1) \\ \downarrow & & \downarrow \cong \\ \text{Pic}(E_2) & \xrightarrow{\cong} & \mathbb{Z} \oplus \mathbb{Z}. \end{array}$$

Consequently, without loss of generality, we can assume that both M_1 and M_2 are stably isomorphic to \mathbb{S}_{E_1} and \mathbb{S}_{E_2} respectively (if not, replace M by $\Sigma^{n,m}M$ for suitable m and n). Let $x \in M$ be an element of smallest degree (this makes sense since the modules are bounded by assumption).

Suppose that Q_0, Q_1, Q_2 and P_2^1 acts trivially on x , then the inclusion

$$x\mathbb{F} \rightarrow M$$

is split. Since this inclusion induces an isomorphism in Margolis Homology (see [Mar83]), it is an isomorphism, showing the result.

Suppose now that one of the operations Q_0, Q_1, Q_2 and P_2^1 acts non-trivially on x . Let us assume this operation belongs to E_1 (a similar argument gives the result when it belongs to E_2). As M_1 is stably isomorphic to \mathbb{S}_{E_1} , x must then belong to a free E_1 -submodule. But x cannot be in the target of any operation for degree reasons (it is in minimal degree and the operation have a positive degree). Thus there is an E_1 -submodule E_1x in M_1 . In particular, an operation in E_2 acts non trivially on Q_0x . As M_2 is stably isomorphic to \mathbb{S}_{E_2} , Q_0x must then belong to a free E_2 -submodule. But again, Q_0x cannot be in the target of any operation in E_2 for degree reasons, giving $E_2 \cdot Q_0x \subset M_2$. Consequently, x generates a $D(2)$ -free submodule, as $Q_2P_2^1Q_1Q_0x \neq 0$. Now, $D(2)$ is an injective submodule (recall that injective, projective and free modules are the same notion over a connective finite dimensional Hopf algebra), so $D(2)$ splits off. This is in contradiction with the hypothesis that M is reduced. \square

4. TWO DESCENTS TO $\text{Pic}(\mathcal{A}(2))$

We will first review the algebraic descent spectral sequence, henceforth will be abbreviated to Alg-DSS, which is our main computational tool for our main result. This spectral sequence was developed in [Ric16] and is inspired from the descent spectral sequence which appeared in [MS14b]. This section recollects the constructions of *loc cit* and reformulate some of the results in our particular case. The key point of this section is Corollary 4.3, which gives a explicit condition under which the relative Picard groups are trivial.

Let $B \subset A$ be a conormal Hopf subalgebra such that algebra $(A//B)^*$ is exterior on one element τ in degree $|\tau| \geq 1$. This is exactly the situation we will encounter later, first when $A = C(2)$ and $B = D(2)$, and, second when $A = \mathcal{A}(2)$ and $B = C(2)$. Given an A -module M_A , let $\text{End}_A(M_A)$ be the space $\text{hom}_A(M_A, M_A)$, i.e. space of homomorphisms from M_A to itself. In particular, $\pi_i(\text{End}_A(M_A)) \cong [M_A, M_A]_{i,0}^A$. In [Ric16], the author considers the moduli space of stable A -modules, over a fixed B -module M_B . This is a topological space $\mathcal{L}_A(M_B)$ whose connected components are in one-to-one correspondence with the set of stable equivalence classes of A -modules M_A such that $UM_A = M_B$. Let M_A be the base point for $\mathcal{L}_A(M_B)$. Then one should note that

$$\Omega\mathcal{L}_A(M_B) \cong GL_1(\text{End}_A(M_A)),$$

which means

$$\pi_i(\mathcal{L}_A(M_B), M_A) = \pi_{i-1}(GL_1(\text{End}_A(M_A)))$$

for $i \geq 1$. In particular, one can give an explicit description of the homotopy groups of $\mathcal{L}_A(M_B)$:

- $\pi_0(\mathcal{L}_A(M_B))$ is the set of stable equivalence classes of A -module whose restriction to $\mathbf{St}(B)$ is M_B ,
- If $\mathcal{L}_A(M_B)$ is nonempty, then $\pi_1(\mathcal{L}_A(M_B), M_A) \cong \text{Aut}_{\mathbf{St}(A)}(M_A)$ where the A -module M_A is chosen to be the basepoint of $\mathcal{L}_A(M_B)$, and,
- for $i \geq 2$, $\pi_i(\mathcal{L}_A(M_B), M) \cong [M, M]_{i-1,0}^A$.

Remark 4.1. In the case when M_B is the unit \mathbb{S}_B in $\mathbf{St}(B)$, $\mathcal{L}_A(\mathbb{S}_B)$ clearly is nonempty as \mathbb{S}_A is a point in $\mathcal{L}_A(\mathbb{S}_B)$, and,

$$\pi_0(\mathcal{L}_A(\mathbb{S}_B)) \cong \text{Pic}(A, B).$$

The higher homotopy groups are

$$\pi_i(\mathcal{L}_A(\mathbb{S}_B), \mathbb{S}_A) \cong [\mathbb{S}_A, \mathbb{S}_A]_{i-1,0}^A.$$

So far we have described the negative homotopy groups in terms of *Ext*-groups in (2). For the unit, \mathbb{S}_A one can use Poincaré duality (see [Ric16, Section 4]) to conclude

$$(4) \quad \pi_i(\mathcal{L}_A(\mathbb{S}_B), \mathbb{S}_A) \cong \text{Ext}_A^{i-2, -|A|}(\mathbb{S}_A, \mathbb{S}_A)$$

where $|A|$ is the maximum internal degree among elements in A . Note that the latter is zero, for degree reasons.

The forgetful functor $\mathbf{Mod}(A) \rightarrow \mathbf{Mod}(B)$ stabilizes in a strong symmetric monoidal functor

$$U : \mathbf{St}(A) \rightarrow \mathbf{St}(B).$$

Furthermore, the functor U has a right adjoint $F_B(A, -)$. Using this adjunction, one can produce an Endomorphism spectral sequence (see [Ric16]), henceforth will be denoted by EndSS,

$$(5) \quad \text{End} E_2^{s,n} \cong \bigoplus_t \text{Ext}_{(A//B)^*}^{n,t}(\mathbb{S}_{(A//B)^*}, \text{Ext}_B^{s-1, t-|B|}(\mathbb{S}_B, \mathbb{S}_B)) \Rightarrow \pi_{s-n}(\text{End}_A(\mathbb{S}_A))$$

which computes the homotopy groups of $\text{End}_A(\mathbb{S}_A)$.

Remark 4.2. Note that, we have the following chain of isomorphisms:

$$\begin{aligned} [\mathbb{S}_B, \mathbb{S}_B]_{s,t}^B &= [U\mathbb{S}_A, \mathbb{S}_B]_{s,t}^B \\ &\cong [\mathbb{S}_A, \mathbf{hom}_B(A, \mathbb{S}_B)]_{s,t}^A \\ &\cong [\mathbb{S}_A, \mathbb{S}_A \otimes (A//B)^*]_{s,t}^A. \end{aligned}$$

The action of $(A//B)^*$ is induced by the action of $(A//B)^*$ on itself in $\text{Ext}_A^{s,t}(\mathbb{S}_A, (A//B)^* \otimes \mathbb{S}_A)$.

EndSS (5) must be compared to the Cartan-Eilenberg spectral sequence (see [Rav86]). The Cartan Eilenberg spectral sequence is a tri-graded spectral sequence

$$E_2^{n,s,t} = \bigoplus_{t'+t''=t} \text{Ext}_{(A//B)^*}^{s,t'}(\mathbb{S}_{(A//B)^*}, \text{Ext}_B^{s',t''}(\mathbb{S}_B, \mathbb{S}_B)) \Rightarrow \text{Ext}_A^{n+s',t}(\mathbb{S}_A, \mathbb{S}_A).$$

which can be extended to compute the bigraded stable homotopy in the stable category

$$(6) \quad \text{cess} E_2^{n,s,t} = \bigoplus_{t'+t''=t} \text{Ext}_{(A//B)^*}^{n,t'}(\mathbb{S}_{(A//B)^*}, [\mathbb{S}_B, \mathbb{S}_B]_{s,t''}^B) \Rightarrow [\mathbb{S}_A, \mathbb{S}_A]_{s-n,t}^A$$

which we will refer to as CESS. Using the fact that $\pi_i(\text{End}(\mathbb{S}_A)) = [\mathbb{S}_A, \mathbb{S}_A]_{i,0}$ along with the Poincaré duality isomorphism, it is easy to see that EndSS (5) is the restriction of CESS (6) to $t = 0$.

Similar to the EndSS (5), we have a spectral sequence

$$E_1^{s,n}(\mathcal{L}_A(M_B)) \Rightarrow \pi_{s-n}(\mathcal{L}_A(M_B)),$$

where s is a homological degree, and n the spectral sequence degree. We call this spectral sequence Algebraic descent spectral sequence or Alg-DSS. The first differential is induced by the product on $(A//B)^*$. Thus we have (see [Ric16, Corollary 8.11])

$$(7) \quad \text{dss} E_2^{s,n} \cong \bigoplus_t \text{Ext}_{(A//B)^*}^{n,t}(\mathbb{S}_{(A//B)^*}, \text{Ext}_B^{s-2, t-|B|}(\mathbb{S}_B, \mathbb{S}_B)) \Rightarrow \pi_{s-n}(\mathcal{L}_A(\mathbb{S}_B)).$$

Taking a minimal resolution of \mathbb{S}_B , one can assume that

$$\text{cess} E_1^{*,s,t} \cong \mathbb{F}[\theta] \otimes \text{Ext}_B^{s, t-|\tau|}(\mathbb{S}_B, \mathbb{S}_B)$$

where $|\theta| = (1, 0, |\tau|)$, since τ is the Bockstein associated to θ . Note that the E_2 -pages of spectral sequences of (5) and (7) are isomorphic up to a shift, hence we may assume that ${}_{dss}E_1^{s,n} \cong {}_{\text{End}}E_1^{s-1,n}$. As a result we get the following lemma.

Lemma 4.3. *The elements in ${}_{dss}E_1^{s,n}$ with $s - n = 0$, are of the form $\theta^n \otimes y$ where*

$$y \in \text{Ext}_B^{n-2, n|\tau|-|B|}(\mathbb{S}_B, \mathbb{S}_B).$$

Many but not all, higher differentials in Alg-DSS (7) can be imported from the stable Cartan Eilenberg spectral sequence (5) using the following comparison tool that has been established in [MS14b, Section 5] (also see [Ric16, Proposition 7.6]).

Proposition 4.4 ([MS14a],[MS14b, 5.2.4, Remark 5.2.5]). *The differential of length r originating at $(s-1, 0, n)$ in ${}_{\text{End}}E_r^{s-1,n}$ coincides with the differential originating at ${}_{dss}E_r^{s,n}$, if $s \geq r+1$.*

Note that if $\theta^q = 0$ in $\text{Ext}_A^{*,*}(\mathbb{S}_A, \mathbb{S}_A)$ then the d_q is the last possible differential in CESS (6) and hence in EndSS (5). Since $\pi_{-1}(\text{End}_A(\mathbb{S}_A)) = 0$ means that all the elements in ${}_{\text{End}}E_1^{n+1,n}$ are either zero or not present in the E_∞ -page of EndSS. Thus by Proposition 4.4 all the in ${}_{dss}E_1^{n,n}$ for $n \geq q+1$ cannot be a nonzero permanent cycle. Hence the potential nonzero elements in $\pi_0(\mathcal{L}_A(M_B))$ are of the form

$$\theta^n \otimes y \in {}_{dss}E_1^{n,n}$$

where $n \leq q$ and y has bidegree $(n-2, n|\tau|-B)$. Suppose $(q+1)|\tau| < |B|$, then y is forced to be trivial whenever $n \leq q$. Thus we can conclude:

Lemma 4.5. *If $B \subset A$ be a normal Hopf subalgebra of a connected Hopf algebra such that*

- $A//B^* = E(\tau)$,
- $\theta^q = 0$ in $\text{Ext}_A^{*,*}(\mathbb{S}_A, \mathbb{S}_A)$, and,
- $(q+1)|\tau| < |B|$,

then $\text{Pic}(A, B) = 0$.

Theorem 4.6. *The morphism of groups*

$$\iota : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Pic}(\mathcal{A}(2)),$$

which sends (n, m) to $\mathbb{S}_{\mathcal{A}(2)}^{n,m}$, is an isomorphism.

Proof. When $A = C(2)$ and $B = D(2)$, then $A//B^* = E(\xi_1^2)$ and ξ_1^2 is Bockstein to h_{11} . Since $h_{11}^3 = 0$ in $\text{Ext}_{C(2)}^{*,*}(\mathbb{S}_{C(2)}, \mathbb{S}_{C(2)})$ (see Proposition A.2), we see that the all the criterias of Lemma 4.5 is satisfied. Hence, $\text{Pic}(C(2), D(2)) = 0$. Since we have already established $\text{Pic}(D(2)) = 0$ (Proposition 3.1), it follows that $\text{Pic}(C(2)) = 0$. This completes our first descent.

For the second descent with $A = \mathcal{A}(2)$ and $B = C(2)$, $A//B^* = E(\xi_1^4)$ and ξ_1^4 is Bockstein to h_{12} which satisfies the relation $h_{12}^3 = 0$ in $\text{Ext}_{\mathcal{A}(2)}^{*,*}(\mathbb{S}_{\mathcal{A}(2)}, \mathbb{S}_{\mathcal{A}(2)})$ (see Proposition A.3). Easy to check that all the criterias of Lemma 4.5 is satisfied, therefore $\text{Pic}(\mathcal{A}(2), C(2)) = 0$, hence $\text{Pic}(\mathcal{A}(2)) \cong \mathbb{Z} \times \mathbb{Z}$. \square

APPENDIX A. VARIOUS EXTENSION GROUPS

The aim of this section is to describe the computations of the bigraded Ext-groups of \mathbb{F} over the Hopf algebras $D(2)$, $C(2)$ and $\mathcal{A}(2)$. The May spectral sequence is the most suitable tool for these sorts of computations which we briefly recall. It is convenient to work with the dual i.e. $D(2)^*$, $C(2)^*$ and $\mathcal{A}(2)^*$, for this purpose.

May spectral sequence is obtained by assigning an additional filtration, called the *May filtration*, to the Hopf algebra such that every element is primitive modulo the filtration. This filtration was introduced by J.P. May in his thesis [May64]. Let the May filtration of ξ_i is $2i - 1$ following Ravenel [Rav86]. As a result we get a filtration on the cobar complexes for $D(2)^*$, $C(2)^*$ and $\mathcal{A}(2)^*$, which produces the trigraded May spectral sequence

$$\begin{aligned} E_1^{s,t,*} &:= \mathbb{F}[h_{10}, h_{20}, h_{21}, h_{30}] &\Rightarrow \text{Ext}_{C(2)}^{s,t}(\mathbb{F}, \mathbb{F}) \\ E_1^{s,t,*} &:= \mathbb{F}[h_{10}, h_{11}, h_{20}, h_{21}, h_{30}] &\Rightarrow \text{Ext}_{C(2)}^{s,t}(\mathbb{F}, \mathbb{F}) \\ E_1^{s,t,*} &:= \mathbb{F}[h_{10}, h_{11}, h_{12}, h_{20}, h_{21}, h_{30}] &\Rightarrow \text{Ext}_{\mathcal{A}(2)}^{s,t}(\mathbb{F}, \mathbb{F}) \end{aligned}$$

where h_{ij} corresponds to the class $[\xi_i^{2^j}]$ in the respective cobar complexes whose tridegree is $|h_{ij}| = (1, 2^j(2^i - 1), 2i - 1)$. The d_1 -differentials in May SS comes from the coproducts structure and higher differentials can be computed using Nakamura's formula [Nak72], which is also described in [BEM14, Section 2]. We display part of the Ext-groups in charts in $(t - s, s)$ -coordinate system where each \bullet represents an \mathbb{F} vector space, vertical line represents multiplication by h_{10} , the slanted lines of slope $\frac{1}{2}$ represent multiplication by h_{11} and dotted lines of slope $\frac{1}{3}$ represent multiplication by h_{12} .

Proposition A.1. *The only non-trivial differentials in the May spectral sequence computing $\text{Ext}_{D(2)}(\mathbb{F}, \mathbb{F})$ is $d_1(h_{30}) = h_{10}h_{21}$. Consequently,*

$$\text{Ext}_{D(2)}^{s,t}(\mathbb{F}, \mathbb{F}) \cong \frac{\mathbb{F}[h_{10}, h_{20}, h_{21}, h_{30}^2]}{(h_{10}h_{21})}.$$

This \mathbb{F} -algebra is represented in figure 2.

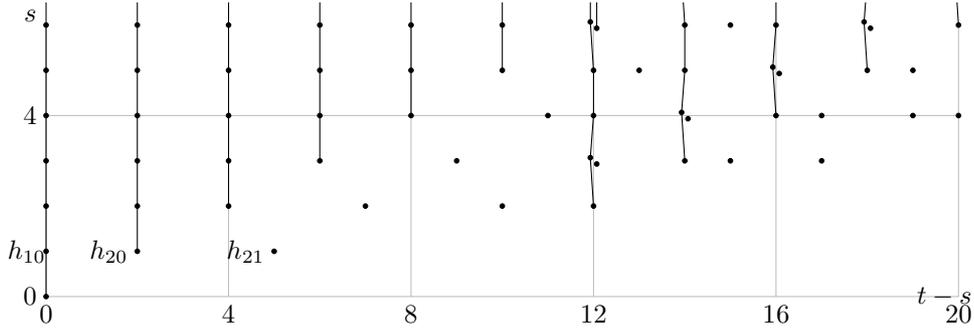


FIGURE 2. The algebra $\text{Ext}_{D(2)}^{s,t}(\mathbb{F}, \mathbb{F})$. An element in degree (s, t) is plotted in $(s, t - s)$.

Proposition A.2. *The only non-trivial differentials in the May spectral sequence computing $\text{Ext}_{C(2)}^{s,t}(\mathbb{F}, \mathbb{F})$ are*

- $d_1(h_{20}) = h_{10}h_{11}$,
- $d_1(h_{30}) = h_{10}h_{21}$,
- $d_2(h_{30}^2) = h_{11}h_{21}^2$, and,

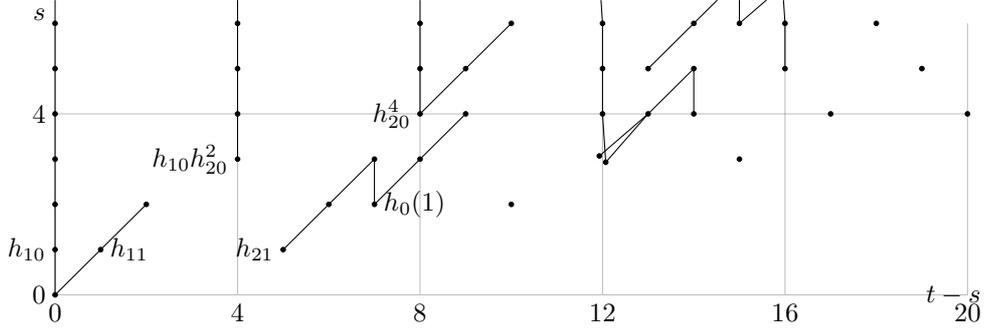


FIGURE 3. The algebra $\text{Ext}_{C(2)}^{s,t}(\mathbb{F}, \mathbb{F})$. An element in degree (s, t) is plotted in $(s, t-s)$. The element denoted $h_0(1)$ is $h_{20}h_{21} + h_{11}h_{30}$.

- $d_2(h_{20}^2) = h_{11}^3$.

This \mathbb{F} -algebra is represented in figure 3.

Proposition A.3. *The only non-trivial differentials in the May spectral sequence computing $\text{Ext}_{\mathcal{A}(2)}^{s,t}(\mathbb{F}, \mathbb{F})$ are*

- $d_1(h_{20}) = h_{10}h_{11}$,
- $d_1(h_{30}) = h_{10}h_{21} + h_{20}h_{12}$,
- $d_1(h_{21}) = h_{11}h_{12}$,
- $d_2(h_{20}^2) = h_{11}^3 + h_{10}^2h_{12}$,
- $d_2(h_{21}^2) = h_{12}^3$,
- $d_2(h_{30}^2) = h_{11}h_{21}^2$,
- $d_2(h_{20}h_{21} + h_{11}h_{30}) = h_{10}h_{12}^2$, and,
- $d_4(h_{30}^4) = h_{12}h_{21}^4$.

This \mathbb{F} -algebra is represented in figure 4.

Remark A.4. Note that the differential $d_2(h_{20}h_{21} + h_{11}h_{30}) = h_{10}h_{12}^2$ is not a straightforward consequence of Nakamura's operations. The interested reader is referred to [Tan70, Proposition 4.2] for the computation. One can compare the previous result to the list of relations in $\text{Ext}_{\mathcal{A}(2)}^{*,*}(\mathbb{F}, \mathbb{F})$ given in [SI67].

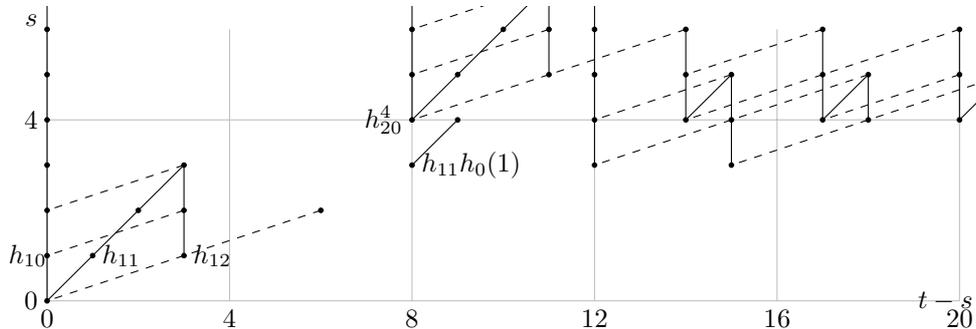


FIGURE 4. The algebra $\text{Ext}_{\mathcal{A}(2)}^{s,t}(\mathbb{F}, \mathbb{F})$. An element in degree (s, t) is plotted in $(s, t-s)$.

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