CHAPTER I

EXTENDED POWERS AND H_∞ RING SPECTRA

by J. P. May

In this introductory chapter, we establish notations to be adhered to throughout and introduce the basic notions we shall be studying. In the first section, we introduce the equivariant half-smash product of a π-space and a π-spectrum, where π is a finite group. In the second, we specialize to obtain the extended powers of spectra. We also catalog various homological and homotopical properties of these constructions for later use. While the arguments needed to make these two sections rigorous are deferred to the sequel (alias [Equiv] or [51]), the claims the reader is asked to accept are all of the form that something utterly trivial on the level of spaces is also true on the level of spectra. The reader willing to accept these claims will have all of the background he needs to follow the arguments in the rest of this volume.

In sections 3 and 4, we define H_∞ ring spectra and H_∞^d ring spectra in terms of maps defined on extended powers. We also discuss various examples and catalog our techniques for producing such structured ring spectra.

§1. Equivariant half-smash products

We must first specify the categories in which we shall work. All spaces are to be compactly generated and weak Hausdorff. Most spaces will be based; $\mathcal{J}$ will denote the category of based spaces.

Throughout this volume, by a spectrum E we shall understand a sequence of based spaces E_i and based homeomorphisms $\tilde{\sigma}_i : E_i \to \Omega E_{i+1}$, the notation σ_i being used for the adjoints $\Sigma E_i \to E_{i+1}$. A map $f: E \to E'$ of spectra is a sequence of based maps $f_i : E_i \to E_i'$ strictly compatible with the given homeomorphisms; f is said to be a weak equivalence if each f_i is a weak equivalence. There results a category of spectra $\mathcal{S}$. There is a cylinder functor $E \wedge I^+$ and a resulting homotopy category $h\mathcal{S}$. The stable category $\overline{h\mathcal{S}}$ is obtained from $h\mathcal{S}$ by adjoining formal inverses to the weak equivalences, and we shall henceforward delete the adjective "weak". $\overline{h\mathcal{S}}$ is equivalent to the other stable categories in the literature, and we shall use standard properties and constructions without further comment. Definitions of virtually all such constructions will appear in the sequel.

Define $h\mathcal{J}$ and $\overline{h\mathcal{J}}$ analogously to $h\mathcal{S}$ and $\overline{h\mathcal{S}}$. For $X \in \mathcal{J}$, define $QX = \operatorname{colim} \Omega^n \Sigma^n X$, the colimit being taken with respect to suspension of maps $S^n \to \Sigma^n X$. Define adjoint functors

$$\Sigma^\infty : \mathscr{J} \to \mathscr{L} \qquad \text{and} \qquad \Omega^\infty : \mathscr{L} \to \mathscr{J}$$

by $\Sigma^\infty X = \{Q\Sigma^i X\}$ and $\Omega^\infty E = E_0$. (This conflicts with the notation used in most of my previous work, where Σ^∞ and Ω^∞ had different meanings and the present Σ^∞ was called Q_∞; the point of the change is that the present Σ^∞ is by now generally recognized to be the most appropriate infinite suspension functor, and the notation Ω^∞ for the underlying infinite loop space functor has an evident mnemonic appeal.) We then have $QX = \Omega^\infty\Sigma^\infty X$, and the inclusion and evaluation maps $\eta: X \to \Omega^n\Sigma^n X$ and $\varepsilon: \Sigma^n\Omega^n Y \to Y$ pass to colimits to give $\eta: X \to \Omega^\infty\Sigma^\infty X$ for a space X and $\varepsilon: \Sigma^\infty\Omega^\infty E \to E$ for a spectrum E. For any homology theory h_*, ε induces the stabilization homomorphism $\tilde{h}_* E_0 \to h_* E$ obtained by passage to colimits from the suspensions associated to the path space fibrations $E_i \to PE_{i+1} \to E_{i+1}$ for $i \geq 0$.

Let π be a finite group, generally supposed embedded as a subgroup of some symmetric group Σ_j. By a based π-space, we understand a left π-space with a basepoint on which π acts trivially. We let $\pi\mathscr{J}$ denote the resulting category. Actually, most results in this section apply to arbitrary compact Lie groups π.

Let W be a free unbased right π-space and form W^+ by adjoining a disjoint basepoint on which π acts trivially. For $X \in \pi\mathscr{J}$, define the "equivariant half-smash product" $W \ltimes_\pi X$ to be $W^+ \wedge_\pi X$, the orbit space of $W \times X / W \times \{*\}$ obtained by identifying $(w\sigma, x)$ and $(w, \sigma x)$ for $w \in W$, $x \in X$, and $\sigma \in \pi$.

In the sequel, we shall generalize this trivial construction to spectra. That is, we shall explain what we mean by a "π-spectrum E" and we shall make sense of "$W \ltimes_\pi E$"; this will give a functor from the category $\pi\mathscr{L}$ of π-spectra to $\mathscr{L}$. For intuition, with $\pi \subset \Sigma_j$, one may think of E as consisting of based π-spaces E_{ji} for $i \geq 0$ together with π-equivariant maps $E_{ji} \wedge S^j \to E_{j(i+1)}$ whose adjoints are homeomorphisms, where π acts on $S^j = S^1 \wedge \dots \wedge S^1$ by permutations and acts diagonally on $E_{ji} \wedge S^j$.

The reader is cordially invited to try his hand at making sense of $W \ltimes_\pi E$ using nothing but the definitions already on hand. He will quickly find that work is required. The obvious idea of getting a spectrum from the evident sequence of spaces $W \ltimes_\pi E_{ji}$ and maps

$$\Sigma(W \ltimes_\pi E_{ji}) \to W \ltimes_\pi (E_{ji} \wedge S^j) \to W \ltimes_\pi E_{j(i+1)}$$

is utterly worthless, as a moment's reflection on homology makes clear (compare II.5.6 below). The quickest form of the definition, which is not the form best suited for proving things, is set out briefly in VIII §8 below. The skeptic is invited to refer to the detailed constructions and proofs of the sequel. The pragmatist is invited to accept our word that everything one might naively hope to be true about $W \ltimes_\pi E$ is in fact true.

The first and perhaps most basic property of this construction is that it generalizes the stabilization of the space level construction. If X is a based π-space, then $\Sigma^{\infty}X$ is a π-spectrum in a natural way.

Proposition 1.1. For based π-spaces X, there is a natural isomorphism of spectra

$$W \ltimes_{\pi} \Sigma^{\infty}X \cong \Sigma^{\infty}(W \ltimes_{\pi} X).$$

The construction enjoys various preservation properties, all of which hold trivially on the space level.

Proposition 1.2 (i) The functor $W \ltimes_{\pi}(?)$ from $\pi\mathscr{S}$ to $\mathscr{S}$ preserves wedges, pushouts, and all other categorical colimits.
(ii) If X is a based π-space and $E \wedge X$ is given the diagonal π action, then $W \ltimes (E \wedge X) \cong (W \ltimes E) \wedge X$ before passage to orbits over π; if π acts trivially on X

$$W \ltimes_{\pi} (E \wedge X) \cong (W \ltimes_{\pi} E) \wedge X$$

(iii) The functor $W \ltimes_{\pi}(?)$ preserves cofibrations, cofibres, telescopes, and all other homotopy colimits.

Taking $X = I^{+}$ in (ii), we see that the functor $W \ltimes_{\pi} (?)$ preserves π-homotopies between maps of π-spectra.

Let F(X,Y) denote the function space of based maps $X \to Y$ and give $F(W^{+},Y)$ the π action $(\sigma f)(w) = f(w\sigma)$ for $f:W \to Y$, $\sigma \in \pi$, and $w \in W$. For π-spaces X and spaces Y, we have an obvious adjunction

$$\mathscr{T}(W \ltimes_{\pi} X,Y) \cong \pi\mathscr{T}(X,F(W^{+},Y)).$$

We shall have an analogous spectrum level adjunction

$$\mathscr{S}(W \ltimes_{\pi} E,D) \cong \pi\mathscr{S}(E,F[W,D))$$

for spectra D and π-spectra E. Since left adjoints preserve colimits, this will imply the first part of the previous result.

Thus the spectrum level equivariant half-smash products can be manipulated just like their simple space level counterparts. This remains true on the calculational level. In particular, we shall make sense of and prove the following result.

Theorem 1.3. If W is a free π-CW complex and E is a CW spectrum with cellular π action, then $W \ltimes_{\pi} E$ is a CW spectrum with cellular chains

$$C_*(W \ltimes_\pi E) \cong C_*W \otimes_\pi C_*E.$$

Moreover, the following assertions hold.

(i) If D is a π-subcomplex of E, then $W \ltimes_\pi D$ is a subcomplex of $W \ltimes_\pi E$ and

$$(W \ltimes_\pi E)/(W \ltimes_\pi D) = W \ltimes_\pi (E/D).$$

(ii) If W^n is the n-skeleton of W, then $W^{n-1} \ltimes_\pi E$ is a subcomplex of $W^n \ltimes_\pi E$ and

$$(W^n \ltimes_\pi E)/(W^{n-1} \ltimes_\pi E) \simeq [(W^n/\pi)/(W^{n-1}/\pi)] \wedge E.$$

(iii) With the notations of (i) and (ii),

$$W^{n-1} \ltimes_\pi D = (W^n \ltimes_\pi D) \cap (W^{n-1} \ltimes_\pi E) \subset W^n \ltimes_\pi E.$$

The calculation of cellular chains follows from (i)-(iii), the simpler calculation of chains for ordinary smash products, and an analysis of the behavior of the π actions with respect to the equivalences of (ii).

So far we have considered a fixed group, but the construction is also natural in π. Thus let $f:\rho \to \pi$ be a homomorphism and let $g:V \to W$ be f-equivariant in the sense that $g(v\sigma) = g(v)f(\sigma)$ for $v \in V$ and $\sigma \in \rho$, where V is a ρ-space and W is a π-space. For π-spectra E, there is then a natural map

$$g \ltimes 1: V \ltimes_\rho (f^*E) \to W \ltimes_\pi E,$$

where f^*E denotes E regarded as a ρ-spectrum by pullback along f.

For $X \in \pi\mathcal{J}$ and $Y \in \rho\mathcal{J}$, we have an obvious adjunction

$$\pi\mathcal{J}(\pi^+ \wedge_\rho Y, X) \cong \rho\mathcal{J}(Y, f^*X).$$

We shall have an analogous extension of action functor which assigns a π-spectrum $\pi \ltimes_\rho F$ to a ρ-spectrum F and an analogous adjunction

$$\pi\mathcal{L}(\pi \ltimes_\rho F, E) \cong \rho\mathcal{L}(F, f^*E).$$

Moreover, the following result will hold.

<u>Lemma 1.4</u>. With the notations above,

$$W \ltimes_\pi (\pi \ltimes_\rho F) = W \ltimes_\rho F.$$

When $\rho = e$ is the trivial group, $\pi \ltimes F$ is the free π-spectrum generated by a spectrum F. Intuitively, $\pi \ltimes F$ is the wedge of copies of F indexed by the elements of π and given the action of π by permutations. Here the lemma specializes to give

$$W \ltimes_\pi (\pi \ltimes F) = W \ltimes F,$$

and the nonequivariant spectrum $W \ltimes F$ is (essentially) just $W^+ \wedge F$. Note that, with $\rho = e$ and V a point in the discussion above, we obtain a natural map

$$\iota : E \to W \ltimes_\pi E$$

depending on a choice of basepoint for W.

For finite groups π and ρ, there are also natural isomorphisms

$$\alpha : (W \ltimes_\pi E) \wedge (V \ltimes_\rho F) \to (W \times V) \ltimes_{\pi \times \rho} (E \wedge F)$$

and, if $\rho \subset \Sigma_j$,

$$\beta : V \ltimes_\rho (W \ltimes_\pi E)^{(j)} \to (V \times W^j) \ltimes_{\rho \int \pi} E^{(j)}$$

for π-spaces W, π-spectra E, ρ-spaces V, and ρ-spectra F. Here $E^{(j)}$ denotes the j-fold smash power of E and $\rho \int \pi$ is the wreath product, namely $\rho \times \pi^i$ with multiplication

$$(\sigma, \mu_1, \ldots, \mu_j)(\tau, \nu_1, \ldots, \nu_j) = (\sigma\tau, \mu_{\tau(1)}\nu_1, \ldots, \mu_{\tau(j)}\nu_j).$$

The various actions are defined in the evident way. These maps will generally be applied in composition with naturality maps of the sort discussed above.

We need one more general map. If E and F are π-spectra and π acts diagonally on $E \wedge F$, there is a natural map

$$\delta : W \ltimes_\pi (E \wedge F) \to (W \ltimes_\pi E) \wedge (W \ltimes_\pi F).$$

All of these maps ι, α, β, and δ are generalizations of their evident space level analogs. That is, when specialized to suspension spectra, they agree under the isomorphisms of Proposition 1.1 with the suspensions of the space level maps. Moreover, all of the natural commutative diagrams relating the space level maps generalize to the spectrum level, at least after passage to the stable category.

§2. Extended powers of spectra

The most important examples of equivariant half-smash products are of the form $W \ltimes_\pi E^{(j)}$ for a spectrum E, where $\pi \subset \Sigma_j$ acts on $E^{(j)}$ by permutations. It requires a little work to make sense of this, and the reader is asked to accept from the sequel that one can construct the j-fold smash power as a functor from $\mathcal{L}$ to $\pi\mathcal{L}$ with all the good properties one might naively hope for. The general properties of these extended powers (or j-adic constructions) are thus direct consequences of the assertions of the previous section. The following consequence of Theorem 1.3 is particularly important.

Corollary 2.1. If W is a free π-CW complex and E is a CW spectrum, then $W \ltimes_\pi E^{(j)}$ is a CW-spectrum with

$$C_*(W \ltimes_\pi E^{(j)}) \cong C_*W \otimes_\pi (C_*E)^j.$$

Thus, with field coefficients, $C_*(W \ltimes_\pi E^{(j)})$ is chain homotopy equivalent to $C_*W \otimes_\pi (H_*E)^j$.

Indeed, $C_*(E^{(j)}) \cong (C_*E)^j$ as a π-complex, where $(C_*E)^j$ denotes the j-fold tensor power. This implies the first statement, and the second statement is a standard, and purely algebraic, consequence (e.g. [68,1.1]).

We shall be especially interested in the case when W is contractible. While all such W yield equivalent constructions, for definiteness we restrict attention to $W = E\pi$, the standard functorial and product-preserving contractible π-free CW-complex (e.g. [70,p.31]). For this W, we define

$$D_\pi E = W \ltimes_\pi E^{(j)}.$$

When $\pi = \Sigma_j$, we write $D_\pi E = D_j E$. Since $E\Sigma_1$ is a point, $D_1 E = E$. We adopt the convention that $D_0 E = E^{(0)} = S$ for all spectra E, where S denotes the sphere spectrum $\Sigma^\infty S^0$.

We adopt analogous notations for spaces X. Thus $D_j X = E\Sigma_j \ltimes_{\Sigma_j} X^{(j)}$, $D_1 X = X$, and $D_0 X = S^0$. Since there is a natural isomorphism $\Sigma^\infty(X^{(j)}) \cong (\Sigma^\infty X)^{(j)}$ of π-spectra, Proposition 1.1 implies the following important consistency statement.

Corollary 2.2. For based spaces X, there is a natural isomorphism of spectra

$$D_\pi \Sigma^\infty X \cong \Sigma^\infty D_\pi X.$$

Corollary 2.1 has the following immediate consequence.

Corollary 2.3. With field coefficients,

$$H_* D_\pi E \cong H_*(\pi;(H_*E)^j).$$

In general, we only have a spectral sequence. Since the skeletal filtrations of $E\pi$ and $B\pi$ satisfy $(E\pi)^n/\pi = (B\pi)^n$, part (ii) of Theorem 1.3 gives a filtration of $D_\pi E$ with successive quotients $[(B\pi)^n/(B\pi)^{n-1}] \wedge E^{(j)}$.

Corollary 2.4. For any homology theory k_*, there is a spectral sequence with $E_2 = H_*(\pi;k_*E^{(j)})$ which converges to $k_*(D_\pi E)$.

This implies the following important preservation properties.

Proposition 2.5. Let T be a set of prime numbers.

(i) If $\lambda:E \to E_T$ is a localization of E at T, then $D_\pi(E_T)$ is T-local and $D_\pi\lambda:D_\pi E \to D_\pi(E_T)$ is a localization at T.

(ii) If $\gamma:E \to \hat{E}_T$ is a completion of E at T, then the completion at T of $D_\pi\gamma:D_\pi E \to D_\pi(\hat{E}_T)$ is an equivalence.

Proof. We refer the reader to Bousfield [21] for a nice treatment of localizations and completions of spectra. By application of the previous corollary with $k_* = \pi_*$, we see that $D_\pi(E_T)$ has T-local homotopy groups and is therefore T-local. (Note that there is no purely homological criterion for recognizing when general spectra, as opposed to bounded below spectra, are T-local.) Taking k_* to be ordinary homology with T-local or mod p coefficients, we see that $D_\pi\lambda$ is a Z_T-homology isomorphism and $D_\pi\gamma$ is a Z_p-homology isomorphism for all $p \in T$. The conclusions follow.

Before proceeding, we should make clear that, except where explicitly stated otherwise, we shall be working in the appropriate homotopy categories $\bar{h}\mathcal{J}$ or $\bar{h}\mathcal{S}$ throughout this volume. Maps and commutative diagrams are always to be understood in this sense.

The natural maps discussed at the end of the previous section lead to natural maps

$$\iota_j:E^{(j)} \to D_jE$$

$$\alpha_{j,k}:D_jE \wedge D_kE \to D_{j+k}E$$

$$\beta_{j,k}:D_jD_kE \to D_{jk}E$$

and

$$\delta_j:D_j(E \wedge F) \to D_jE \wedge D_jF .$$

These are compatible with their obvious space level analogs in the sense that the following diagrams commute.

$$
\begin{array}{ccc}
 & \overset{\iota_j}{\nearrow} & D_j\Sigma^\infty X \\
\Sigma^\infty(X^{(j)}) & & \wr\| \\
 & \underset{\Sigma^\infty\iota_j}{\searrow} & \Sigma^\infty D_jX
\end{array}
\qquad
\begin{array}{ccc}
D_j(\Sigma^\infty X \wedge \Sigma^\infty Y) & \xrightarrow{\delta_j} & D_j\Sigma^\infty X \wedge D_j\Sigma^\infty Y \\
\wr\| & & \wr\| \\
\Sigma^\infty D_j(X\wedge Y) & \xrightarrow{\Sigma^\infty\delta_j} & \Sigma^\infty(D_jX\wedge D_jY)
\end{array}
$$

$$
\begin{array}{ccc}
D_j\Sigma^\infty X \wedge D_k\Sigma^\infty X & \xrightarrow{\alpha_{j,k}} & D_{j+k}\Sigma^\infty X \\
\wr\| & & \wr\| \\
\Sigma^\infty(D_jX \wedge D_kX) & \xrightarrow{\Sigma^\infty\alpha_{j,k}} & \Sigma^\infty D_{j+k}X
\end{array}
\qquad
\begin{array}{ccc}
D_jD_k\Sigma^\infty X & \xrightarrow{\beta_{j,k}} & D_{jk}\Sigma^\infty X \\
\wr\| & & \wr\| \\
\Sigma^\infty D_jD_kX & \xrightarrow{\Sigma^\infty\beta_{j,k}} & \Sigma^\infty D_{jk}X
\end{array}
$$

These maps will play an essential role in our theory. H_∞ ring spectra will be defined in terms of maps $D_jE \to E$ such that appropriate diagrams commute. Just as the notion of a ring spectrum presupposes the coherent associativity and commutativity of the smash product of spectra in the stable category, so the notion of an H_∞ ring spectrum presupposes various coherence diagrams relating the extended powers.

Before getting to these, we describe the specializations of our transformations when one of j or k is zero or one.

Remarks 2.6. When j or k is zero, the specified transformations specialize to identity maps (this making sense since $D_0E = S$ and S is the unit for the smash product) with one very important exception, namely $\beta_{j,0}:D_jS \to S$. these maps play a special role in our theory, and we shall also write $\xi_j = \beta_{j,0}$. Observe that D_jS^0 is just $B\Sigma_j^+$, the union of $B\Sigma_j$ and a disjoint basepoint 0. We have the discretization map $d:B\Sigma_j^+ \to S^0$ specified by $d(0) = 0$ and $d(x) = 1$ for $x \in B\Sigma_j$, and ξ_j is given explicitly as

$$D_jS = D_j\Sigma^\infty S^0 \cong \Sigma^\infty D_jS^0 \xrightarrow{\Sigma^\infty d} \Sigma^\infty S^0 = S.$$

Remarks 2.7. The transformations ι_1, $\beta_{j,1}$, $\beta_{1,j}$, and δ_1 are all given by identity maps, and

$$\alpha_{1,1} = \iota_2:E\wedge E \to D_2E.$$

The last equation is generalized in Lemma 2.11 below.

We conclude this section with eight lemmas which summarize the calculus of extended powers of spectra. Even for spaces, such a systematic listing is long overdue, and every one of the diagrams specified will play some role in our theory. The proofs will be given in the sequel, but in all cases the analogous space level assertion is quite easy to check.

Let $\tau: E \wedge F \to F \wedge E$ denote the commutativity isomorphism in $\bar{h}\mathscr{S}$.

Lemma 2.8. $\{\alpha_{j,k}\}$ is a commutative and associative system, in the sense that the following diagrams commute.

$$\begin{array}{ccc} D_jE \wedge D_kE & & \\ \downarrow \tau & \searrow^{\alpha_{j,k}} & D_{j+k}E \\ D_kE \wedge D_jE & \nearrow_{\alpha_{k,j}} & \end{array} \quad \text{and} \quad \begin{array}{ccc} D_iE \wedge D_jE \wedge D_kE & \xrightarrow{\alpha_{i,j} \wedge 1} & D_{i+j}E \wedge D_kE \\ \downarrow 1 \wedge \alpha_{j,k} & & \downarrow \alpha_{i+j,k} \\ D_iE \wedge D_{j+k}E & \xrightarrow{\alpha_{i,j+k}} & D_{i+j+k}E \end{array}$$

Write $\alpha_{i,j,k}$ for the composite in the second diagram, and so on inductively.

Lemma 2.9. $\{\beta_{j,k}\}$ is an associative system, in the sense that the following diagrams commute.

$$\begin{array}{ccc} D_iD_jD_kE & \xrightarrow{\beta_{i,j}} & D_{ij}D_kE \\ \downarrow D_i\beta_{j,k} & & \downarrow \beta_{ij,k} \\ D_iD_{jk}E & \xrightarrow{\beta_{i,jk}} & D_{ijk}E \end{array}$$

Write $\beta_{i,j,k}$ for the composite, and so on inductively.

Lemma 2.10. Each δ_j is commutative and associative, in the sense that the following diagrams commute.

$$\begin{array}{ccc} D_j(E \wedge F) & \xrightarrow{\delta_j} & D_jE \wedge D_jF \\ \downarrow D_j\tau & & \downarrow \tau \\ D_j(F \wedge E) & \xrightarrow{\delta_j} & D_jF \wedge D_jE \end{array} \quad \text{and} \quad \begin{array}{ccc} D_j(E \wedge F \wedge G) & \xrightarrow{\delta_j} & D_j(E \wedge F) \wedge D_jG \\ \downarrow \delta_j & & \downarrow \delta_j \wedge 1 \\ D_jE \wedge D_j(F \wedge G) & \xrightarrow{1 \wedge \delta_j} & D_jE \wedge D_jF \wedge D_jG \end{array}$$

Continue to write δ_j for the composite in the second diagram, and so on inductively.

Our next two lemmas relate the remaining transformations to the ι_j.

Lemma 2.11. The following diagrams commute.

$$\begin{array}{ccc} E^{(j)} \wedge E^{(k)} & = & E^{(j+k)} \\ \Big\downarrow {\scriptstyle \iota_j \wedge \iota_k} & & \Big\downarrow {\scriptstyle \iota_{j+k}} \\ D_jE \wedge D_kE & \xrightarrow{\alpha_{j,k}} & D_{j+k}E \end{array} \qquad \text{and} \qquad \begin{array}{ccc} (D_kE)^{(j)} & \xrightarrow{\alpha_{k,\dots,k}} & D_{jk}E \\ \Big\downarrow {\scriptstyle \iota_j} & \nearrow {\scriptstyle \beta_{j,k}} & \\ D_jD_kE & & \end{array}$$

Lemma 2.12. The following diagram commutes, where ν_j is the evident shuffle isomorphism

$$\begin{array}{ccc} (E \wedge F)^{(j)} & \xrightarrow{\nu_j} & E^{(j)} \wedge F^{(j)} \\ \Big\downarrow {\scriptstyle \iota_j} & & \Big\downarrow {\scriptstyle \iota_j \wedge \iota_j} \\ D_j(E \wedge F) & \xrightarrow{\delta_j} & D_jE \wedge D_jF \end{array}$$

Our last three lemmas of diagrams are a bit more subtle and appear to be new already on the level of spaces.

Lemma 2.13. The following diagram commutes.

$$\begin{array}{ccc} D_iD_kE \wedge D_jD_kE & \xrightarrow{\beta_{i,k} \wedge \beta_{j,k}} & D_{ik}E \wedge D_{jk}E \\ \Big\downarrow {\scriptstyle \alpha_{i,j}} & & \Big\downarrow {\scriptstyle \alpha_{ik,jk}} \\ D_{i+j}D_kE & \xrightarrow{\beta_{i+j,k}} & D_{ik+jk}E \end{array}$$

Lemma 2.14. The following diagrams commute.

$$\begin{array}{ccccc} D_j(E \wedge F) \wedge D_k(E \wedge F) & & \xrightarrow{\alpha_{j,k}} & & D_{j+k}(E \wedge F) \\ \Big\downarrow {\scriptstyle \delta_j \wedge \delta_k} & & & & \Big\downarrow {\scriptstyle \delta_{j+k}} \\ D_jE \wedge D_jF \wedge D_kE \wedge D_kF & \xrightarrow{1 \wedge \tau \wedge 1} & D_jE \wedge D_kE \wedge D_jF \wedge D_kF & \xrightarrow{\alpha_{j,k} \wedge \alpha_{j,k}} & D_{j+k}E \wedge D_{j+k}F \end{array}$$

and

$$\begin{array}{ccccc} D_jD_k(E \wedge F) & & \xrightarrow{\beta_{j,k}} & & D_{jk}(E \wedge F) \\ \Big\downarrow {\scriptstyle D_j\delta_k} & & & & \Big\downarrow {\scriptstyle \delta_{jk}} \\ D_j(D_kE \wedge D_kE) & \xrightarrow{\delta_j} & D_jD_kE \wedge D_jD_kF & \xrightarrow{\beta_{j,k} \wedge \beta_{j,k}} & D_{jk}E \wedge D_{jk}F \end{array}$$

Lemma 2.15. The following diagram commutes.

$$\begin{array}{ccccc} D_i(D_jE\wedge D_kE) & \xrightarrow{\delta_i} & D_iD_jE\wedge D_iD_kE & \xrightarrow{\beta_{i,j}\wedge\beta_{i,k}} & D_{ij}E\wedge D_{ik}E \\ \Big\downarrow{\scriptstyle D_i\alpha_{j,k}} & & & & \Big\downarrow{\scriptstyle \alpha_{ij,ik}} \\ D_iD_{j+k}E & & \xrightarrow{\qquad\beta_{i,j+k}\qquad} & & D_{ij+ik}E \end{array}$$

When j = k = 1, this diagram specializes to

$$\begin{array}{ccc} D_j(E\wedge E) & \xrightarrow{\delta_j} & D_jE\wedge D_jE \\ \Big\downarrow{\scriptstyle D_j\iota_2} & & \Big\downarrow{\scriptstyle \alpha_{j,j}} \\ D_jD_2E & \xrightarrow{\beta_{j,2}} & D_{2j}E \end{array}$$

(On a technical note, all of these coherence diagrams except those of Lemma 2.15 will commute for the extended powers associated to an arbitrary operad; Lemma 2.15 requires restriction to E_∞ operads.)

§3. H_∞ ring spectra

Recall that a (commutative) ring spectrum is a spectrum E together with a unit map $e:S \to E$ and a product map $\phi:E\wedge E \to E$ such that the following diagrams commute (in the stable category, as always).

$$\begin{array}{ccccc} E\wedge S & \xrightarrow{1\wedge e} & E\wedge E & \xleftarrow{e\wedge 1} & S\wedge E \\ & \searrow\!\!\!= & \Big\downarrow{\scriptstyle\phi} & =\!\!\!\swarrow & \\ & & E & & \end{array} \qquad \begin{array}{ccc} E\wedge E\wedge E & \xrightarrow{\phi\wedge 1} & E\wedge E \\ \Big\downarrow{\scriptstyle 1\wedge\phi} & & \Big\downarrow{\scriptstyle\phi} \\ E\wedge E & \xrightarrow{\phi} & E \end{array} \qquad \begin{array}{ccc} E\wedge E & & \\ \Big\downarrow{\scriptstyle\tau} & \searrow^{\phi} & E \\ E\wedge E & \nearrow_{\phi} & \end{array}$$

In fact, this notion incorporates only a very small part of the full structure generally available.

Definition 3.1. An H_∞ ring spectrum is a spectrum E together with maps $\xi_j:D_j \to E$ for $j \geq 0$ such that ξ_1 is the identity map and the following diagrams commute for $j,k \geq 0$.

$$\begin{array}{ccccc} D_jE \wedge D_kE & \xrightarrow{\alpha_{j,k}} & & & D_{j+k}E \\ \downarrow{\scriptstyle \xi_j \wedge \xi_k} & & & & \downarrow{\scriptstyle \xi_{j+k}} \\ E \wedge E & \xrightarrow{\iota_2} & D_2E & \xrightarrow{\xi_2} & E \end{array} \quad \text{and} \quad \begin{array}{ccc} D_jD_kE & \xrightarrow{\beta_{j,k}} & D_{jk}E \\ \downarrow{\scriptstyle D_j\xi_k} & & \downarrow{\scriptstyle \xi_{jk}} \\ D_jE & \xrightarrow{\xi_j} & E \end{array}$$

A map $f:E \to F$ between H_∞ ring spectra is an H_∞ ring map if $\xi_j \circ D_jf = f \circ \xi_j$ for $j \geq 0$.

This is a valid sharpening of the notion of a ring spectrum in view of the following consequence of Remarks 2.6 and Lemma 2.8.

Lemma 3.2. With $e = \xi_0:S \to E$ and $\phi = \xi_2 \circ \iota_2:E\wedge E \to E$, an H_∞ ring spectrum is a ring spectrum and an H_∞ ring map is a ring map.

There are various variants and alternative forms of the basic definition that will enter into our work. For a first example, we note the following facts.

Proposition 3.3. Let E be a ring spectrum with maps $\xi_j:D_jE \to E$ such that $\xi_0 = e$, $\xi_1 = 1$, and $\phi = \xi_2\iota_2$. If the first diagram of Definition 3.1 commutes, then ξ_j factors as the composite

$$D_jE = D_jE\wedge S \xrightarrow{1\wedge e} D_jE\wedge E \xrightarrow{\alpha_{j,1}} D_{j+1}E \xrightarrow{\xi_{j+1}} E .$$

Conversely, if all ξ_j so factor and the second diagram of Definition 1.1 commutes, then the first diagram also commutes and thus E is an H_∞ ring spectrum.

Proof. The first part is an elementary diagram chase. The second part results from Lemmas 2.8 and 2.11 via a rather lengthy diagram chase.

The definition of an H_∞ ring spectrum, together with the formal properties of extended powers, implies the following important closure and consistency properties of the category of H_∞ ring spectra.

Proposition 3.4. The following statements hold, where E and F are H_∞ ring spectra.
(i) With $\xi_j = \beta_{j,0}:D_jS \to S$, the sphere spectrum S is an H_∞ ring spectrum, and $e:S \to E$ is an H_∞ ring map.
(ii) The smash product $E\wedge F$ is an H_∞ ring spectrum with structural maps the composites

$$D_j(E\wedge F) \xrightarrow{\delta_j} D_jE\wedge D_jF \xrightarrow{\xi_j\wedge\xi_j} E\wedge F;$$

the resulting product is the standard one, $(\phi\wedge\phi)(1\wedge\tau\wedge 1)$.

(iii) The composite $\xi_j \iota_j : E^{(j)} \to E$ is the j-fold iterated product on E and is itself an H_∞ ring map for all j.

Proof. These are elementary diagram chases based respectively on:

(i) Remarks 2.6 and the case k = 0 and E = S of Lemmas 2.9 and 2.13.

(ii) Lemmas 2.12 and 2.14.

(iii) Remarks 2.7 and Lemmas 2.9 and 2.11.

In view of Proposition 2.5, we have the following further closure property of the category of H_∞ ring spectra.

Proposition 3.5. If E is an H_∞ ring spectrum, then its localization E_T and completion $\hat{E}_T$ at any set of primes T admit unique H_∞ ring structures such that $\lambda : E \to E_T$ and $\gamma : E \to \hat{E}_T$ are H_∞ ring maps.

Proof. The assertion is obvious in the case of localization. In the case of completion, $\xi_j : D_j \hat{E}_T \to \hat{E}_T$ can and must be defined as the composite

$$D_j \hat{E}_T \xrightarrow{\gamma} (D_j \hat{E}_T)\hat{}_T \xrightarrow{((D_j\gamma)\hat{}_T)^{-1}} (D_j E)\hat{}_T \xrightarrow{(\xi_j)\hat{}_T} \hat{E}_T .$$

An easy calculation in ordinary cohomology shows that Eilenberg-MacLane spectra are H_∞ ring spectra.

Proposition 3.6. The Eilenberg-MacLane spectrum HR of a commutative ring R admits a unique H_∞ ring structure, and this structure is functorial in R. If E is a connective H_∞ ring spectrum and $i : E \to H(\pi_0 E)$ is the unique map which induces the identity homomorphism on π_0, then i is an H_∞ ring map.

Proof. Corollary 2.1 implies that $\iota_j : F^{(j)} \to D_j F$ induces an isomorphism in R-cohomology in degree 0 for any connective spectrum F. Moreover, by the Hurewicz theorem and universal coefficients, $H^0(F;R)$ may be identified with $\mathrm{Hom}(\pi_0 F, R)$. Thus we can, and by Proposition 3.4(iii) must, define $\xi_j : D_j HR \to HR$ to be that cohomology class which restricts under ι_j to the j-fold external power of the fundamental class or, equivalently under the identification above, to the j-fold product on R. Similarly, the commutativity of the diagrams in Definition 3.1 is checked by restricting to smash powers and considering cohomology in degree 0. The same argument gives the functoriality. For the last statement, the maps $\xi_j D_j i$ and $i\xi_j$ from $D_j E$ to $H(\pi_0 E)$ are equal because they both restrict under ι_j to the cohomology class given by the iterated product $(\pi_0 E)^j \to \pi_0 E$.

We shall continue to write i for its composite with any map $H(\pi_0 E) \to HR$ induced by a ring homomorphism $\pi_0 E \to R$. We think of such a map $i:E \to HR$ as a counit of E. the composite $ie:S \to HR$ is clearly the unit of HR.

In the rest of this section, we consider the behavior of H_∞ ring spectra with respect to the functors Σ^∞ and Ω^∞. Note first that if E is a ring spectrum, then its unit $e:S \to E$ is determined by the restriction of $e_0:QS^0 \to E_0$ to S^0. If the two resulting basepoints 0 and 1 of E_0 lie in the same component, then e is the trivial map and therefore E is the trivial spectrum.

Definition 3.7. An H_∞ space with zero, or $H_{\infty 0}$ space, is a space X with basepoint 0 together with based maps $\xi_j:D_jX \to X$ for $j \geq 0$ such that the diagrams of Definition 3.1 commute with E replaced by X. Note that $\xi_0:S^0 \to X$ gives X a second basepoint 1. An H_∞ space is a space Y with basepoint 1 together with based maps $E\Sigma_j \times_{\Sigma_j} Y^j \to Y$ for $j \geq 0$ such that the evident analogs of the diagrams of Definition 3.1 commute; $Y^+ = Y \amalg \{0\}$ is then an $H_{\infty 0}$ space.

We remind the reader that we are working up to homotopy (i.e., in $\bar{h}\mathcal{J}$). There is a concomitant notion of a (homotopy associative and commutative) H-space with zero, or H_0-space, given by maps $e:S^0 \to X$ and $\phi:X \wedge X \to X$ such that the diagrams defining a ring spectrum commute with E replaced by X. It is immediately obvious that, mutatis mutandis, Lemma 3.2 and Propositions 3.3-3.5 remain valid for spaces. A commutative ring $R = K(R,0)$ is evidently an $H_{\infty 0}$ space, ξ_j being given by the j-fold product with the $E\Sigma_j$ coordinate ignored.

The isomorphisms $D_j\Sigma^\infty X \cong \Sigma^\infty D_j X$ together with the compatibility of the space and spectrum level transformations ι_j, $\alpha_{j,k}$, and $\beta_{j,k}$ under these isomorphisms have the following immediate consequence.

Proposition 3.8. If X is an $H_{\infty 0}$ space, then $\Sigma^\infty X$ is an H_∞ ring spectrum with structural maps

$$\Sigma^\infty \xi_j : D_j\Sigma^\infty X \cong \Sigma^\infty D_j X \to \Sigma^\infty X.$$

The relationship of Ω^∞ to H_∞ ring structures is a bit more subtle since it is not true that $D_j\Omega^\infty E \cong \Omega^\infty D_j E$. However, the evaluation map $\varepsilon:\Sigma^\infty\Omega^\infty E \to E$ induces

$$D_j\varepsilon : \Sigma^\infty D_j\Omega^\infty E \cong D_j\Sigma^\infty\Omega^\infty E \to D_j E,$$

the adjoint $(\Omega^\infty D_j\varepsilon)\eta$ of which is a natural map

$$\zeta_j : D_j\Omega^\infty E \to \Omega^\infty D_j E \quad \text{or} \quad \zeta_j : D_j E_0 \to (D_j E)_0 \ .$$

Proposition 3.9. If E is an H_∞ ring spectrum, then E_0 is an $H_{\infty 0}$ space with structural maps

$$(\xi_j)_0 \circ \zeta_j : D_j E_0 \to E_0 .$$

Proof. We must check that the commutativity of the diagrams of Definition 3.1 for E implies their commutativity for E_0. For the first diagram, it is useful to introduce the natural map

$$\zeta : E_0 \wedge F_0 \xrightarrow{\eta} Q(E_0 \wedge F_0) \cong (\Sigma^\infty E_0 \wedge \Sigma^\infty F_0)_0 \xrightarrow{(\varepsilon \wedge \varepsilon)_0} (E \wedge F)_0$$

for spectra E and F. The relevant diagrams then look as follows

$$\begin{array}{ccccc}
D_j E_0 \wedge D_k E_0 & & \xrightarrow{\alpha_{j,k}} & & D_{j+k} E_0 \\
\downarrow{\scriptstyle \zeta_j \wedge \zeta_k} & & & & \downarrow{\scriptstyle \zeta_{j+k}} \\
(D_j E)_0 \wedge (D_k E)_0 & \xrightarrow{\zeta} & (D_j E \wedge D_k E)_0 & \xrightarrow{(\alpha_{j,k})_0} & (D_{j+k} E)_0 \\
\downarrow{\scriptstyle (\xi_j)_0 \wedge (\xi_k)_0} & & {\scriptstyle (\xi_j \wedge \xi_k)_0} & & \downarrow{\scriptstyle (\xi_{j+k})_0} \\
E_0 \wedge E_0 & \xrightarrow{\zeta} & (E \wedge E)_0 & \longrightarrow & E_0
\end{array}$$

$$E_0 \wedge E_0 \xrightarrow{\iota_2} D_2 E_0 \xrightarrow{\zeta_2} (D_2 E)_0 , \qquad (E \wedge E)_0 \xrightarrow{(\iota_2)_0} (D_2 E)_0 \xrightarrow{(\xi_2)_0} E_0$$

and

$$\begin{array}{ccccc}
D_j D_k E_0 & & \xrightarrow{\beta_{j,k}} & & D_{jk} E_0 \\
\downarrow{\scriptstyle D_j \zeta_k} & & & & \downarrow \\
D_j (D_k E)_0 & \xrightarrow{\zeta_j} & (D_j D_k E)_0 & \xrightarrow{(\beta_{j,k})_0} & (D_{jk} E)_0 \\
\downarrow{\scriptstyle D_j (\xi_k)_0} & & \downarrow{\scriptstyle (D_j \xi_k)_0} & & \downarrow{\scriptstyle (\xi_{jk})_0} \\
D_j E_0 & \xrightarrow{\zeta_j} & (D_j E)_0 & \xrightarrow{(\xi_j)_0} & E_0
\end{array}$$

In the upper diagram, $\zeta_2\iota_2 = (\iota_2)_0\zeta$ by the naturality of η and ι_2 and the compatibility of the space and spectrum level maps ι_2. The commutativity of the top rectangles of both diagrams follows similarly, via fairly elaborate chases, from naturality and compatibility diagrams together with the fact that the composite $\varepsilon\circ\Sigma^\infty\eta:\Sigma^\infty \to \Sigma^\infty\Omega^\infty\Sigma^\infty \to \Sigma^\infty$ is the identity transformation.

The preceding results combine in the following categorical description of the relationship between $H_{\infty 0}$ spaces and H_∞ ring spectra.

Proposition 3.10. If X is an $H_{\infty 0}$ space, then $\eta:X \to \Omega^\infty\Sigma^\infty X$ is a map of $H_{\infty 0}$ spaces. If E is an H_∞ ring spectrum, then $\varepsilon:\Sigma^\infty\Omega^\infty E \to E$ is a map of H_∞ ring spectra. Therefore Σ^∞ and Ω^∞ restrict to an adjoint pair of functors relating the categories of $H_{\infty 0}$ spaces and of H_∞ ring spectra.

The proof consists of easy diagram chases. It follows that if E is an H_∞ ring spectrum, then $\varepsilon_0:QE_0 \to E_0$ is a map of $H_{\infty 0}$ spaces. As we shall explain in the sequel, the significance of this fact is that it implies that the 0^{th} space of an H_∞ ring spectrum is an "H_∞ ring space".

§4. Power operations and H_∞^d ring sprectra

Just as the product of a ring spectrum gives rise to an external product in its represented cohomology theory on spectra and thus to an internal cup product in its represented cohomology theory on spaces, so the structure maps ξ_j of an H_∞ ring spectrum give rise to external and internal extended power operations.

Definitions 4.1. Let E be an H_∞ ring spectrum. For a spectrum Y, define

$$\mathcal{P}_j:E^0Y = [Y,E] \to [D_jY,E] = E^0D_jY$$

by letting $\mathcal{P}_j(h) = \xi_j \circ D_jh$ for $h:Y \to E$. For a based space X, let $\tilde{E}^*X$ denote the reduced cohomology of X and define

$$P_j:\tilde{E}^0X = E^0\Sigma^\infty X \to E^0\Sigma^\infty(B\Sigma_j^+ \wedge X) = \tilde{E}^0(B\Sigma_j^+ \wedge X)$$

by $P_j(h) = (\Sigma^\infty d)^* \mathcal{P}_j(h)$ for $h:\Sigma^\infty X \to E$, where

$$d = 1 \times \Delta:B\Sigma_j^+ \wedge X = E\Sigma_j \ltimes_{\Sigma_j} X \to E\Sigma_j \ltimes_{\Sigma_j} X^{(j)} = D_jX.$$

Of course, the main interest is in the case $j = p$ for a prime p. A number of basic properties of these operations can be read off directly from the definition of an H_∞ ring spectrum, the most important being that $\iota_j^* \mathcal{P}_j(h) = h^j$, where

$h^j \in E^0(Y^{(j)})$ is the external j^{th} power of h, and similarly for the internal operations. McClure will give a systematic study in chapter VIII. While we think of the $\mathcal{P}_j$ as cohomology operations, they can be manipulated to obtain various other kinds of operations. For example, we can define homotopy operations on $\pi_* E$ parametrized by elements of $E_* D_j S^q$.

Definition 4.2. Let E be an H_∞ ring spectrum. For $\alpha \in E_r D_j S^q$, define $\tilde{\alpha}: \pi_q E \to \pi_r E$ by $\tilde{\alpha}(h) = \alpha / \mathcal{P}_j(h)$ for $h \in \pi_q E$. Explicitly, $\tilde{\alpha}(h)$ is the composite

$$S^r \xrightarrow{\alpha} D_j S^q \wedge E \xrightarrow{\mathcal{P}_j(h) \wedge 1} E \wedge E \xrightarrow{\phi} E.$$

These operations will make a fleeting appearance in our study of nilpotency relations in the next chapter, and Bruner will study them in detail in the case $E = S$ in chapter V. McClure will introduce a related approach to homology operations in chapter VIII.

Returning to Definition 4.1 and replacing Y by $\Sigma^j Y$ for any i, we obtain operations $\mathcal{P}_j: E^{-i}Y \to E^0 D_j \Sigma^i Y$. A moment's reflection on the Steenrod operations in ordinary cohomology makes clear that we would prefer to have operations $E^{-i}Y \to E^{-ji} D_j Y$ for all i. However, the twisting of suspension coordinates which obstructs the equivalence of $D_j \Sigma^i Y$ with $\Sigma^{ji} D_j Y$ makes clear that the notion of an H_∞ ring spectrum is inadequate for this purpose. For $Y = \Sigma^\infty X$, one can set up a formalism of twisted coefficients to define one's way around the obstruction, but this seems to me to be of little if any use calculationally. Proceeding adjointly, we think of $E^i Y$ as $[Y, \Sigma^i E]$ and demand structural maps $\xi_j: D_j \Sigma^i E \to \Sigma^{ji} E$ for all integers i rather than just for i = 0. We can then define extended power operations

$$\mathcal{P}_j: E^i Y = [Y, \Sigma^i E] \to [D_j Y, \Sigma^{ji} E] = E^{ji} D_j Y$$

by letting $\mathcal{P}_j(h) = \xi_j \circ D_j h$ for $h: Y \to \Sigma^i E$; internal operations

$$P_j: \tilde{E}^i X = E^i \Sigma^\infty X \to E^{ji}\ \Sigma^\infty(B\Sigma_j^+ \wedge X) = \tilde{E}^{ji}(B\Sigma_j^+ \wedge X)$$

for spaces are given by $P_j(h) = (\Sigma^\infty d)^* \mathcal{P}_j(h)$, as in Definition 4.1.

In practice, this demands too much. One can usually only obtain maps $\xi_j: D_j \Sigma^{di} E \to \Sigma^{dji} E$ for all j and i and some fixed $d > 0$, often 2 and always a power of 2. In favorable cases, one can use twisted coefficients or restriction to cyclic groups to fill in the missing operations, in a manner to be explained by McClure in chapter VIII. The experts will recall that some such argument was already necessary to define the classical mod p Steenrod operations on odd dimensional classes when $p > 2$.

Definition 4.3. Let d be a positive integer. An H_∞^d ring spectrum is a spectrum E together with maps

$$\xi_{j,i}: D_j\Sigma^{di}E \to \Sigma^{dji}E$$

for all $j \geq 0$ and all integers i such that each $\xi_{1,i}$ is an identity map and the following diagrams commute for all $j \geq 0$, $k \geq 0$, and all integers h and i.

$$\begin{array}{ccc} D_j\Sigma^{di}E\wedge D_k\Sigma^{di}E & \xrightarrow{\alpha_{j,k}} & D_{j+k}\Sigma^{di}E \\ \downarrow{\scriptstyle \xi_{j,i}\wedge\xi_{k,i}} & & \downarrow{\scriptstyle \xi_{j+k,i}} \\ \Sigma^{dji}E\wedge E^{dki}E & \xrightarrow{\phi} & \Sigma^{d(j+k)i}E \end{array} \qquad \begin{array}{ccc} D_jD_k\Sigma^{di}E & \xrightarrow{\beta_{j,k}} & D_{jk}\Sigma^{di}E \\ \downarrow{\scriptstyle D_j\xi_{k,i}} & & \downarrow{\scriptstyle \xi_{jk,i}} \\ D_j\Sigma^{dki}E & \xrightarrow{\xi_{j,ki}} & \Sigma^{djki}E \end{array}$$

and

$$\begin{array}{ccccc} D_j(\Sigma^{dh}E\wedge\Sigma^{di}E) & & \xrightarrow{\delta_j} & & D_j\Sigma^{dh}E\wedge D_j\Sigma^{di}E \\ \downarrow{\scriptstyle D_j\phi} & & & & \downarrow{\scriptstyle \xi_{j,h}\wedge\xi_{j,i}} \\ D_j\Sigma^{d(h+i)}E & \xrightarrow{\xi_{j,h+i}} & \Sigma^{dj(h+i)}E & \xleftarrow{\phi} & \Sigma^{djh}E\wedge\Sigma^{dji}E \end{array}$$

Here the maps ϕ are obtained by suspension from the product $\xi_{2,0}\iota_2$ on E. A map $f:E \to F$ between H_∞^d ring spectra is an H_∞^d ring map if $\xi_{j,i}\circ D_j\Sigma^{di}f = \Sigma^{dji}f \circ \xi_{j,i}$ for all j and i.

Remarks 4.4. (i) Taking i = 0, we see that E is an H_∞ ring spectrum. The last diagram is a consequence of the first two when i = 0 but is independent otherwise.
(ii) Since $D_0E = S$ for all spectra E, there is only one map $\xi_{i,0}$, namely the unit $e:S^0 \to E$.
(iii) As in Proposition 3.4(iii), the following diagram commutes.

$$\begin{array}{ccc} (\Sigma^{di}E)^{(j)} & \xrightarrow{\iota_j} & D_j\Sigma^{di}E \\ & {\scriptstyle \phi}\searrow \quad \swarrow{\scriptstyle \xi_{j,i}} & \\ & \Sigma^{dji}E & \end{array}$$

(iv) As in Proposition 3.4(ii), the smash product of an H_∞^d ring spectrum E and an H_∞ ring spectrum F is an H_∞^d ring spectrum with structural maps the composites

$$D_j(\Sigma^{di}E\wedge F) \xrightarrow{\delta_j} D_j\Sigma^{di}E\wedge D_jF \xrightarrow{\xi_{j,i}\wedge\xi_j} \Sigma^{dji}E\wedge F.$$

(v) The last diagram in the definition involves a permutation of suspension coordinates, hence one would expect a sign to appear. However, as McClure will explain in VII.6.1, $\pi_0 E$ necessarily has characteristic two when d is odd.

Given this last fact, precisely the same proof as that of Proposition 3.6 yields the following result.

Proposition 4.5. Let R be a commutative ring. If R has characteristic two, then HR admits a unique and functorial H_∞^1 ring structure. In general, HR admits a unique and functorial H_∞^2 ring structure. If E is a connective H_∞^d ring spectrum and $i:E \to H(\pi_0 E)$ is the unique map which induces the identity homomorphism on π_0, then i is an H_∞^d ring map.

At this point, most of the main definitions are on hand, but only rather simple examples. We survey the examples to be obtained later in the rest of this section.

We have three main techniques for the generation of examples. The first, and most down to earth where it applies, is due to McClure and will be explained in chapter VII. The idea is this. In nature, one does not encounter spectra E with E_i homeomorphic to ΩE_{i+1} but only prespectra T consisting of spaces T_i and maps $\sigma_i:\Sigma T_i \to T_{i+1}$. There is a standard way of associating a spectrum to a prespectrum, and McClure will specify concrete homotopical conditions on the spaces T_{di} and composites $\Sigma^d T_{di} \to T_{d(i+1)}$ which ensure that the associated spectrum is an H_∞^d ring spectrum. Curiously, the presence of d is essential. We know of no such concrete way of recognizing H_∞ ring spectra which are not H_∞^d ring spectra for some $d > 0$.

McClure will use this technique to show that the most familiar Thom spectra and K-theory spectra are H_∞^d ring spectra for the appropriate d. While this technique is very satisfactory where it applies, it is limited to the recognition of H_∞^d ring spectra and demands that one have reasonably good calculational control over the spaces T_{di}. The first limitation is significant since, as McClure will explain, the sphere spectrum, for example, is not an H_∞^d ring spectrum for any d. The second limitation makes the method unusable for generic classes of examples.

Our second method is at the opposite extreme, and depends on the black box of infinite loop space machinery. In [71], Nigel Ray, Frank Quinn, and I defined the notion of an E_∞ ring spectra. Intuitively, this is a very precise point-set level notion, of which the notion of an H_∞ ring spectrum is a cruder and less structured up to homotopy analog. Of course, E_∞ ring spectra determine H_∞ ring spectra by neglect of structure. There are also notions of E_∞ space and H_∞ ring space which bear the same relationship of one to the other. Just as the zeroth space of an H_∞ ring spectrum is an H_∞ ring space, so the zeroth space of an E_∞ ring spectrum is an E_∞ ring space. In general, given an H_∞ ring space, there is not the slightest reason to believe that it is equivalent, or nicely related, to the zeroth space of an H_∞ ring spectrum. However, the machinery of [71,73] shows that E_∞ ring spaces functorially determine E_∞ ring spectra the zeroth spaces of which are, in a suitable sense, ring completions of the original semiring spaces. Precise definitions and proofs of the relationship between E_∞ ring theory and H_∞ ring theory will be given in the sequel.

As explained in detail in [73], which corrects [71], the classifying spaces of categories with suitable internal structure, namely bipermutative categories, are E_∞ ring spaces. Among other examples, there result E_∞ ring structures and therefore H_∞ ring structures on the connective spectra of the algebraic K-theory of commutative rings.

The E_∞ and H_∞ ring theories summarized above are limiting cases of E_n and H_n theories for $n \geq 1$, to which the entire discussion applies verbatim. The full theory of extended powers and structured ring spaces and spectra entails the use of operads, namely sequences $\mathcal{C}$ of suitably related Σ_j-spaces $\mathcal{C}_j$. An action of $\mathcal{C}$ on a spectrum E consists of maps $\xi_j: \mathcal{C}_j \ltimes_{\Sigma_j} E^{(j)} \rightarrow E$ such that appropriate diagrams commute. For an action up to homotopy, the same diagrams are only required to homotopy commute. If each $\mathcal{C}_j$ has the Σ_j-equivariant homotopy type of the configuration space of j-tuples of distinct points in R^n, then $\mathcal{C}$ is said to be an E_n operad. E_n or H_n ring spectra are spectra with actions or actions up to homotopy by an E_n operad. The notions of E_n and H_n ring space require use of a second operad, assumed to be an E_∞ operad, to encode the additive structure which is subsumed in the iterated loop structure on the spectrum level. E_n ring spaces naturally give rise to E_n and thus H_n ring spectra, and interesting examples of E_n ring spaces have been discovered by Cohen, Taylor, and myself [29] in connection with our study of generalized James maps.

Our last technique for recognizing E_n and H_n ring spectra lies halfway between the first two, and may be described as the brute force method. It consists of direct appeal to the precise definition of extended powers of spectra to be given in the sequel. One class of examples will be given by Steinberger's construction of free $\mathcal{C}$-spectra. Another class of examples will be given in Lewis' study of generalized Thom spectra.