BLUMBERG HFP THEOREM

Note Title 10/29/2010

<u>Thm</u> For M a module over $D^{-1}MU^{((G))} = \tilde{\Omega}$ the map $M^G \longrightarrow M^{hG}$ is an equiv.

<u>Cor</u> Periodicity for $\tilde{\Omega}^G$

The map
$$\begin{array}{ccc} M^G & \longrightarrow & M^{hG} \\ \| & & \| \\ \mathrm{Map}_G(*, M) & & \mathrm{Map}(EG, M) \end{array}$$

comes from $* \longleftarrow EG$

M^G is a limit

M^{hG} is a homotopy limit

Similar issues figure in the Sullivan and Segal conjectures.

Will split proof into formal and calculational points

1) Formal part: For a G ring spectrum R with $\Phi^H R \simeq *$ for each nontrivial $H \leq G$ then $M^G \xrightarrow{\sim} M^{hG}$

2) Calculation: Show that $D^{-1}MU^{((G))}$ satisfies this property.

Cofiber seq $EG_+ \to S^0 \to \tilde{E}G$

Smash it with $M \to F(EG_+, M) = F$

$$\begin{tikzcd}
EG_+ \wedge M \arrow[r] \arrow[d, "1"'] & M \arrow[r] \arrow[d, "2"'] & \tilde{E}G \wedge M \arrow[d, "3"'] \\
EG_+ \wedge F \arrow[r] & F \arrow[r] & \tilde{E}G \wedge F
\end{tikzcd}$$

This the "Tate diagram" of Greenlees–May

It is also used in TC (trace theory)

built from $THH^{C_{p^n}}$ $\qquad G = C_{p^n} \qquad G' = C_{p^{n-1}}$

$$\begin{tikzcd}
T_{hG} \arrow[r] \arrow[d] & T^{G} \arrow[r] \arrow[d] & T^{G'} \arrow[d] \\
T_{hG} \arrow[r] & T^{hG} \arrow[r] & \text{Tate}
\end{tikzcd}$$

The map $EG_+ \wedge M \xrightarrow{1} EG_+ \wedge F$ is an equiv (equiv)
because $M \to F$ is ordinary equiv,
so it suffices to show that 3 is
Will show both source and target are
contractible

<u>Prop</u> If R is a ring s.t. $\Phi^H R \simeq *$ for each
$\{e\} \neq H \leq G$, then $\tilde{E}G \wedge M \simeq *$.

<u>Proof</u>: It suffices to show it for $M = R$.
Enough to see $X_H = i_H^* \tilde{E}G \wedge R \simeq *$ for all H.
1) True for $H = \{e\}$
2) Assume true for subgps of H

Let $\mathcal{F}_H$ be family of proper subgps of H

$$(E\mathcal{F}_H)_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{F}_H \quad \text{(isotropy separation)}$$

Smash this with X_H. It follows by induction hyp that $(E\mathcal{F}_H)_+ \wedge X_H \simeq *$

It suffices to show

$$\tilde{E}\mathcal{F}_H \wedge X_H \simeq *$$

which is equiv to $\Phi^H(\tilde{E}G \wedge R) \simeq *$

We know that $\Phi^H(\tilde{E}G \wedge R) \simeq \Phi^H(\tilde{E}G) \wedge \Phi^H(R) \simeq *$

since $\Phi^H(R) \simeq *$. QED

This completes the formal part of the argument. It remains to show

$\Phi^H(D^{-1} MU^{((G))}) \simeq *$ for $\{e\} \neq H \subseteq G$

It depends on the choice of D

Recall $D = \Delta_1^{(8)} N_4^8(\Delta_2^{(4)}) N_2^8(\Delta_4^{(2)})$

The $\Delta_k^{(?)}$ are defined in terms of several generators that are killed by Φ^H

Hence inverting D and applying Φ^H makes the spectrum contractible. QED

This is the easy part of the HKR theorem.