BOUNDING THE $K(p-1)$-LOCAL EXOTIC PICARD GROUP AT $p > 3$

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Abstract. In this paper, we bound the descent filtration of the exotic Picard group κ_n, for a prime number $p > 3$ and $n = p-1$. Our method involves a detailed comparison of the Picard spectral sequence, the homotopy fixed point spectral sequence, and an auxiliary β-inverted homotopy fixed point spectral sequence whose input is the Farrell-Tate cohomology of the Morava stabilizer group. Along the way, we deduce that the K(n)-local Adams-Novikov spectral sequence for the sphere has a horizontal vanishing line at $3n^2+1$ on the E_{2n^2+2}-page.

The same analysis also allows us to express the exotic Picard group of K(n)-local modules over the homotopy fixed points spectrum E_n^{hN}, where N is the normalizer in $\mathbb{G}_n$ of a finite cyclic subgroup of order p, as a subquotient of a single continuous cohomology group $H^{2n+1}(N, \pi_{2n}\mathrm{E_n})$.

Contents

1. Introduction

A key objective of chromatic homotopy theory is to understand the K(n)-local category of spectra $\mathcal{S}\mathrm{p}_{\mathrm{K(n)}}$, at various primes p and heights n, as these are building blocks of spectra from the chromatic point of view. Furthermore, K(n)-local spectra are approachable via descent from their Lubin-Tate homology (aka Morava E-theory). Specifically, there is a K(n)-local even-periodic $\mathcal{E}_\infty$-ring spectrum $\mathrm{E_n}$, whose π_0 is a complete local ring carrying a universal deformation of a height n formal group law in characteristic p [GH04, Lur09].

The automorphism group $\mathbb{G}_n$ of the formal group law[1] acts on $\mathrm{E_n}$ through ring homomorphisms, and the homotopy fixed points of the action recover the K(n)-local sphere [DH04]. In fact, this relationship can be categorified to great effect, exhibiting the K(n)-local category as the homotopy fixed points of the category of $\mathbb{G}_n$-equivariant K(n)-local $\mathrm{E_n}$-modules [Mat16, Mor23].

The invertible objects in $\mathcal{S}\mathrm{p}_{\mathrm{K(n)}}$, in turn, can be thought of as the building blocks of the K(n)-local category. Equipped with the K(n)-local smash product, they form the Picard group

$$\mathrm{Pic}_n = \mathrm{Pic}(\mathcal{S}\mathrm{p}_{\mathrm{K(n)}}) = \{X \in \mathcal{S}\mathrm{p}_{\mathrm{K(n)}} \mid \exists Y \text{ such that } X \otimes Y \simeq \mathrm{S}^0_{\mathrm{K(n)}}\}/\sim,$$

[1] together with the base field, which should contain enough roots of unity

where the equivalence relation is K(n)-local homotopy equivalence.

The investigation into Pic_n was initiated by the groundbreaking work of Hopkins, Mahowald, and Sadofsky [HMS94], in which they observe the wealth of information these groups can contain. Indeed, while the category of (unlocalized) spectra has Picard group $\mathbb{Z}$, consisting of only the integer-dimensional spheres, the structure of Pic_n encodes the existence of p-adic dimensional spheres [HMS94] and determinant spheres [BBGS22], among other beasts. A starting point in the study of invertible K(n)-local spectra is their characterization as such spectra X for which $(E_n)_*X$ is invertible in the category of $\pi_*\mathrm{E_n}-\mathbb{G}_n$ modules (i.e. modules over $\pi_*\mathrm{E_n}$ with compatible $\mathbb{G}_n$ action; these are also called Morava modules). We let Pic_n^{alg} denote the Picard group of Morava modules, and we consider the comparison map

$$\varepsilon : \mathrm{Pic}_n \to \mathrm{Pic}_n^{alg} \tag{1}$$

which sends a spectrum X to $(\mathrm{E_n})_*X = \pi_* L_{K(n)}(\mathrm{E_n} \wedge X)$. Since $\pi_0\mathrm{E_n}$ is a complete local ring, an invertible $\pi_*\mathrm{E_n}$-module is completely determined by whether it is concentrated in even or odd degrees. Thus, most of the information in the algebraic Picard group is encoded by twists of the $\mathbb{G}_n$-action on $\pi_*\mathrm{E_n}$.

The map in (1) is known to be an isomorphism when $2p-2 > n^2+n$ [Pst22], demonstrating the fact that the K(n)-local category is algebraic in these cases. For small primes, ε is not injective, and its kernel κ_n is the group of *exotic* invertible K(n)-local spectra. The few results that are known about the non-trivial exotic Picard groups can be summarized as follows:

- At height $n = 1$, κ_1 is non-zero only if $p = 2$, in which case it is $\mathbb{Z}/2$ [HMS94];
- At height $n = 2$, κ_2 is non-trivial only in two cases:
  - When $p = 3$, $\kappa_2 \cong \mathbb{Z}/3\times\mathbb{Z}/3$ [GHMR15], and
  - When $p = 2$, $\kappa_2 \cong (\mathbb{Z}/8)^2\times(\mathbb{Z}/2)^3$ [BBG+22];
- For all p and $n = p-1$, κ_n contains a non-trivial subgroup of order p [BGHS22, Theorem 14.6];
- For odd primes p, κ_n is product of cyclic p-groups, according to [Hea14, Theorem 4.4.1];
- Conditional upon a homological conjecture, $\kappa_3 = 0$ at the prime 5 [CZ24, Theorem 3.32].

In this paper we will further explore κ_n and related groups when $n = p-1$. This is a boundary case for complication: the Morava stabilizer group at $n = p-1$ has infinite cohomological dimension because it contains finite torsion subgroups, but this is minimally complicated as the order of any finite p-torsion subgroup is exactly p.

The analysis of κ_2 at $p = 2$ in [BBG+22] contains a compendium of almost all the known strategies for approaching κ_n, and in particular, it made clear that understanding two natural filtrations on κ_n can be crucial for clarifying the structure of this group. In this paper we study the descent filtration on κ_n (cf. [BBG+22, Definition 3.28], Definition 6.2). It is most naturally described as the filtration arising from the descent spectral sequence for the Picard spectrum of the K(n)-local category. We review this in more detail in Section 6.

To approach this descent spectral sequence, we first dive into a close study of the K(n)-local Adams-Novikov spectral sequence for the sphere, i.e. the homotopy fixed point spectral sequence[2]

$$H^s(\mathbb{G}_n,\pi_t\mathrm{E_n}) \Rightarrow \pi_{t-s}\mathrm{S}^0_{\mathrm{K(n)}}. \tag{2}$$

While the group cohomology $H^*(\mathbb{G}_n,\pi_*\mathrm{E_n})$ is generally inaccessible, it is well-understood in high cohomological degrees, at least in the case $n = p-1$. Namely, above the p-virtual cohomological dimension of $\mathbb{G}_n$, which is n^2, $H^s(\mathbb{G}_n,\pi_*\mathrm{E_n})$ is isomorphic to the Farrell-Tate cohomology $\widehat{H}^s(\mathbb{G}_n,\pi_*\mathrm{E_n})$ explicitly computed by Symonds [Sym04] for $n = p-1$, which is the height we focus

[2] Here, and everywhere in this paper, group cohomology of profinite groups is always taken continuously.

on. One key computational feature of Farrell-Tate cohomology is that for profinite groups of finite virtual cohomological dimension, whose finite p-Sylow subgroups are cyclic, their Farrell-Tate cohomology reduces to the Farrell-Tate cohomology of the normalizers of cyclic subgroups of order p [Sym04, Theorem 1.3]. This result is a strengthening of Henn's f-isomorphism theorem [Hen98, Theorem 1.4].

In the case of $\mathbb{G}_n$ at height $n=p-1$, any non-trivial finite p-subgroup of $\mathbb{G}_n$ is isomorphic to the cyclic group C_p, and its normalizer N is such that the quotient N/C_p has virtual cohomological dimension n. In fact, N/C_p is an extension of $\mathbb{Z}_p^n$ and a finite group of order prime to p, making its Farrell-Tate cohomology simple enough to compute. Then it is a result of Symonds [Sym04, Theorem 1.1] that there is an isomorphism

$$\widehat{H}^*(\mathbb{G}_n,\pi_*\mathrm{E_n})\cong\widehat{H}^*(N,\pi_*\mathrm{E_n})\cong\widehat{H}^*(F,\pi_*\mathrm{E_n})\otimes\Lambda_{\mathbb{F}_p}(a_0,\dots,a_{n-1}), \tag{3}$$

where F is a maximal finite subgroup containing our C_p. The Tate cohomology of F with coefficients in $\pi_*\mathrm{E_n}$ can be explicitly determined from a well-known computation due to Hopkins and Miller; see [Nav10]. The exterior generators arise from the cohomology of N/C_p. We review these results in Section 3 below.

The Farrell-Tate cohomology in (3) is in fact the E_2-page of the β-localized homotopy fixed point spectral sequence (2), where β is a cohomology class in $H^2(\mathbb{G},\pi_{2pn}\mathrm{E_n})$ detecting the class $\beta_1\in\pi_{2pn-2}\mathrm{S}^0_{\mathrm{K(n)}}$. There is a similar β-inverted homotopy fixed point spectral sequence for a finite subgroup F of $\mathbb{G}$ as well as for the normalizer N. Their relationship and full computation is described in Corollary 4.8.

While the β-inverted spectral sequences converge to zero, they encode crucial information (in high enough cohomological degree) about the homotopy fixed point spectral sequence (2), as well as the analogous homotopy fixed point spectral sequence for the action of N on $\mathrm{E_n}$. A detailed analysis of this information yields the following explicit horizontal vanishing line, which we prove in Section 7.

Theorem A (Theorem 5.2)**.** *Let $n=p-1$ for the prime $p\geq 3$, and let G be N or $\mathbb{G}_n$. There is a horizontal vanishing line $s=2n^2+\mathrm{vcd}(G)+1$ on the E_{2n^2+2}-page of the homotopy fixed points spectral sequence*

$$E_2^{s,t}=H^s(G,\pi_t\mathrm{E_n})\Rightarrow\pi_{t-s}(\mathrm{E}_n^{hG}).$$

Of course, nilpotence technology ensures the existence of a horizontal vanishing line on some finite page of the homotopy fixed point spectral sequence for E_n^{hG} for any closed subgroup $G\subseteq\mathbb{G}_n$, cf. [DH04, Lemma 5.11] and [BGH22, Corollary 2.3.10]. Nonetheless, there are few examples where the exact bound is known, and our result contributes another class of such examples.

Remark 1.1. When $p=3$, we recover the horizontal vanishing line (at $s=13$ on the E_{10}-page) from [GHMR15, Theorem 4.2]. Furthermore, in op.cit, the authors demonstrated that the line is sharp by finding elements in $E_{10}^{12,96+72k}$ that are detected by the Adams-Novikov spectral sequence of the Smith-Toda complex $V(0)$, see the paragraph after [GHMR15, Lemma 4.7]. Unfortunately, our analysis does not prove sharpness for $p>3$, as we do not have fine enough information about the fate of the Tate cohomology classes that could interact with the β-torsion in the homotopy fixed point spectral sequence for the sphere (2) or a suitable generalized Smith-Toda complex.

Remark 1.2. In comparison, if $p-1$ does not divide the height n, then there is a horizontal vanishing line $s=n^2+1$ on the E_2-page. This is because $H^*(\mathbb{G}_n,\pi_*\mathrm{E_n})$ is isomorphic to the Galois fixed points of the cohomology of the small Morava stabilizer group $\mathbb{S}_n$, which has no p-torsion and thus its cohomological dimension is n^2; see for example ([Hen98, Theorem 3.2.1] and [GH22, Proposition 1.13]).

The complete understanding of the homotopy fixed point spectral sequence (2) in high enough degrees also yields useful information for the Picard group Pic_n of the K(n)-local category. Namely, we use the additive-to-Picard comparison of differentials from [MS16] (see Theorem 6.1) to deduce differentials in the Picard spectrum homotopy fixed point spectral sequence. Note that this is made possible in the profinite case via the recent descent results of Mor [Mor23].

To wit, consider the Picard spectrum $\mathfrak{pic}(\mathrm{E_n})$ of the category of K(n)-local $\mathrm{E_n}$-modules. It has a natural $\mathbb{G}_n$-action, and for a closed subgroup G of $\mathbb{G}_n$ such as N when $n = p-1$ or all of $\mathbb{G}_n$, we have a spectral sequence

$$H^s(G, \pi_t \mathfrak{pic}(\mathrm{E_n})) \Rightarrow \pi_{t-s}(\mathfrak{pic}(\mathrm{E_n}))^{hG}. \tag{4}$$

Here, π_0 of the abutment is the Picard group of K(n)-local E_n^{hG}-modules. Thus, the spectral sequence induces a filtration on this Picard group, and the subgroup of filtration $s \geq 2$ is the group κ_n^G of exotic invertible K(n)-local E_n^{hG}-modules; see Definition 6.2. This is in agreement with $\kappa_n = \kappa_n^{\mathbb{G}_n}$ being the kernel of the map ε in (1), since $H^0(G, \pi_0 \mathfrak{pic}(\mathrm{E_n})) = \mathbb{Z}/2$ and $H^1(G, \pi_1(\mathfrak{pic}(\mathrm{E_n}))) \cong H^1(G, (\pi_0 \mathrm{E_n})^\times)$ conspire to build the algebraic Picard group of G-equivariant $\pi_*\mathrm{E_n}$-modules. In particular, κ_n^G itself inherits a filtration, called the descent filtration, and we deduce the following bound on its size. Note that part (1) in the following result comes from knowing that there is only one descent filtration jump in the case of the normalizer subgroup N.

Theorem B (Corollary 7.2)**.** *Let $p \geq 5$ be a prime and let $n = p-1$.*

(1) *Let N be the normalizer of $C_p \subset \mathbb{G}_n$. The exotic Picard group of* K(n)*-local E_n^{hN}-modules is a subquotient of $H^{2n+1}(N, \pi_t \mathrm{E_n})$. In particular, κ_n^N is a finite group of simple p-torsion.*
(2) *The descent filtration on the exotic Picard group κ_n has length at most n^2, and the associated graded is concentrated in $\frac{n}{2}-1$ degrees.*

Conventions. Throughout this paper, we fix an odd prime number p and a height n. To avoid clunky notation, we will omit the subscript n from $\mathrm{E_n}$, denoting it by $\mathbf{E} = \mathrm{E_n}$, and then $\mathbf{E}_t$ will denote $\pi_t \mathrm{E_n}$. We will also omit the subscript from the Morava stabilizer group, thus $\mathbb{G}$ denotes $\mathbb{G}_n$, and the small stabilizer group will be $\mathbb{S}$. Other objects which depend on n might also have notation which does not explicitly include n, but since n is fixed throughout, there is little chance of confusion.

We will work exclusively in a K(n)-local setting, thus all spectra are implicitly or explicitly K(n)-local, and tensor products (i.e. smash products) are implicitly K(n)-localized as needed.

We denote cyclic groups of order m by C_m. The center $\mathbb{Z}_p^\times$ of $\mathbb{G}$ contains a finite subgroup of the roots of unity, which we denote by μ_{p-1}. All group cohomology of profinite groups is continuous.

Acknowledgements. The influence of Agnès Beaudry's ideas on this project is hard to overestimate; it is our pleasure to thank her for her unending generosity and enthusiasm for our work. We also thank Paul Goerss, Hans-Werner Henn, and Peter Symonds for their support and their willingness to share their time and ideas on several aspects of this project. Additionally, Stojanoska thanks Itamar Mor for explaining his descent result from [Mor23] in the case of general closed subgroups of $\mathbb{G}_n$.

During the writing of this article, Lachmann is supported by the German Research Foundation DFG through the Research Training Group GRK 2240. Lima is supported by the GFSD and NSA Graduate Fellowship. Zhang is supported by the Danish National Research Foundation through the Copenhagen Centre for Geometry and Topology (DNRF151). This research has additionally been supported by the National Science Foundation grants DMS-2239362, DMS-2220741, and DMS-2304797. The authors would like to thank the Hausdorff Institute for Mathematics in Bonn for their hospitality during the Women in Topology IV workshop.

2. Preliminaries from chromatic homotopy theory

We begin by recalling some standard notions and notation from chromatic homotopy theory. Fix a formal group law Γ of height n over $\mathbb{F}_{p^n}$ that is already defined over $\mathbb{F}_p$, for example the Honda formal group law with $[p]$-series $[p](x) = x^{p^n}$. The Morava K-theory spectrum $\mathrm{K(n)}$ is a 2-periodic complex-oriented cohomology theory with a formal group law Γ, and coefficients $\mathrm{K(n)}_* = \mathbb{F}_{p^n}[u^{\pm 1}]$, where u is in degree -2.

The Morava stabilizer group $\mathbb{G} = \mathrm{Aut}(\Gamma, \mathbb{F}_{p^n})$ is the group of automorphisms of the pair $(\Gamma, \mathbb{F}_{p^n})$, and the small Morava stabilizer group $\mathbb{S} = \mathrm{Aut}(\Gamma/\mathbb{F}_{p^n})$ is the group of automorphisms of Γ over $\mathbb{F}_{p^n}$. Since Γ is defined over $\mathbb{F}_p$, the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on $\mathbb{S}$ and there is a decomposition

$$\mathbb{G} \cong \mathbb{S} \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p).$$

Denote by $\mathbf{E} = \mathrm{E_n}$ the Morava (or Lubin-Tate) E-theory $\mathbf{E}(\Gamma, \mathbb{F}_{p^n})$. Its coefficient ring

$$\mathbf{E}_* \cong W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$$

is a Laurent polynomial ring on the ring $\mathbf{E}_0$, which classifies deformations of Γ. Indeed, the formal group law of $\mathbf{E}$ is a universal deformation of Γ. The Morava stabilizer group $\mathbb{G}$ acts continuously on $\mathbf{E}_*$, and the Goerss-Hopkins-Miller theorem upgrades this action to an $\mathcal{E}_\infty$-ring action on $\mathbf{E}$ [GH04]. The homotopy fixed points $\mathbf{E}^{h\mathbb{G}}$ with respect to this action recover the $\mathrm{K(n)}$-local sphere $L_{\mathrm{K(n)}}S^0$ [DH04].

2.1. Subgroups of the Morava stabilizer groups.

In this subsection we let $n = p-1$, and introduce the subgroups N and F of the Morava stabilizer groups $\mathbb{G}$, following [Hen07, Section 3.6]. The reader is referred to [Hen07] for proofs and details on this rather brief summary.

The endomorphism ring of Γ can be described as a non-commutative algebra over the Witt vectors of $\mathbb{F}_{p^n}$ in one generator S satisfying $S^n = p$, i.e.

$$\mathrm{End}_{\mathbb{F}_{p^n}}(\Gamma) \cong \mathbb{W}_{\mathbb{F}_{p^n}}\langle S\rangle/(S^n - p).$$

It is the ring of integers of the division algebra $\mathbb{D}$ over $\mathbb{Q}_p$ of dimension n^2 and Hasse invariant $1/n$. The Morava stabilizer group $\mathbb{S}$ is the group of *automorphisms* of Γ, i.e. it has the presentation $\mathbb{S} \cong \mathrm{End}_{\mathbb{F}_{p^n}}(\Gamma)^\times$.

Choose a primitive (p^n-1)-st root of unity ω in $\mathbb{F}_{p^n}^\times \subseteq \mathbb{S}$. Denote by X the element $\omega^{\frac{p-1}{2}} S$ in $\mathbb{S}$, so $X^n = -p$. By [Hen07, Lemma 19], the field $\mathbb{Q}_p(X)$ is a subalgebra of $\mathbb{D}$ isomorphic to $\mathbb{Q}_p(\zeta_p)$ for some primitive p-th root of unity ζ_p in $\mathbb{D}$, and this isomorphism restricts to

$$\mathbb{Z}_p[X]/(X^n + p) \cong \mathbb{Z}_p[\zeta_p].$$

The element ζ_p is an algebraic integer and also a unit in $\mathbb{D}$, thus it is an element of $\mathbb{S}$. We let $C_p = \langle \zeta_p \rangle$; as a subgroup of $\mathbb{G}$ it is unique up to conjugacy. We denote by $N = N_{\mathbb{G}}(C_p)$ the normalizer of the elementary abelian subgroup C_p in $\mathbb{G}$.

Denote by τ the element

$$\tau = \omega^{\frac{p^n-1}{(p-1)^2}} \in \mathbb{S}.$$

The two elements X and $\tau^n X^2$ generate a subgroup of $\mathbb{G}$ denoted by H. This group is isomorphic to $C_{2n} \times C_{n/2}$.

The elements X, ζ_p, and τ generate a finite subgroup of N of order pn^3 which we denote by F. It follows that F is a maximal finite subgroup of $\mathbb{G}$.

Proposition 2.1 ([Hen07, Proposition 20])**.** *The subgroups C_p, H, F, and N of $\mathbb{G}$ are related as follows.*

(1) *There is a short exact sequence*
$$1 \to H \times C_p \times \mathbb{Z}_p^n \to N \to \mathrm{Aut(C_p)} \to 1.$$
(2) *The subgroups H, C_p and $\mathbb{Z}_p^n$ are invariant with respect to the action of* $\mathrm{Aut(C_p)}$*. The action on C_p is the tautological action, while $\mathbb{Z}_p^n$ is isomorphic to the direct sum of the distinct one-dimensional $\mathbb{Z}_p$-representations of* $\mathrm{Aut(C_p)}$.
(3) *There is a short exact sequence*
$$1 \to H \times C_p \to F \to \mathrm{Aut}(C_p) \to 1\,.$$

Remark 2.2. It follows that the quotient N/C_p is a group of cohomological dimension n at the prime p, as it is an extension of $\mathbb{Z}_p^n$ and finite subgroups of order prime to p.

2.2. **Homotopy fixed points spectral sequences.** We begin by briefly summarizing some results of Devinatz-Hopkins [DH04] that relate the K(n)-local $\mathbf{E}$-based Adams-Novikov spectral sequence with the homotopy fixed points spectral sequence.

Construction 2.3. For any spectrum X, and a (homotopy) commutative ring spectrum R we can form a cosimplicial object
$$R\otimes X \rightrightarrows R\otimes R\otimes X \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} R\otimes R\otimes X \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}} \cdots$$
in $\mathcal{S}$p, which is obtained by smashing the Amitsur complex of the unit map $S^0 \to R$ with X. The R-based Adams-Novikov spectral sequence for X can be obtained as the Bousfield-Kan spectral sequence associated to the totalization of this cosimplicial spectrum.

Alternatively, one constructs a "filtered" object giving the same spectral sequence after the E_2-page. Namely, denote by $\overline{R}$ the fiber of the unit map $\mathrm{S}^0_{\mathrm{K(n)}} \to R$; then we have the canonical Adams-Novikov R-resolution
$$T^R_\bullet(X) \to X, \tag{5}$$
where $T^R_m(X) = \overline{R}^{\otimes m+1} \otimes X$, as in [Rav86, Definition 2.2.10]. To avoid any potential for confusion with the notion of resolution from Definition 2.9, we will refer to (5) as the Adams-Novikov *tower* for X.

The classical Adams-Novikov tower is the above based on the complex cobordism spectrum, or p-locally, on the Brown-Peterson spectrum BP. Throughout this paper we will be working K(n)-locally, in which case all the terms in Construction 2.3 should be re-localized after tensoring.

After K(n)-local localization, the E_2-page of the BP-based Adams-Novikov spectral sequence is usefully identified with the E_2-page of the $\mathbf{E}$-based Adams-Novikov spectral sequence, which in turn becomes group cohomology of the stabilizer group. In fact, that is an identification on the level of spectral sequences, per the following result.

Theorem 2.4 ([MR77, DH04])**.** *For G any closed subgroup of $\mathbb{G}$, the spectral sequence associated to the* K(n)*-localized Adams-Novikov tower of the homotopy fixed points spectrum $\mathbf{E}^{hG}$ is strongly convergent with signature*
$$E_2^{s,t}(G,\mathbf{E}) = H^s(G,\mathbf{E}_t) \Rightarrow \pi_{t-s}(\mathbf{E}^{hG}). \tag{6}$$
Furthermore, if X is a dualizable object in the K(n)*-local category, then the spectral sequence obtained by mapping X into the* K(n)*-local Adams-Novikov tower of $\mathbf{E}^{hG}$ is strongly convergent with signature*
$$E_2^{s,t}(G,\mathbf{E}\otimes X) = H^s(G,\mathbf{E}_t(X)) \Rightarrow \pi_{t-s}(\mathbf{E}^{hG}\otimes X). \tag{7}$$

While very little can be said about the homotopy fixed point spectral sequence (6) in general, if the subgroup G contains the subgroup μ_{p-1} of $(p-1)$st roots of unity, which are central in $\mathbb{G}$, then we have the following well-known sparseness result on the E_2-page. We record it here for convenience, as it will be used to reduce the size of some exotic Picard groups below in Section 7.

Proposition 2.5. *Let G be any closed subgroup of $\mathbb{G}$ containing the central subgroup μ_{p-1} of roots of unity. Then for any s, $H^s(G,\mathbf{E}_t)=0$ unless t is divisible by $2(p-1)$.*

Proof. The argument here is the same as the argument in the standard sparsity result for the Adams-Novikov Spectral Sequence. For completion, we will just include the argument as given in Heard's thesis [Hea14, Proposition 4.2.1].

Given our assumption on G, we have the Lyndon-Hochschild-Serre spectral sequence

$$H^i(G/\mu_{p-1}, H^j(\mu_{p-1},\mathbf{E}_t)) \Rightarrow H^{i+j}(G,\mathbf{E}_t).$$

For a given element $g\in\mathbb{G}$, one can find a description of the action of g_* on $\mathbf{E}_*$ in [DH95]. In the case of central elements in $\mathbb{G}$, such as elements in μ_{p-1}, one can be very explicit: if ζ is a generator of μ_{p-1}, then

$$\zeta_* u^k = \zeta u^k \quad\text{and}\quad \zeta u_i = u_i.$$

One should note that since the order of μ_{p-1} is coprime to p, hence invertible in $\mathbf{E}_*$, the group $H^j(\mu_{p-1},\mathbf{E}_t)$ is zero unless $j=0$. So that means we are only left to compute the group $H^0(\mu_{p-1},\mathbf{E}_t)$, which from the action of μ_{p-1} given above, one can see that $H^0(\mu_{p-1},\mathbf{E}_t)$ is nonzero only when t is a multiple of $2(p-1)$, finishing the proof. □

2.3. **Homotopy fixed point spectral sequence for $\mathbf{E}^{hF}$. Suppose now that p is odd, and $n=p-1$.** The homotopy fixed points spectral sequence (6), in case G is a finite subgroup of $\mathbb{G}$, is explicitly well-understood, due to Hopkins and Miller, and first published in [Nav10, Section 2]. The starting point is the following calculation of the E_2-page modulo the transfers in the case $G=C_p$.

Proposition 2.6 (Hopkins-Miller, cf. [Nav10, Theorem 2.1], [HMS17, Proposition 2.6])**.** *There is an exact sequence*

$$\mathbf{E}_t \xrightarrow{tr} H^s(C_p,\mathbf{E}_t) \to \mathbb{F}_{p^n}[\alpha,\beta,\delta^{\pm}]/(\alpha^2)\to 0$$

of bigraded groups, where the (s,t)-bidegrees are $|\alpha|=(1,2n)$, $|\beta|=(2,2pn)$, and $|\delta|=(0,2p)$.

When $F\subset\mathbb{G}$ is a maximal finite subgroup containing C_p, we obtain a similar exact sequence.

The following result follows from the explicit understanding of the homotopy fixed points spectral sequence of $\mathbf{E}^{hF}$, which was first recorded in [Nav10, Section 2], and the fact that the generators on its E_2-page are invariant under the action by the Galois group.

Theorem 2.7 (Hopkins-Miller, cf. [HMS17], Lemma 2.8)**.** *Modulo transfer elements, the E_2-page of the homotopy fixed point spectral sequence*

$$E_2^{s,t}(F,\mathbf{E}) = H^s(F,\mathbf{E}_t) \Longrightarrow \pi_{t-s}\mathbf{E}^{hF}, \tag{8}$$

is given by

$$E_2^{*,*}/(\mathrm{tr}) = H^*(F,\mathbf{E}_*)/(\mathrm{tr}) \cong \mathbb{F}_p[\alpha,\beta,\Delta^{\pm1}]/(\alpha^2)$$

with $|\alpha|=(1,2n)$, $|\beta|=(2,2pn)$, and $|\Delta|=(0,2pn^2)$. Along the line $s=0$, classes are concentrated in degrees $t=t-s$ divisible by $2n$.

The differentials are generated multiplicatively by

$$d_{2n+1}(\Delta)=d_{2p-1}(\Delta)=\alpha\beta^n \qquad \text{and} \qquad d_{2n^2+1}(\Delta^n\alpha)=\beta^{n^2+1}, \tag{9}$$

up to units, with $E_\infty(F,\mathbf{E})=E_{2n^2+2}(F,\mathbf{E})$. The class Δ^p is a permanent cycle and a periodicity generator for $\pi_*\mathbf{E}^{hF}$.

Remark 2.8. The differentials of this spectral sequence are deduced by a comparison with the classical Adams-Novikov spectral sequence, via the composition

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \to H^*(\mathbb{G}, \mathbf{E}_*) \to H^*(F, \mathbf{E}_*).$$

In particular, under this map the element $\beta_1 \in \pi_{2pn-2}(S^0)$ is detected by the permanent cycle β, cf. [Nav10] and [Rav78].

2.4. **Finite resolutions of $\mathbf{E}^{hN}$ and $\mathbf{E}^{h\mathbb{G}}$.** As can be glimpsed from Theorem 2.4, the K(n)-local category of spectra $\mathcal{S}\mathrm{p}_{\mathrm{K(n)}}$ is largely controlled by the continuous cohomology of the Morava stabilizer group $\mathbb{G}$. As a result, the homological properties of $\mathbb{G}$ are reflected in homotopy. For one, the existence of a finite length Adams $\mathbf{E}$-resolution for $\mathrm{S}^0_{\mathrm{K(n)}}$ is closely connected to the existence of a finite length projective resolution of the trivial $\mathbb{G}$-module $\mathbb{Z}_p$. However, that can only happen in case $(p-1)$ does not divide n; see [Hen07, Theorem 4]. Otherwise the small stabilizer group $\mathbb{S}$ has infinite virtual cohomological dimension at p.

In [GHMR05], Goerss, Henn, Mahowald, and Rezk pioneered the study of finite resolutions of the K(n)-local sphere by spectra which are not $\mathbf{E}$-injective or flat over $\mathbf{E}$, but are nonetheless well understood. While [GHMR05] deals with the case of $p = 3$ and $n = 2$, [Hen07] discusses similar resolutions at more general heights and primes. In particular, in op.cit., Henn constructs a resolution of the K(n)-local sphere and the related spectrum $\mathbf{E}^{hN}$ at height $n = p-1$ for arbitrary odd primes p. We review those resolutions here, as they will play an important role in proving Corollary 4.8 below.

Throughout this paper we will use the term "resolution" of spectra which we define below.

Definition 2.9 ([Hen07, Section 3.3.1])**.** A sequence of spectra

$$* \to X \to X_0 \to X_1 \to \cdots \tag{10}$$

is a *resolution* of X if the composite of any two consecutive maps is null-homotopic, and any of the maps $X_i \to X_{i+1}$ for $i \geq 0$ can be written as $X_i \to C_i \to X_{i+1}$ such that each $C_{i-1} \to X_i \to C_i$ is a cofibration for every $i \geq 0$. Here, our convention is that $C_{-1} := X$. We say that the resolution is of *length* n if $C_n \cong X_n$ and $X_i \cong *$ for $i > n$.

Remark 2.10. Note that this definition implies that the resolution can be refined to a tower of spectra

$$\begin{tikzcd} X \arrow[dr] & & F_0 \arrow[ll] \arrow[dr] & & F_1 \arrow[ll] \arrow[dr] & & F_2 \arrow[ll] \arrow[dr] & & \cdots \arrow[ll] \\ & X_0 \arrow[ur, dashed] & & X_1 \arrow[ur, dashed] & & X_2 \arrow[ur, dashed] & & X_3 \arrow[ur, dashed] & \end{tikzcd} \tag{11}$$

in which each F_{i+1} is the homotopy fiber of the map $F_i \to X_i$. Note that $F_i = \Sigma^{-i-1}C_i$ for all $i \geq 0$. This tower gives rise to an associated *resolution spectral sequence*

$$E_1^{s,t} = \pi_t X_s \Longrightarrow \pi_{t-s} X, \tag{12}$$

For the remainder of this section, **let** $n = p-1$ **for an odd prime** p**.** While a finite $\mathbf{E}$-based Adams resolution (i.e. tower) for $\mathrm{S}^0_{\mathrm{K(n)}}$ does not exist, in [Hen07, Section 3.6], Henn constructs a finite resolution for it whose terms are wedge sums of suspensions of homotopy fixed points of $\mathbf{E}$ under the action of the finite subgroup F of $\mathbb{G}$, as well as certain retracts of $\mathbf{E}$.

Note that $\mathbf{E}^{hF}$ is well understood due to the Theorem 2.7.

The first step towards a resolution of $\mathrm{S}^0_{\mathrm{K(n)}}$ is an algebraic resolution of $\mathbb{Z}_p$ as a $\mathbb{Z}_p[[\mathbb{G}]]$-module. To construct this algebraic resolution, Henn considers the following short exact sequence of $\mathbb{Z}_p[[\mathbb{G}]]$-modules [Hen07, Prop. 17],

$$0 \leftarrow \mathbb{Z}_p \xleftarrow{\epsilon} \mathbb{Z}_p\!\uparrow_N^{\mathbb{G}} \xleftarrow{f} K \leftarrow 0 \tag{13}$$

where the map ϵ is the canonical augmentation map from the induced module to the trivial one, and K is simply defined as the kernel of ϵ. Thus, having appropriate resolutions of K and $\mathbb{Z}_p\uparrow_N^{\mathbb{G}}$ would yield a resolution of $\mathbb{Z}_p$, by taking the total complex of the resulting double complex.

On the topological level, the exact sequence (13) is realized by a cofibration

$$\mathrm{S}^0_{\mathrm{K(n)}} \simeq \mathbf{E}^{h\mathbb{G}} \xrightarrow{\epsilon} \mathbf{E}^{hN} \to C. \tag{14}$$

The finite algebraic resolution of $\mathbb{Z}_p\uparrow_N^{\mathbb{G}}$ by permutation modules yields an analogous resolution of $\mathbf{E}^{hN}$. Similarly, the finite projective resolution for K gives rise to a finite topological resolution of C. Here we summarize these results in the following theorem; see [Hen07, Section 3.5, Proposition 17, and Section 3.6] for details.

Theorem 2.11 ([Hen07, Theorems 25 and 26])**.** *Let $p>2$ be a prime number and let $n=p-1$.*

(1) *There is a resolution of length n*

$$X_\bullet : * \to \mathbf{E}^{hN} \to X_0 \to \cdots \to X_n \to * \tag{15}$$

The spectrum X_0 is equivalent to $\mathbf{E}^{hF}$, while for $r>0$ we have

$$X_r \simeq \bigvee_{(i_1,\cdots,i_r)} \Sigma^{2p^2n(i_1+\cdots+i_r)}\mathbf{E}^{hF}$$

where the wedge is taken over all sequences of integers $(i_1,\cdots,i_r)$ with $0\le i_1<i_2\cdots<i_r\le n-1$.

(2) *There is a resolution of finite length $m>n$,*

$$Z_\bullet : * \to \mathrm{S}^0_{\mathrm{K(n)}} \to Z_0 \to \cdots \to Z_m \to * \tag{16}$$

The spectrum Z_0 is equivalent to $\mathbf{E}^{hF}$, for $r>n$ each Z_r is a summand of a finite wedge of $\mathbf{E}$'s, while for $0<r\le n$,

$$Z_r \simeq V_r \vee X_r$$

where the X_r's are as in (1) and V_r is a direct summand of a finite wedge sum of $\mathbf{E}$'s.

From a computational standpoint, the resolution $Z_\bullet$ of $\mathrm{S}^0_{\mathrm{K(n)}}$ may be inefficient, since it lacks an explicit description of all the V_r terms. By construction, these terms are derived from an algebraic resolution of the $\mathbb{Z}_p[[\mathbb{G}]]$-module K, which is generally mysterious and encodes the difference between $H^*(\mathbb{G})$ and $H^*(N)$. Nonetheless, the close relationship between the resolution $X_\bullet$ of $\mathbf{E}^{hN}$ and $Z_\bullet$ of $\mathrm{S}^0_{\mathrm{K(n)}}$ was the main inspiration for us to first attempt to understand the Picard group of K(n)-local $\mathbf{E}^{hN}$-modules, as a step toward that of the K(n)-local category.

Remark 2.12. The length m of the resolution $Z_\bullet$ can be explicitly bounded. Namely, there is a resolution of length n^2 stemming from the fact that there exists a minimal-length algebraic resolution of the $\mathbb{Z}_p[[\mathbb{G}]]$-module K [Sch96, Sym07].

Remark 2.13. Note that in the case $p=3$, the resolution of Theorem 2.11(2) is very different from the duality resolution of [GHMR05]. Nonetheless, just as in [GHMR15], this resolution can be key to understanding the exotic K(n)-local Picard group κ_n.

Remark 2.14. By construction, the terms in the resolution $\mathbf{E}^{hN}\to X_\bullet$ are indexed by a graded exterior algebra $\Lambda(a_0,\dots,a_{n-1})$, cf. [Hen07, Proposition 21], such that X_1 can be thought of as $\bigvee_i a_i\mathbf{E}^{hF}$. More precisely, there is an equivalence

$$\bigvee_{r=0}^{n} X_r \simeq \Lambda(a_0,\dots,a_{n-1})\otimes\mathbf{E}^{hF}.$$

Compare with Remark 3.4 below. Note, however, that since the resolution $X_\bullet$ is not constructed to have any multiplicative properties, this is only an additive equivalence which nonetheless underlies the multiplicative structure in the Farrell-Tate cohomology of N.

3. Farrell-Tate cohomology with coefficients in $\mathbf{E}_*$

A first obstruction to understanding the K(n)-local $\mathbf{E}$-based Adams-Novikov, or homotopy fixed point spectral sequence (6), is computing its E_2-page, namely the continuous group cohomology of a closed subbroup of the Morava stabilizer group with coefficients in $\mathbf{E}_*$. While a complete determination is out of reach, we can get partial information by passing to the Farrell-Tate cohomology, which is controlled by the maximal finite subgroups.[3] For $G = \mathbb{G}$ or N at the heights $n = p-1$, this results in a full and explicit computation in high degrees (more precisely, above the virtual cohomological dimension). The topological companion of this comparison is discussed in Section 4. In this section we will recall the background and relevant computations at $n = p-1$.

First, let us briefly recall some properties of the Farrell-Tate cohomology of profinite groups of finite virtual cohomological dimension; for details, the reader can consult [Sch96, Sym07]. Especially Scheiderer's approach closely follows the classical case for (infinite) discrete groups, details of which can be found in [Bro94, X.2-3].

Suppose G is a profinite group, satisfying the FP_∞ finiteness criterion over $\mathbb{Z}_p$; in other words, there exists a resolution of the trivial module $\mathbb{Z}_p$ by finitely generated projective $\mathbb{Z}_p[[G]]$-modules. We also assume that the virtual cohomological dimension d of G is finite. It is well-established that the Morava stabilizer groups $\mathbb{G}$ (and thus any of its closed subgroups), at any height and prime, satisfy these conditions.

In this situation, the trivial G-module $\mathbb{Z}_p$ has a complete resolution $F_\bullet$ by finitely generated projectives, and for a discrete or compact $\mathbb{Z}_p[[G]]$-module M, the Tate-Farrell cohomology $\widehat{H}^*(G,M)$ can be defined as the cohomology of the complex of continuous G-homomorphisms from $F_\bullet$ to M [Sch96, Definition 4.1]. When G is finite, this is exactly the standard definition of Tate cohomology, which in positive degrees agrees with ordinary cohomology. The analogue of that latter fact in the positive virtual dimension case fact is the following comparison result.

Theorem 3.1. [Sch96, Sym07] *Let G be a profinite group with $\mathrm{vcd}(G) = d < \infty$. Suppose that M is either a discrete or compact $\mathbb{Z}_p[[G]]$-module. Then the canonical map*

$$H^s(G,M) \to \widehat{H}^s(G,M)$$

is an isomorphism for $s > d$.

Example 3.2. The natural action of $\mathbb{G}$ on $\mathbf{E}_*$ makes $\mathbf{E}_*$ a compact $\mathbb{Z}_p[[\mathbb{G}]]$-module. Restricting to N makes $\mathbf{E}_*$ a compact $\mathbb{Z}_p[[N]]$-module.

Theorem 3.3. [Sym04, Theorem 1.1] *There is an isomorphism of bigraded algebras*

$$\widehat{H}^*(\mathbb{G},\mathbf{E}_*) \cong \widehat{H}^*(N,\mathbf{E}_*) \cong \mathbb{F}_p[\alpha,\beta^{\pm 1},\Delta^{\pm 1}]/(\alpha^2) \otimes_{\mathbb{F}_p} \Lambda(a_0,\dots,a_{n-1}). \tag{17}$$

The bidegrees of the generators are given in the following table

	s	t
α	*1*	$2n$
β	2	$2pn$
Δ	*0*	$2pn^2$
a_i	1	$2p^2ni$

[3]For the precise statement, see [Sym04, Theorem 1.3] or [Sym07, Theorem 7.3].

in which having a bidegree (s,t) corresponds to being an element of $\widehat{H}^s(G,\mathbf{E}_t)$.

Remark 3.4. The classes α,β,Δ are chosen so that they map to their namesakes in the Tate cohomology of the finite subgroup F

$$\widehat{H}^*(F,E_*)\cong\mathbb{F}_p[\alpha,\beta^{\pm1},\Delta^{\pm1}]/(\alpha^2). \tag{18}$$

Note that our choice of generators is *different* from Symonds's in [Sym04, Proposition 2.3], but is compatible with the Hopkins-Miller computation Theorem 2.7. We choose the exterior generators a_i to be in one to one correspondence with each copy of $\Sigma^{2p^2ni}\mathbf{E}^{hF}$ in the term X_1 of the resolution of $\mathbf{E}^{hN}$ in Theorem 2.11. Our a_i's can be obtained from Symonds' x_i's via multiplication by powers of $\beta\Delta\in\widehat{H}^2(\mathbb{G},\mathbf{E}_{2p^2n})$. More precisely, for $0\le i\le n-1$, we have $a_i=x_i\beta^i\Delta^i$.

Proposition 3.5. *Let G be N or $\mathbb{G}$. The natural map $H^*(G,\mathbf{E}_*)\to\widehat{H}^*(G,\mathbf{E}_*)$ can be identified with the β-inversion map*

$$\varphi_G:H^*(G,\mathbf{E}_*)\to\beta^{-1}H^*(G,\mathbf{E}_*).$$

In particular, the β-inverted group cohomology $\beta^{-1}H^*(G,\mathbf{E}_*)$ is isomorphic to the Tate cohomology $\widehat{H}^*(G,\mathbf{E}_*)$, and the map φ_G is an isomorphism in cohomological degrees above the virtual cohomological dimension n^2.

Proof. Given Theorem 3.3, it suffices to prove the case $G=N$.

The statement follows by tracing through the proof of [Sym04, Theorem 1.1]. First, note that for the cyclic group C_p, the β-inversion map φ_{C_p} identifies $\beta^{-1}H^*(C_p,\mathbf{E}_*)$ with $\widehat{H}^*(C_p,\mathbf{E}_*)$; this is classical and the interested reader can find details in loc.cit., for example. As in loc.cit., there is a spectral sequence

$$H^*(N/C_p,\widehat{H}^*(C_p,\mathbf{E}_*))\Rightarrow\widehat{H}^*(N,\mathbf{E}_*), \tag{19}$$

whose E_2-term is now identified with $\beta^{-1}H^*(N/C_p,H^*(C_p,M))$, since β is an N-invariant permanent cycle. Thus we identify (19) with the β-inverted Lyndon-Hochschild-Serre spectral sequence, and the claim follows. □

4. The β-inverted spectral sequences

The homotopy fixed points spectral sequences for $\mathbf{E}^{hN}$ and $\mathbf{E}^{h\mathbb{G}}$ are difficult to understand fully or directly. However, we saw in Theorem 3.3 that the Farrell-Tate cohomology, unlike the ordinary continuous cohomology, is readily computable. By Proposition 3.5, the passage from ordinary to Farrell-Tate cohomology amounts to inverting the class β. Thus, while we lose information about β-torsion classes, the Farrell-Tate cohomology retains information about ordinary cohomology classes which are not killed by powers of β.

By Remark 3.4 and Theorem 2.7, the class β detects its namesake in $\pi_*\mathbf{E}^{hF}$, which is well-known to be (up to a unit) the Hurewicz image of the element $\beta_1\in\pi_{2pn-2}(S^0)$; see for example [Rav78]. Since β_1 is nilpotent, its inversion results in the zero spectrum, so in particular $\beta_1^{-1}\mathbf{E}^{hG}$ is contractible. Nonetheless, the β-inverted homotopy fixed point spectral sequence can be used to determine differentials in the non-β-inverted one, despite its convergence to zero. The strategy of using β-inverted or Tate spectral sequence to deduce information is well-established; see for example [Sto12]. Our argument is inspired by the analogous one at $p=3$ in [GHMR15, Section 4.2], where the β-inverted homotopy fixed point spectral sequence and the resolution spectral sequence are played against each other.

The exterior classes in the Farrell-Tate cohomology in Theorem 3.3 are closely related to the generators (i.e. shifted units of the summands) of the spectra X_i in the resolution of $\mathbf{E}^{hN}$ in Theorem 2.11; see Remark 2.14. Understanding their behavior in the homotopy fixed point spectral sequence will be done by comparison to their behavior in the resolution spectral sequence (12). We will review how such comparisons can be done in the first subsection, and then we pass to inverting β in the next.

4.1. **Combining the two towers for $\mathbf{E}^{hN}$ and $\mathbf{E}^{h\mathbb{G}}$.** Combining Henn's finite resolutions of $\mathbf{E}^{hN}$ and $\mathbf{E}^{h\mathbb{G}}$ from Theorem 2.11 with their Adams-Novikov resolutions as in Construction 2.3, results in squares of spectral sequences, which will help us relate their differentials. This was done in [BGH22], see the construction on page 401 of op.cit. Since we will further invert β in this square of spectral sequences, we review the construction.

Construction 4.1. Let G be one of N or $\mathbb{G}$, and let $F_\bullet(G)$ denote the tower

$$F_\bullet(G) \to \mathbf{E}^{hG}$$

constructed as in Remark 2.10 from the corresponding finite resolution in Theorem 2.11. For each $F_r(G)$, let $F_{r,\bullet}(G) \to F_r(G)$ denote its canonical $\mathbf{E}$-based K(n)-local Adams-Novikov tower, as in (5). Naturality of the Adams-Novikov tower makes $F_{\bullet,\bullet}(G)$ into what could be called a double filtration of spectra.

This double filtration gives rise to a square of spectral sequences, by filtering in the two different directions, following the procedure in [Mil81, Section 3]. We can first take the E_1-terms in the finite resolution direction. For specificity, note these are the X_i's (resp. Z_i's) in the resolution from Theorem 2.11, and the corresponding d_1-differentials are the maps in these resolutions. Naturality of Adams-Novikov towers now implies that we get maps between the Adams-Novikov towers of these E_1-terms. Thus the Adams-Novikov differentials will also commute with the finite resolution d_1-differentials. Altogether, taking first the resolution direction E_1-page, and then the Adams-Novikov E_2-page, yields

$$\Lambda(a_0, \ldots, a_{n-1}) \otimes H^*(F, \mathbf{E}_*) \tag{20}$$

in the case of $G = N$, due to Remark 2.14 and Theorem 2.4. In the case of $G = \mathbb{G}$, there are additional summands coming from the spectra V_r in part (2) of Theorem 2.11.

The expression (20) is the beginning of two spectral sequences: assembling it in the Adams-Novikov filtration gives rise to the homotopy fixed point spectral sequence, whereas assembling it in the finite resolution direction gives an algebraic resolution spectral sequence. The abutments of these spectral sequences in turn are the starting pages of two more spectral sequences.

Altogether, in the case of $G = N$, this becomes

$$\begin{tikzcd} H^*(N, \mathbf{E}_*) \arrow[rr, Rightarrow, "\text{HFP}"] & & \pi_*(\mathbf{E}^{hN}) \\ \Lambda(a_0, \ldots, a_{n-1}) \otimes H^*(F, \mathbf{E}_*) \arrow[u, Rightarrow] \arrow[rr, Rightarrow, "\text{HFP}"] & & \Lambda(a_0, \ldots, a_{n-1}) \otimes \pi_*(\mathbf{E}^{hF}). \arrow[u, Rightarrow] \end{tikzcd} \tag{21}$$

The horizontal arrows denote the homotopy fixed points spectral sequences arising from the Adams-Novikov tower direction, while the vertical ones arise from the finite resolution in Construction 4.1.

There is a similar square of spectral sequences for $\mathbb{G}$ instead of N, but its starting corner is slightly more mysterious, due to the V_r summands in part (2) of Theorem 2.11. Indeed, we have the following square

$$\begin{tikzcd} H^*(\mathbb{G}, \mathbf{E}_*) \arrow[rr, Rightarrow, "\text{HFP}"] & & \pi_*(\mathbf{E}^{h\mathbb{G}}) \\ \begin{array}{c} H^*(F, \mathbf{E}_*) \otimes \Lambda(a_0, \ldots, a_{n-1}) \\ \oplus H^0(\mathbb{G}, \bigoplus_{r=0}^m Q_r) \end{array} \arrow[u, Rightarrow] \arrow[rr, Rightarrow, "\text{HFP}"] & & \begin{array}{c} \pi_*(\mathbf{E}^{hF}) \otimes \Lambda(a_0, \ldots, a_{n-1}) \\ \oplus H^0(\mathbb{G}, \bigoplus_{r=0}^m Q_r) \end{array} \arrow[u, Rightarrow] \end{tikzcd}, \tag{22}$$

where each Q_r is a finitely generated projective $\mathbb{Z}_p[[\mathbb{G}]]$-module related to the V_r from Theorem 2.11. Our Q_r is denoted by Q'_r in [Hen07, Theorem 26].

Remark 4.2. In both (21) and (22), the exterior algebra consists of permanent cycles for the bottom horizontal spectral sequence almost tautologically, due to the naturality of the Adams-Novikov differentials.

4.2. **The β-inverted homotopy fixed point spectral sequences.** Now we turn to studying what happens after inverting β in the two squares of spectral sequences (21) and (22). While the β-inverted homotopy fixed point spectral sequence for the finite group F has a solid footing as a Tate spectral sequence [GM95], we explain a construction of the others before working with them.

As discussed above, the element $\beta_1 \in \pi_{2pn-2}S^0$ is detected by a cohomology class

$$\beta \in E_2^{2,2pn}(\mathbb{G}, \mathbf{E}) \cong H^2(\mathbb{G}, \mathbf{E}_{2pn}).$$

Let $\tilde{\beta}$ be a lift of β to the E_1-page of the K(n)-local Adams-Novikov spectral sequence constructed from the K(n)-localized tower $T_\bullet(S^0) = T_\bullet^{\mathbf{E}}(S^0) \to S^0$ in (5). Since $\tilde{\beta}$ is a permanent cycle, it defines a map $\tilde{\beta} : S^{2pn} \to T_2(S^0) = \overline{\mathbf{E}}^{\otimes 3}$.

For any X then, the "inclusion" $\overline{\mathbf{E}} \to S^0$ allows us to extend $\tilde{\beta}$ to a map of Adams-Novikov towers

$$\tilde{\beta} : T_\bullet(X) \to T_{\bullet+2}(X),$$

where we have suppressed the shift of internal degree from the notation. The β-localized Adams-Novikov tower of X is the colimit

$$\beta^{-1}T_\bullet(X) = \operatorname{colim}_{\tilde{\beta}} T_\bullet(X).$$

It is a $\mathbb{Z}$-indexed diagram of spectra, which is natural in X, and which gives rise to a four-quadrant spectral sequence whose E_2-page is the β-inverted E_2-page arising from $T_\bullet(X)$. Furthermore, the localization map $T_\bullet(X) \to \beta^{-1}T_\bullet(X)$ gives rise to a map of spectral sequences with corresponding multiplicative properties.

While the convergence properties of whole plane spectral sequences are generally tricky (see [Boa99, Section 8]), the diagram $\beta^{-1}T_\bullet(X)$ gives rise to a conditionally convergent spectral sequence since $T_\bullet(X)$ does. In our case, some power of the cohomology class β is a target of an Adams-Novikov differential in the spectral sequence for the sphere. This implies that the β-localized spectral sequence for any X will collapse to zero at a finite stage, and strong convergence follows from [Boa99, Theorem 8.10].

Now consider the "double complex" $F_{\bullet,\bullet}(G)$ from Construction 4.1. It can also be viewed as a diagram of Adams-Novikov towers

$$T_\bullet(F_d(G)) \to \cdots \to T_\bullet(F_1(G)) \to T_\bullet(F_0(G)),$$

and hence it gives rise to a diagram of β-inverted towers

$$\beta^{-1}T_\bullet(F_d(G)) \to \cdots \to \beta^{-1}T_\bullet(F_1(G)) \to \beta^{-1}T_\bullet(F_0(G)).$$

This diagram, in turn, gives rise to a square of spectral sequences as in (21) and (22) above, and we record the corresponding result as follows.

Theorem 4.3. *For $G = N$ and $\mathbb{G}$, there exists a commutative square of strongly convergent spectral sequences*

$$\begin{tikzcd} \beta^{-1}H^*(G,\mathbf{E}_*) \cong \widehat{H}^*(G,\mathbf{E}_*) \arrow[rr, Rightarrow, "\beta^{-1}\mathrm{HFP}"] & & \pi_*(\beta_1^{-1}\mathbf{E}^{hG}) = 0 \\ \widehat{H}^*(F,E_*)\otimes\Lambda(a_0,\dots,a_{n-1}) \arrow[u, Rightarrow, "\beta^{-1}\mathrm{Alg}"] \arrow[r, Rightarrow, "\beta^{-1}\mathrm{HFP}"] & \pi_*(\beta_1^{-1}\mathbf{E}^{hF})\otimes\Lambda(a_0,\dots,a_{n-1}) = 0 \arrow[ur, Rightarrow] & \end{tikzcd} . \tag{23}$$

Proof. The only thing that remains is to identify the terms. First, note that when $G = \mathbb{G}$, the contribution from the projective modules Q_r in (22) are killed after β-inversion since β is in positive cohomological dimension. Now apply Proposition 3.5 to identify the β-inverted ordinary group cohomology with the Farrell-Tate cohomology. □

Remark 4.4. In the β-inverted Adams-Novikov spectral sequence along the bottom of (23), the exterior algebra $\Lambda(a_0,\dots,a_{n-1})$ consists of permanent cycles by Remark 4.2, and thus the spectral sequence can be thought of as the $\Lambda(a_0,\dots,a_{n-1})$-tensored β-inverted F-homotopy fixed point spectral sequence. The latter is identified with the Tate spectral sequence for the action of F on $\mathbf{E}$, and it appears in Heard's thesis [Hea14] and in his preprint [Hea15]. We record it here as a corollary of the Hopkins-Miller computation from Theorem 2.7.

Proposition 4.5. *The β-inverted homotopy fixed point spectral sequence for $\mathbf{E}^{hF}$ takes the form*

$$\beta^{-1}E_2^{s,t}(F,\mathbf{E}) = \widehat{H}^s(F;\mathbf{E}_t) \cong \mathbb{F}_p[\alpha,\beta^{\pm1},\Delta^{\pm1}]/(\alpha^2) \Rightarrow \pi_{t-s}(\beta_1^{-1}\mathbf{E}^{hF}) = 0. \tag{24}$$

Its differentials are multiplicatively generated by the formulas in (9), and the spectral sequence collapses to zero on the E_{2n^2+2}-page.

Proof. Since β_1 is detected by β on the E_2-page, inverting β_1 in the homotopy fixed points spectral sequence

$$E_2^{s,t}(F,\mathbf{E}) = H^s(F,\mathbf{E}_t) \Rightarrow \pi_{t-s}(\mathbf{E}^{hF})$$

is the same as inverting the element β on the E_2-page, which is $\mathbb{F}_p[\alpha,\beta,\Delta^{\pm1}]/(\alpha^2)$ modulo transfer elements. It follows from Theorem 2.7 and the fact that transfer elements are β-torsion that the β-inverted homotopy fixed points spectral sequence has E_2-page $\mathbb{F}_p[\alpha,\beta^{\pm1},\Delta^{\pm1}]/(\alpha^2)$. The spectral sequence collapses to 0 on the E_{2n^2+2}-page, since the unit is hit by a differential; namely $d_{2n^2+1}(\alpha\beta^{-1-n^2}\Delta^n) = 1$. □

Remark 4.6. Note that α, β, and Δ^p are permanent cycles, and we can give explicit formulas for all differentials in the spectral sequence (24). Namely, we have that $d_{2n+1}(\beta^m\Delta^k) = k\alpha\beta^{m+n}\Delta^{k-1}$, which is non-zero if and only if k is not divisible by p, while all the α-multiples are $(2n+1)$-cycles since $\alpha^2 = 0$. Thus, the E_{2n+2} page is generated by classes of form $\beta^m\Delta^{pk}$ and $\alpha\beta^m\Delta^{n+pl}$, for some integers m,k,l. Then $d_{2n^2+1}(\alpha\beta^m\Delta^{n+pk}) = \beta^{m+n^2+1}\Delta^{pk}$ wipes away all the classes on E_{2n^2+1}.

Next, we investigate the spectral sequence $\beta^{-1}\mathrm{Alg}$ in square (23).

Proposition 4.7. *The left vertical spectral sequence $\beta^{-1}\mathrm{Alg}$ in diagram (23) has no non-trivial differentials.*

Proof. This follows from Theorem 3.3. There is a clear bijection between the E_2-page and the E_∞-page, and any non-zero differentials would contradict Theorem 3.3. □

As a corollary of Remark 4.4 and Proposition 4.7, we conclude that all higher differentials in the β-inverted spectral sequence

$$\beta^{-1}E_2^{s,t}(G,\mathbf{E}) = \beta^{-1}H^*(G,\mathbf{E}_*) = \widehat{H}^s(G,\mathbf{E}_t) \Rightarrow \pi_{t-s}(\beta_1^{-1}\mathbf{E}^{hG})$$

i.e. the top horizontal spectral sequence in (23), come from the Tate spectral sequence from (4.5), i.e. the bottom horizontal spectral sequence in (23). We record this conclusion as the following result.

Corollary 4.8. *Let G be one of N or $\mathbb{G}$. The β-inverted homotopy fixed points spectral sequence*

$$\beta^{-1}E_2^{s,t}(G,\mathbf{E}) = \widehat{H}^s(G,\mathbf{E}_t) \Rightarrow \pi_{t-s}(\beta_1^{-1}\mathbf{E}^{hG}) = 0$$

splits as a direct sum of shifts of the Tate spectral sequence (24) for $\beta_1^{-1}\mathbf{E}^{hF}$ indexed over the monomial basis of the exterior algebra $\Lambda(a_0,\dots,a_{n-1})$.

In particular, the spectral sequence collapses to zero on the E_{2n^2+2} page.

5. Differentials detected by the β-inverted homotopy fixed point spectral sequence

Let $n=p-1$, and let G denote either $\mathbb{G}$ or N. In this section, we import differentials in the homotopy fixed points spectral sequence $E_*^{*,*}(G,\mathbf{E})$ (6), above the virtual cohomological dimension of G, using our complete knowledge of the β-inverted homotopy fixed points spectral sequence from Corollary 4.8.

To do so, we consider the map of homotopy fixed point spectral sequences

$$\begin{array}{ccc} E_2^{*,*}(G,\mathbf{E}) = H^*(G,\mathbf{E}_*) & \xRightarrow{\text{HFP}} & \pi_*(\mathbf{E}^{hG}) \\ \big\downarrow \varphi & & \big\downarrow \\ \beta^{-1}E_2^{*,*}(G,\mathbf{E}) = \beta^{-1}H^*(G,\mathbf{E}_*) \cong \widehat{H}^*(G,\mathbf{E}_*) & \xRightarrow{\beta^{-1}\text{HFP}} & \pi_*(\beta_1^{-1}\mathbf{E}^{hG}). \end{array} \tag{25}$$

Lemma 5.1. *Let $r\geq 2$, and suppose that $x\in E_r^{s,t}(G,\mathbf{E})$ is a class on the E_r-page of the homotopy fixed points spectral sequence*

$$E_r^{s,t}(G,\mathbf{E}) \Rightarrow \pi_{t-s}(\mathbf{E}^{hG}).$$

Suppose that $s>d=\mathrm{vcd}(G)$. Then $d_r(x)=y\in E_r^{s+r,t+r-1}$ if and only if $d_r(\varphi(x))=\varphi(y)\in\beta^{-1}E_r^{s+r,t+r-1}$.

Proof. The comparison map φ in (25) is an isomorphism for $s>\mathrm{vcd}(G)=d$ by Proposition 3.5. Hence any differential whose source and target have cohomological degrees greater than d along the bottom spectral sequence pulls back isomorphically via φ to the differential in the homotopy fixed points spectral sequence along the top. □

This immediately yields a general statement about vanishing lines of the homotopy fixed points spectral sequence. While we will not need this result in the analysis of the K(n)-local Picard groups, we record it here as it is of independent interest.

Theorem 5.2. *Let $n=p-1$ for the prime $p\geq 3$, and let G be N or $\mathbb{G}$. There is a horizontal vanishing line $s=2n^2+\mathrm{vcd}(G)+1$ on the E_{2n^2+2}-page of the homotopy fixed points spectral sequence*

$$E_2^{s,t}=H^s(G,\mathbf{E}_t)\Rightarrow\pi_{t-s}(\mathbf{E}^{hG}).$$

In other words, $E_r^{s,t}(G,\mathbf{E})=0$ for $s\geq 2n^2+\mathrm{vcd}(G)+1$, all t, and $r\geq 2n^2+2$.

Remark 5.3. In particular, the case $G=\mathbb{G}$ and Theorem 2.4 give that the K(n)-local Adams-Novikov spectral sequence for the sphere has a horizontal vanishing line $s=3n^2+1$ at the E_{2n^2+2}-page.

Proof. Let $d=\mathrm{vcd}(G)$, and suppose x is a class in $E_r^{s,t}(G,\mathbf{E})$, with $s>2n^2+d+1$. By Lemma 5.1, $d_r(x)$ is determined by $d_r(\varphi(x))$ in the β-inverted homotopy fixed point spectral sequence.

(1) First, assume that $d_r(\varphi(x))\neq 0$. Then $r\leq 2n^2+1$, and $d_r(x)\neq 0$, so x does not survive to the E_{2n^2+2}-page.

(2) Now suppose $d_k(\varphi(x)) = 0$ for all $k \geq r$. Then, $\varphi(x)$ must be a boundary, i.e. there exists k and $z \in \beta^{-1}E_k^{s-k,t-k+1}(G,\mathbf{E})$ such that $d_k(z) = \varphi(x)$. From Lemma 5.1 and Proposition 4.5, we conclude that k must be one of $2n+1$ or $2n^2+1$. Thus, the cohomological degree $s-k$ of z is at least $s-(2n^2+1) > d$. Applying Lemma 5.1 again, we conclude that x is the target of a differential in the homotopy fixed point spectral sequence, and so again it does not survive to the E_{2n^2+2}-page.

□

Now we turn to a finer analysis of classes on the vertical line $t-s=-1$ in the homotopy fixed point spectral sequence $E_*^{*,*}(G,\mathbf{E})$, as those groups provide an upper bound for the filtration quotients of the exotic Picard group of K(n)-local $\mathbf{E}^{hG}$-modules, by Section 6.2 below.

First we record some elementary facts, accounting the supply of classes in $\widehat{H}^{t+1}(G,\mathbf{E}_t)$ above the virtual cohomological dimension. For the rest of this section, we let $p \geq 5$, so that $n^2 > 2n+1$.

Proposition 5.4. *Suppose that G is N or $\mathbb{G}$, and $p \geq 5$.*

(1) *For $s > vcd(G)$ we have that the groups $H^s(G,\mathbf{E}_t) \cong \widehat{H}^s(G,\mathbf{E}_t)$ are zero, unless $t = 2n\epsilon + 2pnl$ for some $\epsilon \in \{0,1\}$ and $l \in \mathbb{Z}$.*
(2) *In the β-inverted homotopy fixed points spectral sequence, let x be a class in $\beta^{-1}E_2^{t+1,t}(G,\mathbf{E}) \cong \widehat{H}^{t+1}(G,\mathbf{E}_t)$ with $n^2 \leq t \leq 4pn$. If x survives to the $E_{2n+2} \cong E_{2n^2+1}$-page, it cannot be the target of a d_{2n^2+1}-differential.*

Proof. (1) This follows from Theorem 3.3 and degree considerations. Note that we are not claiming much here, an arbitrary element in $\widehat{H}^*(G,\mathbf{E}_*)$ has the form $\alpha^\epsilon\beta^m\Delta^k a_0^{\epsilon_0}\ldots a_{n-1}^{\epsilon_{n-1}}$ and has topological degree $t = 2n\epsilon + 2pnl$, where $l = m + nk + p\sum_{i=0}^{n-1} i\epsilon_i$.

(2) If $n^2 \leq t = 2n\epsilon + 2pnl \leq 4pn$, then $l = 1$ and $\epsilon = 0$ or 1. If $\epsilon = 1$, then x has the form $\alpha\beta^m\Delta^k a_0^{\epsilon_0}\cdots a_{n-1}^{\epsilon_{n-1}}$, which can not be the target of a d_{2n^2+1}-differential by Remark 4.6 and Corollary 4.8. If $\epsilon = 0$, then again by Remark 4.6, x has the form $\beta^m\Delta^{kp} a_0^{\epsilon_0}\cdots a_{n-1}^{\epsilon_{n-1}}$, where the variables $m,k \in \mathbb{Z}$, and $\epsilon_i \in \{0,1\}$ for $i = 0,\ldots,n-1$ satisfy

$$2m + \sum_{i=0}^{n-1} \epsilon_i = 2pn+1, \text{ and} \tag{26}$$

$$m + pnk + p\sum_{i=0}^{n-1} i\epsilon_i = 1. \tag{27}$$

From (27), we deduce that $m-1$ is divisible by p. Setting $m-1 = ph$ with $h \in \mathbb{Z}$ and plugging into (26), we obtain that $\sum_{i=0}^{n-1}\epsilon_i$ equals $2p(n-h)-1$. But each ϵ_i is either 0 or 1, implying that $0 \leq 2p(n-h)-1 \leq n = p-1$, which is impossible. □

6. K(n)-local Picard groups

In this section we let $n \geq 1$ be an arbitrary height again, and introduce the objects of main interest in this paper, namely the various K(n)-local Picard groups. Recall that in wide generality, the Picard group of a (small enough) symmetric monoidal category $\mathscr{C}$ is the group of isomorphism classes of invertible objects in $\mathscr{C}$, equipped with the monoidal tensor operation. In chromatic homotopy theory, the Picard group of the K(n)-local stable homotopy category is usually denoted Pic_n, and often called the Hopkins' Picard group honoring the fact that Mike Hopkins first observed how rich its structure can be.

One of the original tools for studying Pic_n is the fundamental exact sequence [HMS94]

$$0 \to \kappa_n \to \mathrm{Pic}_n \xrightarrow{\varepsilon} \mathrm{Pic}_n^{alg} \cong H^1(\mathbb{G}, \mathbf{E}_*^\times), \tag{28}$$

determined by the map ε that sends an invertible K(n)-local spectrum X to its (K(n)-local) $\mathbf{E}$-homology, which is a graded invertible $\mathbf{E}_*$-module with a compatible $\mathbb{G}$-action. Since $\mathbf{E}_0$ is a complete local ring and $\mathbf{E}_* = \mathbf{E}_0[u^{\pm1}]$, invertible $\mathbf{E}_*$-modules (without the $\mathbb{G}$-action) are determined by whether they are concentrated in even or odd degrees. In other words, the Picard group $\mathrm{Pic}(\mathbf{E}_*)$ is $\mathbb{Z}/2$. Denoting by Pic_n^0 the kernel of the natural map $\mathrm{Pic}_n \to \mathrm{Pic}(\mathbf{E}_*) \cong \mathbb{Z}/2$, the sequence (28) is refined to the following form

$$0 \to \kappa_n \to \mathrm{Pic}_n^0 \xrightarrow{\varepsilon} H^1(\mathbb{G}, (\mathbf{E}_0)^\times), \tag{29}$$

which is often easier to work with since $\mathbf{E}_0$ is not a graded ring.

The isomorphism $\mathrm{Pic}_n^{alg} \cong H^1(\mathbb{G}, (\mathbf{E}_*)^\times)$ [HMS94, Proposition 8.4] gives cohomological description of this algebraic Picard group. While complete calculations of Pic_n^{alg} are few and far between (see, for example [Kar10] for a computation of Pic_2^{alg} at the prime 3), understanding Pic_n^{alg} may be best suited as a problem in arithmetic geometry [HG94], and homotopy theorists tend to focus their attention on the complementary information contained in κ_n. This is what we will do here as well.

The group κ_n is simply defined as the kernel of ε, and is called the exotic K(n)-local Picard group. Its elements are those invertible K(n)-local spectra X whose $\mathbf{E}$-homology is $\mathbb{G}$-equivariantly isomorphic to $\mathbf{E}_*$, thus they are exotic in the sense that they are not seen by the algebra of their $\mathbf{E}$-homology (i.e. by their Morava modules).

While our main interest is in the (exotic) Picard groups of the various K(n)-local categories of interest, for many purposes it is useful to think of them as the connected components of the respective Picard spaces, or π_0 of the Picard spectra. Given a presentable symmetric monoidal category $\mathscr{C}$, its Picard spectrum $\mathfrak{pic}(\mathscr{C})$ is the connective spectrum obtained by delooping the ∞-groupoid of invertible objects of $\mathscr{C}$. See, for example, [MS16] or [GL21] for more details.

As is usual, if R is a commutative ring spectrum, we denote by $\mathrm{Pic}(R)$ and $\mathfrak{pic}(R)$ the Picard group and spectrum of the category of R-modules. In fact, when R is a K(n)-local ring spectrum, such as the K(n)-local sphere, the Lubin-Tate spectrum $\mathbf{E}$, or any homotopy fixed point spectrum $\mathbf{E}^{hG}$, *we will denote by* $\mathrm{Pic}(R)$ *and* $\mathfrak{pic}(R)$ *the Picard group and spectrum of the category of* K(n)*-local* R*-modules.*

6.1. **Descent for Picard groups.** The main appeal of studying $\mathrm{Pic}(\mathscr{C})$ as $\pi_0\mathfrak{pic}(\mathscr{C})$ is the amenability of the Picard spectrum to descent techniques. In [GL21, Theorem 6.31], see also [MS16, Section 3.3], Picard spectrum descent was established for faithful finite Galois extensions. More recently, [Mor23, LZ23] studied profinite Galois descent for Picard spectra in the K(n)-local setting. We will be mostly referencing [Mor23], as Mor's approach is more suitable for our intended applications.

Let R be a ring spectrum. Then the homotopy groups of $\mathfrak{pic}(R)$, the Picard spectrum of R-modules, are given by

$$\pi_t\mathfrak{pic}(R) = \begin{cases} \mathrm{Pic}(R), \text{ for } t=0; \\ \pi_0(R)^\times, \text{ for } t=1; \\ \pi_{t-1}(R), \text{ for } t>1. \end{cases}$$

Suppose that $A \to B$ is a faithful G-Galois extension for a finite group G, which in particular implies that $A \simeq B^{hG}$. Then $\mathfrak{pic}(A)$ is equivalent to the connective cover of $\mathfrak{pic}(B)^{hG}$ [MS16, Section 3.3], and there is an associated homotopy fixed point spectral sequence, also called the Picard spectral sequence

$$E_2^{s,t}(G, \mathfrak{pic}(B)) = H^s(G, \pi_t\mathfrak{pic}(B)) \Rightarrow \pi_{t-s}\mathfrak{pic}(B)^{hG}. \tag{30}$$

Restricting to $t-s \geq 0$, this spectral sequence computes $\pi_*(\mathfrak{pic}(A))$, which yields $\mathrm{Pic}(A) \cong \pi_0(\mathfrak{pic}(A))$. In particular, this gives a natural filtration of $\mathrm{Pic}(A) \cong \pi_0 \mathfrak{pic}(B)^{hG}$, whose filtration quotients are $E_\infty^{s,s}(G, \mathfrak{pic}(B))$ for $s \geq 0$. We will come back to this filtration below in Section 6.2.

Furthermore, Mathew-Stojanoska obtained a general comparison tool ([MS16, 5.2.4]) to deduce differentials in the Picard spectral sequence (30) (in a range) from those in the homotopy fixed points spectral sequence

$$E_2^{s,t}(G, B) = H^s(G, \pi_t(B)) \Rightarrow \pi_{t-s}(B^{hG}). \tag{31}$$

The key observation is that there are equivalences

$$\Sigma\tau_{[m,2m-1]}B \simeq \tau_{[m+1,2m]}\mathfrak{pic}(B), \tag{32}$$

for any $m \geq 2$ that are natural in the spectrum B. If B is equipped with a G-action, then this equivalence is compatible with the G-action, so one can compare the differentials in (31) with those in (30) in a suitable range.

The quintessential Galois extension of chromatic homotopy theory, namely $S^0_{\mathrm{K(n)}} \to \mathbf{E}$, has a profinite Galois group. The recent work [Mor23] (see also [LZ23]) establishes an analogous descent equivalence $\mathfrak{pic}(S^0_{\mathrm{K(n)}}) \simeq \tau_{\geq 0}\mathfrak{pic}(\mathbf{E})^{h\mathbb{G}}$, giving rise to an associated spectral sequence (30) with E_2-page the continuous $\mathbb{G}$-cohomology of the homotopy groups of $\mathfrak{pic}(\mathbf{E})$. In fact, Mor's work also allows us to conclude that for any closed subgroup G of $\mathbb{G}$, there is an equivalence

$$\mathfrak{pic}(\mathbf{E}^{hG}) \simeq \tau_{\geq 0}\mathfrak{pic}(\mathbf{E})^{hG}. \tag{33}$$

Thus we obtain an associated spectral sequence (30), whose E_2-page is the continuous cohomology of G; see also [LZ23, Corollary 3.3.14]. The natural equivalences (32) then allow us to generalize Mathew-Stojanoska's comparison tool.

Theorem 6.1 ([Mor23, Theorem A.IV][LZ23, Theorem B, Corollary 3.3.14]). *Let G be any closed subgroup of $\mathbb{G}$, and consider the homotopy fixed point spectral sequences $E_r^{*,*}(G, \mathbf{E})$ (as in (31)) with differentials $d_{r,+}$, and $E_r^{*,*}(G, \mathfrak{pic}(\mathbf{E}))$ (as in (30)) with differentials $d_{r,\diamondsuit}$.*

Let x be an element in $E_2^{s,t}(G, \mathbf{E})$, with $t \geq 2$, and let $x_\diamondsuit$ be the corresponding element in

$$E_2^{s,t+1}(G, \mathfrak{pic}\mathbf{E}) \cong E_2^{s,t}(G, \mathbf{E}).$$

Given $2 \leq r \leq t$, assume that x survives to $E_r^{s,t}(G, \mathbf{E})$, i.e. for all $q < r$, $d_{q,+}(x) = 0$ and x is not in the image of $d_{q,+}$. Then, $x_\diamondsuit$ survives to $E_r^{s,t+1}(G, \mathfrak{pic}(\mathbf{E}))$ and $d_{r,\diamondsuit}(x_\diamondsuit)$ is identified with $d_{r,+}(x)$.

6.2. **The descent filtration on exotic Picard groups.** The Picard group of $\mathbf{E}^{hG}$ inherits a natural filtration $f_s\mathrm{Pic}(\mathbf{E}^{hG})$ from the descent spectral sequence, which in the case of $G = \mathbb{G}$ is closely related to (28) and (29), as well as the descent filtration on κ_n from [GHMR15, Construction 3.2], [BBG+22, Section 3.3], or [CZ24, Section 1.2, 1.3]. To be more precise, note that $\mathrm{Pic}(\mathbf{E}) \cong \mathrm{Pic}(\mathbf{E}_*) = \mathbb{Z}/2$, generated by the suspension shift. Thus, for any subgroup G of $\mathbb{G}$, we have $H^0(G, \pi_0\mathfrak{pic}(\mathbf{E})) \cong \mathbb{Z}/2$, and the bottom of the filtration is an exact sequence

$$0 \to f_1\mathrm{Pic}(\mathbf{E}^{hG}) \to \mathrm{Pic}(\mathbf{E}^{hG}) \to E_\infty^{0,0}(G, \mathfrak{pic}(\mathbf{E})) \cong E_2^{0,0}(G, \mathfrak{pic}(\mathbf{E})) \cong \mathbb{Z}/2 \to 0,$$

where $f_1\mathrm{Pic}(\mathbf{E}^{hG})$ consists of those invertible $\mathbf{E}^{hG}$-modules X for which $X \otimes_{\mathbf{E}^{hG}} \mathbf{E}$ is concentrated in even degrees. In particular, $f_1\mathrm{Pic}(\mathbf{E}^{h\mathbb{G}}) = \mathrm{Pic}_n^0$.

The next step in the filtration is the exact sequence

$$0 \to f_2\mathrm{Pic}(\mathbf{E}^{hG}) \to f_1\mathrm{Pic}(\mathbf{E}^{hG}) \to E_\infty^{1,1}(G, \mathfrak{pic}(\mathbf{E})) \subseteq H^1(G, \mathbf{E}_0^\times),$$

which is precisely (29) when $G = \mathbb{G}$. Thus, $\kappa_n = f_2\mathrm{Pic}(\mathbf{E}^{h\mathbb{G}})$. In line with this example, we make the following definition.

Definition 6.2. Let G be a closed subgroup of the Morava stabilizer group $\mathbb{G}$. The group κ_n^G of exotic elements in the Picard group $\mathrm{Pic}(\mathbf{E}^{hG})$ is $f_2\mathrm{Pic}(\mathbf{E}^{hG})$. Equivalently, κ_n^G consists of $X \in \mathrm{Pic}(\mathbf{E}^{hG})$ such that $\pi_*(X \otimes_{\mathbf{E}^{hG}} \mathbf{E})$ is G-equivariantly equivalent to $\mathbf{E}_*$, and it sits in an exact sequence

$$0 \to \kappa_n^G \to \mathrm{Pic}(\mathbf{E}^{hG}) \to H^1(G, \mathbf{E}_*^\times). \tag{34}$$

Furthermore, the descent filtration on κ_n^G is $\kappa_{n,s}^G = f_s\mathrm{Pic}(\mathbf{E}^{hG})$ for $s \geq 2$.

Note that by Definition 6.2, $\kappa_{n,s}^G$ comes with a map $\kappa_{n,s}^G \to E_s^{s,s}(G, \mathfrak{pic}(\mathbf{E}))$, and we have a comparison of the target group with $E_s^{s,s-1}(G, \mathbf{E})$ according to Theorem 6.1. In particular, the latter is a subquotient of $H^s(G, \mathbf{E}_{s-1})$.

If the group G contains the central roots of unity μ_{p-1}, the sparsity result of Proposition 2.5 implies a corresponding sparsity of the descent filtration on κ_n^G. We record it here for future reference. In this paper, we will use it in the proof of Corollary 7.2.

Lemma 6.3. *Assume the closed subgroup G of $\mathbb{G}$ contains the central subgroup $\mu_{p-1} \in \mathbb{G}$. Then the associated graded of the descent filtration on κ_n^G is concentrated in degrees congruent to 1 modulo $2(p-1)$.*

Remark 6.4. The relationship of the descent filtration quotients $\kappa_{n,s}^G/\kappa_{n,s+1}^G$ to $H^s(G, \mathbf{E}_{s-1})$ can be used to describe the filtration without reference to the Picard spectral sequence (30). Instead, one studies the differential pattern of the Adams-Novikov spectral sequence (7) for a representative exotic invertible K(n)-local spectrum. For details on this approach, the reader is referred to [BBG+22, Section 3.3].

Example 6.5. From [MS16, Theorem 7.1.2], we conclude that at the prime 2 we have $\kappa_1^{C_2} = \mathbb{Z}/2$. From the computation in the proof of [MS16, Theorem 8.1.3], we deduce that at the prime 3, $\kappa_2^F = \mathbb{Z}/3$, for a maximal finite subgroup of F of $\mathbb{G}$. Further, the proof of [HMS17, Theorem 4.1] shows that if G is a finite group containing C_p, then $\kappa_{p-1}^G = \mathbb{Z}/p$.

At the prime 2, larger groups appear; for example, for a maximal finite subgroup $F = G_{48}$, we conclude that $\kappa_2^F = \mathbb{Z}/8$ from [MS16, Theorem 8.2.2]. The group $\kappa_2^{C_4}$ at $p=2$ has order 4, according to the proof of [BBHS20, Proposition 7.4].

Remark 6.6. Note that while κ_n^G is not unrelated to the subgroup filtration on κ_n from [BBG+22, Section 3.1], it is different. In particular, if $G_1 \subseteq G_2$ are nested closed subgroups of $\mathbb{G}$, then we have a map $\kappa_n^{G_2} \to \kappa_n^{G_1}$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & \kappa_n^{G_2} \arrow[r] \arrow[d] & \mathrm{Pic}(\mathbf{E}^{hG_2}) \arrow[r] \arrow[d] & H^1(G_2, \mathbf{E}_*^\times) \arrow[d] \\
0 \arrow[r] & \kappa_n^{G_1} \arrow[r] & \mathrm{Pic}(\mathbf{E}^{hG_1}) \arrow[r] & H^1(G_1, \mathbf{E}_*^\times)
\end{tikzcd}$$

commute, in which the middle and right-most maps are the natural ones. However, $\kappa_n^{G_2}$ need not be a subgroup of $\kappa_n^{G_1}$, and in fact none of the vertical maps need be inclusions. For a closed sugroup G of $\mathbb{G}$, the group $\kappa_n(G)$ of [BBG+22] is closely related to the kernel of $\kappa_n = \kappa_n^{\mathbb{G}} \to \kappa_n^G$.

7. Bounding the descent filtration

We are finally ready to apply the tools we have developed above and deduce certain differentials in the Picard spectral sequence (30) with $B = \mathbf{E}$, **at height** $n = p-1$, and where G is $\mathbb{G}$ or N. As a result, we obtain a bound on the descent filtration of κ_n^N and κ_n, as well as an explicit bound on the size of κ_n^N.

Using Theorem 6.1 will allow us to compare an appropriate range of the Picard spectral sequence with the homotopy fixed point spectral sequence for the G action on $\mathbf{E}$. We have partial but crucial information about the latter in Proposition 5.4.

Theorem 7.1. *Suppose that $p \geq 5$, and let G be $\mathbb{G}$ or N. Let $x_{\spadesuit}$ be a class of bidegree $(t+1, t+1)$ in the E_2 page of the homotopy fixed point spectral sequence*

$$E_2^{*,*}(G, \mathfrak{pic}(\mathbf{E})) \Rightarrow \pi_* \mathfrak{pic}(\mathbf{E})^{hG}. \tag{35}$$

Assume that $t \geq 1$, and that $x_{\spadesuit}$ is a non-trivial permanent cycle. Then

(1) *when $G = N$, t equals $2n$, while*
(2) *when $G = \mathbb{G}$, t is less than n^2.*

Proof. First of all, note that for $m \geq 1$, we have an isomorphism $E_2^{s,m+1}(G, \mathfrak{pic}(\mathbf{E})) \cong E_2^{s,m}(G, \mathbf{E})$, so for each $y \in E_2^{s,m}(G, \mathbf{E})$, denote by $y_{\spadesuit}$ the corresponding class in $E_2^{s,m+1}(G, \mathfrak{pic}(\mathbf{E}))$. Since $x_{\spadesuit}$ has a companion class in $x \in E_2^{t+1,t}(G, \mathbf{E})$, by the sparseness result of Proposition 2.5, we conclude that if $x_{\spadesuit}$ is to be non-trivial, t must be at least $2n$.

(1) Suppose $G = N$ and $x_{\spadesuit} \in E_2^{t+1,t+1}(G, \mathfrak{pic}(\mathbf{E}))$ is a non-zero permanent cycle in cohomological degree above $\mathrm{vcd}(N) = n$. By looking at its companion $x \in E_2^{t+1,t}(G, \mathbf{E})$, we deduce that t has form $2n\epsilon + 2pnl$ by part (1) of Proposition 5.4. We need to show that if $t > 2n$, then $x_{\spadesuit}$ will be in the image of a differential. The formula $t = 2n\epsilon + 2pnl$ implies that if $t > 2n$, then $t \geq 2pn = 2n^2 + 2n$.

By Theorem 6.1, and the assumption that $x_{\spadesuit}$ is a permanent cycle, we conclude that $d_r(x) = 0$ for $r < 2n^2 + 2n$. Now Lemma 5.1 implies that the image $\varphi(x)$ of x in the β-inverted homotopy fixed point spectral sequence is a d_r-cycle for $r < 2n^2 + 2n$. In light of Corollary 4.8, this means $\varphi(x)$ is a permanent cycle. So, $\varphi(x)$ must be in the image of a differential, i.e. there exists either $\tilde{y} \in \beta^{-1} E_{2n+1}^{t-2n,t-2n}(G, \mathbf{E})$ such that $d_{2n+1}(\tilde{y}) = \varphi(x)$, or $\tilde{z} \in \beta^{-1} E_{2n^2+1}^{t-2n^2,t-2n^2}(G, \mathbf{E})$ such that $d_{2n^2+1}(\tilde{z}) = \varphi(x)$. In either case, the cohomological degree of the class hitting $\varphi(x)$ is at least $2n$, thus invoking Lemma 5.1 again gives that x is the target of either a d_{2n+1} or a d_{2n^2+1} differential.

Suppose $d_{2n+1}(y) = x$ for some $y \in E_{2n+1}^{t-2n,t-2n}(G, \mathbf{E})$. Then its topological degree $t - 2n$ is at least $2n^2$, so by Theorem 6.1, the companion class $y_{\spadesuit}$ hits $x_{\spadesuit}$. If $x_{\spadesuit}$ is not in the image of d_{2n+1}, we conclude that there exists $z \in E_{2n^2+1}^{t-2n^2,t-2n^2}(G, \mathbf{E})$ such that $d_{2n^2+1}(z) = x$. Part (2) of Proposition 5.4 now implies that $t > 4n^2$. Invoking Theorem 6.1, since the topological degree $t - 2n^2$ of z is at least $2n^2 + 1$, we conclude that $d_{2n^2+1}(z_{\spadesuit}) = x_{\spadesuit}$.

Altogether, we have that if $x_{\spadesuit}$ is a non-trivial permanent cycle in $E_*^{t+1,t+1}(N, \mathfrak{pic}(\mathbf{E}))$, and $t \geq 1$, then $t = 2n$.

(2) Now we turn to the case $G = \mathbb{G}$, which we argue in a similar fashion. We only need to show that if $t > n^2 = \mathrm{vcd}(G)$, then a permanent cycle $x_{\spadesuit} \in E_2^{t+1,t+1}(G, \mathfrak{pic}(\mathbf{E}))$ must be hit by a differential. The companion class in $E_2^{t+1,t}(G, \mathbf{E})$ will have topological degree $t = 2n\epsilon + 2pnl$ by part (1) of Proposition 5.4, implying that $l \geq 1$, i.e. $t \geq 2n^2 + 2n$.

The rest of the argument proceeds exactly as in the case of $G = N$. We argue that x is a permanent cycle, by comparison to the β-inverted homotopy fixed point spectral sequence. If $\varphi(x)$ is in the image of a d_{2n+1}-differential, we import this differential to the Picard spectral sequence using Lemma 5.1 and Theorem 6.1. If not, then it is in the image of a d_{2n^2+1}-differential, but in this case we conclude $t > 4n^2$ by Proposition 5.4, which in turn allows us to import this differential to the Picard spectral sequence by another application of Theorem 6.1. □

With this result in hand, we are ready to read off the implications for the K(n)-local exotic Picard group. Recall that κ_n^G is the subgroup of $\mathrm{Pic}(\mathbf{E}^{hG})$ of filtration 2 (and above) in the spectral sequence

(35), and its descent filtration is the one inherited from this spectral sequence; cf. Definition 6.2. Thus, part (1) of Theorem 7.1 implies that κ_n^N in fact equals $\kappa^N_{n,2n+1}$ and further, that $\kappa^N_{n,2n+2}=0$.

When G is the whole group $\mathbb{G}$, part (2) of Theorem 7.1 gives that $\kappa_{n,n^2+1}=\kappa^{\mathbb{G}}_{n,n^2+1}=0$. While this much less precise, the sparsity result from Lemma 6.3 implies that the filtration quotients of κ_n are concentrated in degrees congruent to 1 modulo $2n$.

Corollary 7.2. *Suppose $p\geq 5$ and $n=p-1$. Let N be the normalizer of $C_p\subset\mathbb{G}$.*

(1) *The exotic Picard group of* K(n)*-local $\mathbf{E}^{hN}$-modules is a subquotient of $H^{2n+1}(N,\mathbf{E}_{2n})$. In particular, κ_n^N is a finite group of simple p-torsion.*
(2) *The descent filtration on the exotic Picard group κ_n has length at most n^2, and its associated graded is concentrated in degrees congruent to* 1 *modulo* $2n$. *More precisely,*

$$\mathrm{gr}_*\kappa_n\cong\bigoplus_{m=1}^{n/2-1}E_\infty^{2nm+1,2nm}(\mathbb{G},\mathfrak{pic}(\mathbf{E})),$$

and each $E_\infty^{2nm+1,2nm}(\mathbb{G},\mathfrak{pic}(\mathbf{E}))$ is a subquotient of $H^{2nm+1}(\mathbb{G},\mathbf{E}_{2nm})$.

Example 7.3. When $p=5$, part (2) gives that κ_n itself is concentrated in a single degree and is a subquotient of $H^9(\mathbb{G},\mathbf{E}_8)$

Remark 7.4. Unlike $H^{2n+1}(\mathbb{G},\mathbf{E}_{2n})$, the cohomology group $H^{2n+1}(N,\mathbf{E}_{2n})$ is not particularly mysterious. A combinatorial description of an $\mathbb{F}_p$-basis of $H^{2n-1}(N,\mathbf{E}_{2n})\cong\widehat{H}^{2n-1}(N,\mathbf{E}_{2n})$ can be obtained as follows. By Theorem 3.3, a generator with internal degree $2n$ has to be of the form $x=\alpha(\beta^n\Delta^{-1})^k x_0^{\epsilon_0}\cdots x_{n-1}^{\epsilon_{n-1}}$, where $k\in\mathbb{Z}$ and $\epsilon,\epsilon_i\in\{0,1\}$. Hence we want to find all tuples $(k,\epsilon_0,\dots,\epsilon_{n-1})$ such that the cohomological degree of x is

$$1+2nk+\sum_{i=1}^{n-1}(2i+1)\epsilon_i=2n+1. \tag{36}$$

It follows that there is an even number of ϵ_i's that are 1, say those indexed by $0\leq i_1<\cdots<i_{2l}\leq n-1$. Then (36) can be rewritten as

$$-l+\sum_{r=1}^{2l}i_r\equiv 0\mod n. \tag{37}$$

In other words, the dimension of $H^{2n-1}(N,\mathbf{E}_{2n})$ as an $\mathbb{F}_p$-vector space is given by the sum, as l ranges from 0 to $\frac{n-1}{2}$, of the number of $2l$-tuples $0\leq i_1<\cdots<i_{2l}\leq n-1$ satisfying (37).

Remark 7.5. In an alternative approach, one could use obstruction theory on Henn's resolution

$$\mathbf{E}^{hN}\to X_0\to X_1\to\cdots\to X_n$$

of $\mathbf{E}^{hN}$ (Theorem 2.11) to bound the size of κ_n^N by modifying the argument in [Hea14, 4.4.1.(iii)]. Consider the homomorphism $\kappa_n^N\to\kappa_n^F$ sending Y to $Y\otimes_{\mathbf{E}^{hN}}\mathbf{E}^{hF}$. An upper bound for the kernel is given by the amount of obstructions to lifting a non-exotic $\mathbf{E}^{hF}$-module $Y\otimes_{\mathbf{E}^{hN}}\mathbf{E}^{hF}$ to a non-exotic $\mathbf{E}^{hN}$-module Y via the spectral sequence associated to the resolution of $\mathbf{E}^{hN}$ tensored with Y. The obstructions live in π_sX_{s+1} where $0\leq s\leq n-1$. Then a straightforward combinatorial argument shows that the size of obstructions is precisely $H^{2n+1}(N,\mathbf{E}_{2n})$, which recovers the upper bound of κ_n^N given in Corollary 7.2.

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