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The stable Adams conjecture and higher associative structures on Moore spectra

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The first author would like to dedicate this article in the loving memory of his father Jayanta Bhattacharya (*November* 2, 1957–*May* 2, 2021)

Abstract

In this paper, we provide a new proof of the stable Adams conjecture. Our proof constructs a canonical null-homotopy of the stable J-homomorphism composed with a virtual Adams operation, by applying the K-theory functor to a multinatural transformation. We also point out that the original proof of the stable Adams conjecture is incorrect and present a correction. This correction is crucial to our main application. We settle the question on the height of higher associative structures on the mod p^k Moore spectrum $\mathrm{M}_p(k)$ at odd primes. More precisely, for any odd prime p, we show that $\mathrm{M}_p(k)$ admits a Thomified $\mathbb{A}_n$-structure if and only if $n < p^k$. We also prove a weaker result for $p = 2$.

Contents

1. Introduction

The celebrated result commonly referred to as "the Adams conjecture" establishes the fact that for a given *odd* prime p, and for any integer q serving

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as a topological generator of the p-adic units $\hat{\mathbb{Z}}_p^\times$, the following composite map of spaces is null homotopic (when localized at p):

$$\mathrm{J} \circ (\psi^q - 1) : \mathrm{BU} \longrightarrow \mathrm{BU} \longrightarrow \mathrm{BSL}_1(\mathbb{S}), \tag{1.1}$$

where ψ^q represents the corresponding Adams operation, and J represents the complex J-homomorphism from the classifying space of infinite unitary group to the classifying space of the group-like $\mathbb{E}_\infty$-space of stable self homotopy equivalences of degree 1 of the sphere. A similar statement holds for the prime $p = 2$ with BU replaced by BO, $\mathrm{BSL}_1(\mathbb{S})$ replaced by $\mathrm{BGL}_1(\mathbb{S})$, J replaced by the real J-homomorphism and $q = 3$.

A host of important consequences ensuing from this result are well known to any practitioner of algebraic topology; see [Ada63] and subsequent articles.

Notice that all the spaces and maps involved in the statement of the Adams conjecture (1.1) are infinite loop spaces and infinite loop maps (when localized at a prime p). However, the original conjecture did not demand that the composite be null as an infinite loop map. Demanding that the composite be null as an infinite loop map is known as the *infinite loop Adams conjecture* or the *stable Adams conjecture.* This stable enhancement of the Adams conjecture has its own set of important consequences [MP78], [May78a], [CLM76, II.10].

The Adams conjecture was solved by Quillen [Qui71] and Sullivan [Sul74] using different techniques. Quillen made use of modular character theory to approximate the p-completion of BU (or BO at the prime 2) using classifying spaces of discrete groups $\mathrm{Gl}_n(\overline{\mathbb{F}}_q)$, where $\overline{\mathbb{F}}_q$ is the algebraic closure of the field of order q and q is prime to p. On the other hand, Sullivan's key idea was to use étale homotopy theory to interpret Adams operations as elements of the absolute Galois group $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ acting on the profinite completion of BU.

In 1977, Friedlander and R. Seymour [FS77] announced two solutions to the stable Adams conjecture. However, the paper was later retracted due to a fatal flaw in the arguments of one of the proofs. The other solution was later elaborated in [Fri80] and remains the only accepted proof of the stable Adams conjecture to this date. Unfortunately, however, Friedlander's proof now also appears to be incorrect, leaving the literature with an awkward gap; see Appendix A.

In this paper, we provide two *different* solutions to the stable Adams conjecture. In our first solution we realize the Adams operation as a map of permutative categories. We then realize the J-homomorphism as a multifunctor and produce a canonical null-homotopy refining the stable Adams conjecture. In fact, this null-homotopy is constructed in the form of a multinatural transformation exploiting the important work of Elmendorf and Mandell [EM06], [EM09]. This approach avoids results from [Fri80]. Our second proof can be regarded as a correction to [Fri80] as it uses the classification theorem [Fri80, Th. 6.1] that Friedlander devised to resolve the stable Adams conjecture.

Our correction to the stable Adams conjecture is critical to our main application — detecting homotopy coherence of associativity (in the sense of Stasheff [Sta63a], [Sta63b]) of multiplicative structures on the mod p^i Moore spectrum $\mathrm{M}_p(i)$, a problem that is at least 50 years old!

Remark 1.2. The stable enhancement of the Adams conjecture is not true if in (1.1), we replace BU by BO, choose J to be the real J-homomorphism and localize at the prime 2. This was proven by Madsen [Mad75] by studying the Dyer-Lashof algebra structures.

Notation 1.3. We use the following notation through out the paper:

- $\mathcal{S}\mathrm{p}$ – the category of symmetric spectra.
- ku – the (-1)-connected cover of the periodic complex K-theory KU.
- bu – the 0-connected cover of ku.
- $\underline{\mathsf{ku}}_{(p)}$ – the fiber of the composite map $\mathsf{ku}_{(p)} \to \mathrm{H}\mathbb{Z}_{(p)} \twoheadrightarrow \mathrm{H}(\mathbb{Z}_{(p)}/\mathbb{Z})$, where the first map is the zeroth Postnikov approximation of $\mathsf{ku}_{(p)}$. Thus, $\pi_{2i}\underline{\mathsf{ku}}_{(p)} \cong \pi_{2i}\mathsf{ku}_{(p)} \cong \mathbb{Z}_{(p)}$ for all $i > 0$, and $\pi_0\underline{\mathsf{ku}}_{(p)} \cong \mathbb{Z}$.
- $\mathsf{pic}(\mathrm{R})$ – the Picard spectrum of an $\mathbb{E}_\infty$-ring R.
- $\mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})$ – the even Picard spectrum of the p-local sphere, i.e., the spectrum associated to the order 2 subgroup of $\pi_0\mathsf{pic}(\mathbb{S}_{(p)}) \cong \mathbb{Z}$ (see Examples 2.7 and 2.8).
- $\mathsf{pic}^{\mathrm{ev}}(\hat{\mathbb{S}}_p)$ – The even Picard spectrum of the p-complete sphere $\hat{\mathbb{S}}_p$.
- $\mathrm{GL}_1(\mathrm{R})$ – The space of units of the ring spectrum R.
- $\mathrm{SL}_1(\mathrm{R})$ – The identity component of $\mathrm{GL}_1(\mathrm{R})$.
- $\mathrm{bgl}_1(\mathrm{R})$ – The spectrum obtained by delooping $\mathrm{BGL}_1(\mathrm{R})$ for an $\mathbb{E}_\infty$-ring R.
- $\mathrm{bsl}_1(\mathrm{R})$ – The 1-connected cover of $\mathrm{bgl}_1(\mathrm{R})$.
- $\mathfrak{J} : \underline{\mathsf{ku}}_{(p)} \to \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})$ – The stable J-homomorphism (defined in Section 5).

Notation 1.4. For the rest of the paper, p will be used to denote any prime. For a fixed p, let q denote another prime such that

$$q \equiv \begin{cases} l \mod p^2 & \text{if } p \text{ is odd,} \\ -1 \mod 4 & \text{if } p = 2, \end{cases}$$

where l is a generator of $(\mathbb{Z}/p^2\mathbb{Z})^\times$. By Dirichlet's theorem there exists infinitely many choices for q for a given p. With these properties, q is a topological generator of $\hat{\mathbb{Z}}_p^\times$, when p is odd. If $p = 2$, q is a topological generator of an infinite subgroup of $\hat{\mathbb{Z}}_2^\times \cong \mathbb{Z}/2 \times \hat{\mathbb{Z}}_2$ isomorphic to $\hat{\mathbb{Z}}_2$.

The bulk of the work in this paper is to prove the following theorem.

Theorem 1.5 (Unreduced stable Adams conjecture). *The composite map*

$$\underline{\mathsf{ku}}_{(p)} \xrightarrow{\psi^q - 1} \underline{\mathsf{ku}}_{(p)} \xrightarrow{\mathfrak{J}} \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)}) \tag{1.6}$$

is null-homotopic, where ψ^q *denotes the corresponding Adams operation.*

The above null-homotopy can be chosen canonically giving rise to an extension

$$\tilde{\mathfrak{J}} : \mathrm{Cof}(\psi^q - 1) \longrightarrow \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)}).$$

In Section 6, we prove that

THEOREM 1.7. *The map induced by $\tilde{\mathfrak{J}}$ on $\pi_1(-)$,*

$$\pi_1(\tilde{\mathfrak{J}}) : \mathbb{Z} \cong \pi_1(\mathrm{Cof}(\psi^q - 1)) \longrightarrow \pi_1(\mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})) \cong \mathbb{Z}_{(p)}^{\times},$$

sends 1 *to* q.

Theorem 1.7 embodies a crucial fact that contradicts [Fri80]. We address the matter in more detail in Appendix A.

We also study a *reduced version* of the stable Adams conjecture, which is exactly the stable enhancement of the Adams conjecture in its original form (1.1). The statement of the reduced version of the stable Adams conjecture remains unchanged (see [Fri80, p. 109]) regardless of the modifications (in particular Theorem 1.7) in the statement of unreduced Adams conjecture proposed in this paper.

THEOREM 1.8 (Reduced stable Adams conjecture). *The composition of the maps*

$$\mathsf{bu}_{(p)} \xrightarrow{\psi_0^q - 1} \mathsf{bu}_{(p)} \xrightarrow{\mathfrak{J}_0} \mathrm{bsl}_1(\mathbb{S})_{(p)}$$

is null-homotopic.

As an application of Theorem 1.7 we study the problem of higher associativity of Moore spectra. Let $\mathrm{M}_p(i)$ denote the Moore spectrum given by the cofiber of the degree p^i-map on the sphere spectrum $\mathbb{S}$. Here is a brief historical account regarding the development of this problem of higher associative multiplication on $\mathrm{M}_p(i)$.

By 1960, it was well known to the experts that $\mathrm{M}_2(1)$ cannot support a unital multiplication, i.e., an $\mathbb{A}_2$-structure. This is an easy application of the Cartan's formula for Steenrod operations. Perhaps the first non-trivial result was due to Toda [Tod68] when he proved that the multiplication on $\mathrm{M}_3(1)$ is not homotopy associative, i.e., $\mathrm{M}_3(1)$ does not admit an $\mathbb{A}_3$-structure. Soon after it was noticed that the work of Kraines [Kra66] and Kochmann [Koc72] can be combined to generalize Toda's result to show that $\mathrm{M}_p(1)$ admits an $\mathbb{A}_{p-1}$-structure that does not extend to an $\mathbb{A}_p$-structure. In 1982, Oka [Oka84] showed that $\mathrm{M}_2(i)$ admits an $\mathbb{A}_3$-structure for $i > 1$. Aside from these sporadic results, the question of $\mathbb{A}_n$-structures on $\mathrm{M}_p(i)$ has proved to be intractable and remained open until the recent work in [Bha20].

Now we summarize the key idea in [Bha20] with minor modifications. Notice that $\mathrm{M}_p(i)$ is the Thom spectrum associated to a map

$$f_{p,i} : \mathrm{S}^1 \longrightarrow \mathrm{BGL}_1(\hat{\mathbb{S}}_p)$$

representing the class $1 + p^i u \in \pi_1(\hat{\mathbb{S}}_p) \cong \hat{\mathbb{Z}}_p$, where $u \in \hat{\mathbb{Z}}_p^\times$. If the map $f_{p,i}$ is an $\mathbb{A}_n$-map, then $\mathrm{M}_p(i)$ inherits an $\mathbb{A}_n$-structure. Motivated by this fact, we make the following definition.

Definition 1.9. A *Thomified* $\mathbb{A}_n$*-structure* on the Moore spectrum $\mathrm{M}_p(i)$ is an $\mathbb{A}_n$-structure induced by an $\mathbb{A}_n$-map

$$f_{p,i} : \mathrm{S}^1 \longrightarrow \mathrm{BGL}_1(\hat{\mathbb{S}}_p)$$

of degree $1 + p^i u \in \pi_1(\hat{\mathbb{S}}_p) \cong \hat{\mathbb{Z}}_p$ for some $u \in \hat{\mathbb{Z}}_p^\times$.

A result of Stasheff [Sta63a], [Sta63b] implies $f_{p,i}$ is an $\mathbb{A}_n$-map if and only if there exists a stable lift (up to homotopy) in the diagram

$$\begin{tikzcd} \Sigma\mathbb{S} \arrow[r, "f_{p,i}"] \arrow[d] & \mathrm{bgl}_1(\hat{\mathbb{S}}_p). \\ \Sigma^{-1}\mathbb{CP}^n \arrow[ur, dashed, "f_{p,i}^{(n)}"'] & \end{tikzcd} \tag{1.10}$$

In [Bha20], the author studied an Atiyah-Hirzebruch spectral sequence and obtained a lower bound on i, dependent on p and n, which guaranteed an $\mathbb{A}_n$-structure on $\mathrm{M}_p(i)$.

In this paper, we resolve [Bha20, Conj. 4.12], which predicts that $f_{p,i}$ factors through the J-homomorphism. Indeed, by Theorem 1.7 we get $f_{p,i}$ as the composite

$$f_{p,i} : \Sigma\mathbb{S} \xrightarrow{\epsilon_p(i)} \mathrm{Cof}(\psi^q - 1) \xrightarrow{\tilde{\mathfrak{J}}} \mathrm{bgl}_1(\mathbb{S}_{(p)}) \longrightarrow \mathrm{bgl}_1(\hat{\mathbb{S}}_p), \tag{1.11}$$

where

$$\epsilon_p(i) = \begin{cases} (p-1)p^{i-1} & \text{if } p \text{ is odd,} \\ 2^{i-2} & \text{if } p = 2. \end{cases} \tag{1.12}$$

This leads to a sharp answer to the problem of higher associativity of Moore spectra, at least at odd primes. In Section 7, we prove

THEOREM 1.13. *When p is an odd prime, $\mathrm{M}_p(i)$ admits a Thomified $\mathbb{A}_n$-structure if and only if $n < p^i$. For $i > 1$, $\mathrm{M}_2(i+1)$ admits a Thomified $\mathbb{A}_{2^i-1}$-structure that does not extend to a Thomified $\mathbb{A}_{2^{i+1}}$-structure.*

Remark 1.14. At an odd prime, the obstruction to extending the Thomified $\mathbb{A}_{p^i-1}$-structure on $\mathrm{M}_p(i)$ to a Thomified $\mathbb{A}_{p^i}$-structure is an element in $\pi_{2p^i-3}(\mathrm{M}_p(i))$ represented by a generator of the same degree in the image of J.

Convention 1. Throughout the paper, p-completion, p-localization and $\mathbb{Q}$-localization of spectra will refer to the Bousfield localization [Bou79] at $\mathrm{M}_p(1)$, $\mathbb{S}_{(p)}$ and $\mathrm{H}\mathbb{Q}$ respectively. In the context of spaces, we will prefer to work with Bousfield-Kan localization [BK72] since the constructions needed in this paper require explicit point-set/simplicial models. These constructions make use of the natural map (3.11) of simplicial sets whose existence is guaranteed for Bousfield-Kan localization (see [Goe98]).

Convention 2. Since we have made essential use of étale homotopy theory in Section 3, that section, and the first part of Section 4, is developed in the language of simplicial sets and simplicial schemes. However, our applications are most naturally described in the category of topological spaces. Therefore, after Section 4.3 we switch from simplicial sets to topological spaces.

A sketch-proof of Theorem 1.5. Our first and the most important step (also see (1.17)) is to construct a family of ψ^q-equivariant p-local spherical fibrations

$$\{\pi_i : \mathrm{SBGl}_i \longrightarrow \mathrm{BGl}_i : i \in \mathbb{N}\} \tag{1.15}$$

such that π_i is equivalent to the p-localization of the fibration $\mathrm{B}(*, \mathrm{Gl}_i(\mathbb{C}), \mathrm{S}^{2i}) \to \mathrm{BGl}_i(\mathbb{C})$. Therefore, the fiber of π_i is equivalent to $\mathrm{S}^{2i}_{(p)}$.

Using the Moore loop space of BGl_i as a model for $\mathrm{Gl}_i(\mathbb{C})_{(p)}$, we construct a permutative category $\mathbf{Gl}_{\mathbb{C},p}$ and a monoidal functor

$$\Psi^q : \mathbf{Gl}_{\mathbb{C},p} \longrightarrow \mathbf{Gl}_{\mathbb{C},p}$$

with the property that on applying the K-theory functor, we get
(i) $\mathrm{K}(\mathbf{Gl}_{\mathbb{C},p}) \simeq \underline{\mathsf{ku}}_{(p)}$, and
(ii) $\mathrm{K}(\Psi^q) \simeq \psi^q$, the q-th Adams operation.

There also exists a permutative category $\mathbf{Gl}_{\mathrm{S}^2_{(p)}}$ such that $\mathrm{K}(\mathbf{Gl}_{\mathrm{S}^2_{(p)}}) \simeq \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})$. The family of fibrations $\{\pi_i : i \geq 1\}$ produces a functor $\mathrm{J} : \mathbf{Gl}_{\mathbb{C},p} \to \mathbf{Gl}_{\mathrm{S}^2_{(p)}}$, but unfortunately, it is not guaranteed to be a monoidal functor (see (5.5)). However, the family of fibrations $\{\pi_i : i \geq 1\}$ constructs for us a multifunctor

$$\widehat{\mathrm{J}} : \upsilon\mathbf{Gl}_{\mathbb{C},p} \longrightarrow \upsilon\mathbf{Gl}_{\mathrm{S}^2_{(p)}},$$

where υ denotes the forgetful functor from the category of permutative categories to the category of multicategories (see (2.3)). The K-theory functor of [EM09] (denoted by K^{EM}), which constructs spectra starting from multicategories, then produces the stable J-homomorphism $\mathfrak{J}$ (as in Theorem 1.5).

The ψ^q-equivariance of π_i leads to a weak-equivalence

$$\hat{\psi}^q_i : \mathrm{S}^{2i}_{(p)} \simeq \mathrm{Fib}(\pi_i) \longrightarrow \mathrm{Fib}(\pi_i) \simeq \mathrm{S}^{2i}_{(p)},$$

which is a degree q^i map (Corollary 4.39). The family of maps $\{\hat{\psi}^q_i : i \in \mathbb{N}\}$ can be assembled to form a multinatural transformation

$$\eta : \widehat{\mathrm{J}} \simeq \widehat{\mathrm{J}} \circ \upsilon\Psi^q. \tag{1.16}$$

Thus, we get an explicit null-homotopy of Theorem 1.5 by applying the functor K^{EM} (see Theorem 2.13) or the functor $\mathrm{K} \circ \phi$ (see Theorem 2.12 and Remark 5.7).

In Appendix A, we provide another proof of Theorem 1.5, using a classification theory of X-fibrations in $\mathcal{F}$-$\mathcal{T}$op [Fri80, Th. 6.1].

Remark 1.17. The family of fibrations (1.15) that we construct in this paper is the p-local analog of a p-completed family considered in [Fri80]. This construction allows us to avoid various technical issues arising from the fact that p-completions are not closed under smash product.

Organization of the paper. In Section 2, we review the construction of the K-theory functor. In Section 3, we summarize some of the fundamental results in étale homotopy theory that we use in this paper.

In Section 4 we construct a p-local spherical fibration which is the key to the proof of Theorem 1.5 — *the unreduced stable Adams conjecture.* In Section 5, we prove Theorems 1.5 and 1.8.

In Section 6, we prove Theorem 1.7, a result which is crucial, not only to the study of $\mathbb{A}_n$-structures on $\mathrm{M}_p(i)$, but also to the comparison of our solution to that in [Fri80]. In Section 7, we study Thomified $\mathbb{A}_n$-structures on $\mathrm{M}_p(i)$ and prove Theorem 1.13.

In Appendix A, we discuss [Fri80] and the errors therein. Using [Fri80, Th. 6.1], we provide another solution to the stable Adams conjecture.

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We would also like to acknowledge our debt to some of the beautiful papers [Fri82], [FM84], [Fri80] of Friedlander.

Last, but not least, we would like to thank the referee for bringing the error in [Fri80] to our notice through an elegant argument. In fact, Theorem 6.5, Lemma 6.8 and Corollary 6.9 is due to him/her (see Remarks A.2 and A.3). We are also grateful to the referee for several other helpful comments and remarks which led to the improvement of this paper.

2. Permutative categories, multicategories and K-theory

In this section, we review the construction of the K-theory functor, starting with the work of Segal [Seg74] and ending with the work of Elmendorf and Mandell [EM06], [EM09].

Let $\mathcal{F}$ denote the category of finite pointed sets. For any category $\mathcal{C}$, let $\mathcal{F}$-$\mathcal{C}$ be the category of functors from $\mathcal{F}$ to $\mathcal{C}$. In [Seg74], Segal constructed a functor

$$\Phi : \mathcal{F}\text{-}\mathcal{T}\mathrm{op} \longrightarrow \mathrm{Ho}(\mathcal{S}\mathrm{p}),$$

where $\mathrm{Ho}(-)$ stands for the associated homotopy category, and he showed that many interesting spectra (such as ku , ko, $\mathbb{S}$ among others) can be constructed using the functor Φ.

A few years later, J.P. May [May78b] showed that one can construct spectra starting from permutative categories (see definition below). More precisely, J.P. May constructed a sequence of functors

$$\mathcal{P}\mathrm{C} \xrightarrow{\mu} \mathcal{F}\text{-}\mathcal{P}\mathrm{C} \xrightarrow{\mathrm{B}} \mathcal{F}\text{-}\mathcal{T}\mathrm{op}, \tag{2.1}$$

where $\mathcal{P}\mathrm{C}$ is the category of small permutative categories and B is the usual bar construction of categories.

In [EM06], authors refined the functor Φ to obtain a K-theory functor

$$\mathrm{K}^{\mathrm{seg}} : \mathcal{P}\mathrm{C} \longrightarrow \mathcal{S}\mathrm{p},$$

where $\mathcal{S}\mathrm{p}$ is the modern pointset category of symmetric spectra. The work in [EM06], [EM09] resulted in a new K-theory functor

$$\mathrm{K} : \mathcal{P}\mathrm{C} \xrightarrow{\upsilon} \mathcal{M}\mathrm{ult} \xrightarrow{\mathrm{K}^{\mathrm{EM}}} \mathcal{S}\mathrm{p}$$

that factors through the category of small multicategories. They showed

Theorem 2.2 (Elmendorf-Mandell). *For any small permutative category $\mathbf{P}$, $\mathrm{K}^{\mathrm{seg}}(\mathbf{P})$ and $\mathrm{K}(\mathbf{P})$ are weakly equivalent.*

A permutative category $\mathbf{P}$ is a symmetric monoidal category in which associativity (including unitality) holds strictly. A multicategory $\mathbf{M}$ consists of a set of objects, the data of n-morphisms $\mathbf{M}_n(a_1, \ldots, a_n; b)$ that admits an action of Σ_n, a multiproduct structure

$$\begin{array}{c}\mathbf{M}_n(b_1, \ldots, b_n; c) \times \mathbf{M}_{k_1}(a_{11}, \ldots, a_{1k_1}; b_1) \times \cdots \times \mathbf{M}_{k_1}(a_{11}, \ldots, a_{1k_1}; b_1) \\ \downarrow \Gamma \\ \mathbf{M}_{k_1+\cdots+k_n}(a_{11}, \ldots, a_{nk_n}; c)\end{array}$$

and a unit $\mathbf{0_M}$, which satisfy the conditions listed in [EM06, Def. 2.1]. An n-morphism in $\mathbf{M}_n(a_1, \ldots, a_n; b)$ should be interpreted as a map whose source is the n-tuple of objects $(a_1, \ldots, a_n)$ and target is the object b. There is a forgetful functor

$$\upsilon : \mathcal{P}\mathrm{C} \longrightarrow \mathcal{M}\mathrm{ult} \tag{2.3}$$

that assigns to every permutative category $\mathbf{P}$ its underlying multicategory $\upsilon\mathbf{P}$ where

$$\upsilon\mathbf{P}_n(a_1, \ldots, a_n; b) := \mathbf{P}(a_1 \oplus \cdots \oplus a_n; b).$$

Its left adjoint, i.e., the free functor

$$\phi : \mathcal{M}\mathrm{ult} \longrightarrow \mathcal{P}\mathrm{C},$$

constructs a permutative category $\phi\mathbf{M}$ from the multicategory $\mathbf{M}$. The category $\phi\mathbf{M}$ has as its objects the free monoid on objects of $\mathbf{M}$. Explicit description of morphism sets can be found in the proof of [EM09, Th. 4.2]. The ϕ-υ adjunction is comonadic [EM09, Th. 4.3].

Example 2.4. Let $\mathbf{Gl}_{\mathbb{C}}$ be the permutative category whose objects are $\{\mathbb{C}^n : n \in \mathbb{N}\} \cong \mathbb{N}$ and morphisms are

$$\mathbf{Gl}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) = \begin{cases} \mathrm{Gl}_n(\mathbb{C}) & \text{if } m = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

The monoidal product is the "block-diagonal sum." Upon applying the K-theory functor we obtain the spectrum ku. By changing the coefficients to real numbers, one can construct $\mathbf{Gl}_{\mathbb{R}} \in \mathcal{P}\mathrm{C}$ whose K-theory is the spectrum ko.

In the following examples, we let $\mathrm{G}(\mathrm{X}, \mathrm{X}')$ denote the space of weak-equivalences between X and X′ with the assumption that $\mathrm{G}(\mathrm{X}, \mathrm{X}') = \emptyset$ if $\mathrm{X} \not\simeq \mathrm{X}'$.

Example 2.5. Let X be a pointed topological space. Let $\mathbf{Gl}_{\mathrm{X}}$ denote the permutative category whose objects are pairs

$$\{(n, \mathrm{Y}) : \ n \in \mathbb{N} \text{ and Y is weakly equivalent to } \mathrm{X}^{\wedge n} \ \}$$

and morphisms are

$$\mathbf{Gl}_{\mathrm{X}}((n, \mathrm{Y}), (m, \mathrm{Y}')) = \begin{cases} \mathrm{G}(\mathrm{Y}, \mathrm{Y}') & \text{if } m = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

The monoidal structure is induced by the smash product in $\mathcal{T}\mathrm{op}_*$.

Remark 2.6. Let $\mathbf{N}$ denote the permutative category with objects $(\mathbb{N}, +)$ whose morphism set consists of identity maps only. Note that $\mathrm{K}(\mathbf{N}) \simeq \mathrm{H}\mathbb{Z}$. There is a monoidal functor

$$\mathbf{Gl}_{\mathrm{X}} \longrightarrow \mathbf{N}$$

that is identity on objects and trivial projection on morphisms. This map induces an isomorphism on the zeroth homotopy groups and thus $\pi_0(\mathrm{K}(\mathbf{Gl}_{\mathrm{X}})) \cong \mathbb{Z}$.

Example 2.7. Note that $\mathrm{K}(\mathbf{Gl}_{\mathrm{S}^1})$ is the Picard spectrum $\mathsf{pic}(\mathbb{S})$. The spectrum $\mathrm{K}(\mathbf{Gl}_{\mathrm{S}^2})$ is the even Picard spectrum $\mathsf{pic}^{\mathrm{ev}}(\mathbb{S})$. The K-theory functor applied to the obvious monoidal functor

$$\mathbf{Gl}_{\mathrm{S}^2} \hookrightarrow \mathbf{Gl}_{\mathrm{S}^1},$$

which sends $(\mathrm{S}^2)^{\wedge n}$ to $(\mathrm{S}^1)^{\wedge 2n}$, results in a map

$$\mathsf{pic}^{\mathrm{ev}}(\mathbb{S}) \longrightarrow \mathsf{pic}(\mathbb{S}),$$

which is multiplication by 2 on the zeroth homotopy. The 0-connected cover of both $\mathsf{pic}^{\mathrm{ev}}(\mathbb{S})$ and $\mathsf{pic}(\mathbb{S})$ are equivalent to the spectrum $\mathrm{bgl}_1(\mathbb{S})$.

Example 2.8. Similar to the above example, we also have $\mathrm{K}(\mathbf{Gl}_{\mathrm{S}^1_{(p)}}) \simeq \mathsf{pic}(\mathbb{S}_{(p)})$ and $\mathrm{K}(\mathbf{Gl}_{\mathrm{S}^2_{(p)}}) \simeq \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})$.

Example 2.9. We denote by $\mathrm{SG}(\mathrm{X}^{\wedge n}, \mathrm{X}^{\wedge n})$ the identity component of $\mathrm{G}(\mathrm{X}^{\wedge n}, \mathrm{X}^{\wedge n})$.

Let $\mathbf{Sl}_{\mathrm{X}}$ denote the category with objects $\{\mathrm{X}^{\wedge n} : n \in \mathbb{N}\} \cong \mathbb{N}$ and morphisms

$$\mathbf{Sl}_{\mathrm{X}}(\mathrm{X}^{\wedge n}, \mathrm{X}^{\wedge m}) = \begin{cases} \mathrm{SG}(\mathrm{X}^{\wedge n}, \mathrm{X}^{\wedge n}) & \text{if } m = n, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathbf{Sl}_{\mathrm{X}}$ can be given a structure of a permutative category using the smash product provided all coordinate-wise permutations $\lambda_\sigma : \mathrm{X}^{\wedge n} \to \mathrm{X}^{\wedge n}$ (where $\sigma \in \Sigma_n$) is an element of $\mathrm{SG}(\mathrm{X}^{\wedge n}, \mathrm{X}^{\wedge n})$.

Remark 2.10. When $\mathrm{X} = \mathrm{S}^1$, $\mathbf{Sl}_{\mathrm{X}}$ is not a permutative category. This is because the non-trivial permutation of the two-fold smash product $\mathrm{S}^1 \wedge \mathrm{S}^1$ is of degree -1. However, $\mathbf{Sl}_{\mathrm{S}^2}$ is a permutative category.

Remark 2.11. Whenever $\mathbf{Sl}_{\mathrm{X}}$ is a permutative category, there is an evident monoidal functor from $\mathbf{Sl}_{\mathrm{X}} \to \mathbf{Gl}_{\mathrm{X}}$. In fact, $\mathrm{K}(\mathbf{Sl}_{\mathrm{X}})$ is equivalent to the fiber of a map $\mathrm{K}(\mathbf{Gl}_{\mathrm{X}}) \to \Sigma\mathrm{H}(\pi_1\mathrm{K}(\mathbf{Gl}_{\mathrm{X}}))$ that induces an isomorphism on fundamental groups.

The category $\mathcal{P}\mathrm{C}$ of small permutative categories can be given a multicategory structure that is enriched over $\mathcal{C}\mathrm{at}$ (the category of small categories). Likewise $\mathcal{M}\mathrm{ult}$, the category of small multicategories, is a symmetric monodial category. Let K^{EM} be a lax monodial functor, and let K be a multifunctor. Combining the main results of [EM06] and [EM09] we get, among other things, that both K^{EM} and K admit enrichment over simplicial sets. In particular, we have

THEOREM 2.12 (Elmendorf-Mandell). *Let* $\mathrm{F}, \mathrm{G} : \mathbf{P} \to \mathbf{Q}$ *be two lax monoidal functors that are strict on units. Then a lax monoidal natural transformation* $\eta : \mathrm{F} \to \mathrm{G}$ *produces a homotopy*

$$\mathrm{K}(\eta) : \mathrm{K}(\mathrm{F}) \simeq \mathrm{K}(\mathrm{G})$$

on applying the K*-theory functor.*

THEOREM 2.13 (Elmendorf-Mandell). *Let* $\mathrm{F}, \mathrm{G} : \mathbf{M} \to \mathbf{N}$ *be two multifunctors. Then a multinatural transformation* $\eta : \mathrm{F} \to \mathrm{G}$ *produces a homotopy*

$$\mathrm{K}^{\mathrm{EM}}(\eta) : \mathrm{K}^{\mathrm{EM}}(\mathrm{F}) \simeq \mathrm{K}^{\mathrm{EM}}(\mathrm{G})$$

on applying the K*-theory functor.*

Since we use Theorem 2.13 to resolve the stable Adams conjecture, we quickly review the definition of a multifunctor and a multinatural transformation. A multifunctor between two multicategories $\mathrm{F} : \mathbf{M} \to \mathbf{N}$ consists of

- a function from the objects of $\mathbf{M}$ to the objects of $\mathbf{N}$ such that $\mathrm{F}(\mathbf{0}_{\mathbf{M}}) = \mathbf{0}_{\mathbf{N}}$, and
- for all objects b and a tuple $(a_1, \ldots, a_n)$, a function

  $$\mathbf{M}_n(a_1, \ldots, a_n; b) \longrightarrow \mathbf{M}'_n(\mathrm{F}(a_1), \ldots, \mathrm{F}(a_n); \mathrm{F}(b))$$

  that preserves Σ_n-action, units and multiproduct structures.

Given two multifunctors $\mathrm{F}, \mathrm{G} : \mathbf{M} \to \mathbf{N}$, a multinatural transformation $\eta : \mathrm{F} \to \mathrm{G}$ consists of a collection of maps

$$\eta_a : \mathrm{F}(a) \to \mathrm{G}(a)$$

for every object $a \in \mathbf{M}$ so that the diagram

$$\begin{tikzcd}
\mathbf{M}(a_1, \ldots, a_n; b) \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{G}"'] & \mathbf{N}(\mathrm{F}(a_1), \ldots, \mathrm{F}(a_n); \mathrm{F}(b)) \arrow[d, "(\eta_b)_*"] \\
\mathbf{N}(\mathrm{G}(a_1), \ldots, \mathrm{G}(a_n); \mathrm{G}(b)) \arrow[r, "(\eta_{a_1}, \ldots, \eta_{a_n})^*"'] & \mathbf{N}(\mathrm{F}(a_1), \ldots, \mathrm{F}(a_n); \mathrm{G}(b))
\end{tikzcd}$$

commutes for all tuples $(a_1, \ldots, a_n)$ and all objects b of $\mathbf{M}$.

The functor $\mathrm{K}^{\mathrm{seg}}$ has the advantage that it constructs an Ω-spectrum, i.e., for a permutative category $\mathbf{P}$ and all $n \in \mathbb{N}$, we have an equivalence

$$\mathrm{B}(\mathrm{K}^{\mathrm{seg}}(\mathbf{P}))_{[n]} \xrightarrow{\simeq} \mathrm{K}^{\mathrm{seg}}(\mathbf{P})_{[n+1]},$$

where $(\mathrm{K}^{\mathrm{seg}}(\mathbf{P}))_{[n]}$ is the n-th space of the spectrum $\mathrm{K}^{\mathrm{seg}}(\mathbf{P})$. This particular property of the functor $\mathrm{K}^{\mathrm{seg}}$ allows us to track the homotopy type of the resultant spectrum. On the other hand, K^{EM} has the advantage that it constructs spectra out of multicategories (which are arguably less restrictive than permutative categories), but it does not necessarily produce an Ω-spectrum, making it difficult to track the homotopy type of the constructed spectrum. However, if a multicategory is the underlying multicategory of a permutative category, then from the ϕ-υ adjunction we get

$$\mathrm{K}(\mathbf{P}) = \mathrm{K}^{\mathrm{EM}}(\upsilon\mathbf{P}) \simeq \mathrm{K}(\phi\upsilon\mathbf{P}). \tag{2.14}$$

One should be careful about the fact that $\mathrm{K}^{\mathrm{EM}}(\mathbf{M})$ may not be equivalent to $\mathrm{K}(\phi\mathbf{M})$ without the hypothesis $\mathbf{M} \cong \upsilon\mathbf{P}$ for some $\mathbf{P} \in \mathcal{P}\mathrm{C}$.

3. A brief review of étale homotopy theory

Étale homotopy theory began with the work of Artin and Mazur [AM69], where they constructed a functor

$$\acute{\mathrm{E}}\mathrm{t}^{\mathrm{AM}} : \mathsf{Sch} \longrightarrow \text{Pro-Ho}(\mathsf{s}\mathcal{S}\mathsf{et})$$

from the category of schemes to the procategory of the homotopy category of simplicial sets, which assigns a scheme to its étale homotopy type. Friedlander [Fri82] developed the notion of rigid étale cover for simplicial schemes to produce a refinement

$$\acute{\mathrm{E}}\mathrm{t} : \mathsf{sSch} \longrightarrow \text{Pro-}\mathsf{s}\mathcal{S}\mathsf{et}$$

such that $\acute{\mathrm{E}}\mathrm{t}^{\mathrm{AM}}$ is the composite

$$\acute{\mathrm{E}}\mathrm{t}^{\mathrm{AM}} : \mathsf{Sch} \hookrightarrow \mathsf{sSch} \xrightarrow{\acute{\mathrm{E}}\mathrm{t}} \text{Pro-}\mathsf{s}\mathcal{S}\mathsf{et} \twoheadrightarrow \text{Pro-Ho}(\mathsf{s}\mathcal{S}\mathsf{et}).$$

Remark 3.1. Note that any simplicial set can be viewed as a constant pro-simplicial set by virtue of a fully faithful functor $c : \mathsf{s}\mathcal{S}\mathsf{et} \longrightarrow \text{Pro-}\mathsf{s}\mathcal{S}\mathsf{et}$.

The étale cohomology of a scheme V with constant coefficients coincides with the singular cohomology of $\acute{\mathrm{E}}\mathrm{t}(\mathrm{V})$ (which is computed as the direct limit of singular cohomology groups induced by an inverse system representing $\acute{\mathrm{E}}\mathrm{t}(\mathrm{V})$). More precisely,

$$\mathrm{H}^*_{\acute{\mathrm{e}}\mathrm{t}}(\mathrm{V}, \mathrm{C}_{\mathrm{A}}) \cong \mathrm{H}^*(\acute{\mathrm{E}}\mathrm{t}(\mathrm{V}); \mathrm{A})$$

for any finite abelian group A. If the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ of a field $\mathbb{F}$ is finite, then the étale homotopy type of $\mathsf{Spec}(\mathbb{F})$ is the classifying space of its absolute Galois group, i.e.,

$$\acute{\mathrm{E}}\mathrm{t}(\mathsf{Spec}\,\mathbb{F}) \simeq \mathrm{BGal}(\overline{\mathbb{F}}/\mathbb{F})$$

as pro-simplicial sets. In general, $\acute{\mathrm{E}}\mathrm{t}(\mathsf{Spec}\,\overline{\mathbb{F}})$ is contractible with an action of $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$. If $\mathbb{K}$ is a field extension of $\mathbb{F}$, $\mathrm{V}^{\mathbb{F}}_\bullet$ is a simplicial scheme over $\mathbb{F}$ and

$$\mathrm{V}^{\mathbb{K}}_\bullet := \mathrm{V}^{\mathbb{F}}_\bullet \underset{\mathsf{Spec}\,\mathbb{F}}{\times} \mathsf{Spec}\,\mathbb{K},$$

then $\acute{\mathrm{E}}\mathrm{t}(\mathrm{V}^{\mathbb{K}}_\bullet)$ admits an action of $\mathrm{Aut}(\mathbb{K}/\mathbb{F})$.

For a scheme $\mathrm{V}^{\mathbb{F}}$ and a field extension $\mathbb{K}$ of $\mathbb{F}$, we let

$$\mathrm{V}(\mathbb{K}) := \mathrm{Hom}_{\mathsf{Sch}|_{\mathbb{K}}}(\mathsf{Spec}\,\mathbb{K}, \mathrm{V}^{\mathbb{K}})$$

denote the set of $\mathbb{K}$-points. The action of $\mathrm{Gal}(\mathbb{K}/\mathbb{F})$ on $\mathrm{V}(\mathbb{K})$ is by conjugation

$$\sigma_* f(-) := \sigma f(\sigma^{-1}(-)). \tag{3.2}$$

Likewise, for a simplicial scheme $\mathrm{V}^{\mathbb{F}}_\bullet$, we let $\mathrm{V}(\mathbb{K})_\bullet \in \mathsf{s}\mathcal{S}\mathsf{et}$ denote the simplicial set obtained by taking the $\mathbb{K}$-points levelwise.

By applying the étale homotopy functor Ét, we get a natural $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$-invariant map of pro-simplicial sets

$$\nu : \mathrm{V}(\overline{\mathbb{F}})_\bullet \longrightarrow \mathrm{Hom}_{\mathrm{Pro}\text{-}\mathsf{s}\mathcal{S}\mathsf{et}}(\acute{\mathrm{E}}\mathrm{t}(\mathsf{Spec}\,\overline{\mathbb{F}}), \acute{\mathrm{E}}\mathrm{t}(\mathrm{V}_\bullet^{\overline{\mathbb{F}}})). \tag{3.3}$$

Notation 3.4. For any field $\mathbb{K}$, we let

$$\acute{\mathrm{E}}\mathrm{t}_{\mathbb{K}}(-) := \mathrm{Hom}_{\mathrm{Pro}\text{-}\mathsf{s}\mathcal{S}\mathsf{et}}(\acute{\mathrm{E}}\mathrm{t}(\mathsf{Spec}\,\mathbb{K}), \acute{\mathrm{E}}\mathrm{t}(-)).$$

Remark 3.5. When $\mathbb{K}$ is algebraically closed, the functor $\acute{\mathrm{E}}\mathrm{t}_{\mathbb{K}}$ can replace Ét because $\acute{\mathrm{E}}\mathrm{t}(\mathsf{Spec}\,\mathbb{K})$ is contractible and there is a natural equivalence

$$\acute{\mathrm{E}}\mathrm{t}(\mathrm{V}_\bullet^{\mathbb{K}}) \xrightarrow{\simeq} \acute{\mathrm{E}}\mathrm{t}_{\mathbb{K}}(\mathrm{V}_\bullet^{\mathbb{K}}). \tag{3.6}$$

Further, when $\mathbb{K} = \overline{\mathbb{F}}$ and $\mathrm{V}_\bullet^{\mathbb{F}}$ is defined over $\mathbb{F}$, then the natural equivalence (3.6) is also a $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$-equivariant map.

Given an algebraic group $\mathrm{G}^{\mathbb{F}}$ over $\mathbb{F}$, its bar complex $\mathrm{BG}^{\mathbb{F}}$ is a simplicial scheme. The General Isomorphism Conjecture (GIC) of Friedlander and Mislin [FM84] asserts that (3.3) will induce an isomorphism

$$\mathrm{H}^*_{\mathrm{et}}(\mathrm{BG}^{\mathbb{F}}, \mathrm{C}_{\mathbb{Z}/n\mathbb{Z}}) \cong \mathrm{H}^*(\mathrm{BG}(\mathbb{F}); \mathbb{Z}/n\mathbb{Z}) \tag{3.7}$$

for any connected linear algebraic group scheme when $\mathbb{F}$ is algebraically closed and n is invertible in $\mathbb{F}$. They prove GIC when $\mathbb{F} = \overline{\mathbb{F}}_q$, the algebraic closure of the field of order q (see [FM84, Prop. 2.3, Proposition 2.4]).

Theorem 3.8 (Friedlander-Mislin). *A connected linear algebraic group* $\mathrm{G}^{\overline{\mathbb{F}}_q}$ *satisfies* GIC.

When $\mathbb{F} = \mathbb{C}$ with the usual topology, then the isomorphism of (3.7) is known to be true for a larger class of simplicial schemes. It should be noted that if $\mathbb{C}$ is given the usual topology, then $\mathrm{V}(\mathbb{C})_\bullet$ is a simplicial space and not a simplicial set. However, we obtain a simplicial set by considering the singular simplex of the geometric realization of $\mathrm{V}(\mathbb{C})_\bullet$

$$\mathrm{V}(\mathbb{C})^{\mathrm{top}} := \mathrm{Sing}|\mathrm{V}(\mathbb{C})_\bullet|.$$

The generalized Riemann existence theorem [AM69, Th. 12.9] as exposed by Friedlander in the context of simplicial schemes [Fri82, Th. 8.4], says that

Theorem 3.9 (Generalized Riemann Existence Theorem). *Let* $(\mathrm{V}_\bullet^{\mathbb{C}}, \mathrm{v})$ *be a pointed, connected simplicial scheme over* $\mathbb{C}$ *of finite type. Then there is a weak equivalence*

$$\mathrm{V}(\mathbb{C})^{\mathrm{top}} \simeq \acute{\mathrm{E}}\mathrm{t}_{\mathbb{C}}(\mathrm{V}_\bullet^{\mathbb{C}}, \mathrm{v}) \tag{3.10}$$

in $\mathrm{Pro}\text{-}\mathsf{s}\mathcal{S}\mathsf{et}_*$.

A weak equivalence, as in (3.10), is equivalent to saying $\acute{\mathrm{E}}\mathrm{t}_{\mathbb{C}}(\mathrm{V}_\bullet^{\mathbb{C}}, \mathrm{v})$ is the pro-finite completion of $\mathrm{V}(\mathbb{C})_\bullet^{\mathrm{top}}$. The pro-finite completion functor

$$\widehat{(\cdot)} : \mathsf{s}\mathcal{S}\mathsf{et} \longrightarrow \text{Pro-}\mathsf{s}\mathcal{S}\mathsf{et}$$

is the left adjoint of the limit functor

$$\| \cdot \| : \text{Pro-}\mathsf{s}\mathcal{S}\mathsf{et} \longrightarrow \mathsf{s}\mathcal{S}\mathsf{et}$$

that sends a pro-simplicial set to its limiting simplicial set. Morel [Mor96] constructed the p-pro-finite completion $(\cdot)_p$ (a functorial p-completion within Pro-$\mathsf{s}\mathcal{S}\mathsf{et}$) such that the composite functor

$$\widehat{(\cdot)}_p : \mathsf{s}\mathcal{S}\mathsf{et} \xrightarrow{\widehat{(\cdot)}} \text{Pro-}\mathsf{s}\mathcal{S}\mathsf{et} \xrightarrow{(\cdot)_p} \text{Pro-}\mathsf{s}\mathcal{S}\mathsf{et}$$

can be compared with the p-completion functor of Bousfield and Kan [BK72]. For any $\mathrm{Y} \in \mathsf{s}\mathcal{S}\mathsf{et}$, we get a natural map

$$\mu : \mathrm{Y}_p^{\hat{}} \longrightarrow \|\widehat{\mathrm{Y}}_p\|, \tag{3.11}$$

which is an equivalence if $\mathrm{H}_k(\mathrm{Y}; \mathbb{Z}/p\mathbb{Z})$ is finite for all k (see [Goe98, Cor. 3.16]). Thus, we may conclude

THEOREM 3.12. *Let* $(\mathrm{V}_\bullet^{\mathbb{C}}, \mathrm{v})$ *be a pointed, connected simplicial scheme over* $\mathbb{C}$ *of finite type. Further, if* $\mathrm{H}^*(\mathrm{V}(\mathbb{C})^{\mathrm{top}}, \mathbb{Z}/p\mathbb{Z})$ *is of finite type, then*

(1) $(\mathrm{V}(\mathbb{C})^{\mathrm{top}})_p^{\hat{}} \simeq \|\acute{\mathrm{E}}\mathrm{t}_{\mathbb{C}}(\mathrm{V}_\bullet^{\mathbb{C}})_p\|$, *and*
(2) $\mathrm{H}^*(\mathrm{V}(\mathbb{C})^{\mathrm{top}}; \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{H}_{\mathrm{et}}^*(\mathrm{V}_\bullet^{\mathbb{C}}, \mathrm{C}_{\mathbb{Z}/p\mathbb{Z}})$.

Let $\mathbb{W}\overline{\mathbb{F}}_q$ denote the ring of Witt vectors over $\overline{\mathbb{F}}_q$. Then we have a zigzag of ring maps

$$\overline{\mathbb{F}}_q \overset{\pi}{\twoheadleftarrow} \mathbb{W}\overline{\mathbb{F}}_q \overset{\iota}{\hookrightarrow} \mathbb{C}, \tag{3.13}$$

where π is the quotient map that annihilates the unique maximal ideal of $\mathbb{W}\overline{\mathbb{F}}_q$ and ι is a choice of Brauer embedding. Friedlander [Fri82, §8] expressed the comparison results of Artin and Mazur [AM69, §12] in terms of simplicial schemes to obtain the following result: If $\mathrm{V}_\bullet^{\mathbb{W}\overline{\mathbb{F}}_q}$ is a smooth proper connected simplicial scheme over $\mathsf{Spec}\,\mathbb{W}\overline{\mathbb{F}}_q$, then the maps in the zigzag induced by (3.13) and (3.6),

$$\acute{\mathrm{E}}\mathrm{t}_{\overline{\mathbb{F}}_q}(\mathrm{V}_\bullet^{\overline{\mathbb{F}}_q})_p \xleftarrow{\simeq} \acute{\mathrm{E}}\mathrm{t}(\mathrm{V}_\bullet^{\overline{\mathbb{F}}_q})_p \xrightarrow{\simeq} \acute{\mathrm{E}}\mathrm{t}(\mathrm{V}_\bullet^{\mathbb{W}\overline{\mathbb{F}}_q})_p \xleftarrow{\simeq} \acute{\mathrm{E}}\mathrm{t}(\mathrm{V}_\bullet^{\mathbb{C}})_p \xrightarrow{\simeq} \acute{\mathrm{E}}\mathrm{t}_{\mathbb{C}}(\mathrm{V}_\bullet^{\mathbb{C}})_p,$$

are weak equivalences in Pro-$\mathsf{s}\mathcal{S}\mathsf{et}$. Combining this with Theorems 3.8 and 3.12, Friedlander and Mislin proved

THEOREM 3.14 ([FM84, Th. 1.4]). *When* G *is an integral group scheme such that* $\mathrm{G}(\mathbb{C})^{\mathrm{top}}$ *is a reductive complex Lie group, then*

$$\mathrm{BG}(\overline{\mathbb{F}}_q)_p^{\hat{}} \simeq (\mathrm{BG}(\mathbb{C})^{\mathrm{top}})_p^{\hat{}}. \tag{3.15}$$

Example 3.16. The general linear group $\mathrm{Gl}_i(\mathbb{C})$ and its maximal torus

$$\mathbb{T}^i(\mathbb{C}) := \mathrm{Gl}_1(\mathbb{C})^{\times i} \subset \mathrm{Gl}_i(\mathbb{C})$$

are examples of complex reductive Lie groups and defined over $\mathsf{Spec}(\mathbb{Z})$. Thus (3.15) holds when G is chosen to be $\mathrm{Gl}_i^{\mathbb{Z}}$ and $\mathbb{T}^i_{\mathbb{Z}}$.

Remark 3.17. The Teichmüller lift results in a group homomorphism

$$\mathsf{e} : \overline{\mathbb{F}}_q^{\times} \longrightarrow \mathbb{W}\overline{\mathbb{F}}_q^{\times} \xrightarrow{\iota^{\times}} \mathbb{C}^{\times}$$

sending elements of $\overline{\mathbb{F}}_q^{\times}$ within roots of unity. Therefore, we have a map of groups

$$\mathbb{T}^i(\mathsf{e}) : \mathbb{T}^i(\overline{\mathbb{F}}_q) \longrightarrow \mathbb{T}^i(\mathbb{C}) \longrightarrow \mathbb{T}^i(\mathbb{C})^{\mathrm{top}}.$$

It is known that the induced map on the p-completion of the classifying space $\mathrm{B}(\mathbb{T}^i(\mathsf{e}))_p^{\hat{}}$ is a weak equivalence. Thus when $\mathrm{G} = \mathbb{T}^i$, the weak equivalence established by (3.15) can also be obtained by an explicit map, namely, $\mathrm{B}(\mathbb{T}^i(\mathsf{e}))_p^{\hat{}}$. A proof can be found in [FM84].

PROPOSITION 3.18. *Let* $\mathrm{N}_i^{\mathbb{Z}} = \Sigma_i \ltimes \mathbb{T}^i_{\mathbb{Z}}$ *denote the discrete extension of the torus* $\mathbb{T}^i_{\mathbb{Z}}$. *Then*

$$\mathrm{BN}_i(\mathsf{e}) : \mathrm{BN}_i(\overline{\mathbb{F}}_q) \longrightarrow \mathrm{BN}_i(\mathbb{C})^{\mathrm{top}}$$

is an equivalence after p*-completion.*

Proof. Note that the spaces $\mathrm{BN}_i(\overline{\mathbb{F}}_q)$ and $\mathrm{BN}_i(\mathbb{C})^{\mathrm{top}}$ map to $\mathrm{B}\Sigma_i$,

$$\begin{tikzcd} \mathrm{BN}_i(\overline{\mathbb{F}}_q) \arrow[rr, "\mathrm{BN}_i(\mathsf{e})"] \arrow[dr] & & \mathrm{BN}_i(\mathbb{C})^{\mathrm{top}} \arrow[dl] \\ & \mathrm{B}\Sigma_i, & \end{tikzcd}$$

with fibers $\mathrm{B}\mathbb{T}^i(\overline{\mathbb{F}}_q)$ and $\mathrm{B}\mathbb{T}^i(\mathbb{C})^{\mathrm{top}}$ respectively. Thus, using Remark 3.17 and a Serre spectral sequence argument, we conclude that the map $\mathrm{BN}_i(\mathsf{e})$ induces an isomorphism in $\mathrm{H}\mathbb{F}_p$-coefficients, and hence, by [BK72, I.5.5] we get the result. □

4. A family of ψ^q-equivariant spherical fibrations

The main goal of this section is to construct the family (1.15) of ψ^q-equivariant p-local spherical fibrations and explore some of its properties. Our construction of the fibration π_i can be broken down into three steps:

(i) Construct a ψ^q-equivariant p-completed spherical fibration
$$\hat{\pi}_i : \mathrm{SBGl}_i\hat{} \longrightarrow \mathrm{BGl}_i\hat{}$$
with fiber equivalent to the p-completed sphere $\hat{\mathrm{S}}^{2i}_p$.

(ii) Construct ψ^q-equivariant spherical fibration
$$\nu_i : \mathrm{SBN}_i \longrightarrow \mathrm{BN}_i$$
so that we have a ψ^q-equivariant map
$$(\nu_i)_{\mathbb{Q}} \longrightarrow (\hat{\pi}_i)_{\mathbb{Q}}$$
of fibrations.

(iii) Use ψ^q-equivariant arithmetic fracture squares, in the sense of [BK72, VI.8.1], to construct π_i.

The first two steps make use of étale homotopy theory.

4.1. *Constructing p-complete spherical fibrations.* Let $\star := \mathsf{Spec}\,\mathbf{0}$ denote the empty scheme. For any ring R, let $\mathbb{A}^i_{\mathrm{R}} - 0$ denote the scheme representing the i-plane without the origin. Define the simplicial scheme
$$\mathrm{S}^{2i}_{\mathrm{R}} := \star \cup (\mathbb{A}^i_{\mathrm{R}} - 0) \times \Delta[1] \cup \star,$$
where we choose and fix one of the two copies of $\star$ as the basepoint (see Remarks 4.2 and 4.3), and let
$$\mathrm{SBGl}^{\mathbb{C}}_{i,\bullet} := \mathrm{B}(\star, \mathrm{Gl}^{\mathbb{C}}_i, \mathrm{S}^{2i}_{\mathbb{C}}).$$
There is a natural map of simplicial schemes
$$\alpha_i : \mathrm{SBGl}^{\mathbb{C}}_{i,\bullet} \longrightarrow \mathrm{BGl}^{\mathbb{C}}_{i,\bullet} \tag{4.1}$$
that admits a section σ_i, i.e., $\alpha_i \circ \sigma_i$ is the identity map on $\mathrm{BGl}^{\mathbb{C}}_{i,\bullet}$.

Remark 4.2. For a group scheme G, and schemes X and Y that are paired with G appropriately, the two-sided bar construction $\mathrm{B}(\mathrm{X}, \mathrm{G}, \mathrm{Y})$ is a simplicial scheme. If X (or Y) is the empty-scheme $\star$, one should interpret $\mathrm{B}(\star, \mathrm{G}, \mathrm{Y})$ (likewise $\mathrm{B}(\mathrm{X}, \mathrm{G}, \star)$) as the simplicial scheme obtained by deleting X (likewise Y) in the resolution of $\mathrm{B}(\mathrm{X}, \mathrm{G}, \mathrm{Y})$ (see [Fri82, Exam. 1.2]). With this convention, $\mathrm{BG}_\bullet$ is indeed $\mathrm{B}(\star, \mathrm{G}, \star)$.

Remark 4.3. The simplicial scheme $\mathrm{B}(\star, \mathrm{Gl}_i^{\mathrm{R}}, \mathrm{S}^{2i}_{\mathrm{R}})$ should be interpreted as the pushout in the diagram

$$\begin{tikzcd} \mathrm{B}(\star, \mathrm{Gl}_i^{\mathrm{R}}, (\mathbb{A}^i_{\mathrm{R}} - 0) \times \partial\Delta[1]) \arrow[r] \arrow[d] & \mathrm{B}(\star, \mathrm{Gl}_i^{\mathrm{R}}, \partial\Delta[1]) \arrow[d, dashed] \\ \mathrm{B}(\star, \mathrm{Gl}_i^{\mathrm{R}}, (\mathbb{A}^i_{\mathrm{R}} - 0) \times \Delta[1]) \arrow[r, dashed] & \mathrm{B}(\star, \mathrm{Gl}_i^{\mathrm{R}}, \mathrm{S}^{2i}_{\mathrm{R}}). \end{tikzcd}$$

A choice of basepoint $\star \to \mathrm{S}^{2i}_{\mathrm{R}}$ is really a choice of section

$$\sigma_i : \mathrm{BGl}^{\mathrm{R}}_{i,\bullet} \longrightarrow \mathrm{SBGl}^{\mathbb{C}}_{i,\bullet}$$

of α_i.

Remark 4.4. There is a natural map

$$\tau^{\mathbb{Z}}_{i,j} : \mathrm{S}^{2i}_{\mathbb{Z}} \times \mathrm{S}^{2i}_{\mathbb{Z}} \longrightarrow \mathrm{S}^{2i+2j}_{\mathbb{Z}}$$

induced by

(1) the usual map of schemes $\tau_{i,j} : (\mathbb{A}^i_{\mathbb{Z}} - 0) \underset{\mathsf{Spec}\,\mathbb{Z}}{\times} (\mathbb{A}^j_{\mathbb{Z}} - 0) \longrightarrow (\mathbb{A}^{i+j}_{\mathbb{Z}} - 0)$, and
(2) a map of simplicial sets $\mathsf{c} : \Delta[1] \times \Delta[1] \longrightarrow \Delta[1]$, which we described below.

Note that the n-th set of $\Delta[1]_n$ is $\hom_{\Delta}(\mathbf{n}, \mathbf{1})$, where

$$\mathbf{n} := \{0 < 1 < \cdots < n\}$$

is the totally ordered set with $n+1$ elements and can be viewed as a category. The map c is induced by the functor

$$\mathbf{1} \times \mathbf{1} \longrightarrow \mathbf{1},$$

which sends $(0,0), (1,0), (0,1)$ to 0 and $(1,1)$ to 1. Therefore, the restriction of $\tau^{\mathbb{Z}}_{i,j}$ to either $\mathrm{S}^{2i}_{\mathbb{Z}}$ or $\mathrm{S}^{2j}_{\mathbb{Z}}$ is a trivial map.

Notation 4.5. Set $\mathrm{SBGl}_i^{\hat{}} := \|\acute{\mathrm{E}}\mathrm{t}_{\mathbb{C}}(\mathrm{SBGl}^{\mathbb{C}}_{i,\bullet})_p\|$, $\mathrm{BGl}_i^{\hat{}} := \|\acute{\mathrm{E}}\mathrm{t}_{\mathbb{C}}(\mathrm{BGl}^{\mathbb{C}}_{i,\bullet})_p\|$ and $\hat{\pi}_i := \|\acute{\mathrm{E}}\mathrm{t}_{\mathbb{C}}(\alpha_i)_p\|$.

LEMMA 4.6. *The fiber of the map*

$$\hat{\pi}_i : \mathrm{SBGl}_i^{\hat{}} \longrightarrow \mathrm{BGl}_i^{\hat{}}$$

is weakly equivalent to $\hat{\mathrm{S}}^{2i}_p$, *the* p*-completion of* $2i$*-sphere.*

Proof. By definition, the map α_i of (4.1) at the level of complex points gives rise to a spherical fibration over $\mathrm{BGl}_i(\mathbb{C})$ with fiber S^{2i}. By [BK72, II.4.8], p-completing the map α_i on the level of complex points also gives rise to a fibration with fiber being $\hat{\mathrm{S}}^{2i}_p$. By Theorem 3.12, we may identify this fibration with the étale homotopy type as stated in the lemma. □

Now we would like to discuss the "ψ^q-equivariance" of the map $\hat{\pi}_i$. Since $\hat{\alpha}_i$ is induced by a map of simplicial schemes over $\mathbb{Z}$, $\hat{\pi}_i$ is equivariant with respect to the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on $\mathrm{SBGl}_i^{\hat{}}$ and $\mathrm{BGl}_i^{\hat{}}$. Any element

$$\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}),$$

whose cyclotomic character is $\frac{1}{q}$, can be regarded as the q-th Adams operation; see [Sul74] as well as (4.7). Therefore, $\hat{\pi}_i$ is a "ψ^q-equivariant map."

Remark 4.7. For the multiplicative group scheme $\mathrm{Gl}_1^{\mathbb{Z}} := \mathsf{Spec}\,\mathbb{Z}[x^{\pm}]$ and any field $\mathbb{F}$, the action of $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{F})$ (as defined in (3.2)) on

$$\mathrm{Gl}_1(\mathbb{K}) \cong \mathbb{K}^\times$$

is given by $\sigma_*(z) = \sigma^{-1}(z)$. If $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ has cyclotomic character $\chi(\sigma) = \frac{1}{q}$, then $\chi(\sigma^{-1}) = q$ and it sends a root of unity ζ to its q-th power ζ^q. Therefore,

$$\sigma_* : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$$

agree with the Frobenius automorphism $\mathrm{Fr}_q : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$ under the map

$$\mathsf{e} : \overline{\mathbb{F}}_q^\times \longrightarrow \mathbb{W}\overline{\mathbb{F}}_q^\times \xrightarrow{\iota^\times} \mathbb{C}^\times.$$

This fact will play a crucial role in the construction of (1.15) as in Section **??**.

4.2. *Constructing p-local spherical fibrations.* The space $\mathrm{BN}_i(\mathbb{C})^{\mathrm{top}}$ is rationally equivalent to $\mathrm{BGl}_i(\mathbb{C})^{\mathrm{top}}$, and the q-th Adams operation

$$\psi_i^q : \mathrm{BN}_i(\mathbb{C})^{\mathrm{top}} \longrightarrow \mathrm{BN}_i(\mathbb{C})^{\mathrm{top}}$$

is induced by the q-th power map on $\mathbb{C}^{\mathrm{top}}$. However, $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ may not act continuously on $\mathrm{BN}_i(\mathbb{C})^{\mathrm{top}}$ (as it acts discontinuously on $\mathbb{C}$ with the usual topology). Thus, $\mathrm{BN}_i(\mathbb{C})^{\mathrm{top}}$ may not necessarily admit a "ψ^q-equivariant map" to $\mathrm{BGl}_i^{\hat{}}$. Therefore, we define

Definition 4.8. Let BN_i be the pullback in the diagram

$$\begin{tikzcd} \mathrm{BN}_i \arrow[r, dashed, "\gamma_i"] \arrow[d, dashed] & \mathrm{BN}_i(\overline{\mathbb{F}}_q)_p^{\hat{}} \arrow[d, "\mathrm{BN}_i(\mathsf{e})_p^{\hat{}}", "\simeq"'] \\ \mathcal{R}\mathrm{BN}_i(\mathbb{C})^{\mathrm{top}} \arrow[r, "\rho_i"] & (\mathrm{BN}_i(\mathbb{C})^{\mathrm{top}})_p^{\hat{}}, \end{tikzcd} \tag{4.9}$$

where $\rho_i : \mathcal{R}\mathrm{BN}_i(\mathbb{C}) \to \mathrm{BN}_i(\mathbb{C})_p^{\hat{}}$ denotes the functorial fibrant replacement of the natural map from $\mathrm{BN}_i(\mathbb{C})^{\mathrm{top}}$ to its p-completion.

Note that the automorphism of $\mathrm{BN}_i(\overline{\mathbb{F}}_q)$ induced by the Frobenius Fr_q agree with the automorphism ψ_i^q under $\mathrm{BN}_i(\mathsf{e})$. Thus, Fr_q and ψ_i^q together induce an automorphism of BN_i which, due to a lack of a better name, will also be denoted by

$$\psi_i^q : \mathrm{BN}_i \longrightarrow \mathrm{BN}_i$$

and will be referred to as "the q-th Adams operation" on BN_i. By construction

$$\mathrm{BN}_i \simeq \mathrm{BN}_i(\mathbb{C})^{\mathrm{top}}$$

and there is a natural map

$$\begin{tikzcd}
\tilde{\gamma}_i : \mathrm{BN}_i \arrow[r, "\gamma_i"] & \mathrm{BN}_i(\overline{\mathbb{F}}_q)\hat{_p} \arrow[d, "\mathrm{BN}_i(\mathsf{e})\hat{_p}"] & & \\
& \mathrm{BN}_i(\mathbb{C})\hat{_p} \arrow[r] & \mathrm{BGl}_i(\mathbb{C})\hat{_p} \arrow[d, "\mu"] & \\
& & \|\widehat{\mathrm{BGl}_i(\mathbb{C})_p}\| \arrow[r] & \mathrm{BGl}_i\hat{}\,,
\end{tikzcd} \tag{4.10}$$

where the last map is induced by the universal property of pro-finite completions.

Remark 4.11. The action of any element $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ of cyclotomic character $\frac{1}{q}$ on $\mathrm{BGl}_i\hat{}$ agrees with the action of ψ_i^q on BN_i along the map $\tilde{\gamma}_i$; also see (4.7).

Notation 4.12. For the rest of the paper we choose and fix an element

$$\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$$

of cyclotomic character $\frac{1}{q}$.

Definition 4.13. Define BGl_i as the pullback in the diagram

$$\begin{tikzcd}
\mathrm{BGl}_i \arrow[r, dashed] \arrow[d, dashed] & \mathrm{BGl}_i\hat{} \arrow[d] \\
\mathcal{R}(\mathrm{BN}_i)_{\mathbb{Q}} \arrow[r, "\mathcal{R}(\tilde{\gamma}_i)_{\mathbb{Q}}"] & (\mathrm{BGl}_i\hat{})_{\mathbb{Q}},
\end{tikzcd} \tag{4.14}$$

where $\mathcal{R}(\tilde{\gamma}_i)_{\mathbb{Q}} : \mathcal{R}(\mathrm{BN}_i)_{\mathbb{Q}} \to (\mathrm{BGl}_i\hat{})_{\mathbb{Q}}$ is the functorial fibrant replacement of the map $(\tilde{\gamma}_i)_{\mathbb{Q}}$.

It is immediate from the construction that BGl_i is weakly equivalent to $\mathrm{BGl}_i(\mathbb{C})^{\mathrm{top}}_{(p)}$. It follows from Remark 4.11 that the automorphism ψ_i^q of BN_i together with the action of σ on $\mathrm{BGl}_i\hat{}$, induces an automorphism of BGl_i which we also denote by

$$\psi_i^q : \mathrm{BGl}_i \longrightarrow \mathrm{BGl}_i. \tag{4.15}$$

We refer to it as the "q-th Adams operation" on BGl_i.

Our next goal is to construct the total space SBGl_i using an arithmetic fracture square similar to (4.14). Therefore, we consider the simplicial scheme

$$\mathrm{SBN}^{\mathrm{R}}_{i,\bullet} := \mathrm{B}(\star, \mathrm{N}^{\mathrm{R}}_{i,\bullet}, ({\mathrm{S}^2}_{\mathrm{R}})^{\wedge i}).$$

Proposition 4.16. *The natural map induced by* $\mathsf{e} : \overline{\mathbb{F}}_q^{\times} \to \mathbb{C}^{\times}$ *defined in Remark* 3.17

$$\mathrm{SBN}_i(\mathsf{e}) : \mathrm{SBN}_i(\overline{\mathbb{F}}_q) \longrightarrow \mathrm{SBN}_i(\mathbb{C})^{\mathrm{top}}$$

is an isomorphism after p-completion.

Proof. Both the groups $\mathrm{N}_i(\overline{\mathbb{F}}_q)$ and $\mathrm{N}_i(\mathbb{C})^{\mathrm{top}}$ map to Σ_i with kernels $\mathbb{T}^i(\overline{\mathbb{F}}_q)$ and $\mathbb{T}^i(\mathbb{C})^{\mathrm{top}}$ respectively. Hence, both the spaces $\mathrm{SBN}_i(\overline{\mathbb{F}}_q)$ and $\mathrm{SBN}_i(\mathbb{C})^{\mathrm{top}}$ map to $\mathrm{B}\Sigma_i$

$$\begin{tikzcd} \mathrm{SBN}_i(\overline{\mathbb{F}}_q) \arrow[rr, "\mathrm{SBN}_i(\mathsf{e})"] \arrow[dr] & & \mathrm{SBN}_i(\mathbb{C})^{\mathrm{top}} \arrow[dl] \\ & \mathrm{B}\Sigma_i & \end{tikzcd}$$

with fibers $\mathrm{B}(\star, \mathbb{T}^i(\overline{\mathbb{F}}_q), \mathrm{S}^2_{\overline{\mathbb{F}}_q}(\overline{\mathbb{F}}_q)^{\wedge i})$ and $\mathrm{B}(\star, \mathbb{T}^i(\mathbb{C}), \mathrm{S}^2_{\mathbb{C}}(\mathbb{C})^{\wedge i})^{\mathrm{top}}$. Therefore, by [BK72, I.5.5], it is enough to show that the map

$$\mathrm{B}(\star, \mathbb{T}^i(\overline{\mathbb{F}}_q), \mathrm{S}^{2i}_{\overline{\mathbb{F}}_q}(\overline{\mathbb{F}}_q)) \longrightarrow \mathrm{B}(\star, \mathbb{T}^i(\mathbb{C}), \mathrm{S}^{2i}_{\mathbb{C}}(\mathbb{C}))^{\mathrm{top}}$$

induces an isomorphism in $\mathrm{H}\mathbb{F}_p$-homology.

Since $\mathrm{Gl}_1^{\mathrm{R}} \cong \mathbb{A}^1_{\mathrm{R}} - 0$ as a $\mathrm{Gl}_1^{\mathrm{R}}$-torsor over $\mathsf{Spec}\,\mathrm{R}$, the $\mathrm{Gl}_1(\mathrm{R})$-equivariant cellular structure of ${\mathrm{S}^2}_{\mathrm{R}}(\mathrm{R})$ is given by the pushout diagram

$$\begin{tikzcd} \mathrm{Gl}_1(\mathrm{R}) \times \partial\Delta[1] \arrow[r] \arrow[d] & \partial\Delta[1] \arrow[d, dashed] \\ \mathrm{Gl}_1(\mathrm{R}) \times \Delta[1] \arrow[r, dashed] & {\mathrm{S}^2}_{\mathrm{R}}(\mathrm{R}) \end{tikzcd}$$

for any ring R. Thus, the equivariant cells of $({\mathrm{S}^2}_{\overline{\mathbb{F}}_q}(\overline{\mathbb{F}}_q))^{\wedge i}$ and $({\mathrm{S}^2}_{\mathbb{C}}(\mathbb{C}))^{\wedge i}$ are in bijection and the isotropy subgroup of a cell in ${\mathrm{S}^2}_{\overline{\mathbb{F}}_q}(\overline{\mathbb{F}}_q)^{\wedge i}$ includes in the isotropy subgroup of the corresponding cell in ${\mathrm{S}^2}_{\mathbb{C}}(\mathbb{C})^{\wedge i}$ as an approximation of the form $\mathbb{T}^j(\overline{\mathbb{F}}_q) \to \mathbb{T}^j(\mathbb{C})$ for some $j \leq i$. Taking the bar construction converts this to the inclusion on the level of classifying spaces that is an isomorphism in $\mathrm{H}\mathbb{F}_p$-homology; see (3.15) and Remark 3.17. Thus, if we filter both $\mathrm{SBN}_i(\overline{\mathbb{F}}_q)$ and $\mathrm{SBN}_i(\mathbb{C})^{\mathrm{top}}$ using the $\mathbb{T}^i(\cdot)$-equivariant cellular filtration of ${\mathrm{S}^2}_{(\cdot)}(\cdot)^{\wedge i}$, then the map induced by $\mathrm{SBN}_i(\mathsf{e})$ on each filtration quotient is an isomorphism in $\mathrm{H}\mathbb{F}_p$-homology. Since, $({\mathrm{S}^2}_{\mathrm{R}}(\mathrm{R}))^{\wedge i}$ consists of finitely many $\mathbb{T}^i(\mathrm{R})$-equivariant cells, an inductive argument proves the result. $\square$

Definition 4.17. Define the space SBN_i as the pullback along in the diagram

$$\begin{tikzcd} \mathrm{SBN}_i \arrow[r, dashed, "\mathrm{S}\gamma_i"] \arrow[d, dashed] & \mathrm{SBN}_i(\overline{\mathbb{F}}_q)\hat{_p} \arrow[d, "\mathrm{SBN}_i(\mathsf{e})\hat{_p}"] \\ \mathcal{R}\mathrm{SBN}_i(\mathbb{C})^{\mathrm{top}} \arrow[r, "\mathrm{S}\rho_i"] & (\mathrm{SBN}_i(\mathbb{C})^{\mathrm{top}})\hat{_p}, \end{tikzcd} \tag{4.18}$$

where $\mathrm{S}\rho_i : \mathcal{R}\mathrm{SBN}_i(\mathbb{C})^{\mathrm{top}} \to (\mathrm{SBN}_i(\mathbb{C})^{\mathrm{top}})\hat{_p}$ denotes a functorial fibrant replacement of the natural map from $\mathrm{SBN}_i(\mathbb{C})^{\mathrm{top}}$ to its p-completion.

The automorphism of $\mathcal{R}\mathrm{SBN}_i(\mathbb{C})^{\mathrm{top}}$ induced by the q-th power map and the automorphism of $\mathrm{SBN}_i(\overline{\mathbb{F}}_q)$ induced by the Frobenius map Fr_q agree along $\mathrm{SBN}_i(\mathbb{C})^{\mathrm{top}}$. Let the common automorphism be denoted by

$$\psi_i^q : \mathrm{SBN}_i \longrightarrow \mathrm{SBN}_i,$$

and refer to it as the "q-th Adams operation" on SBN_i. Straightforward from the construction, we get a ψ^q-equivariant map

$$\nu_i : \mathrm{SBN}_i \longrightarrow \mathrm{BN}_i$$

whose fiber is weakly equivalent to S^{2i}. Now, we have a map

$$\begin{tikzcd} \mathrm{S}\tilde{\gamma}_i : \mathrm{SBN}_i \arrow[r, "\mathrm{S}\gamma_i"] & \mathrm{SBN}_i(\overline{\mathbb{F}}_q)\hat{_p} \arrow[d, "\mathrm{SBN}_i(\mathsf{e})\hat{_p}"] & & \\ & \mathrm{SBN}_i(\mathbb{C})\hat{_p} \arrow[r] & \mathrm{SBGl}_i(\mathbb{C})\hat{_p} \arrow[d, "\mu"] & \\ & & \|\widehat{\mathrm{SBGl}_i(\mathbb{C})}_p\| \arrow[r] & \mathrm{SBGl}_i\hat{}, \end{tikzcd} \tag{4.19}$$

where the last map is induced by the universal property of pro-finite completions.

Definition 4.20. Define the space SBGl_i as the pullback along the diagram

$$\begin{tikzcd} \mathrm{SBGl}_i \arrow[r, dashed] \arrow[d, dashed] & \mathrm{SBGl}_i\hat{} \arrow[d] \\ \mathcal{R}(\mathrm{SBN}_i)_{\mathbb{Q}} \arrow[r, "\mathcal{R}(\mathrm{S}\tilde{\gamma}_i)_{\mathbb{Q}}"] & (\mathrm{SBGl}_i\hat{})_{\mathbb{Q}}, \end{tikzcd} \tag{4.21}$$

where $\mathcal{R}(\mathrm{S}\tilde{\gamma}_i)_{\mathbb{Q}} : \mathcal{R}(\mathrm{SBN}_i)_{\mathbb{Q}} \to (\mathrm{SBGl}_i\hat{})_{\mathbb{Q}}$ is the functorial fibrant replacement of $(\mathrm{S}\tilde{\gamma}_i)_{\mathbb{Q}}$.

Since the automorphism ψ^q of SBN_i commutes with the action of σ on $\mathrm{SBGl}_i^{\wedge}$ along the composite (4.10), SBGl_i is also equipped with an automorphism

$$\psi_i^q : \mathrm{SBGl}_i \longrightarrow \mathrm{SBGl}_i,$$

which we refer to as the "q-th Adams operation" on SBGl_i. It follows from the construction of SBGl_i and BGl_i that there is a ψ^q-equivariant map

$$\pi_i : \mathrm{SBGl}_i \longrightarrow \mathrm{BGl}_i. \tag{4.22}$$

The map π_i admits a ψ^q-equivariant section, i.e., a map

$$\sigma_i : \mathrm{BGl}_i \longrightarrow \mathrm{SBGl}_i,$$

such that $\pi_i \circ \sigma_i = \mathbb{1}_{\mathrm{BGl}_i}$, for reasons that are explained in Remark **??**.

Lemma 4.23. *The fiber of the map π_i is equivalent to the p-local $2i$-sphere.*

Proof. From Definition 4.20, $\mathrm{Fib}(\pi_i)$ fits into the homotopy pullback square

$$\begin{tikzcd}
\mathrm{Fib}(\pi_i) \arrow[r] \arrow[d] & \mathrm{Fib}(\pi_i^{\wedge}) \arrow[d] \\
\mathrm{Fib}((\nu_i)_{\mathbb{Q}}) \arrow[r] & \mathrm{Fib}((\pi_i^{\wedge})_{\mathbb{Q}}).
\end{tikzcd}$$

It is easy to see $\mathrm{Fib}((\nu_i)_{\mathbb{Q}}) \simeq \mathrm{S}^{2i}_{\mathbb{Q}}$. From Lemma 4.6, $\mathrm{Fib}(\pi_i^{\wedge}) \simeq \hat{\mathrm{S}}^{2i}_p$, further

$$\mathrm{Fib}((\pi_i^{\wedge})_{\mathbb{Q}}) \simeq \mathrm{Fib}(\pi_i^{\wedge})_{\mathbb{Q}} \simeq (\hat{\mathrm{S}}^{2i}_p)_{\mathbb{Q}},$$

and hence, $\mathrm{Fib}(\pi_i) \simeq \mathrm{S}^{2i}_{(p)}$. $\square$

We end this subsection proving the following result.

Theorem 4.24. *For all $i, j \geq 0$, there exists a ψ^q-equivariant commutative diagram*

$$\begin{tikzcd}
\mathrm{SBGl}_i \times \mathrm{SBGl}_j \arrow[r, "\omega_{i,j}"] \arrow[d, "\pi_i \times \pi_j"'] & \mathrm{SBGl}_{i+j} \arrow[d, "\pi_{i+j}"] \\
\mathrm{BGl}_i \times \mathrm{BGl}_j \arrow[r, "\mu_{i,j}"] & \mathrm{BGl}_{i+j}
\end{tikzcd} \tag{4.25}$$

induced by the block-diagonal sum map. Further, the families $\{\mu_{i,j} : i \geq 0, j \geq 0\}$ and $\{\omega_{i,j} : i \geq 0, j \geq 0\}$ satisfy the external associativity condition

$$\mu_{i+j,k} \circ (\mu_{i,j} \times \mathbb{1}_{\mathrm{BGl}_{k,p}}) = \mu_{i,j+k} \circ (\mathbb{1}_{\mathrm{BGl}_{i,p}} \times \mu_{j,k}),$$
$$\omega_{i+j,k} \circ (\omega_{i,j} \times \mathbb{1}_{\mathrm{BGl}_{k,p}}) = \omega_{i,j+k} \circ (\mathbb{1}_{\mathrm{BGl}_{i,p}} \times \omega_{j,k}).$$

Proof. Notice that the maps $\tau_{i,j}$ and c of Remark 4.4 satisfy the external associativity condition

$$\tau_{i+j,k} \circ (\tau_{i,j} \underset{\mathsf{Spec}\,\mathbb{Z}}{\times} \mathbb{1}_{(\mathbb{A}^k_{\mathbb{Z}}-0)}) = \tau_{i,j+k} \circ (\mathbb{1}_{(\mathbb{A}^i_{\mathbb{Z}}-0)} \underset{\mathsf{Spec}\,\mathbb{Z}}{\times} \tau_{j,k}),$$
$$\mathsf{c} \circ (\mathsf{c} \times \mathbb{1}_{\Delta[1]}) = \mathsf{c} \circ (\mathbb{1}_{\Delta[1]} \times \mathsf{c}).$$

This, along with the commutative diagram

$$\begin{tikzcd} \mathrm{Gl}_i \underset{\mathsf{Spec}\,\mathbb{Z}}{\times} \mathrm{Gl}_j \underset{\mathsf{Spec}\,\mathbb{Z}}{\times} (\mathbb{A}^i_{\mathbb{Z}} - 0) \underset{\mathsf{Spec}\,\mathbb{Z}}{\times} (\mathbb{A}^j_{\mathbb{Z}} - 0) \arrow[r] \arrow[d] & \mathrm{Gl}_{i+j} \underset{\mathsf{Spec}\,\mathbb{Z}}{\times} (\mathbb{A}^{i+j}_{\mathbb{Z}} - 0) \arrow[d] \\ (\mathbb{A}^i_{\mathbb{Z}} - 0) \underset{\mathsf{Spec}\,\mathbb{Z}}{\times} (\mathbb{A}^j_{\mathbb{Z}} - 0) \arrow[r] & (\mathbb{A}^{i+j}_{\mathbb{Z}} - 0) \end{tikzcd}$$

implies the commutative diagram of simplicial schemes

$$\begin{tikzcd} \mathrm{SBGl}^{\mathbb{Z}}_{i,\bullet} \times \mathrm{SBGl}^{\mathbb{Z}}_{j,\bullet} \arrow[r, "\omega^{\mathbb{Z}}_{i,j}"] \arrow[d, "\pi^{\mathbb{Z}}_i \times \pi^{\mathbb{Z}}_j"'] & \mathrm{SBGl}^{\mathbb{Z}}_{i+j,\bullet} \arrow[d, "\pi^{\mathbb{Z}}_{i+j}"] \\ \mathrm{BGl}^{\mathbb{Z}}_{i,\bullet} \times \mathrm{BGl}^{\mathbb{Z}}_{j,\bullet} \arrow[r, "\mu^{\mathbb{Z}}_{i,j}"] & \mathrm{BGl}^{\mathbb{Z}}_{i+j,\bullet} \end{tikzcd} \tag{4.26}$$

such that the families $\{\omega^{\mathbb{Z}}_{i,j} : i, j \geq 1\}$ and $\{\mu^{\mathbb{Z}}_{i,j} : i, j \geq 1\}$ satisfy external associativity conditions. Applying the functor $\|\acute{\mathrm{E}}\mathrm{t}_{\mathbb{C}}((\cdot) \underset{\mathsf{Spec}\,\mathbb{Z}}{\times} \mathsf{Spec}\,\mathbb{C})_p\|$ to the above diagram one constructs a p-completed version of the diagram in (4.25). An easy diagram chase leads to the commutative diagram

$$\begin{tikzcd} \mathrm{SBN}_i \times \mathrm{SBN}_j \arrow[r] \arrow[d] & \mathrm{SBN}_{i+j} \arrow[d] \\ \mathrm{BN}_i \times \mathrm{BN}_j \arrow[r] & \mathrm{BN}_{i+j} \end{tikzcd}$$

such that its $\mathbb{Q}$-localization maps to the diagram obtained by applying $\|\acute{\mathrm{E}}\mathrm{t}(\cdot)_p\|_{\mathbb{Q}}$ to (4.26). Thus we can form an arithmetic fracture square of the diagram (4.26) and the result follows. ∎

4.3. *Some properties of the fiber of* π_i. We now discuss some of the important consequences of Theorem 4.24. An explicit point-set definition for the fiber of a map is important for the completeness of our arguments.

Convention 3. For the sake of simplicity of arguments, henceforth we work only in $\mathcal{T}\mathrm{op}$ and avoid $\mathsf{s}\mathcal{S}\mathsf{et}$.

Recall that the Moore path space of a space $\mathrm{B} \in \mathcal{T}\mathrm{op}_*$ is defined as

$$\mathrm{P(B)} := \{(t, f : \mathbb{R}_{\geq 0} \to \mathrm{B}) : t \geq 0, f(0) = * \text{ and } f(s) = f(t) \text{ for all } s \geq t\},$$

where $*$ is the base-point. There is a natural evaluation map

$$\mathsf{ev} : \mathrm{P}(\mathrm{B}) \longrightarrow \mathrm{B}$$

such that $(t, f) \mapsto f(t)$. For the rest of the paper, we let $\Omega(\mathrm{B})$ denote the Moore loop space

$$\Omega(\mathrm{B}) := \mathsf{ev}^{-1}(*).$$

Definition 4.27. For $\mathrm{B} \in \mathcal{T}\mathrm{op}_*$ and a map $\pi : \mathrm{E} \to \mathrm{B}$ in $\mathcal{T}\mathrm{op}$, define the *fiber* $\mathrm{Fib}(\pi) \in \mathcal{T}\mathrm{op}$ as the pullback

$$\begin{tikzcd} \mathrm{Fib}(\pi) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{E} \arrow[d, "\pi"] \\ \mathrm{P}(\mathrm{B}) \arrow[r] & \mathrm{B}. \end{tikzcd}$$

Explicitly, $\mathrm{Fib}(\pi) := \{(e, (t, f)) : \pi(e) = f(t)\} \subset \mathrm{E} \times \mathrm{P}(\mathrm{B})$. If π admits a section, i.e., a map $\mathrm{s} : \mathrm{B} \to \mathrm{E}$ such that $\pi \circ \mathrm{s} = \mathbb{1}_{\mathrm{B}}$, then its *reduced fiber* $\widetilde{\mathrm{Fib}}(\pi) \in \mathcal{T}\mathrm{op}_*$ is defined as the cofiber

$$\widetilde{\mathrm{Fib}}(\pi) := \mathrm{Cof}(\mathsf{s}' : \mathrm{P}(\mathrm{B}) \to \mathrm{Fib}(\pi)),$$

where $\mathsf{s}'(t, f) = ((\mathsf{s} \circ \mathsf{ev})(t, f), (t, f))$ and the collapsed image of $\mathrm{P}(\mathrm{B})$ is set as the basepoint.

A straightforward consequence of Definition 4.27 is the following lemma.

Lemma 4.28. *Let* B *be a pointed topological space, and let* $\pi : \mathrm{E} \to \mathrm{B}$ *be a sectioned map. Then there exists a strictly associative map of monoids*

$$a : \Omega(\mathrm{B}) \longrightarrow \mathrm{G}(\widetilde{\mathrm{Fib}}(\pi), \widetilde{\mathrm{Fib}}(\pi))$$

induced by the concatenation of paths, which is a map $\mathrm{P}(\mathrm{B}) \times \mathrm{P}(\mathrm{B}) \to \mathrm{P}(\mathrm{B})$.

Notation 4.29. For the rest of the paper we let $\tilde{\mathrm{S}}^{2i}_{(p)}$ denote the fiber $\widetilde{\mathrm{Fib}}(|\pi_i|)$ when $i \geq 1$. The q-th Adams operation on $\tilde{\mathrm{S}}^{2i}_{(p)}$ induced by the ψ^q-invariance of the map π_i will be denoted by

$$\hat{\psi}^q_i : \tilde{\mathrm{S}}^{2i}_{(p)} \longrightarrow \tilde{\mathrm{S}}^{2i}_{(p)}. \tag{4.30}$$

When $i = 0$, we set $\tilde{\mathrm{S}}^0_{(p)} := \mathrm{S}^0$ and declare $\hat{\psi}^q_0 = \mathbb{1}_{\mathrm{S}^0}$.

Notation 4.31. For $i \geq 1$, we abbreviate $\Omega|\mathrm{BGl}_i|$ to $\widetilde{\mathrm{Gl}}_i$, let

$$\tilde{\mu}_{i,j} := \Omega(|\mu_{i,j}|) : \widetilde{\mathrm{Gl}}_i \times \widetilde{\mathrm{Gl}}_j \longrightarrow \widetilde{\mathrm{Gl}}_{i+j}$$

denote the "block-diagonal sum," and let

$$\tilde{\psi}^q_i := \Omega(|\psi^q|) : \widetilde{\mathrm{Gl}}_i \longrightarrow \widetilde{\mathrm{Gl}}_i$$

denote the q-th Adams operation on $\widetilde{\mathrm{Gl}}_i$. When $i = 0$, we designate $\widetilde{\mathrm{Gl}}_0$ as the trivial group, $\tilde{\mu}_{i,0}$ and $\tilde{\mu}_{0,j}$ as the identity maps, and $\tilde{\psi}^q_0$ as the unique self-map of $\widetilde{\mathrm{Gl}}_0$.

Immediate from Lemma 4.28, we have a map

$$\iota_i : \widetilde{\mathrm{Gl}}_i \longrightarrow \mathrm{G}(\tilde{\mathrm{S}}^{2i}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)})$$

of strictly associative H-spaces with strict units. Let $\tilde{\iota}_i : \widetilde{\mathrm{Gl}}_i \times \tilde{\mathrm{S}}^{2i}_{(p)} \longrightarrow \tilde{\mathrm{S}}^{2i}_{(p)}$ denote the adjoint of ι_i. Clearly, the map $\tilde{\iota}_i$ is ψ^q-equivariant:

$$\begin{tikzcd} \widetilde{\mathrm{Gl}}_i \times \tilde{\mathrm{S}}^{2i}_{(p)} \arrow[r, "\tilde{\iota}_i"] \arrow[d, "\tilde{\psi}^q_i \times \hat{\psi}^q_i"'] & \tilde{\mathrm{S}}^{2i}_{(p)} \arrow[d, "\hat{\psi}^q_i"] \\ \widetilde{\mathrm{Gl}}_i \times \tilde{\mathrm{S}}^{2i}_{(p)} \arrow[r, "\tilde{\iota}_i"] & \tilde{\mathrm{S}}^{2i}_{(p)}. \end{tikzcd} \tag{4.32}$$

Furthermore, Theorem 4.24 implies that for all $i \geq 0$ and $j \geq 0$, we have maps

$$\tilde{\rho}_{i,j} : \tilde{\mathrm{S}}^{2i}_{(p)} \times \tilde{\mathrm{S}}^{2j}_{(p)} \longrightarrow \tilde{\mathrm{S}}^{2i+2j}_{(p)} \tag{4.33}$$

that are ψ^q-equivariant,

$$\begin{tikzcd} \tilde{\mathrm{S}}^{2i}_{(p)} \times \tilde{\mathrm{S}}^{2j}_{(p)} \arrow[r, "\tilde{\rho}_{i,j}"] \arrow[d, "\hat{\psi}^q_i \times \hat{\psi}^q_j"'] & \tilde{\mathrm{S}}^{2i+2j}_{(p)} \arrow[d, "\hat{\psi}^q_{i+j}"] \\ \tilde{\mathrm{S}}^{2i}_{(p)} \times \tilde{\mathrm{S}}^{2j}_{(p)} \arrow[r, "\tilde{\rho}_{i,j}"] & \tilde{\mathrm{S}}^{2i+2j}_{(p)}, \end{tikzcd} \tag{4.34}$$

externally associative,

$$\tilde{\rho}_{i+j,k} \circ (\tilde{\rho}_{i,j} \times \mathbb{1}_{\tilde{\mathrm{S}}^{2k}_{(p)}}) = \tilde{\rho}_{i,j+k} \circ (\mathbb{1}_{\tilde{\mathrm{S}}^{2i}_{(p)}} \times \tilde{\rho}_{j,k}), \tag{4.35}$$

and satisfy the ψ^q-equivariant diagram

$$\begin{tikzcd} \widetilde{\mathrm{Gl}}_i \times \widetilde{\mathrm{Gl}}_j \times \tilde{\mathrm{S}}^{2i}_{(p)} \times \tilde{\mathrm{S}}^{2j}_{(p)} \arrow[r, "\tilde{\iota}_i \times \tilde{\iota}_j"] \arrow[d, "\tilde{\mu}_{i,j} \times \tilde{\rho}^{i+j}"'] & \tilde{\mathrm{S}}^{2i}_{(p)} \times \tilde{\mathrm{S}}^{2j}_{(p)} \arrow[d, "\tilde{\rho}^{i+j}"] \\ \widetilde{\mathrm{Gl}}_{i+j} \times \tilde{\mathrm{S}}^{2i+2j}_{(p)} \arrow[r, "\tilde{\iota}_{i+j}"] & \tilde{\mathrm{S}}^{2i+2j}_{(p)}. \end{tikzcd} \tag{4.36}$$

It will follow, essentially from Remark 4.4, that the maps $\tilde{\rho}_{i,j}$ can be extended ψ^q-equivariantly to $\tilde{\mathrm{S}}^{2i}_{(p)} \wedge \tilde{\mathrm{S}}^{2j}_{(p)}$, which also satisfies (4.34), (4.35) and (4.36) with obvious modifications. More precisely, we prove

LEMMA 4.37. *There exists a family of ψ^q-equivariant maps*

$$\mathfrak{F}\mathsf{am} := \{\rho_{i,j} : \tilde{\mathrm{S}}^{2i}_{(p)} \wedge \tilde{\mathrm{S}}^{2j}_{(p)} \to \tilde{\mathrm{S}}^{2i+2j}_{(p)} : i \geq 0, j \geq 0\}$$

such that

(1) $\rho_{i,0} = \rho_{0,i} = \mathbb{1}_{\tilde{S}^{2i}_{(p)}}$,
(2) $\rho_{i,j}$ *is a weak equivalence for all* $i, j \geq 0$,
(3) $\rho_{i+j,k} \circ (\rho_{i,j} \wedge \mathbb{1}_{\tilde{S}^{2k}_{(p)}}) = \rho_{i,j+k} \circ (\mathbb{1}_{\tilde{S}^{2i}_{(p)}} \wedge \rho_{j,k})$, *and*
(4) *the diagram*

$$\begin{tikzcd}
\widetilde{\mathrm{Gl}}_i \times \widetilde{\mathrm{Gl}}_j \times \tilde{S}^{2i}_{(p)} \wedge \tilde{S}^{2j}_{(p)} \arrow[r, "\tilde{\iota}_i \wedge \tilde{\iota}_j"] \arrow[d, "\mu_{i,j} \times \rho_{i,j}"'] & \tilde{S}^{2i}_{(p)} \wedge \tilde{S}^{2j}_{(p)} \arrow[d, "\rho_{i,j}"] \\
\widetilde{\mathrm{Gl}}_{i+j} \times \tilde{S}^{2i+2j}_{(p)} \arrow[r, "\tilde{\iota}_{i+j}"'] & \tilde{S}^{2i+2j}_{(p)}
\end{tikzcd}$$

commutes.

Proof. In the commutative diagram of ψ^q-equivariant sectioned maps

$$\begin{tikzcd}
\mathrm{SBGl}_\epsilon \arrow[r, "\tilde{\lambda}_\epsilon"] \arrow[d, "\pi_\epsilon"] & \mathrm{SBGl}_i \times \mathrm{SBGl}_j \arrow[r, "\omega_{i,j}"] \arrow[d, "\pi_i \times \pi_j"] & \mathrm{SBGl}_{i+j} \arrow[d, "\pi_{i+j}"] \\
\mathrm{BGl}_\epsilon \arrow[r, "\lambda_\epsilon"] \arrow[u, dashed, bend left, "\sigma_\epsilon"] & \mathrm{BGl}_i \times \mathrm{BGl}_j \arrow[r, "\mu_{i,j}"'] \arrow[u, dashed, bend left, "\sigma_i \times \sigma_j"] & \mathrm{BGl}_{i+j}, \arrow[u, dashed, bend left, "\sigma_{i+j}"]
\end{tikzcd} \tag{4.38}$$

where $\epsilon \in \{i, j\}$, the composite $\omega_{i,j} \circ \tilde{\lambda}_\epsilon$ factors through σ_{i+j} (see Remark 4.4). Consequently, the right commutative square in (4.38) satisfies (4.41). By setting

$$(\pi_1, \pi_2, \pi_3, \pi_{1,2}, \pi_{2,3}, \pi_{1,2,3}) = (\pi_i, \pi_j, \pi_k, \pi_{i+j}, \pi_{j+k}, \pi_{i+j+k})$$

in (4.40), we get (1), (2) and (3). Since all the maps in (4.38) are ψ^q-equivariant, the maps $\rho_{i,j}$ is also a ψ^q-equivariant map.

It follows from (4.38) and Lemma 4.28 that the composite

$$\tilde{S}^{2i}_{(p)} \vee \tilde{S}^{2j}_{(p)} \longrightarrow \tilde{S}^{2i}_{(p)} \times \tilde{S}^{2j}_{(p)} \longrightarrow \tilde{S}^{2i+2j}_{(p)}$$

is equivariantly contractible with respect to the action of $\widetilde{\mathrm{Gl}}_i \times \widetilde{\mathrm{Gl}}_j$, and hence the condition (4). □

Corollary 4.39. *The q-th Adams operation* $\hat{\psi}^q_i : \tilde{S}^{2i}_{(p)} \to \tilde{S}^{2i}_{(p)}$ *is a map of degree* q^i*, hence a weak equivalence.*

Proof. The case $i = 1$ follows from the formal property that the q-th Adams operation converts a line bundle $\mathbb{L}$ to its q-th tensor power $\mathbb{L}^{\otimes q}$. This is encoded in the fact that the q-th Adams operation on $\mathrm{BGl}_1(\mathbb{C})^{\mathrm{top}}$ is induced by the q-th power map of $\mathrm{Gl}_1(\mathbb{C})^{\mathrm{top}}$. The general case follows from the ψ^q-equivariance of the map $\rho_{i,j}$ (see Lemma 4.37),

$$\begin{tikzcd}
\tilde{\mathrm{S}}^{2i}_{(p)} \wedge \tilde{\mathrm{S}}^{2j}_{(p)} \arrow[r, "\rho_{i,j}"] \arrow[d, "\hat{\psi}^q_i \wedge \hat{\psi}^q_j"'] & \tilde{\mathrm{S}}^{2i+2j}_{(p)} \arrow[d, "\hat{\psi}^q_{i+j}"] \\
\tilde{\mathrm{S}}^{2i}_{(p)} \wedge \tilde{\mathrm{S}}^{2j}_{(p)} \arrow[r, "\rho_{i,j}"] & \tilde{\mathrm{S}}^{2i+2j}_{(p)},
\end{tikzcd}$$

and an inductive argument. ☐

Proposition 4.40. *Suppose that there exists a commutative diagram of sectioned maps*

$$\begin{tikzcd}
\mathrm{E}_1 \times \mathrm{E}_2 \arrow[r, "\omega_{1,2}"] \arrow[d, "\pi_1 \times \pi_2"] & \mathrm{E}_{1,2} \arrow[d, "\pi_{1,2}"'] \\
\mathrm{B}_1 \times \mathrm{B}_2 \arrow[r, "\mu_{1,2}"] \arrow[u, dashed, bend left, "s_1 \times s_2"] & \mathrm{B}_{1,2} \arrow[u, dashed, bend right, "s_{1,2}"']
\end{tikzcd}$$

such that

$$\omega(\pi_1^{-1}(b_1) \times s_2(b_2) \cup s_1(b_1) \times \pi_2^{-1}(b_2)) = s(\mu(b_1, b_2)). \tag{4.41}$$

Then there exists a natural map

$$\hat{\omega}_{1,2} : \widetilde{\mathrm{Fib}}(\pi_1) \wedge \widetilde{\mathrm{Fib}}(\pi_2) \longrightarrow \widetilde{\mathrm{Fib}}(\pi_{1,2}).$$

Further, if there are commutative diagrams of sectioned maps

$$\begin{tikzcd}
\mathrm{E}_2 \times \mathrm{E}_3 \arrow[r, "\omega_{2,3}"] \arrow[d, "\pi_2 \times \pi_3"] & \mathrm{E}_{2,3} \arrow[d, "\pi_{2,3}"'] \\
\mathrm{B}_2 \times \mathrm{B}_3 \arrow[r, "\mu_{2,3}"] \arrow[u, dashed, bend left, "s_2 \times s_3"] & \mathrm{B}_{2,3} \arrow[u, dashed, bend right, "s_{2,3}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{E}_{1,2} \times \mathrm{E}_3 \arrow[r, "\omega_{(1,2),3}"] \arrow[d, "\pi_{1,2} \times \pi_3"] & \mathrm{E}_{1,2,3} \arrow[d, "\pi_{1,2,3}"'] \\
\mathrm{B}_{1,2} \times \mathrm{B}_2 \arrow[r, "\mu_{(1,2),3}"] \arrow[u, dashed, bend left, "s_{1,2} \times s_3"] & \mathrm{B}_{1,2,3} \arrow[u, dashed, bend right, "s_{1,2,3}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{E}_1 \times \mathrm{E}_{2,3} \arrow[r, "\omega_{1,(2,3)}"] \arrow[d, "\pi_1 \times \pi_{2,3}"] & \mathrm{E}_{1,2,3} \arrow[d, "\pi_{1,2,3}"'] \\
\mathrm{B}_1 \times \mathrm{B}_{2,3} \arrow[r, "\mu_{1,(2,3)}"] \arrow[u, dashed, bend left, "s_1 \times s_{2,3}"] & \mathrm{B}_{1,2,3} \arrow[u, dashed, bend right, "s_{1,2,3}"']
\end{tikzcd}$$

that satisfy (4.41) *and*

$$\omega_{(1,2),3} \circ (\omega_{1,2} \times \mathbb{1}_{\mathrm{E}_3}) = \omega_{1,(2,3)} \circ (\mathbb{1}_{\mathrm{E}_1} \times \omega_{2,3}),$$
$$\mu_{(1,2),3} \circ (\mu_{1,2} \times \mathbb{1}_{\mathrm{B}_3}) = \mu_{1,(2,3)} \circ (\mathbb{1}_{\mathrm{B}_1} \times \mu_{2,3}),$$

then

$$\hat{\omega}_{(1,2),3} \circ (\hat{\omega}_{1,2} \wedge \mathbb{1}_{\widetilde{\mathrm{Fib}}(\pi_3)}) = \hat{\omega}_{1,(2,3)} \circ (\mathbb{1}_{\widetilde{\mathrm{Fib}}(\pi_1)} \wedge \hat{\omega}_{2,3}).$$

Proof. By assumption, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Fib}(\pi_1) \times \mathrm{Fib}(\pi_2) \arrow[r, "\omega'"] \arrow[d, "\pi_1' \times \pi_2'"] & \mathrm{Fib}(\pi) \arrow[d, "\pi'"] \\
\mathrm{P}(\mathrm{B}_1) \times \mathrm{P}(\mathrm{B}_2) \arrow[r, "\mu'"] \arrow[u, dashed, bend left, "s_1' \times s_2'"] & \mathrm{P}(\mathrm{B}) \arrow[u, dashed, bend right, "s'"']
\end{tikzcd}$$

such that

$$\omega'((\pi_1')^{-1}(x_1) \times s_2'(x_2) \cup s_1'(x_1) \times (\pi_2')^{-1}(x_2)) = s'(\mu'(x_1, x_2)).$$

Consequently, the continuous map

$$\hat{\omega}_{1,2} : \widetilde{\mathrm{Fib}}(\pi_1) \wedge \widetilde{\mathrm{Fib}}(\pi_2) \longrightarrow \widetilde{\mathrm{Fib}}(\pi)$$

which sends $([f_1, t_1], [f_2, t_2])$ to $[\omega'(f_1, f_2), \max\{t_1 + t_2, 1\}]$, where $0 \leq t_i \leq 1$ is the cone coordinate with the basepoint at $t_i = 1$, is well-defined. Rest of the assertions can be easily verified from the above formula. $\square$

5. A canonical solution to the stable Adams conjecture

We first construct a new permutative category $\mathbf{Gl}_{\mathbb{C},p}$ equipped with a monoidal functor

$$\Psi^q : \mathbf{Gl}_{\mathbb{C},p} \longrightarrow \mathbf{Gl}_{\mathbb{C},p}$$

such that $\mathrm{K}(\mathbf{Gl}_{\mathbb{C},p}) \simeq \underline{\mathsf{ku}}_{(p)}$ and $\mathrm{K}(\Psi^q)$ is equivalent to the q-th Adams operation on $\underline{\mathsf{ku}}_{(p)}$. Then we construct a multifunctor

$$\widehat{\mathrm{J}} : \upsilon\mathbf{Gl}_{\mathbb{C},p} \longrightarrow \upsilon\mathbf{Gl}_{\mathrm{S}^2_{(p)}} \tag{5.1}$$

and declare $\mathrm{K}^{\mathrm{EM}}(\widehat{\mathrm{J}})$ to be the map $\mathfrak{J}$ of Theorem 1.5. Finally, we observe that the collection of maps $\{\hat{\psi}_i : i \in \mathbb{N}\}$ of (4.31) produces the multinatural transformation of (1.16). Thus, invoking Theorem 2.13 we produce a canonical null-homotopy that resolves Theorem 1.5 — the *unreduced stable Adams conjecture.*

Definition 5.2. Let $\mathbf{Gl}_{\mathbb{C},p}$ denote the permutative category whose objects are the natural numbers $(\mathbb{N}, +)$ and morphisms are

$$\mathrm{Mor}_{\mathbf{Gl}_{\mathbb{C},p}}(i,j) := \begin{cases} \widetilde{\mathrm{Gl}}_i & \text{if } i = j, \\ \emptyset & \text{otherwise,} \end{cases}$$

with $\tilde{\mu}_{i,j}$ (see (4.31)) as the monoidal product on morphisms.

LEMMA 5.3. *There is a strict monoidal functor $\Psi^q : \mathbf{Gl}_{\mathbb{C},p} \to \mathbf{Gl}_{\mathbb{C},p}$ such that*

$$\mathrm{K}(\Psi^q) : \underline{\mathsf{ku}}_{(p)} \longrightarrow \underline{\mathsf{ku}}_{(p)}$$

is the q-th Adams operation.

Proof. Define Ψ^q as the map that is identity on objects and maps $f \in \mathrm{Mor}_{\mathbf{Gl}_{\mathbb{C},p}}(i,i) = \widetilde{\mathrm{Gl}}_i$ to $\tilde{\psi}^q_i(f)$ for all $i \in \mathbb{N}$, where $\tilde{\psi}^q_i$ is the map defined in (4.31). It can be readily checked that $\Psi^q(\mathbb{1}_i) = \mathbb{1}_i$. Further, the map $\mu_{i,j}$ (see (4.25)) is ψ^q-equivariant, so we have

$$\tilde{\mu}_{i,j} \circ (\tilde{\psi}^q_i \times \tilde{\psi}^q_j) = \tilde{\psi}^q_{i+j} \circ \tilde{\mu}_{i,j}.$$

Hence, the functor Ψ^q is strictly monoidal.

The fact that $\mathrm{K}(\Psi^q)$ induces the q-th Adams operation is essentially a well-known observation of Sullivan [Sul74]. □

Although one would ideally like to construct the stable J-homomorphism

$$\mathfrak{J} : \underline{\mathsf{ku}}_{(p)} \longrightarrow \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)}).$$

by applying the K-theory functor to a monoidal functor $\mathrm{J} : \mathbf{Gl}_{\mathbb{C},p} \to \mathbf{Gl}_{\mathrm{S}^2_{(p)}}$ of permutative categories, the obvious functor J, which sends i to $\tilde{\mathrm{S}}^{2i}_{(p)}$ on objects and

$$\iota_i : \widetilde{\mathrm{Gl}}_i \longrightarrow \mathrm{G}(\tilde{\mathrm{S}}^{2i}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)}) \tag{5.4}$$

on morphisms, may not be a monoidal functor; see (5.5). Therefore, we forget down to $\mathcal{M}$ult.

Remark 5.5. In order for J to be monoidal, we need a family of maps

$$\{\rho_{i,j} : \tilde{\mathrm{S}}^{2i}_{(p)} \wedge \tilde{\mathrm{S}}^{2j}_{(p)} \to \tilde{\mathrm{S}}^{2i+2j}_{(p)} : i,j \geq 0\}$$

that satisfy

(A) $\rho_{i,j}$ is invertible, i.e., it is a homeomorphism;
(B) $\rho_{i,0} = \rho_{0,i} = \mathbb{1}_{\tilde{\mathrm{S}}^{2i}_{(p)}}$; and
(C) $\rho_{i,j+k} \circ (\mathbb{1}_{\tilde{\mathrm{S}}^{2i}_{(p)}} \wedge \rho_{i,j}) = \rho_{i+j,k} \circ (\rho_{i,j} \wedge \mathbb{1}_{\tilde{\mathrm{S}}^{2k}_{(p)}})$.

While the family $\mathfrak{Fam}$ of Lemma 4.37 satisfies (B) and (C), it may not satisfy (A).

Lemma 4.37 immediately gives rise to the multifunctor $\widehat{\mathrm{J}}$ of formula (5.1), which maps the object i to $\tilde{\mathrm{S}}^{2i}_{(p)}$, and on n-morphisms, sends $x \in \widetilde{\mathrm{Gl}}_i = \upsilon\mathbf{Gl}_{\mathbb{C},p}(i_1, \ldots, i_n; i)$ to the composite map

$$\tilde{\mathrm{S}}^{2i_1}_{(p)} \wedge \cdots \wedge \tilde{\mathrm{S}}^{2i_n}_{(p)} \xrightarrow{\rho_{i_1,\ldots,i_n}} \tilde{\mathrm{S}}^{2i}_{(p)} \xrightarrow{\iota_i(x)} \tilde{\mathrm{S}}^{2i}_{(p)}.$$

Let $(i_1, \ldots, i_n) \in \mathbb{N}^{\times k}$, and define $\rho_{i_1,\ldots,i_k}$ inductively using the formula

$$\rho_{i_1,\ldots,i_n} := \rho_{i_1, i_2+\cdots+i_n} \circ (\mathbb{1}_{\tilde{\mathrm{S}}^{2i_1}_{(p)}} \wedge \rho_{i_2,\ldots,i_n}).$$

Likewise, define the map $\tilde{\mu}_{i_1,\dots,i_n} : \widetilde{\mathrm{Gl}}_{i_1} \times \cdots \times \widetilde{\mathrm{Gl}}_{i_n} \to \widetilde{\mathrm{Gl}}_{i_1+\cdots+i_n}$ using the formula

$$\tilde{\mu}_{i_1,\dots,i_n} := \tilde{\mu}_{i_1,i_2+\cdots+i_n} \circ (\mathbb{1}_{\widetilde{\mathrm{Gl}}_{i_1}} \wedge \tilde{\mu}_{i_2,\dots,i_n}).$$

The fact that $\widehat{\mathrm{J}}$ is a multifunctor follows from the observations that

(i) the map $\rho_{i_1,\dots,i_n}$ is a weak equivalence with $\rho_i = \mathbb{1}_{\tilde{\mathrm{S}}^{2i}_{(p)}}$,

(ii) $\rho_{i_1,\dots,i_n} \circ (\rho_{i_{11},\dots i_{1k_1}} \wedge \cdots \wedge \rho_{i_{n1},\dots i_{nk_n}}) = \rho_{i_{11},\dots,i_{nk_n}}$ whenever $i_{j1} + \cdots + i_{jk_j} = i_j$ for all $1 \le j \le n$; and

(iii) there is a commutative diagram

$$\begin{array}{ccc}
\widetilde{\mathrm{Gl}}_{i_1} \times \cdots \times \widetilde{\mathrm{Gl}}_{i_n} \times \tilde{\mathrm{S}}^{2i_1}_{(p)} \wedge \cdots \wedge \tilde{\mathrm{S}}^{2i_2}_{(p)} & \xrightarrow{\tilde{\iota}_{i_1} \wedge \dots \wedge \tilde{\iota}_{i_n}} & \tilde{\mathrm{S}}^{2i_1}_{(p)} \wedge \cdots \wedge \tilde{\mathrm{S}}^{2i_2}_{(p)} \\
\Big\downarrow{\scriptstyle \mu_{i_1,\dots,i_n} \times \rho_{i_1,\dots,i_n}} & & \Big\downarrow{\scriptstyle \rho_{i_1,\dots,i_n}} \\
\widetilde{\mathrm{Gl}}_{i_1+\cdots+i_n} \times \tilde{\mathrm{S}}^{2i_1+\cdots+2i_n}_{(p)} & \xrightarrow[\tilde{\iota}_{i_1+\cdots+i_n}]{} & \tilde{\mathrm{S}}^{2i_1+\cdots+2i_n}_{(p)}.
\end{array}$$

Proof of Theorem 1.5 (Unreduced stable Adams conjecture). Theorem 2.12 implies that it is enough to produce a multinatural transformation $\eta : \widehat{\mathrm{J}} \to \widehat{\mathrm{J}} \circ \upsilon\Psi^q$. We declare

$$\eta_i := \hat{\psi}^q_i : \widehat{\mathrm{J}}(i) = \tilde{\mathrm{S}}^{2i}_{(p)} \to (\widehat{\mathrm{J}} \circ \upsilon\Psi^q)(i) = \tilde{\mathrm{S}}^{2i}_{(p)}.$$

In order for η to be a multinatural transformation, the diagram

$$\begin{array}{ccc}
\widetilde{\mathrm{Gl}}_i & \xrightarrow{\rho^*_{i_1,\dots,i_k}(\iota_i)} & \mathrm{G}(\tilde{\mathrm{S}}^{2i_1}_{(p)} \wedge \cdots \wedge \tilde{\mathrm{S}}^{2i_k}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)}) \\
\Big\downarrow{\scriptstyle \rho^*_{i_1,\dots,i_k}(\iota_i \circ \tilde{\psi}^q_i)} & & \Big\downarrow{\scriptstyle (\eta_i)_*} \\
\mathrm{G}(\tilde{\mathrm{S}}^{2i_1}_{(p)} \wedge \cdots \wedge \tilde{\mathrm{S}}^{2i_k}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)}) & \xrightarrow[(\eta_{i_1} \wedge \cdots \wedge \eta_{i_k})^*]{} & \mathrm{G}(\tilde{\mathrm{S}}^{2i_1}_{(p)} \wedge \cdots \wedge \tilde{\mathrm{S}}^{2i_k}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)})
\end{array} \tag{5.6}$$

must commute. Assume $i = i_1 + \cdots + i_k$; otherwise the spaces involved in the diagram (5.6) are empty. From the ψ^q-equivariance of the map $\rho_{i,j}$ we get

$$\begin{array}{ccc}
\tilde{\mathrm{S}}^{2i}_{(p)} \wedge \tilde{\mathrm{S}}^{2j}_{(p)} & \xrightarrow{\rho_{i,j}} & \tilde{\mathrm{S}}^{2i+2j}_{(p)} \\
\Big\downarrow{\scriptstyle \eta_i \wedge \eta_j} & & \Big\downarrow{\scriptstyle \eta_{i+j}} \\
\tilde{\mathrm{S}}^{2i}_{(p)} \wedge \tilde{\mathrm{S}}^{2j}_{(p)} & \xrightarrow{\rho_{i,j}} & \tilde{\mathrm{S}}^{2i+2j}_{(p)},
\end{array}$$

and therefore,

$$\rho_{i_1,\dots,i_k} \circ (\eta_{i_1} \wedge \cdots \wedge \eta_{i_n}) = \eta_i \circ \rho_{i_1,\dots,i_k}.$$

Further, from the commutative diagram

$$\begin{tikzcd} \widetilde{\mathrm{Gl}}_i \times \tilde{\mathrm{S}}^{2i}_{(p)} \arrow[r, "\tilde{\iota}_i"] \arrow[d, "\tilde{\psi}^q \times \eta_i"'] & \tilde{\mathrm{S}}^{2i}_{(p)} \arrow[d, "\eta_i"] \\ \widetilde{\mathrm{Gl}}_i \times \tilde{\mathrm{S}}^{2i}_{(p)} \arrow[r, "\tilde{\iota}_i"] & \tilde{\mathrm{S}}^{2i}_{(p)}, \end{tikzcd}$$

we conclude

$$\iota_i(\psi^q_i(\cdot))(\eta_i(\cdot\cdot)) = \eta_i(\iota_i(\cdot))(\cdot\cdot).$$

Thus (5.6) commutes:

$$\begin{aligned}(\iota_i(\tilde{\psi}^q(\cdot)) \circ \rho_{i_1,\dots,i_k} \circ (\eta_{i_1} \wedge \cdots \wedge \eta_{i_k}))(\cdot\cdot) &= (\iota_i(\tilde{\psi}^q(\cdot)) \circ \eta_i \circ \rho_{i_1,\dots,i_k})(\cdot\cdot) \\ &= (\iota_i(\tilde{\psi}^q(\cdot)) \circ \eta_i)(\rho_{i_1,\dots,i_k}(\cdot\cdot)) \\ &= \eta_i(\iota_i(\cdot))(\rho_{i_1,\dots,i_k}(\cdot\cdot)). \end{aligned}$$ □

Remark 5.7. By choosing

$$\begin{aligned} \underline{\mathsf{ku}}_{(p)} &:= \mathrm{K}^{\mathrm{seg}}(\phi\upsilon\mathbf{Gl}_{\mathbb{C}}), & \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)}) &:= \mathrm{K}^{\mathrm{seg}}(\phi\upsilon\mathbf{Gl}_{\mathrm{S}^2_{(p)}}), \\ \psi^q &:= \mathrm{K}^{\mathrm{seg}}(\phi\upsilon\Psi^q), & \mathfrak{J} &:= \mathrm{K}^{\mathrm{seg}}(\phi\widehat{\mathrm{J}}), \end{aligned}$$

we can make sure that the maps

$$\psi^q : \underline{\mathsf{ku}}_{(p)} \longrightarrow \underline{\mathsf{ku}}_{(p)}$$

$$\mathfrak{J} : \underline{\mathsf{ku}}_{(p)} \longrightarrow \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})$$

are maps of Ω-spectrum; see Theorem 2.2 and (2.14). Then $\mathrm{K}^{\mathrm{seg}}(\phi(\eta))$ is a homotopy $\mathfrak{J} \simeq \mathfrak{J} \circ \psi^q$ that solves Theorem 1.5.

Proof of Theorem 1.8. By definition, $\mathsf{bu}_{(p)}$ is the fiber of a ring map

$$\underline{\mathsf{ku}}_{(p)} \longrightarrow \mathrm{H}\mathbb{Z}$$

representing a generator of $\mathrm{H}\mathbb{Z}^0\underline{\mathsf{ku}}_{(p)}$. Likewise, $\mathrm{bgl}_1(\mathbb{S}_{(p)})$ is the fiber of a map

$$\mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)}) \longrightarrow \mathrm{H}\mathbb{Z}.$$

Since ψ^q induces the identity map on π_0, we have a lift

$$\psi^q_0 : \mathsf{bu}_{(p)} \longrightarrow \mathsf{bu}_{(p)}$$

by an argument using the five lemma. An identical argument lifts $\mathfrak{J}$ to a map $\mathfrak{J}_0' : \mathsf{bu}_{(p)} \to \mathrm{bgl}_1(\mathbb{S}_{(p)})$. Thus we have a homotopy commutative diagram

$$\begin{tikzcd}
\mathsf{bu}_{(p)} \arrow[r, "\psi_0^q - 1"] \arrow[d] & \mathsf{bu}_{(p)} \arrow[r, "\mathfrak{J}_0'"] \arrow[d] & \mathrm{bgl}_1(\mathbb{S}_{(p)}) \arrow[d, "\iota"] \\
\underline{\mathsf{ku}}_{(p)} \arrow[r, "\psi^q - 1"'] & \underline{\mathsf{ku}}_{(p)} \arrow[r, "\mathfrak{J}"'] & \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})
\end{tikzcd}$$

and consequently $\iota \circ \mathfrak{J}_0' \circ (\psi_0^q - 1) \simeq *$. In fact, $\mathfrak{J}_0' \circ (\psi_0^q - 1) \simeq *$; if not, then it must factor through the fiber

$$\mathrm{Fib}(\iota) \simeq \Sigma^{-1}\mathrm{H}\mathbb{Z}$$

via an essential map. This contradicts the fact that $\mathrm{H}\mathbb{Z}^{-1}\mathsf{bu}_{(p)} = 0$.

Note that $\mathrm{bsl}_1(\mathbb{S})_{(p)}$ is the fiber of $\mathrm{bgl}_1(\mathbb{S}_{(p)}) \to \Sigma\mathrm{H}\mathbb{Z}_{(p)}^{\times}$. Thus we have a lift of $\mathfrak{J}_0'$, namely $\mathfrak{J}_0$,

$$\begin{tikzcd}
 & & \mathrm{bsl}_1(\mathbb{S})_{(p)} \arrow[d] & \\
\mathsf{bu}_{(p)} \arrow[r, "\psi_0^q - 1"'] & \mathsf{bu}_{(p)} \arrow[r, "\mathfrak{J}_0'"'] \arrow[ur, dashed, "\mathfrak{J}_0"] & \mathrm{bgl}_1(\mathbb{S}_{(p)}) \arrow[r] & \Sigma\mathrm{H}\mathbb{Z}_{(p)}^{\times},
\end{tikzcd}$$

as $(\mathrm{H}\mathbb{Z}_{(p)}^{\times})^1\mathsf{bu}_{(p)} \cong 0$. Further, $\mathfrak{J}_0 \circ (\psi_0^q - 1) \simeq *$ as $(\mathrm{H}\mathbb{Z}_{(p)}^{\times})^0\mathsf{bu}_{(p)} \cong 0$. □

6. The J-homomorphism and fundamental groups

Note $\pi_0(\psi^q) : \pi_0\underline{\mathsf{ku}}_{(p)} \to \pi_0\underline{\mathsf{ku}}_{(p)}$ is the identity map as Adams operations on vector bundles do not change the virtual dimension of the bundle. Thus by running the long exact sequence

$$\begin{tikzcd}
0 \arrow[d, phantom, "\cong" rotate=90] & 0 \arrow[d, phantom, "\cong" rotate=90] & \mathbb{Z} \arrow[d, phantom, "\cong" rotate=90] \\
\pi_1\underline{\mathsf{ku}}_{(p)} \arrow[r, "\psi^q - 1"] & \pi_1\underline{\mathsf{ku}}_{(p)} \arrow[r] & \pi_1(\mathrm{Cof}(\psi^q - 1)) \arrow[dll, out=0, in=180] \\
\pi_0\underline{\mathsf{ku}}_{(p)} \arrow[r, "\psi^q - 1"'] \arrow[d, phantom, "\cong" rotate=90] & \pi_0\underline{\mathsf{ku}}_{(p)} \arrow[r, "\cong"] \arrow[d, phantom, "\cong" rotate=90] & \pi_0(\mathrm{Cof}(\psi^q - 1)) \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$

associated to the cofiber sequence $\underline{\mathsf{ku}}_{(p)} \to \underline{\mathsf{ku}}_{(p)} \to \mathrm{Cof}(\psi^q - 1)$, we get

$$\pi_1(\mathrm{Cof}(\psi^q - 1)) \cong \mathbb{Z}.$$

By Theorem 1.5, $\mathfrak{J} \circ (\psi^q - 1) \simeq *$ and thus we have an extension

$$\tilde{\mathfrak{J}} : \mathrm{Cof}(\psi^q - 1) \longrightarrow \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)}).$$

The main purpose of this section is to understand the effect of $\tilde{\mathfrak{J}}$ on the fundamental groups and prove Theorem 1.7.

By Remark 5.7, we may assume that $\underline{\mathsf{ku}}_{(p)}$ and $\mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})$ are Ω-spectrum. By construction, there is a map in $\mathrm{Ho}(\mathcal{T}\mathrm{op})$ from $\mathrm{B}\widetilde{\mathrm{Gl}}_i$ to the i-th component of the zeroth space $\mathrm{K}(\mathbf{Gl}_{\mathbb{C},p})_{[0]}$. Thus we get a map (in $\mathrm{Ho}(\mathcal{S}\mathrm{p})$)

$$r_i : \Sigma^\infty(\mathrm{B}\widetilde{\mathrm{Gl}}_i)_+ \longrightarrow \underline{\mathsf{ku}}_{(p)},$$

such that $\pi_0(r_i) : \mathbb{Z} \to \mathbb{Z}$ sends 1 to i. Similarly, we also have a map

$$s_i : \Sigma^\infty \mathrm{BG}(\tilde{\mathrm{S}}^{2i}_{(p)})_+ \longrightarrow \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})$$

such that $\pi_0(s_i) : \mathbb{Z} \to \mathbb{Z}$ sends 1 to i and $\pi_1(s_i) : \mathbb{Z}^\times_{(p)} \to \mathbb{Z}^\times_{(p)}$ is an isomorphism. Let

$$\mathrm{Cyl}(\psi^q, \mathrm{B}\widetilde{\mathrm{Gl}}_i) := \mathrm{hocolim} \left\{ \mathrm{B}\widetilde{\mathrm{Gl}}_i \underset{\mathbb{1}_{\mathrm{B}\widetilde{\mathrm{Gl}}_i}}{\overset{\mathrm{B}\tilde{\psi}^q_i}{\rightrightarrows}} \mathrm{B}\widetilde{\mathrm{Gl}}_i \right\}$$

denote the mapping cylinder. By construction, we have a commutative diagram

$$\begin{tikzcd}
\Sigma^\infty(\mathrm{B}\widetilde{\mathrm{Gl}}_i)_+ \arrow[r, "\Sigma^\infty_+(\mathrm{B}\tilde{\psi}^q_i - 1)"] \arrow[d, "r_i"'] & \Sigma^\infty(\mathrm{B}\widetilde{\mathrm{Gl}}_i)_+ \arrow[r] \arrow[d, "r_i"'] & \Sigma^\infty \mathrm{Cyl}(\psi^q, \mathrm{B}\widetilde{\mathrm{Gl}}_i)_+ \arrow[d, "\tilde{r}_i"] \\
\underline{\mathsf{ku}}_{(p)} \arrow[r, "\psi^q - 1"'] & \underline{\mathsf{ku}}_{(p)} \arrow[r] & \mathrm{Cof}(\psi^q - 1),
\end{tikzcd}$$

where the rows are cofiber sequences. By comparing the long exact sequences of homotopy groups associated to each row above, we deduce that

(6.1)
$$\pi_1(\tilde{r}_i) : \mathbb{Z} \cong \pi_1(\Sigma^\infty \mathrm{Cyl}(\psi^q, \mathrm{B}\widetilde{\mathrm{Gl}}_i)_+) \longrightarrow \pi_1(\mathrm{Cof}(\psi^q - 1)) \cong \mathbb{Z}_{(p)}$$

sends 1 to i.

If we view $\widetilde{\mathrm{Gl}}_i$ as well as $\mathrm{G}(\tilde{\mathrm{S}}^{2i}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)})$ as categories with one object, then the map ι_i of (5.4) (which defines the J-homomorphism) is a functor and the map $\eta_i = \hat{\psi}^q_i$ is a natural transformation between ι_i and $\iota_i \circ \tilde{\psi}^q_i$. Thus we have a homotopy $\mathrm{B}\iota_i \simeq \mathrm{B}\iota_i \circ \mathrm{B}\tilde{\psi}^q_i$, and consequently, an induced map

$$\bar{\iota}_i : \mathrm{Cyl}(\psi^q, \mathrm{B}\widetilde{\mathrm{Gl}}_i) \longrightarrow \mathrm{BG}(\tilde{\mathrm{S}}^{2i}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)})$$

such that the diagram

$$
\begin{tikzcd}
\Sigma^\infty \mathrm{Cyl}(\psi^q, \mathrm{B}\widetilde{\mathrm{Gl}}_i)_+ \arrow[r, "\Sigma^\infty_+ \bar{\iota}_i"] \arrow[d, "r_i"'] & \Sigma^\infty \mathrm{BG}(\tilde{\mathrm{S}}^{2i}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)})_+ \arrow[d, "s_i"] \\
\mathrm{Cof}(\psi^q - 1) \arrow[r, "\tilde{\mathfrak{J}}"] & \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})
\end{tikzcd}
\tag{6.2}
$$

commutes in $\mathrm{Ho}(\mathcal{S}\mathrm{p})$.

LEMMA 6.3. *The map induced by* $\bar{\iota}_i$ *on the fundamental groups*

$$\pi_1(\bar{\iota}_i) : \mathbb{Z} \cong \pi_1 \mathrm{Cyl}(\psi^q, \mathrm{B}\widetilde{\mathrm{Gl}}_i) \longrightarrow \pi_1 \mathrm{BG}(\tilde{\mathrm{S}}^{2i}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)}) \cong \mathbb{Z}^\times_{(p)}$$

sends n *to* $\deg(\hat{\psi}^q_i)^n$.

Proof. The Hurewicz theorem implies

$$\mathrm{H}\mathbb{Z}_1 \mathrm{Cyl}(\psi^q, \mathrm{B}\widetilde{\mathrm{Gl}}_i) \cong \pi_1 \mathrm{Cyl}(\psi^q, \mathrm{B}\widetilde{\mathrm{Gl}}_i) \cong \mathbb{Z}$$

and

$$\mathrm{H}\mathbb{Z}_1 \mathrm{BG}(\tilde{\mathrm{S}}^{2i}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)}) \cong \pi_1 \mathrm{BG}(\tilde{\mathrm{S}}^{2i}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)}) \cong \mathbb{Z}^\times_{(p)}.$$

Thus it is enough to show the result for $\mathrm{H}\mathbb{Z}_1(\bar{\iota}_i)$.

Let $\mathbf{1}$ be the unit interval category, with objects $\{0,1\}$ and $\alpha : 0 \to 1$ denoting the only non-trivial morphism. As a simplicial set, $\mathrm{Cyl}(\psi^q, \mathrm{B}\widetilde{\mathrm{Gl}}_i)$ is a quotient of $\mathrm{B}(\mathbf{1} \times \widetilde{\mathrm{Gl}}_i)$. More explicitly, its n-th space is the set

$$\mathrm{L}_n = \{[0 \xrightarrow{g_1} \cdots \xrightarrow{g_{i-1}} 0 \xrightarrow{\alpha} 1 \xrightarrow{g_{i+1}} \cdots \xrightarrow{g_n} 1] : g_i \in \widetilde{\mathrm{Gl}}_i\}/\sim,$$

where the equivalence relation is generated by

$$[0 \xrightarrow{\tilde{\psi}^q_i(g)} 0 \xrightarrow{\alpha} 1] \sim [0 \xrightarrow{\alpha} 1 \xrightarrow{g} 1].$$

The homology of the chain-complex $\mathbb{Z}[\mathrm{L}_*]$ is isomorphic to $\mathrm{H}\mathbb{Z}_* \mathrm{Cyl}(\psi^q, \mathrm{B}\widetilde{\mathrm{Gl}}_i))$, and $\mathrm{H}\mathbb{Z}_1 \mathrm{Cyl}(\psi^q, \mathrm{B}\widetilde{\mathrm{Gl}}_i))$ is generated by the class $[\alpha]$.

The n-th space of the simplicial set $\mathrm{BG}(\tilde{\mathrm{S}}^{2i}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)})$ is

$$\mathrm{W}_n = \{\tilde{\mathrm{S}}^{2i}_{(p)} \xrightarrow{f_1} \cdots \xrightarrow{f_n} \tilde{\mathrm{S}}^{2i}_{(p)} : f_i \in \mathrm{G}(\tilde{\mathrm{S}}^{2i}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)})\},$$

and homology of the chain complex $\mathbb{Z}[\mathrm{W}_*]$ is isomorphic to $\mathrm{H}\mathbb{Z}_* \mathrm{BG}(\tilde{\mathrm{S}}^{2i}_{(p)}, \tilde{\mathrm{S}}^{2i}_{(p)})$. The map $\bar{\iota}_i$ on the n-th space is given by

$$
\begin{gathered}
[0 \xrightarrow{g_1} \cdots \xrightarrow{g_{i-1}} 0 \xrightarrow{\alpha} 1 \xrightarrow{g_{i+1}} \cdots \xrightarrow{g_n} 1] \\
\big\downarrow \\
[\tilde{\mathrm{S}}^{2i}_{(p)} \xrightarrow{\iota_i(\tilde{\psi}^q_i(g_1))} \cdots \xrightarrow{\iota_i(\tilde{\psi}^q_i(g_{i-1}))} \tilde{\mathrm{S}}^{2i}_{(p)} \xrightarrow{\hat{\psi}^q_i} \tilde{\mathrm{S}}^{2i}_{(p)} \xrightarrow{\iota_i(g_{i+1})} \cdots \xrightarrow{\iota_i(g_n)} \tilde{\mathrm{S}}^{2i}_{(p)}].
\end{gathered}
$$

In particular, $[\alpha] \mapsto [\hat{\psi}^q_i]$, and hence the result. $\square$

Proof of Theorem 1.7. As a consequence of Corollary 4.39, Lemma 6.3 and (6.1), we get the following diagram when we hit (6.2) by $\pi_1(-)$:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "1\mapsto q^i"] \arrow[d, "1\mapsto i"'] & \mathbb{Z}_{(p)}^\times \arrow[d, equal] \\ \mathbb{Z}_{(p)} \arrow[r, "\pi_1(\tilde{\mathfrak{J}})"'] & \mathbb{Z}_{(p)}^\times. \end{tikzcd} \tag{6.4}$$

It follows that $\pi_1(\tilde{\mathfrak{J}})$ must send 1 to q. $\square$

6.1. *The indeterminacy of* $\pi_1(\tilde{\mathfrak{J}})$. Note that Theorem 1.7 depends on the explicit null-homotopy, namely, $\mathrm{K}^{\mathrm{seg}}(\phi(\eta))$ (see Remark 5.7), as it is needed in the construction of the extension $\tilde{\mathfrak{J}}$. Therefore, a different null-homotopy

$$\eta' : \mathfrak{J} \simeq \mathfrak{J} \circ \psi^q$$

can result in an extension

$$\tilde{\mathfrak{J}}' : \mathrm{Cof}(\psi^q - 1) \longrightarrow \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})$$

different from $\tilde{\mathfrak{J}}$ (in $\mathrm{Ho}(\mathcal{S}\mathrm{p})$). Thus, as in Theorem 1.7, $\pi_1(\tilde{\mathfrak{J}}')$ may not send 1 to q. Our next goal is to prove the following theorem.

THEOREM 6.5. *If $\tilde{\mathfrak{J}}'$ fits into the homotopy commutative diagram*

$$\begin{tikzcd} \underline{\mathsf{ku}}_{(p)} \arrow[r, "\psi^q-1"] & \underline{\mathsf{ku}}_{(p)} \arrow[r] \arrow[d, "\mathfrak{J}"] & \mathrm{Cof}(\psi^q - 1) \arrow[dl, "\tilde{\mathfrak{J}}'"] \\ & \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)}), & \end{tikzcd} \tag{6.6}$$

then $\pi_1(\mathfrak{J}')(1) = \pm q \in \mathbb{Z}_{(p)}^\times$ *if p is odd, and* $\pi_1(\mathfrak{J}')(1) = q$ *if* $p = 2$.

By Remark 2.11 there exists a map

$$\delta : \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)}) \longrightarrow \Sigma \mathrm{H}\mathbb{Z}_{(p)}^\times$$

that induces isomorphism on fundamental groups. Now notice that in the following sequence of maps,

$$\underline{\mathsf{ku}}_{(p)} \xrightarrow{\psi^q-1} \underline{\mathsf{ku}}_{(p)} \xrightarrow{\mathfrak{J}} \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)}) \xrightarrow{\delta} \Sigma \mathrm{H}\mathbb{Z}_{(p)}^\times, \tag{6.7}$$

$\mathfrak{J} \circ (\psi^q - 1) \simeq *$ by Theorem 1.5, and $\delta \circ \mathfrak{J} \simeq *$ as $\mathrm{H}\mathbb{Z}^1 \underline{\mathsf{ku}}_{(p)} \cong 0$. Thus, the Toda bracket

$$\langle \delta, \mathfrak{J}, \psi^q - 1 \rangle \subset [\Sigma \underline{\mathsf{ku}}_{(p)}, \Sigma \mathrm{H}\mathbb{Z}_{(p)}^\times] \cong \mathbb{Z}_{(p)}^\times$$

is well defined.

LEMMA 6.8. *Let $\rho : \mathrm{Cof}(\psi^q - 1) \to \Sigma\underline{\mathsf{ku}}_{(p)}$ denote the connecting map. There exists a map $\tilde{\mathfrak{J}}'$ satisfying the diagram (6.6) and the equation*

$$\pi_1(\delta \circ \tilde{\mathfrak{J}}')(1) = k$$

if and only if there exists an element $\gamma \in \langle \delta, \mathfrak{J}, \psi^q - 1\rangle$ such that $\pi_1(\gamma \circ \rho)(1) = k$.

Proof. It follows from the definition of Toda bracket that for any choice of $\tilde{\mathfrak{J}}'$, there exists an element $\gamma \in \langle \delta, \mathfrak{J}, \psi^q - 1\rangle$, and vice versa, that fits in a diagram

$$\begin{tikzcd}
\underline{\mathsf{ku}}_{(p)} \arrow[r, "\psi^q - 1"] & \underline{\mathsf{ku}}_{(p)} \arrow[r, "\mathfrak{J}"] \arrow[d] & \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)}) \arrow[r, "\delta"] & \Sigma \mathrm{H}\mathbb{Z}_{(p)}^{\times}, \\
& \mathrm{Cof}(\psi^q - 1) \arrow[ur, dashed, "\tilde{\mathfrak{J}}'"'] \arrow[d, "\rho"'] & & \\
& \Sigma\underline{\mathsf{ku}}_{(p)} \arrow[uurr, dashed, "\gamma"'] & &
\end{tikzcd}$$

and hence the result. ◻

From the above lemma and Theorem 1.7, we get the following corollary.

COROLLARY 6.9. *The Toda bracket $\langle \delta, \mathfrak{J}, \psi^q - 1\rangle \subset \mathbb{Z}_{(p)}^{\times}$ contains q.*

Proof of Theorem 6.5. By Lemma 6.8, it is enough to show that the indeterminacy of the bracket $\langle \delta, \mathfrak{J}, \psi^q - 1\rangle$, which is the double coset

$$\delta \circ [\Sigma\underline{\mathsf{ku}}_{(p)}, \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})] + [\Sigma\underline{\mathsf{ku}}_{(p)}, \Sigma \mathrm{H}\mathbb{Z}_{(p)}^{\times}] \circ \Sigma(\psi^q - 1) \subset [\Sigma\underline{\mathsf{ku}}_{(p)}, \Sigma \mathrm{H}\mathbb{Z}_{(p)}^{\times}] \cong \mathbb{Z}_{(p)}^{\times}$$

lies in the torsion subgroup $\{\pm 1\} \subset \mathbb{Z}_{(p)}^{\times}$. Since ψ^q acts as the identity map on $\pi_0(\underline{\mathsf{ku}}_{(p)})$, it follows that

$$[\Sigma\underline{\mathsf{ku}}_{(p)}, \Sigma \mathrm{H}\mathbb{Z}_{(p)}^{\times}] \circ \Sigma(\psi^q - 1) = \{0\}.$$

Notice that $[\Sigma\underline{\mathsf{ku}}_{(p)}, \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})] \cong [\Sigma\underline{\mathsf{ku}}_{(p)}, \mathrm{bgl}_1(\mathbb{S}_{(p)})]$ and that the composition (6.10)

$$\begin{tikzcd}
\Sigma\underline{\mathsf{ku}}_{(p)} \arrow[r, "\Sigma(\psi^q - 1)"] & \Sigma\underline{\mathsf{ku}}_{(p)} \arrow[r, "\gamma"] & \mathrm{bgl}_1(\mathbb{S}_{(p)}) \arrow[d, phantom, "\wr|"] & & \\
& & \Sigma\, \mathrm{gl}_1(\mathbb{S}_{(p)}) \arrow[r] & \Sigma\, \mathrm{gl}_1(\mathrm{KU}_p^{\hat{}}) \arrow[r, "\ell"] & \Sigma \mathrm{KU}_p^{\hat{}},
\end{tikzcd}$$

where ℓ is logarithm map of [Rez06], must be trivial. This is because the composite induces zero-map on π_k for $k > 1$ as the higher homotopy groups of $\mathrm{bgl}_1(\mathbb{S}_{(p)})$ are torsion and the homotopy groups of $\Sigma \mathrm{KU}_p^{\hat{}}$ are torsion free and periodic. Thus π_1 of the composition map in (6.10) factors through $\ker(\pi_1(\ell))$, which is isomorphic to the torsion subgroup in $\mathbb{Z}_p^\times$ (because ℓ is an equivalence after K(1)-localization). Since the map

$$\mathrm{bgl}_1(\mathbb{S}_{(p)}) \longrightarrow \Sigma\,\mathrm{gl}_1(\mathrm{KU}_p^{\wedge})$$

is injective on π_1, it follows that

$$\delta \circ [\Sigma\underline{\mathsf{ku}}_{(p)}, \mathsf{pic}^{\mathrm{ev}}(\mathbb{S}_{(p)})] \subset \{\pm 1\} = (\mathrm{Tor}\,\mathbb{Z}_p^\times) \cap \mathbb{Z}_{(p)}^\times.$$

When $p = 2$, the indeterminacy is in fact the trivial group: If there is a map $\gamma : \Sigma\underline{\mathsf{ku}}_{(2)} \to \mathrm{bgl}_1(\mathbb{S}_{(2)})$ such that

$$\pi_1(\gamma)(1) = -1 \in \pi_1(\mathrm{bgl}_1(\mathbb{S}_{(2)})),$$

then the composite

$$\mathrm{S}^2 \xrightarrow{\Sigma\eta} \mathrm{S}^1 \xrightarrow{-1} \mathrm{bgl}_1(\mathbb{S}_{(2)})$$

must factor through $\Sigma\underline{\mathsf{ku}}_{(2)}$, and therefore, must be trivial. This is a contradiction to the fact that $0 \neq \eta \in \pi_2(\mathrm{bgl}_1(\mathbb{S}_{(2)}))$. ☐

Remark 6.11. If we consider the p-complete version of the diagram (6.7),

$$\mathsf{ku}_p^{\wedge} \xrightarrow{\hat{\psi}^q - 1} \mathsf{ku}_p^{\wedge} \xrightarrow{\mathfrak{J}_p^{\wedge}} \mathsf{pic}^{\mathrm{ev}}(\hat{\mathbb{S}}_p)_p^{\wedge} \xrightarrow{\delta_p^{\wedge}} (\Sigma\mathrm{H}\mathbb{Z}_p^\times)_p^{\wedge},$$

then an argument identical to that in the proof of Theorem 6.5 shows that there is no indeterminacy in this case, i.e.,

$$\langle \delta_p^{\wedge}, \mathfrak{J}_p^{\wedge}, \hat{\psi}^q - 1\rangle = \{q\}.$$

7. Thomified $\mathbb{A}_n$-structure of Moore spectra

Having established the unreduced stable Adams conjecture, let us turn our attention to our main application — detecting Thomified $\mathbb{A}_n$-structures of $\mathrm{M}_p(i)$. The goal of this section is to prove Theorem 1.13.

By (1.10) and (1.11), we obtain an $\mathbb{A}_n$-structure on $\mathrm{M}_p(i)$ if the diagram

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\epsilon_p(i)"] \arrow[d] & \Sigma^{-1}\mathrm{Cof}(\psi^q - 1) \\ \Sigma^{-2}\mathbb{CP}^n \arrow[ur, dashed] & \end{tikzcd} \tag{7.1}$$

admits a solution in $\mathrm{Ho}(\mathcal{S}\mathrm{p})$; see [Bha20, §4].

THEOREM 7.2. *A map $f : \mathbb{S} \to \Sigma^{-1}\mathrm{Cof}(\psi^q - 1)$ representing the class $p^{i-1}k$, where k is prime to p, extends to a map from $\Sigma^{-2}\mathbb{CP}^{p^i-1}$ but does not extend to a map from $\Sigma^{-2}\mathbb{CP}^{p^i}$.*

Proof. Let $\mathrm{KU}_{(p)} := \beta^{-1}\underline{\mathsf{ku}}_{(p)}$, where β is the Bott class in degree 2. Recall that

$$[\mathbb{CP}^n_+, \mathrm{KU}_{(p)}]_* \cong \mathbb{Z}_{(p)}[\![e]\!][\beta^{\pm}]/(e^{n+1}),$$

where $e = \gamma - 1$ is a class in degree 0 and γ is the tautological line bundle over $\mathbb{CP}^n$. Under the natural map induced by $\underline{\mathsf{ku}}_{(p)} \longrightarrow \mathrm{KU}_{(p)}$, the elements of $\underline{\mathsf{ku}}^2_{(p)}\mathbb{CP}^n \cong \underline{\mathsf{ku}}^2_{(p)}\mathbb{CP}^n_+$ can be identified with a subgroup

$$\mathrm{W}_n \subset \mathrm{KU}^2_{(p)}\mathbb{CP}^n_+ \cong \mathbb{Z}_{(p)}[\![e]\!]/(e^{n+1})\{\beta^{-1}\}$$

such that $\mathrm{KU}^2_{(p)}\mathbb{CP}^n_+/\mathrm{W}_n$ is isomorphic to $\mathbb{Z}_{(p)}[e]/(e^2, \mathbb{Z}\{e\})$. Thus, we may write an element in $\underline{\mathsf{ku}}^2_{(p)}\mathbb{CP}^n_+$ as

$$y := \beta^{-1}f(e)$$

such that $f(0) = 0$ and $f'(0) \in \mathbb{Z}$.

Now consider the cofiber sequence of spectra:

$$\Sigma^{-1}\mathrm{Cof}(\psi^q - 1) \longrightarrow \underline{\mathsf{ku}}_{(p)} \longrightarrow \underline{\mathsf{ku}}_{(p)}.$$

Mapping $\Sigma^{-2}\mathbb{CP}^n$ into this sequence gives rise to a long exact sequence, reducing the question to finding a class $y \in \underline{\mathsf{ku}}^2_{(p)}\mathbb{CP}^n$ that restricts to $p^{i-1}k \in \underline{\mathsf{ku}}^2_{(p)}\mathbb{CP}^1$ and is fixed under ψ^q. Since $\psi^q\beta = q\beta$ and

$$\psi^q(e) = \psi^q(\gamma - 1) = \gamma^q - 1 = (1+e)^q - 1,$$

$f(e)$ must satisfy

$$f((1+e)^q - 1) = qf(e).$$

By Lemma 7.3, we know that rationally f must be of the form

$$f(e) = c\sum_{k=1}^{n} \frac{(-1)^{j+1}}{j}e^j \in \mathbb{Q}[\![e]\!]/(e^{n+1}).$$

Moreover, $c = p^{i-1}k$ as y must restrict to $p^{i-1}k \in \underline{\mathsf{ku}}^2_{(p)}\mathbb{CP}^1 \cong \mathbb{Z}$. From the above formula, it is clear that $f \in \mathrm{W}_n \subset \mathbb{Z}_{(p)}[\![e]\!]/(e^{n+1})$ if and only if $n < p^i$. Hence, the result. $\square$

LEMMA 7.3. *Fix $m > 1$ and $r > 1$. If $f \in \mathbb{Q}[\![x]\!]/(x^m)$ satisfies the relation*

$$f((1+x)^r - 1) \equiv rf(x) \mod x^m, \tag{7.4}$$

then $f(x) \equiv c\ln(1+x) \mod x^m$ for some constant $c \in \mathbb{Q}$.

Proof. Putting $x = 0$, we get $f(0) = 0$. Now let $f(x) \equiv \Sigma_{i=1}^{m-1} a_i x^i \bmod x^m$ and consider the formal difference

$$f((1+x)^r - 1) - rf(x) \equiv d_1x + d_2x^2 + \cdots + d_mx^{m-1} \mod x^m.$$

It is easy to see that $d_1 = 0$, $d_2 = (r^2 - r)a_2 - \binom{r}{2}a_1$ and in general, d_k, for $k < m$, is a linear combination of $a_1, \ldots, a_k$, where coefficient of a_k is $r^k - r$.

When $r > 1$ and f satisfies (7.4), the value of a_k for $k > 1$ are decided by the value of a_1. Hence, $f(x)$ is uniquely determined by a_1, the coefficient of x. Since $a_1 \ln(1+x) \mod x^m$ satisfies (7.4) with a_1 as the coefficient of x, the result follows. □

An argument almost identical to that of Theorem 7.2 leads to

THEOREM 7.5. *A map $f : \mathbb{S} \to \mathrm{L}_{\mathrm{K}(1)}\Sigma^{-1}\mathrm{Cof}(\psi^q - 1)$ representing the class $p^{i-1}\lambda$, where $\lambda \in \hat{\mathbb{Z}}_p^\times$, extends to a map from $\Sigma^{-2}\mathbb{CP}^{p^i-1}$ but does not extend to a map from $\Sigma^{-2}\mathbb{CP}^{p^i}$.*

Proof of Theorem 1.13. Recall from (1.12) that when p is an odd prime $\epsilon_p(i) = (p-1)p^{i-1}$ and when $p = 2$, we have $\epsilon_2(i+1) = 2^{i-1}$. Theorem 7.2 implies that an extension of (7.1) exists if and only if $i < n$. Thus, $\mathrm{M}_p(i)$ admits a Thomified $\mathbb{A}_{p^i-1}$-structure when p odd, and at $p = 2$, $\mathrm{M}_2(i+1)$ admits a Thomified $\mathbb{A}_{2^i-1}$-structure.

The "non-existential part" of Theorem 1.13, at odd primes, follows from Theorem 7.5 and the fact that $\mathrm{L}_{\mathrm{K}(1)}\Sigma^{-1}\mathrm{Cof}(\psi^q - 1) \simeq \mathrm{L}_{\mathrm{K}(1)}\hat{\mathbb{S}}_p$. To see this, consider the composite

(7.6)

$$\begin{tikzcd}
\gamma_i : \mathbb{S} \arrow[r, "\epsilon_p(i)"] & \Sigma^{-1}(\mathrm{Cof}(\psi^q - 1)_{\geq 1}) \arrow[d, "\tilde{\mathfrak{J}}_{\geq 1}"] & & \\
& \Sigma^{-1}\,\mathrm{bgl}_1(\mathbb{S}_{(p)}) \simeq \mathrm{gl}_1(\mathbb{S}_{(p)}) \arrow[r] & \mathrm{gl}_1(\hat{\mathbb{S}}_p) \arrow[d] & \\
& & \mathrm{L}_{\mathrm{K}(1)}\,\mathrm{gl}_1(\hat{\mathbb{S}}_p) \arrow[r, "\ell"] & \mathrm{L}_{\mathrm{K}(1)}\hat{\mathbb{S}}_p,
\end{tikzcd}$$

where $\mathrm{Cof}(\psi^q-1)_{\geq 1}$ is the 0-connected cover of $\mathrm{Cof}(\psi^q-1)$ and ℓ is the Rezk's logarithm map. By [Kuh89], ℓ is a weak equivalence, and therefore the map γ_i of (7.6) belongs to the class

$$p^i\lambda \in \pi_0\mathrm{L}_{\mathrm{K}(1)}\hat{\mathbb{S}}_p,$$

for some $\lambda \in \hat{\mathbb{Z}}_p^\times$. Thus, a Thomified $\mathbb{A}_{p^i}$-structure on $M_p(i)$, i.e., a solution to (1.10) with $n = p^i$, would contradict Theorem 7.5.

At $p = 2$, $\mathrm{L}_{\mathrm{K}(1)}\Sigma^{-1}\mathrm{Cof}(\psi^q - 1)$ is not equivalent to the K(1)-localization of $\hat{\mathbb{S}}_2$. Therefore, we do not know if $\mathrm{M}_2(i+1)$ supports a Thomified $\mathbb{A}_{2^i}$-structure or not. However, we will show that $\mathrm{M}_2(i+1)$ cannot support a Thomified $\mathbb{A}_{2^i+1}$-structure.

By [Kuh89],

$$\mathrm{L}_{\mathrm{K}(1)}\,\mathrm{gl}_1(\hat{\mathbb{S}}_p) \xrightarrow[\ell]{\simeq} \mathrm{L}_{\mathrm{K}(1)}\hat{\mathbb{S}}_p \simeq \mathrm{Fib}(\psi^q_{\mathbb{R}} - 1 : \mathrm{KO}_2^{\hat{}} \to \mathrm{KO}_2^{\hat{}}),$$

where $\psi^q_{\mathbb{R}}$ is the q-th real Adams operation. Various formulas in [Rez06] imply that γ_{i+1} composed with

$$\mathsf{c} : \mathrm{L}_{\mathrm{K}(1)}\hat{\mathbb{S}}_p \longrightarrow \mathrm{L}_{\mathrm{K}(1)}\Sigma^{-1}\mathrm{Cof}(\psi^q - 1)$$

belongs to the class $2^i\lambda \in \pi_0\mathrm{L}_{\mathrm{K}(1)}\Sigma^{-1}\mathrm{Cof}(\psi^q - 1)$. Thus, by Theorem 7.5, a solution to (1.10) does not exist if $n = 2^{i+1}$. Hence, $\mathrm{M}_2(i+1)$ cannot support a Thomified $\mathbb{A}_{2^{i+1}}$-structure. □

Remark 7.7. There may exist "exotic" $\mathbb{A}_n$-structures on $\mathrm{M}_p(i)$ that are not Thomified. Our argument does not address such structures. It will be very interesting to see if there exist exotic $\mathbb{A}_n$-structures on $\mathrm{M}_p(i)$ and, if possible, enumerate them.

Appendix A. Comparison with the work of Friedlander

In [Fri80], Friedlander describes an approach to a p-completed version of the stable Adams conjecture based on the theory of fibrations of Gamma spaces ($\mathcal{F}$-$\mathcal{T}$op).

In the first part of [Fri80] (Sections 1 through 6), Friedlander develops the theory of X-fibrations of Gamma spaces (see (A.5)), and his main result is a classification theorem for X-fibrations [Fri80, Th. 6.1]. This is an elegant idea. Indeed, assuming the validity of [Fri80, Th. 6.1] (which we have no reason to doubt) we shall outline a proof below of the p-local stable Adams conjecture, taking X to be the localization of the 2-sphere $\mathrm{X} = \mathrm{S}^2_{(p)}$.

Subsequent sections of [Fri80] (Sections 7 through 10) extend the theory of X-fibrations to the theory of *completed* X-fibrations and prove a corresponding classification theory for completed X-fibrations [Fri80, Th. 7.9]. Applying the classification theorem for p-completed S^2-fibrations allows Friedlander to claim the following p-completed version of the stable Adams conjecture.

THEOREM A.1 ([Fri80, Th. 10.4]). *The following sequence of maps is null-homotopic*

$$\hat{\mathsf{ku}}_p \xrightarrow{\hat{\psi}^q - 1} \hat{\mathsf{ku}}_p \xrightarrow{\mathfrak{J}^1_p} \mathsf{pic}^{\mathrm{ev}}_1(\hat{\mathbb{S}}_p),$$

where $\mathfrak{J}^1_p$ is the (*canonical*) *lift of $\mathfrak{J}_p^{\hat{}}$ to the fiber $\mathsf{pic}^{\mathrm{ev}}_1(\hat{\mathbb{S}}_p)$ of the covering map*

$$\delta_p^{\hat{}} : \mathsf{pic}^{\mathrm{ev}}(\hat{\mathbb{S}}_p)_p^{\hat{}} \longrightarrow (\Sigma\mathrm{H}\mathbb{Z}_p^{\times})_p^{\hat{}}.$$

Remark A.2. Notice that (A.1) stands in contradiction to the conclusion of (6.11). Indeed, the above theorem would imply that

$$\{1\} \in \langle \delta_p^{\hat{}}, \mathfrak{J}_p^{\hat{}}, \hat{\psi}^q - 1\rangle,$$

which clearly contradicts (6.11). We thank the referee for pointing out this apparent contradiction. In fact, the referee offered us an *alternate* contradiction to (A.1) based on the following general fact (we leave the proof of this fact to the interested reader): Assume p is odd and that one has a map

$$\tilde{\mathfrak{J}}'_p : \mathrm{Cof}(\hat{\psi}^q - 1) \longrightarrow \mathsf{pic}^{\mathrm{ev}}(\hat{\mathbb{S}}_p),$$

with the property that the composite of $\tilde{\mathfrak{J}}'_p$ with the Rezk logarithm

$$\ell : \mathsf{pic}^{\mathrm{ev}}(\hat{\mathbb{S}}_p) \longrightarrow \mathrm{L}_{\mathrm{K}(1)}\mathbb{S}^1$$

is an isomorphism on non-negative *even* homotopy groups. Then $\ell \circ \tilde{\mathfrak{J}}'_p$ is in fact an isomorphism on *all* non-negative homotopy groups on p-completion. Applying this observation to π_1 gives rise to another contradiction in Friedlander's (A.1). We suspect that the orientability assumptions required to develop the theory of *completed* X-fibrations are the most likely source of this contradiction (based on the fact that completions of fibrations fail to be fibrations in general).

Remark A.3. It should be noted that Friedlander also states a 0-connected version of (A.1) (i.e., a reduced p-complete stable Adams conjecture) in the introduction to [Fri80]. This version does not trigger a contradiction as above, and it appears to be valid as stated.

In order to show how one may prove Theorem 1.5 using the first six sections of [Fri80], let us begin by recalling the notion of an X-fibration of Gamma spaces. Let $\mathbf{N}$ denote the permutative category of natural numbers as introduced in Remark 2.6. The n-th space of the Gamma space $\mathcal{N} := (\mu \circ \mathrm{B})(\mathbf{N})$ (where μ and B are functors as in (2.1)) is the discrete space

$$\mathcal{N}_n \cong \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n\text{-times}},$$

where the functors μ and B are as in (2.1). Note that if $\mathcal{B} \in \mathcal{F}\text{-}\mathcal{T}\mathrm{op}$ such that we have a map $\mathcal{B} \to \mathcal{N}$, then the n-th space of $\mathcal{B}$ is a disjoint union

$$\mathcal{B}_n = \coprod_{I \in \mathbb{N}^{\times n}} \mathcal{B}_I.$$

Let X be an object in $\mathcal{T}\mathrm{op}_*$. An X-fibrations is a map $\pi : \mathcal{E} \to \mathcal{B}$ over $\mathcal{N}$ such that

$$\pi_I : \mathcal{E}_I \longrightarrow \mathcal{B}_I$$

is a sectioned map such that $\mathrm{Fib}(\pi_I) \simeq \mathrm{X}^{\wedge i_1} \times \cdots \times \mathrm{X}^{\wedge i_n}$ (where $I = (i_1, \ldots, i_n)$), along with additional criteria as listed in [Fri80, Def. 3.2].

Definition A.4. Two X-fibrations $\pi : \mathcal{E} \to \mathcal{B}$ and $\pi' : \mathcal{E}' \to \mathcal{B}$ are equivalent if there is a map $f : \mathcal{E} \to \mathcal{E}'$ over $\mathcal{B}$ such that

$$f_I : \mathcal{E}_I \to \mathcal{E}'_I$$

are fiberwise weak equivalences.

Friedlander showed that the object $\mathcal{B}\mathbf{Gl}_{\mathrm{X}} := (\mathrm{B} \circ \mu)(\mathbf{Gl}_{\mathrm{X}})$ in $\mathcal{F}\text{-}\mathcal{T}\mathrm{op}$ is the classifying space for X-fibrations [Fri80, Th. 6.1]; i.e., there is an isomorphism of sets

$$\{\text{X-fibrations over } \mathcal{B}\}/(\sim) \quad \cong \quad [\mathcal{B}, \mathcal{B}\mathbf{Gl}_{\mathrm{X}}], \tag{A.5}$$

where $[-,-]$ refers to the homotopy classes of maps in $\mathcal{F}\text{-}\mathcal{T}\mathrm{op}$.

Remark A.6. The universal X-fibration is constructed as follows. Consider the permutative category $\mathbf{UGl}_{\mathrm{X}}$ whose objects are pairs (n, y) such that $n \in \mathbb{N}$ and y is a point in some Y that is weakly equivalent to $\mathrm{X}^{\wedge n}$, and morphisms

$$\mathrm{Mor}_{\mathbf{UGl}_{\mathrm{X}}}((n, y), (m, y')) \subset \mathrm{G}(\mathrm{Y}, \mathrm{Y}')$$

consist of those maps that send y to y' if $m = n$, empty otherwise. Note that there is a functor

$$\pi_u : \mathbf{UGl}_{\mathrm{X}} \longrightarrow \mathbf{Gl}_{\mathrm{X}}$$

such that $\pi_u((n, y)) = (n, \mathrm{Y})$ if $y \in \mathrm{Y}$. This functor admits a section that sends a space (n, Y) to $(n, *)$, where $*$ is the basepoint of Y. The object

$$\mathcal{B}\mathbf{UGl}_{\mathrm{X}} := (\mathrm{B} \circ \mu)(\mathbf{UGl}_{\mathrm{X}}) \in \mathcal{F}\text{-}\mathcal{T}\mathrm{op}$$

is the "total space" of the universal X-fibration

$$\pi_{\mathcal{U}} := \mathrm{B} \circ \mu(\pi_u) : \mathcal{B}\mathbf{UGl}_{\mathrm{X}} \longrightarrow \mathcal{B}\mathbf{Gl}_{\mathrm{X}}.$$

Note that the $\pi_{\mathcal{U}}$ is a map over $\mathcal{N}$.

Using (A.5), one can construct the J-homorphism in the category $\mathcal{F}\text{-}\mathcal{T}\mathrm{op}$ by constructing an $\mathrm{S}^2_{(p)}$-fibration over $(\mathrm{B} \circ \mu)(\mathbf{Gl}_{\mathbb{C},p})$. Consider the permutative category $\mathbf{SGl}_{\mathbb{C},p}$ whose objects are the points of $\coprod_{i \in \mathbb{N}} \tilde{\mathrm{S}}^{2i}_{(p)}$ (see (4.30)) and whose morphisms are those elements

$$f \in \mathrm{Mor}_{\mathbf{SGl}_{\mathbb{C},p}}(x, x') \subset \widetilde{\mathrm{Gl}}_i$$

for which $\tilde{\iota}_i(f, x) = x'$. It is understood that $\mathrm{Mor}_{\mathbf{SGl}_{\mathbb{C},p}}(x, x') = \emptyset$ if $x \in \tilde{\mathrm{S}}^{2i}_{(p)}$ and $x' \in \tilde{\mathrm{S}}^{2j}_{(p)}$ where $i \neq j$. If we declare the monoidal product as

$$x \oplus x' = \tilde{\rho}_{i,j}(x, x')$$

on objects and

$$f \oplus g = \mu_{i,j}(f, g)$$

on morphisms, then it follows from (4.35) and (4.36) that $\mathbf{SGl}_{\mathbb{C},p} \in \mathcal{P}\mathrm{C}$.

There is a functor $\pi_{\mathbb{C}} : \mathbf{SGl}_{\mathbb{C},p} \to \mathbf{Gl}_{\mathbb{C},p}$ which sends $x \mapsto i$ if $x \in \tilde{\mathrm{S}}^{2i}_{(p)}$. The functor $\pi_{\mathbb{C}}$ admits a section $\mathrm{s}_{\mathbb{C}} : \mathbf{Gl}_{\mathbb{C},p} \to \mathbf{SGl}_{\mathbb{C},p}$ that sends $i \mapsto *_i$, where $*_i$ is the basepoint of $\tilde{\mathrm{S}}^{2i}_{(p)}$. On applying the functor $\mathrm{B} \circ \mu$ we get an $\tilde{\mathrm{S}}^{2}_{(p)}$-fibration

$$\mathcal{B}\pi_{\mathbb{C}} : \mathcal{B}\mathbf{SGl}_{\mathbb{C},p} \longrightarrow \mathcal{B}\mathbf{Gl}_{\mathbb{C},p}.$$

Thus by the classification theorem of Friedlander's (A.5) we obtain the J-homomorphism

$$\mathrm{J} : \mathcal{B}\mathbf{Gl}_{\mathbb{C},p} \longrightarrow \mathcal{B}\mathbf{Gl}_{\tilde{\mathrm{S}}^2_{(p)}}$$

in $\mathrm{Ho}(\mathcal{F}\text{-}\mathcal{T}\mathrm{op})$.

Also note that the maps $\{\hat{\psi}^q_i : i \in \mathbb{N}\}$ produce a monoidal functor

$$\mathbf{S}(\Psi^q) : \mathbf{SGl}_{\mathbb{C},p} \longrightarrow \mathbf{SGl}_{\mathbb{C},p}$$

such that we have a commutative diagram

$$\begin{tikzcd}
\mathbf{SGl}_{\mathbb{C},p} \arrow[r, "\mathbf{S}(\Psi^q)"] \arrow[d, "\pi_{\mathbb{C}}"] & \mathbf{SGl}_{\mathbb{C},p} \arrow[d, "\pi_{\mathbb{C}}"] \\
\mathbf{Gl}_{\mathbb{C},p} \arrow[r, "\Psi^q"'] & \mathbf{Gl}_{\mathbb{C},p}
\end{tikzcd}$$

in $\mathcal{P}\mathrm{C}$. On applying the functor $\mathrm{B} \circ \mu$, we get a map of X-fibrations

$$\begin{tikzcd}
\mathcal{B}\mathbf{SGl}_{\mathbb{C},p} \arrow[r, "\mathcal{B}\mathbf{S}(\Psi^q)"] \arrow[d, "\mathcal{B}\pi_{\mathbb{C}}"'] & \mathcal{B}\mathbf{SGl}_{\mathbb{C},p} \arrow[r] \arrow[d, "\mathcal{B}\pi_{\mathbb{C}}"] & \mathcal{B}\mathbf{UGl}_{\tilde{\mathrm{S}}^2_{(p)}} \arrow[d] \\
\mathcal{B}\mathbf{Gl}_{\mathbb{C},p} \arrow[r, "\mathcal{B}(\Psi^q)"] & \mathcal{B}\mathbf{Gl}_{\mathbb{C},p} \arrow[r, "\mathrm{J}"'] & \mathcal{B}\mathbf{Gl}_{\tilde{\mathrm{S}}^2_{(p)}}.
\end{tikzcd}$$

The map $\mathcal{B}\mathbf{S}(\Psi^q)$ is fiberwise a weak equivalence because of Corollary 4.39. Thus by (A.5),

$$\mathrm{J} \simeq \mathrm{J} \circ \mathcal{B}(\Psi^q)$$

in $\mathrm{Ho}(\mathcal{F}\text{-}\mathcal{T}\mathrm{op})$. By applying the Segal functor Φ, we obtain another proof of Theorem 1.5.

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