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A SPECTRUM WHOSE Z_p COHOMOLOGY IS THE ALGEBRA OF REDUCED p^{th} POWERS

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§1. INTRODUCTION

LET p be a prime, A_p the mod p Steenrod Algebra, $Q_O \in A_p$ the Bockstein cohomology operation and (Q_O) the two sided ideal generated by Q_O. Note, when $p \neq 2$, $A_p/(Q_O)$ is isomorphic to the subalgebra of A_p generated by $\mathscr{P}^i$, $i \geqq 0$. The main objective of this paper is to construct a spectrum [6] X such that, as an A_p module, $H^*(X; Z_p) \approx A_p/(Q_O)$.

Throughout this paper all cohomology groups will have Z_p coefficients unless otherwise stated. All spectra will be 0-connected. We will make various constructions on spectra, for example, forming fibrations and Postnikov systems, just as one does with topological spaces. For the details of this see [6]. If one wishes, one may read "spectrum" as N-connected topological space, N a large integer, add N to all dimensions in sight, and read all theorems as applicable in dimensions less than $2N$.

Let $\mathscr{R}$ be the set of sequences of integers $(r_1, r_2, \ldots)$ such that $r_i \geqq 0$ and $r_i = 0$ for almost all i. If $R = (r_1, r_2, \ldots)$, let $\dim R = \sum 2r_i(p^i - 1)$ and $l(R) = \sum r_i$. Let V_s be the graded free abelian group generated by $R \in \mathscr{R}$ such that $l(R) = s$. Let $K(V_s)$ be the Eilenberg MacLane spectrum such that $\pi(K(V_s)) = V_s$. Let $\alpha_R \in H^*(K(V_s); Z)$ be the generator corresponding to $R \in \mathscr{R}$. In [1] and [2] Milnor defined for each $R \in \mathscr{R}$ an element $\mathscr{P}^R \in A_p$ (including the case $p = 2$). Let $c : A_p \to A_p$ be the canonical antiautomorphism. Our main result is the following:

THEOREM 1.1. *There is a sequence of spectra* X_s, $s = 0, 1, 2, \ldots$ *and elements* $1_s \in H^O(X_s)$ *satisfying the following conditions:* $X_O = K(V_O)$. 1_O *is* $\alpha_{(O,O,\ldots)}$ *reduced mod p.* X_s *is a fibration over* X_{s-1} *with fibre* $K(V_s)$. $1_s = \pi_s^* 1_{s-1}$, *where* $\pi_s : X_s \to X_{s-1}$ *is the projection. If* $\tau_s : H^*(K(V_s); Z) \to H^*(X_{s-1}; Z)$ *is the transgression,*

$$p^{s-1}\tau_s(\alpha_R) = \delta c(\mathscr{P}^R)1_{s-1},$$

where δ *is the Bockstein operation associated with* $0 \to Z \to Z \to Z_p \to 0$. (*Note* $\tau_s(\alpha_R)$ *are the k-invariants.*) *For* $s > 0$, (Q_O) *is the kernel of the homomorphism* $A_p \to H^*(X_s)$ *given by* $\alpha \to \alpha 1_s$. $H^*(X_s) = (A_p/(Q_O))1_s$ *in dimensions less than* $2(s + 1)(p - 1)$.

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We prove 1.1 in §3.

Let $X = \varprojlim X_s$. Since $K(V_s)$ is $2s(p-1)-1$ connected, $H^q(X) \approx H^q(X_s)$ for $q < 2(s+1)(p-1)$. Also $\pi_i(K(V_s)) = 0$ for i odd and hence

$$\pi_i(X_s) = \sum_{t \leqq s} \pi_i(K(V_t)).$$

These facts together with (1.1) give the following:

COROLLARY 1.2. *As an A_p module, $H^*(X) \approx A_p/(Q_O) \cdot \pi(X)$ is isomorphic to the free abelian group generated by $\mathscr{R}$.*

(1.1) also yields:

THEOREM 1.3. *If Y is a spectrum such that $H^*(Y; Z)$ has no p-torsion and $H^*(Y)$ is a free $A_p/(Q_O)$ module with generators $y_r \in H^{n_i}(Y)$, then there is a map $f: Y \to \Pi S^{n_i} X$ ($S^{n_i} X$ is the n_i fold suspension) such that $f^*: H^*(\Pi S^{n_i} X) \approx H^*(Y)$. In particular, $\pi(Y) \approx V \otimes U$ modulo C_p, where C_p is the class of finite groups of order prime to p and V and U are the graded free abelian groups generated by $\mathscr{R}$ and $\{y_i\}$, respectively.*

Proof. Let $1 \in H^O(X)$ be the generator over A_p and let $1^n \in H^n(S^n X)$ and $1^n_s \in H^n(S^n X_s)$ be the n-fold suspensions of 1 and 1_s, respectively. There are maps $g_i: Y \to S^{n_i} X_O$ such that $g^*(1^{n_i}_O) = y_i$. Since $p^s \tau(\alpha_R) = p\delta c(\mathscr{P}^R) 1_{s-1} = 0$, all k-invariants in the construction of X have order a power of p. Hence g_i can be lifted to a map $f_i: Y \to S^{n_i} X$ such that $f_i^* 1^{n_i} = y_i$. Let $f = \Pi f_i$. Clearly, $f^*: H^*(\Pi S^{n_i} X) \approx H^*(Y)$.

In [2] Milnor showed that $H^*(MU)$, for all p, and $H^*(MSO)$, for $p \neq 2$, are free $A_p/(Q_O)$ modules. $H^*(MU; Z)$, for all p, and $H^*(MSO; Z)$, for odd p, have no p-torsion. Thus (1.3) implies the following result of Milnor [2] and Novikov [3]:

COROLLARY 1.4. *The U-cobordism groups have no torsion and the oriented cobordism groups have no odd torsion.*

In [5] Thom proved the following result. Let $z \in H_n(K)$, where K is a finite CW-complex. Let L be the N-dual of K, N large. Let $u \in H^{N-n-1}(L)$ be the Alexander dual of z. Then there is an oriented manifold M and a map $f: M \to K$ such that $f_*([M]) = z$ if and only if there is a map $g: L \to MSO(N-n-1)$ such that $g^*(\alpha_{N-n-1}) = u$, where α_{N-n-1} is the Thom class. We show how our result relates to the existence of such an f.

The following is an easy corollary of (1.3).

THEOREM 1.5. *Let Y be a spectrum satisfying the hypotheses of* (1.3) *and such that $H^O(Y) \approx Z_p$ is generated by α (for example, $Y = MU$, MSO). Let $Y(q)$ be the q^{th} term in the spectrum Y and $\alpha_q \in H^q(Y(q))$ the class corresponding to α. Let L be a finite CW complex and $u \in H^q(L)$ with $2q > \dim L$. Then there is a map $g: L \to Y(q)$ such that $q^*(\alpha_q) = u$ if and only if there is a map $h: L \to X(q)$ such that $h^*(1(q)) = u$. In particular, if $H(L; Z)$ has no p torsion, $g: L \to Y(q)$ always exists.*

Theorem 1.5 might suggest the following conjecture. Let L and u be as in (1.5). Then there is a map $h: L \to X(q)$ such that $h^*(1(q)) = u$ if and only if $\delta_s c(\mathscr{P}^R) u = 0$ for all R and s, where δ_s is s^{th} order higher order Bockstein (i.e. $c(\mathscr{P}^R) u \in \operatorname{Im}(H^*(L_1 Z) \to H^*(L))$ for all

R). In §4 we give a counterexample to this conjecture and thus a counterexample to the main theorem of [4].†

§2. PRELIMINARIES CONCERNING THE STEENROD ALGEBRA

In [1] and [2] Milnor defined elements Q_i and $\mathscr{P}^R \in A_p$ for $i = 0, 1, 2, \ldots$, and $R \in \mathscr{R}$ and proved the following facts about them.

If $U, V \in \mathscr{R}$, $U - V \in \mathscr{R}$ is defined if $u_i \geqq v_i$ and is equal to $(u_1 - v_1, u_2 - v_2, \ldots)$. $\Delta_j \in \mathscr{R}$ denotes the sequence with 1 in the j^{th} place and zeros elsewhere.

(2.1) $\dim Q_i = 2p^i - 1$. $\dim \mathscr{P}^R = \dim R$.

(2.2) $\{Q_i\}$ is the basis for a Grassmann subalgebra, A_O of A_p, i.e. $Q_i Q_j = 0$ if and only if $i = j$ and $Q_i Q_j + Q_j Q_i = 0$.

(2.3) A_p is a free right A_O-module and $\{\mathscr{P}^R\}$ is a Z_p basis for $A_p/(Q_O)$.

(2.4) $(Q_O) = A_p Q_O + A_p Q_1 + A_p Q_2 + \cdots$.

(2.5) $\mathscr{P}^R Q_O = Q_O \mathscr{P}^R + \sum Q_j \mathscr{P}^{R-\Delta j}$.

(2.6) If $c\colon A_p \to A_p$ is the canonical antiautomorphism, $c(Q_i) = -Q_i$.

LEMMA 2.6. *$Q_O c(\mathscr{P}^R) = \sum c(\mathscr{P}^{R-\Delta_j}) Q_O c(\mathscr{P}^{\Delta_j}) + aQ_O$ where $a \in A_p$.*

Proof. Applying c to (2.5) gives

$$Q_O c(\mathscr{P}^R) = c(\mathscr{P}^R) Q_O + \sum c(\mathscr{P}^{R-\Delta_j}) Q_j.$$

Taking $R = \Delta_j$, this yields:

$$Q_j = Q_O c(\mathscr{P}^{\Delta_j}) - c(\mathscr{P}^{\Delta_j}) Q_O.$$

Combining these two formulas gives (2.6).

Recall V_s is the graded abelian group generated by $R \in \mathscr{R}$ such that $l(R) = s$. Let $M_s = A_p/A_p Q_O \otimes V_s$ and let $d_s\colon M_s \to M_{s-1}$ be the A_p homomorphism of degree $+1$ given by

$$\begin{aligned} d_s(1 \otimes R) &= \sum Q_j \otimes (R - \Delta_j) \\ &= Q_O \sum c(\mathscr{P}^{\Delta_j}) \otimes (R - \Delta^j). \end{aligned}$$

Let $\alpha\colon M_O \to A_p/(Q_O)$ be given $\alpha(1 \otimes (O, O, \ldots)) = 1$.

LEMMA 2.7. *The following is exact:*

$$\to M_s \xrightarrow{d_s} M_{s-1} \to \cdots \to M_O \xrightarrow{\alpha} A_p/(Q_O) \to 0.$$

Proof. Let B be the Grassman algebra $A_O/\{Q_O\}$. Then B is a Grassman algebra generated by Q_i, $i > 0$. It is well known that the following is a B-free acyclic resolution of Z_p:

$$\to B \otimes V_s \xrightarrow{d_s} B \otimes V_{s-1} \to \cdots \to B \otimes V_O \xrightarrow{\beta} Z_p \to 0 \tag{2.8}$$

† The fact that the proof of the main theorem in [4] is not convincing was noted by A. DOLD: *Math. Rev.* **27** (1964), 2994.

where $d_s(1 \otimes R) = \sum Q_j \otimes (R - \Delta_j)$ and $\beta(1 \otimes (O, O, \ldots)) = 1$. $B \otimes V_s$ is an A_O-module. Note

$$A_p \oplus_{A_O} Z_p = A_p/\Sigma\, A_p Q_i = A_p/(Q_O)$$

Applying the functor $A_p \otimes_{A_O}$ to (2.8) yields the sequence in (2.7). But A_p is a free A_O-module and hence $A_p \otimes_{A_O}$ preserves exactness.

§3. PROOF (1.1)

In this section, if u is a cohomology class with integer coefficients, $\bar{u}$ will denote its reduction mod p.

We will need the following easily proved lemma.

LEMMA 3.1. *If $F \xrightarrow{i} E \xrightarrow{\pi} B$ is a fibration of spectra, $\bar{\tau} : H^*(F) \to H^*(E)$ is the transgression, $v \in H^*(F)$, $u \in H^*(B; Z)$ and $\bar{\tau}(v) = \bar{u}$, then there is a $w \in H^*(E; Z)$ such that $\pi^* u = pw$ and $i^* w = \delta v$, where δ is the Bockstein operation associated to $0 \to Z \to Z \to Z_p \to 0$.*

Note $H^*(K(V_s)) \approx A_p/A_p Q_O \otimes V_s = M_s$. We identify $H^*(K(V_s))$ and M_s.

We construct by induction on $s = 0, 1, 2, \ldots$ a sequence of spectra X_s, elements $1_s \in H^O(X_s)$ and $k_R^s \in H^*(X_s; Z)$, for $R \in \mathscr{R}$ and $l(R) > s$, and homomorphisms $\bar{\tau}_{s+1} : M_{s+1} \to H^*(X_s)$ satisfying the following conditions:

(3.2). If $s > 0$, X_s is a fibration over X_{s-1} with fibre $K(V_s)$. $1_s = \pi^* 1_{s-1}$ where π_s is the projection. $\tau_s(\alpha_R) = k_R^{s-1}$ where $\tau_s : H^*(K(V_s); Z) \to H^*(X_{s-1}; Z)$ is the transgression and $\alpha_R \in H^*(K(V_s); Z)$ is the generator corresponding to $R \in \mathscr{R}$, $l(R) = s$.

(3.3). $\bar{\tau}_{s+1}(1 \otimes R) = \bar{k}_R^s$.

(3.4). If $s > 0$ and $\alpha \in A_p$, then $\alpha 1_s = 0$ if and only if $\alpha \in (Q_O)$. Coker $\bar{\tau}_{s+1} = (A_p/(Q_O))1_s$.

(3.5). The following sequence is exact.

$$M_{s+2} \xrightarrow{d_{s+2}} M_{s+1} \xrightarrow{\bar{\tau}_{s+1}} H^*(X_s).$$

(3.6). $p^s k_R^s = \delta c(\mathscr{P}^R) 1_s$ for $l(R) > s$.

(3.7). If $l(R) > s$

$$\bar{k}_R^s = \sum c(\mathscr{P}^U) \bar{k}_{R-U}^s$$

where the sum ranges over U such that $R - U$ is defined and $l(R - U) = s + 1$.

Note (3.2), (3.4) and (3.6) imply (1.1).

Let $X_O = K(V_O)$, $1_O = 1 \otimes (O, O, \ldots)$, $k_R^O = \delta c(P^R) 1_O$ and $\bar{\tau}_1 = d_1$. (3.3), (3.4) and (3.5) follow from (2.7). (3.6) is immediate. (3.7) is (2.6).

Suppose X_{s-1}, 1_{s-1}, k_R^{s-1} and $\bar{\tau}_s$ have been defined and satisfy (3.2)–(3.7). (3.2) defines X_s and 1_s. If $l(R) > s$

$$\bar{\tau}_s\Big(\sum_{l(R-U)=s} c(\mathscr{P}^U) \otimes (R - U)\Big) = \sum_{l(R-U)=s} c(P^U) \bar{k}_{R-U}^{s-1}$$
$$= \bar{k}_R^{s-1}.$$

Therefore by (3.1), there are elements $k_R^s \in H^*(X_s; Z)$ such that

$$\pi_s^* k_R^{s-1} = p k_R^s$$
$$i^* k_R^s = \delta \sum_{l(R-U)=s} c(\mathscr{P}^U) \otimes (R - U).$$

Let $\bar{\tau}_{s+1}$ be defined by (3.3).

$$\begin{aligned} p^s k_R^s &= \pi_s^*(p^{s-1} k_R^{s-1}) \\ &= \pi_s^*(\delta c(\mathscr{P}^R) 1_{s-1}) \\ &= \delta c(\mathscr{P}^R) 1_s. \end{aligned}$$

Hence (3.6) is satisfied.

Consider the following diagram:

$$\begin{array}{ccccccc} & & M_{s+2} & \xrightarrow{d_{s+2}} & M_{s+1} & & \\ & & & & \bar{\tau}_{s+1} \downarrow & \searrow^{d_{s+1}} & \\ & & & & & & M_s \xrightarrow{\bar{\tau}_s} H^*(X_{s-1}) \\ & & & & & \nearrow_{i^*} & \\ M_s & \xrightarrow{\bar{\tau}_s} & H^*(X_{s-1}) & \xrightarrow{\pi_{*s}} & H^*(X_s) & & \end{array}$$

The two horizontal sequences are exact. $i^*\bar{\tau}_{s+1} = d_{s+1}$, for if $l(R) = s + 1$

$$\begin{aligned} i^*\bar{\tau}_{s+1}(1 \otimes R) &= i^* \bar{k}_R^s \\ &= Q_O \sum c(\mathscr{P}^{\Delta_j}) \otimes (R - \Delta_j) \\ &= d_{s+1}(1 \otimes R). \end{aligned}$$

Kernel $i^* = (A_p/(Q_O))1_s$ since coker $\bar{\tau}_s = (A_p/(Q_O))1_{s-1}$.

We now prove (3.7).

$$\begin{aligned} i^* \sum_{l(R-U)=s+1} c(\mathscr{P}^U) \bar{k}_{R-U}^s &= \sum_{l(R-U)=s+1,j} c(\mathscr{P}^U) Q_O c(\mathscr{P}^{\Delta_j}) \otimes (R - U - \Delta_j) \\ &= \sum_{l(V)=s} c(\mathscr{P}^V) \otimes (R - V) \\ &\doteq i^* \bar{k}_R^s. \end{aligned}$$

The second equality follows from (2.6). Kernel i^* contains only even dimensional elements and $\bar{k}_R^s$ has odd dimension. Therefore (3.7) holds. From the above diagram one sees that to verify (3.4) and (3.5) it is sufficient to show that $\bar{\tau}_{s+1} d_{s+2} = 0$. If $l(R) = s + 2$,

$$\begin{aligned} \bar{\tau}_{s+1} d_{s+2}(1 \otimes R) &= \bar{\tau}_{s+1} \sum_j Q_O c(\mathscr{P}^{\Delta_j}) \otimes (R - \Delta_j) \\ &= Q_O \sum c(\mathscr{P}^{\Delta_j}) \bar{k}_{R-\Delta_j}^s \\ &= Q_O \bar{k}_R^s \\ &= 0. \end{aligned}$$

This completes the inductive step.

§4. APPLICATIONS OF THEOREM 1.1

In this section we give our counterexample to the conjecture stated in §1.

We construct a counterexample for the case $p = 3$. Let $L^{q+4} = S^q \cup e^{q+4}$ attached by a map of degree 3. Let $f: S^{q+7} \to L^{q+4}$ be an element of order 9. Extend $3f$ to a map $h: S^{q+7} \cup_9 e^{q+8} \to L^{q+4}$. Define $L = L^{q+9} = L^{q+4} \cup_h C(s^{q+7} \cup_9 e^{q+8})$. Let $u \in H^q(L)$ be a generator, and let $q > 9$. Then u satisfies the hypotheses of conjecture given in §1, i.e. $c(\mathscr{P}^R)(u) \in \operatorname{Im}(H^*(L; Z) \to H^*(L))$ for all $R \in \mathscr{R}$ as $c(\mathscr{P}^{\Delta_1})(u)$ and $c(\mathscr{P}^O)(u)$ are the only non-zero $c(\mathscr{P}^R)(u)$. Let $g: L \to K(Z, q)$ be such that $u = g^*(\iota) \bmod 3$. $K(Z, q) = X_O(q)$, and we wish to show that g does not factor through $X_2(q)$. The cells of $X_1(q)$ which contribute to the mod 3 homology of $X_1(q)$ in dimensions $\leqq q + 10$ are the following: $W = S^q \cup e^{q+4} \cup_h C(S^{q+7} \cup_9 e^{q+8})$, where $\bar{h}: S^{9+7} \cup_9 e^{q+8} \to L^{q+4}$ extends f and we assume $h = 3\bar{h}$. Let $\bar{g}: L \to W$ be the obvious map. $H^{q+9}(W; Z) = Z_9$, let k be agenerator. This is part of the k-invariant for $X_2(q)$. Then $\bar{g}^*(k) = 3v$, v a generator of $H^{q+9}(L; Z) \approx Z_9$. Let $\bar{\bar{g}}: L \to W$ be another map such that $\pi_1 \bar{\bar{g}} \simeq \pi_1 \bar{g} \simeq g$. Then $\bar{\bar{g}} = \bar{g} - ix$ where $x: L \to K(Z, q+4)$ and $i: K(Z, q+4) \to X_1(q)$. Now $(ix)^*(k) = \delta \mathscr{P}^1 x^*(l_{q+4}) = 0$ as $\mathscr{P}^1(x^*(l_{q+4})) = 0$. Thus $\bar{\bar{g}}^*(k) = \bar{g}^*(k) \neq 0$ and $\bar{\bar{g}}$ cannot be extended to $X_2(q)$.

We conclude by conjecturing that Theorem 1.3 can be generalized. That is, if Y is a spectrum such that $H^*(Y)$ is a free $A_p/(Q_O)$ module, then a knowledge of the Bockstein spectral sequence of $H^*(Y)$ would determine $\pi(Y)$. A case of interest is $Y = MSPL$.

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