WHITEHEAD PRODUCTS AND COHOMOLOGY OPERATIONS

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1. Introduction

LET $\iota \in \pi_n(S^n)$ be a generator and let Y be the space formed by attaching a $2n$-cell to S^n by a map representing $[\iota, \iota]$. Let $u \in H^n(Y; Z)$ and $v \in H^{2n}(Y; Z)$ be generators of the respective groups. It is well known that, if n is even, $u^2 = 2v$. If n is odd, it can easily be shown (for example, by considering the map from Y to the symmetric square of S^n) that every primary cohomology operation

$$\alpha\colon\ H^n(Y; G_1) \to H^{2n}(Y; G_2)$$

is zero, and hence that Y cannot be distinguished from $S^n \vee S^{2n}$ by primary cohomology operations.

In this paper we construct non-stable, *modulo* 2, secondary cohomology operations which are non-zero on u when $n+1$ is not a power of two. Using results by Adams (**1**) we construct tertiary operations non-zero on u when $n+1 = 2^r$ $(r > 3)$. Since $[\iota, \iota] = 0$ for $n = 1, 3, 7$, these are all possible cases. L. Kristensen (**5**) has also considered the secondary operations that we use.

We construct these cohomology operations in the usual way from Postnikov systems (**2**). For example, when $n+1 \neq 2^r$, we construct a quadruple $(K(Z_2, n), \phi_1, E, \phi_2)$, where E is a fibre space over $K(Z_2, n)$ with fibre a product of $K(Z_2, i)$'s $(n < i < 2n)$ and k-invariant ϕ_1, while $\phi_2 \in H^{2n}(E; Z_2)$. Hereafter all cohomology groups will have coefficients in Z_2, and $H^j(X)$ will be identified with the homotopy classes of maps of X into $K(Z_2, j)$. ϕ_2 gives rise to a cohomology operation Φ as follows: Let X be a space and $x \in H^n(X)$. Φ is defined on x if $\phi_1 x = 0$ and, if defined, $\Phi(x)$ consists of all elements of the form $\bar{x}^*(\phi_2) \in H^{2n}(X)$, where $\bar{x}$ is a lifting of x to a map of X into E.

As a corollary of the above we obtain examples of spaces with non-trivial Whitehead squares. When n is even, one can easily construct a space X such that

$$\pi_i(X)\begin{cases} \approx Z & (i = n, 2n-1), \\ = 0 & (\text{otherwise}), \end{cases}$$

and $[\alpha, \alpha] \neq 0$ for any non-zero α of $\pi_n(X)$. Suppose that n is odd; then

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the fact that all primary operations are zero in Y implies that there is no space X with a non-trivial Whitehead square and with non-trivial homotopy groups only in dimensions n and $2n-1$. (Take $G_1 = \pi_n(X)$, $G_2 = \pi_{2n-1}(X)$, and α to be the k-invariant of X.) Our methods provide an upper bound less than n to the minimum number of non-trivial homotopy groups that an $n-1$ connected space must have in order that $[\alpha, \alpha] \neq 0$ for some α in $\pi_n(X)$. For example, if $n \equiv 1 \pmod 4$, there is a space X such that

$$\pi_i(X) \begin{cases} \approx Z_2 & (i = n,\ 2n-2), \\ \approx Z_4 & (i = 2n-1), \\ = 0 & (\text{otherwise}), \end{cases}$$

and $[\alpha, \alpha] \neq 0$, where $\alpha \in \pi_n(X)$ is the generator. (For the construction of this space see the remark at the end of § 2.) In general such spaces can be constructed as follows: let Φ and $(K(Z_2, n), \phi_1, E, \phi_2)$ be as above; let X be the fibration over E with fibre $K(Z_2, 2n-1)$ and k-invariant ϕ_2.

THEOREM 1.1. *If* $\Phi(u) \neq 0$, $u \in H^n(Y)$, *then* $[\alpha, \alpha] \neq 0$, *where* α *is the generator of* $\pi_n(X)$.

Proof. Let $f\colon S^n \to X$ represent α. If $[\alpha, \alpha] = 0$, f can be extended to a map of Y into X. But $\Phi(u) \neq 0$ is exactly the statement that such an extension does not exist.

In order to find an operation Φ such that $\Phi(u) \neq 0$ it is only necessary to find a Φ which is quadratic in the following sense:

THEOREM 1.2. *Suppose* Φ *a cohomology operation such that for any space* X, *if* Φ *is defined on* x, y *in* $H^n(X)$, *it is defined on* $x+y$ *and*

$$\Phi(x+y) = \Phi(x)+\Phi(y)+xy$$

modulo the indeterminacy of Φ, *where* xy *means cup product. If* $\Phi(u)$ *is defined* (mod 0), *then* $\Phi(u) \neq 0$.

Proof. Let $g\colon S^n \times S^n \to Y$ be an extension of the folding map of $S^n \vee S^n$ onto S^n; g exists because $[\iota, \iota] = 0$ in Y. Let $v \in H^n(S^n)$ be a generator. Then

$$\begin{aligned} g^*\Phi(u) &= \Phi(g^*u) \\ &= \Phi(v \otimes 1 + 1 \otimes v) \\ &= \Phi(v) \otimes 1 + 1 \otimes \Phi(v) + v \otimes v \\ &= v \otimes v \\ &\neq 0. \end{aligned}$$

In § 2 we show that Φ is quadratic if ϕ_2 is not primitive, and that a non-primitive ϕ_2 can be constructed from a non-stable relation between Sq^i's obtained as follows. If $n+1 \neq 2^r$, Sq^{n+1} is decomposable in the *modulo* 2 Steenrod algebra. On classes x of dimension n this relation, less the Sq^{n+1} term, gives a relation because $Sq^{n+1}x = 0$. Finally in § 3 we discuss the case when $n+1 = 2^r$.

2. A secondary cohomology operation

Suppose that
$$Sq^{n+1} = \sum_{i=1}^{l} a_i' a_i, \tag{2.1}$$
where a_i' and a_i are in the *modulo* 2 Steenrod algebra and $\dim a_i' > 0$ and $\dim a_i > 0$. When $n+1$ is not a power of two, such relations exist. For example (3)
$$Sq^{n+1} = Sq^j Sq^{n+1-j} + \sum_{i>0} \binom{n-j-i}{j-2i} Sq^{n+1-i} Sq^i,$$
where $n+1 = j+2^k$ and $0 < j < 2^k$. Let
$$K = \prod_{i=1}^{l} K(Z_2, \dim a_i + n)$$
and let ϕ_1: $K(Z_2, n) \to K$ be a map such that $\phi_1^*(\alpha_i) = a_i(\alpha)$, where $\alpha_i \in H^{\dim a_i+n}(Z_2, \dim a_i+n)$ and $\alpha \in H^n(Z_2, n)$ are generators. Let E be the fibre space over $K(Z_2, n)$ with fibre ΩK and k-invariant ϕ_1. Let j: $\Omega K \to E$ be the inclusion map. For any $x \in H^i(X)$, ${}^1x \in H^{i-1}(\Omega X)$ will denote its suspension. There is an element $\phi_2 \in H^{2n}(E)$ such that
$$j^*(\phi_2) = \sum a_i'({}^1\alpha_i)$$
since $\sum a_i'({}^1\alpha_i)$ transgresses into
$$\sum a_i' a_i(\alpha) = Sq^{n+1}(\alpha) = 0.$$
Note that, since $a_i(\alpha)$ is a suspension for each i, the fibration
$$\Omega K \xrightarrow{j} E \xrightarrow{p} K(Z_2, n)$$
can be obtained by applying the loop functor to a fibration
$$K \xrightarrow{{}^{-1}j} {}^{-1}E \xrightarrow{{}^{-1}p} K(Z_2, n+1)$$
with k-invariant ${}^{-1}\phi_1$ such that ${}^1({}^{-1}\phi_1) = \phi_1$.

LEMMA 2.2. *ϕ_2 is not primitive.*

Proof. By a theorem due to G. W. Whitehead (4) it is sufficient to show that ϕ_2 is not a suspension. Suppose that $\phi_2 = {}^1\psi$. Then
$$\begin{aligned} {}^1({}^{-1}j^*\psi) &= j^*\phi_2 \\ &= \sum a_i'({}^1\alpha_i) \\ &= {}^1(\sum a_i'(\alpha_i)). \end{aligned}$$

${}^{-1}E$ is $[(n+1)-1]$-connected and the classes ${}^{-1}j^*(\psi)$ and $\sum a_i'\alpha_i$ are $(2n+1)$-dimensional. Hence

$$ {}^{-1}j^*(\psi) = \sum a_i'(\alpha_i). $$

But in the spectral sequence for ${}^{-1}p$, $\sum a_i'(\alpha_i)$ transgresses to

$$ \sum a_i' a_i({}^{-1}\alpha) = Sq^{n+1}({}^{-1}\alpha) = ({}^{-1}\alpha)^2 \neq 0. $$

This is a contradiction, and hence ϕ_2 is not a suspension.

Let Φ be the secondary cohomology operation associated with $(K(Z_2, n), \phi_1, E, \phi_2)$. For any space X and x in $H^n(X)$, $\Phi(x)$ is defined if

$$ x \in \bigcap \operatorname{Ker} a_i $$

and takes values in $\quad H^{2n}(X)/\sum a_i' H^{2n-i}(X).$

Lemma 2.3. *If Φ is defined on x and y, then Φ is defined on $x+y$ and*

$$ \Phi(x+y) = \Phi(x)+\Phi(y)+xy $$

modulo the indeterminacy of Φ.

Proof. Let $\gamma\colon E\times E \to E$ and $\mu\colon K(Z_2, n)\times K(Z_2, n) \to K(Z_2, n)$ be the loop multiplications. By (2.2), ϕ_2 is not primitive. Therefore

$$ \gamma^*(\phi_2) = \phi_2 \otimes 1 + 1 \otimes \phi_2 + p^*\alpha \otimes p^*\alpha $$

since $p^*\alpha \otimes p^*\alpha$ is the only non-zero term in

$$ H^{2n}(E\times E) \,/\, [H^{2n}(E)\otimes 1 + 1 \otimes H^{2n}(E)]. $$

Let $\bar{x}$, $\bar{y}\colon X \to E$ be liftings of $x, y\colon X \to K(Z_2, n)$ and let $\Delta\colon X \to X\times X$ be the diagonal map. Then

$$ \begin{aligned} p\gamma(\bar{x}\times\bar{y})\Delta &= \mu(x\times y)\Delta \\ &= x+y, \end{aligned} $$

and therefore $\gamma(\bar{x}\times\bar{y})\Delta$ is a lifting of $x+y$. *Modulo* the indeterminacy of Φ,

$$ \begin{aligned} \Phi(x+y) &= (\gamma(\bar{x}\times\bar{y})\Delta)^*\phi_2 \\ &= ((\bar{x}\times\bar{y})\Delta)^*(\phi_2\otimes 1 + 1\otimes\phi_2 + p^*\alpha\otimes p^*\alpha) \\ &= \Phi(x)+\Phi(y)+xy. \end{aligned} $$

Hence by (2.3) and (1.2) we have theorems:

Theorem 2.4. *If $n+1$ is not a power of 2, $Y = S^n \cup_f e^{2n}$, where $f \in [\iota, \iota]$, and $u \in H^n(Y)$ is a generator, then $\Phi(u) \neq 0$.*

Theorem 2.5. *If l is the integer given in (2.1), there is an $n-1$ connected space with $l+2$ non-trivial homotopy groups such that $[\alpha, \alpha] \neq 0$ for some $\alpha \in \pi_n(X)$.*

Remark. The example cited in § 1 of a space X such that $\pi_i(X) = 0$ $(i \neq n,\ 2n-2,\ 2n-1,\ n \equiv 1 \pmod 4)$ can be obtained from the relation

$$ Sq^{n+1} = Sq^2 Sq^{n-1} + Sq^1(Sq^{n-1}Sq^1). $$

3. The case $n+1 = 2^r$

When $n+1 = 2^r$ $(r > 3)$, Sq^{n+1} is not decomposable in the *modulo* 2 Steenrod algebra, but Adams has shown in (**1**) that

$$Sq^{n+1} = \sum a_{ij}\,\phi_{ij},$$

where ϕ_{ij} are stable secondary operations defined on classes u for which $Sq^{2^t}(u) = 0$ $(t = 0, 1, \ldots, r-1)$. Just as in § 2 one can define an unstable tertiary operation Φ associated with the above relation.

THEOREM 3.1. *If* $u \in H^n(Y)$, *then* $\Phi(u) \neq 0$.

The proof is completely analogous to the proof of 2.4 and is left to the reader.

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