

NORMS FOR COMPACT LIE GROUPS IN EQUIVARIANT STABLE HOMOTOPY THEORY

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ABSTRACT. We propose a construction of an analogue of the Hill-Hopkins-Ravenel relative norm N_H^G in the context of a positive dimensional compact Lie group G and closed subgroup H . We explore expected properties of the construction. We show that in the case when G is the circle group (the unit complex numbers), the proposed construction here agrees with the relative norm constructed by Angeltveit, Gerhardt, Lawson, and the authors using the cyclic bar construction. Our construction is based on a new perspective on equivariant factorization homology, using framings to convert from actions of one group to another.

INTRODUCTION

The work of Hill-Hopkins-Ravenel on the Kervaire invariant problem has reinvigorated interest in the foundations of equivariant stable homotopy theory. In particular, the multiplicative norm construction, a key technical tool in their work, provides a spectral version of the Evens transfer from group cohomology.

The theory of the norm now seems to complete our understanding of the multiplicative structure on the (genuine) equivariant stable category when G is finite: additively, the equivariant stable category is characterized by the existence of transfers, or equivalently, the Wirthmuller isomorphism; the multiplicative structure is similarly characterized by the presence of compatible systems of multiplicative norm functors. The combinatorics of the transfers and norms are controlled by the structure of the G - E_∞ operad, and the additive and multiplicative structures are linked by the $(\Sigma_+^\infty, \mathfrak{gl}_1)$ adjunction.

When G is a compact Lie group, the situation is significantly more subtle and less well-understood. The additive structure of the G -stable category is still controlled by transfers, or equivalently, Wirthmuller isomorphisms. However, the classical multiplicative structures appear to be different. Specifically, the multiplicative structure of a commutative ring G -spectrum (= G - E_∞ ring spectrum) can be described in terms of an operad, but we know that the additive structure cannot be described in this way. Moreover, the theory of the norm in this setting is incomplete, insofar as the construction appears to only make sense for subgroups of finite index in G .

This raises the question of what the norm for compact Lie groups should mean in general. A first clue is provided by the defining adjunction for the norm on commutative ring G -spectra; in this setting, the norm N_H^G is the left adjoint to the

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forgetful functor from G -spectra to H -spectra. These adjoints exist and are homotopically meaningful for any subgroup H in G . Therefore, we have norm functors for at least commutative ring G -spectra. To explore this further, we consider the simplest possible example: $G = S^1$ and the subgroup is the trivial subgroup. In this case, we recover a familiar object: $N_e^{S^1} R$ is precisely $THH(R)$ [2]. This description now makes sense not just for commutative ring spectra but more generally for associative ring spectra.

While $THH(R)$ admits several constructions, its identification in terms of factorization homology of S^1 with values in R suggests an approach to the construction of more general norms using factorization homology. In the case of norms of the form N_e^G , it suggests a construction in terms of a genuine equivariant structure on the factorization homology of G ; it also indicates the type of algebras that could admit such a norm. The construction of a norm N_H^G requires an equivariant version of a factorization homology construction.

The purpose of this paper is to propose and outline an approach to equivariant norms in terms of factorization homology. Specifically, we describe a factorization homology style construction of positive dimensional norms. We discuss aspects of its expected homotopy type in terms of geometric fixed point data and lay out a series of conjectures about the properties of such norms. In the case of the circle group, we show that the proposed definition here agrees with the definition in [2]; the conjectures that pertain to this context are theorems, proved *ibid*. Moreover, whereas [2] only treated the homotopy type of $THH(R)$ as well-defined in the \mathcal{F} -model structure (where weak equivalences are detected on passage to fixed points for finite subgroups of G), the work here identifies the full genuine equivariant homotopy type in the context of the norm.

The factorization homology we use to construct the norm is not equivariant factorization homology as it is usually construed. Our use of factorization homology and its design in Section 3 has the goal of using G -equivariant H -framed manifolds to mediate a conversion of H -equivariant orthogonal spectra (with extra structure) to G -equivariant orthogonal spectra. We are not constructing a general theory of G -equivariant factorization homology here. Nevertheless, our construction appears to reproduce the standard versions of genuine G -equivariant factorization homology for V -framed G -manifolds in the literature. To illustrate this, we show how to obtain a version of G -equivariant factorization homology of V -framed G -manifolds in Section 6 as a special case of the theory of Section 3.

The work here on the norm depends on current work in progress of some subsets of the authors which breaks into two projects. The first, which we cite as [CFH] studies non-equivariant factorization homology from the perspective of making as much of the structure as possible covariant with compact Lie group actions. A mostly complete draft exists but is not publicly available; we state the results that depend on it here as **theorems** (but we acknowledge that they should more properly be labeled as **conjectures**). The second project, which we refer to as [PMI] will study generalizations of the norm construction for finite groups of the form

$$I'(A \wedge_{\Sigma_q \wr H} IX^{(q)})$$

where I is a genuine equivariant H -spectrum, A is a $G \times (\Sigma_q \wr H)$ -space, and I, I' are certain point-set change of universe functors (for some compact Lie groups H, G). In the case when H is a subgroup of a finite group G and A is the set of (numbered) coset representatives of H in G (plus a disjoint base point), this

construction is precisely the norm $N_H^G X$, but for other H , G , and A , it produces more general functors from H -spectra to G -spectra. The purpose of [PMI] is to study the equivariant homotopy theory of such functors and specifically to verify the expected formulas for geometric fixed point spectra (generalizing the norm diagonal formulas). A basic result we will use is the following generalization of [6, B.104, B.146]:

Let G be a compact Lie group, A a Σ_q -free $G \times \Sigma_q$ -CW complex, U a complete G -universe, X' a cofibrant orthogonal spectrum and X either a cofibrant orthogonal spectrum or a cofibrant commutative ring orthogonal spectrum. If $X' \rightarrow X$ is a weak equivalence of orthogonal spectra, then

$$I_{UG}^U(A \wedge_{\Sigma_q} X'^{(q)}) \longrightarrow I_{UG}^U(A \wedge_{\Sigma_q} X^{(q)})$$

is a weak equivalence of orthogonal G -spectra indexed on U .

Most of the work of [PMI] beyond the statement above is preliminary, and anything discussed below that depends on more complicated results or more complicated constructions of this type is labeled as a “conjecture”.

Only the work in Sections 2–4 depend on the unpublished work [CFH] and [PMI]. Sections 5 and 6 depend only on Construction 3.19; they are completely independent of the work in progress and non-conjectural.

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1. UNIVERSES IN EQUIVARIANT STABLE HOMOTOPY THEORY

The purpose of this section is to review the role that “universes” play in the orthogonal spectrum models of equivariant stable homotopy theory. Although constructions like the Hill-Hopkins-Ravenel norm were anticipated for many years [5, §3–4], [10, 1.5], it took a surprisingly long time for the precise definition to appear because describing the correct derived functor requires playing off two distinct perspectives on the role of the universe. This phenomenon permeates our work here, and we use this section to carefully explain the situation and our perspective and terminology. We make no particular claim to originality in this section.

Let G be a compact Lie group. A G -universe is a countably infinite dimensional vector space with linear G -action and a G -invariant inner product satisfying the following properties: (i) U contains a copy of the trivial representation, and (ii) if a given finite dimensional representation occurs in U as a G -stable vector subspace, then U contains a countable direct sum of copies of that representation. The inner product space $\mathbb{R}^\infty = \bigcup \mathbb{R}^n$ with the usual inner product and trivial action is a G -universe, and up to isomorphism, every G -universe contains it as a sub inner product space. We use the notation $V < U$ to mean that V is a G -stable finite dimensional vector subspace of U (which then inherits the structure of a G -invariant

inner product space), and for $W < U$, we write $V < W$ to mean that V is a G -stable vector subspace of W (not necessarily proper). In that case, we write $W - V$ for the orthogonal complement of V in W , and for any $V < U$, we write S^V for the one-point compactification of V and $\Sigma^V(-)$ for the V -suspension $(-) \wedge S^V$.

The classical view is that given a universe, we get a category of spectra (or “prespectra” for some authors) *indexed on U* : a spectrum T indexed on U consists of a based G -space $T(V)$ for every $V < U$ and a structure G -map $\sigma_{V,W}: \Sigma^{W-V}T(V) \rightarrow T(W)$ for every $V < W < U$ such that $\sigma_{V,V} = \text{id}$ and when $V < W < X < U$, $\Sigma_{V,X} = \Sigma_{W,X} \circ \Sigma^{X-W}\Sigma_{V,W}$, that is, the diagram

$$\begin{array}{ccc} \Sigma^{X-W}\Sigma^{W-V}T(V) & \xrightarrow{\Sigma^{X-W}\sigma_{V,W}} & \Sigma^{X-W}T(W) \\ \cong \uparrow & & \downarrow \sigma_{W,X} \\ \Sigma^{X-V}T(V) & \xrightarrow{\sigma_{V,X}} & T(X) \end{array}$$

commutes. We then define homotopy groups and weak equivalences (for example as in [12, III§3]), and inverting these weak equivalences gives the U -universe G -equivariant stable category. In fact, the U -universe G -equivariant stable category is the homotopy category of a (G -topological) model structure on the category of spectra indexed on U ; we do not need the details but they can be found for example in [12, III§4].

Various universes lead to different equivariant stable categories. The role of the universe is to provide G -vector spaces that compact G -manifolds M (e.g., orbits) can embed in; which orbits equivariantly embed in the universe precisely determine the equivariant stable homotopy theory. Put another way, the universe controls which equivariant transfers exist. In the two extreme cases, when U contains only the trivial representation (a *trivial universe*), e.g., $U = \mathbb{R}^\infty$, and when U contains every representation (a *complete universe*), the equivariant stable categories are inequivalent whenever G is not the trivial (one point) group. The U -universe equivariant stable category is often called the *naive equivariant stable category* when U is a trivial universe and the *genuine equivariant stable category* when U is a complete universe. More generally, the adjective *naive* means indexed on a trivial universe and the adjective *genuine* means indexed on a complete universe.

The same approach applies to G -equivariant orthogonal spectra. Given a G -universe U , we define a G -equivariant orthogonal spectra indexed on U to consist of:

- (i) For every $V < U$, a based G -space $T(V)$;
- (ii) For every $V, W < U$, a G -map $\alpha_{V,W}: O(V, W)_+ \wedge T(V) \rightarrow T(W)$, where $O(V, W)$ denotes the G -space of (non-equivariant) isometric isomorphisms from V to W ; and
- (iii) For every $V < W < U$, a G -map $\sigma_{V,W}: \Sigma^{W-V}T(V) \rightarrow T(W)$

satisfying the obvious compatibility relations: the data of (ii) make T a G -topologically enriched functor on the G -topological category with objects $V < U$ and maps $O(-, -)$, the data of (iii) makes T into a spectrum indexed on U , and the maps in (iii) are $O(V, V) \times O(W, W)$ -equivariant. (This formulation is slightly different from the formulation in the standard reference [12, II§2], but gives an equivalent category.) We define the weak equivalences to be the weak equivalences of the underlying G -equivariant prespectra indexed on U ; inverting these weak equivalences, the forgetful functor to the U -universe G -equivariant stable category

is an equivalence. Thus, we can treat the U -universe G -equivariant stable category as the localization of the category of G -equivariant orthogonal spectra indexed on U at its natural weak equivalences.

The category of G -equivariant orthogonal spectra indexed on U has a (G -topological) model structure with fibrations and weak equivalences created by the forgetful functor to G -equivariant spectra indexed on U . We use the existence of this structure, but not the details, which can be found for example in [12, III§4].

The theory of G -equivariant orthogonal spectra admits another interpretation first articulated by Mandell-May in [12, V§1] and popularized by Schwede in [16, §2]. Let \mathcal{S} denote the category of (non-equivariant) orthogonal spectra indexed on \mathbb{R}^∞ , and let \mathcal{S}_U^G be the category of G -equivariant orthogonal spectra indexed on U . Let \mathcal{S}^{BG} be the category of G -objects in \mathcal{S} : an object in \mathcal{S}^{BG} consists of an object T of \mathcal{S} and an associative and unital action map $G_+ \wedge T \rightarrow T$; then \mathcal{S}^{BG} and $\mathcal{S}_{\mathbb{R}^\infty}^G$ are essentially the same categories and are certainly at least canonically isomorphic. The observation of [12, V.1.5] is that even though the point-set categories \mathcal{S}_U^G and \mathcal{S}^{BG} are not isomorphic for a non-trivial universe U , they are equivalent: given a spectrum T indexed on \mathbb{R}^∞ with G -action and any n -dimensional G -inner product space V , let

$$T(V) = O(\mathbb{R}^n, V)_+ \wedge_{O(n)} T(\mathbb{R}^n).$$

Then $T(V)$ is a based G -space and the collection $\{T(V) \mid V < U\}$ has the canonical structure of a G -equivariant orthogonal spectrum indexed on U ; we write $I_{\mathbb{R}^\infty}^U T$ for this object. We can also go the other way, if T is a G -equivariant orthogonal spectrum indexed on U and V is any n -dimensional G -stable subspace of U , the G -spaces

$$O(V, \mathbb{R}^n)_+ \wedge_{O(V)} T(V)$$

(where $O(V) = O(V, V)$) are all canonically isomorphic: any non-equivariant isometric isomorphism $f: V \rightarrow V'$ induces the same isomorphism

$$O(V, \mathbb{R}^n)_+ \wedge_{O(V)} T(V) \longrightarrow O(V', \mathbb{R}^n)_+ \wedge_{O(V')} T(V').$$

This gives a functor $I_U^{\mathbb{R}^\infty}$ from \mathcal{S}_U^G to $\mathcal{S}_{\mathbb{R}^\infty}^G$ or \mathcal{S}^{BG} . The functors $I_{\mathbb{R}^\infty}^U$ and $I_U^{\mathbb{R}^\infty}$ are inverse equivalences of categories. More generally, for any pair of G -universes U, U' , formulas of this type define inverse equivalences $I_{U'}^U, I_U^{U'}$ between the point-set categories \mathcal{S}_U^G and $\mathcal{S}_{U'}^G$. We call these *point-set change of universe functors*.

Since the point-set change of universe functor $I_U^{\mathbb{R}^\infty}$ is an equivalences of categories, we can use it to transport the model structure on G -equivariant orthogonal spectra indexed on U (\mathcal{S}_U^G) to G -equivariant orthogonal spectra indexed on \mathbb{R}^∞ ($\mathcal{S}_{\mathbb{R}^\infty}^G$) or G -objects in orthogonal spectra \mathcal{S}^{BG} . If U is a trivial universe, then this agrees with the intrinsic model structure on $\mathcal{S}_{\mathbb{R}^\infty}^G$; if U is non-trivial, then this model structure is different and the homotopy category is inequivalent. When we view $\mathcal{S}_{\mathbb{R}^\infty}^G$ as G -objects in orthogonal spectra (\mathcal{S}^{BG}), none of these model structures are the “usual” model structure, which would have as its weak equivalences the weak equivalences of the underlying orthogonal spectra. (These are commonly called the *Borel equivalences* and they are the weak equivalences of a *Borel model structure*; we never use these weak equivalences or this model structure in this paper.)

Each universe U therefore produces on \mathcal{S}^{BG} a model structure that we call the *U -universe model structure*. We call its weak equivalences the *U -universe weak equivalences*. The U -universe model structure on \mathcal{S}^{BG} is Quillen equivalent (via

$I_{\mathbb{R}^\infty}^U$ and $I_U^{\mathbb{R}^\infty}$) to the category of G -equivariant orthogonal spectra indexed on U , and in particular the homotopy category of the U -universe model structure is equivalent to the U -universe G -equivariant stable category. One can take the perspective then that there is (up to equivalence) only one point-set category of G -equivariant orthogonal spectra and that the role of universes is to define the weak equivalences, or more rigidly, the model structure.

The construction of norms intrinsically uses the perspective that the point-set category of G -equivariant orthogonal spectra indexed on a complete universe is just the category of G -objects in orthogonal spectra. This paper contains a number of point-set constructions that only make sense for G -objects in orthogonal spectra but that we argue (or conjecture) are homotopically meaningful in U -universe model structures for a complete universe U . We have chosen to indicate this by using point-set change of universe functors to specify inside a point-set construction what our homotopical perspective on the weak equivalences is. We start in spectra indexed on a universe U , use the point-set change of universe functor $I_U^{\mathbb{R}^\infty}$ to \mathbb{R}^∞ , do a point-set construction, and finally do the point-set change of universe functor $I_{\mathbb{R}^\infty}^U$ to index on a universe U' ; this indicates that we expect the overall construction to convert U -universe weak equivalences to U' -universe weak equivalences at least for nice (e.g., U -universe cofibrant) input.

Remark 1.1. Although the point-set change of universe functor $I_U^{\mathbb{R}^\infty}$ admits a right derived functor, the construction of the Hill-Hopkins-Ravenel norm (and our constructions here) use it in a rhetorical and non-homotopical way. As such, it is amazing that the overall construction results in a functor that has a homotopical interpretation, admitting a left derived functor.

2. THE ABSOLUTE CASE

We begin with the absolute case of the norm $N_e^G X$ which takes non-equivariant input and produces a genuine equivariant G -spectrum. Fix a compact Lie group G with manifold dimension d and fix a complete G -universe U satisfying $U^G = \mathbb{R}^\infty$. We also fix a basis of the tangent space of G at the identity and use left invariant vector fields to specify a basis for the tangent space at every point, specifying a parallelization of G . The left action of G on itself is then through maps of parallelized manifolds and embeds G as a subgroup of the topological group of automorphisms of G as a parallelized manifold (see Example 3.6).

We assume a continuous point-set factorization homology functor

$$\int: \mathcal{E}_e \times \mathcal{S}[\mathcal{D}^d] \longrightarrow \mathcal{S},$$

where \mathcal{E}_e is the (topological) category of parallelized d -manifolds (with maps the parallelized embeddings; see Definition 3.3), \mathcal{S} is (as indicated in Section 1) the (topological) category of orthogonal spectra, \mathcal{D}^d is the Boardman-Vogt little d -disk operad, and $\mathcal{S}[\mathcal{D}^d]$ is the (topological) category of \mathcal{D}^d -algebras in \mathcal{S} . Given such a functor, we can make the following point-set definition. In it, (as indicated in Section 1) \mathcal{S}_U^G denotes the category of G -equivariant orthogonal spectra indexed on the universe U .

Definition 2.1. Define a point-set functor $N_e^G: \mathcal{S}[\mathcal{D}^d] \rightarrow \mathcal{S}_U^G$ by

$$N_e^G X = I_{U^G}^U \int_G X$$

where we regard $\int_G X$ as a G -object in \mathcal{S} by the natural left G -action on G . As indicated in Section 1, $I_{U^G}^U$ denotes the point-set change of universe functor from $\mathbb{R}^\infty = U^G$ to U .

Definition 2.1 is a point-set definition that depends inherently on the point-set construction \int . It is not clear nor do we claim that this has any homotopical properties for an arbitrary functor \int . However, bar constructions for factorization homology tend to have good (non-equivariant) homotopical properties that can be leveraged to study the homotopical properties of the construction N_e^G . In particular, the construction we consider in [CFH] results in a functor N_e^G that has many of the properties expected of an equivariant norm, as we now begin to explain.

Before starting an in depth discussion, we need to address a particular subtlety that arises in the equivariant theory for compact Lie groups that does not arise in the non-equivariant theory or the equivariant theory for finite groups. We can work with unital \mathcal{D}^d -algebras or non-unital \mathcal{D}^d -algebras. Standard constructions of factorization homology can take as input a unital \mathcal{D}^d -algebra or a non-unital \mathcal{D}^d -algebra (an algebra over the non-unital little d -disk operad, where we replace $\mathcal{D}^d(0) = 0$ with $\mathcal{D}^d(0) = \emptyset$). For a unital \mathcal{D}^d -algebra X , there is a natural map from the non-unital construction $\int_M^{nu} X$ to the unital construction $\int_M X$ that is always a homotopy equivalence non-equivariantly (but the homotopy inverse and homotopy data are not natural in M). This map is generally not an equivariant weak equivalence for G positive dimensional: when X is the sphere spectrum \mathbb{S} , $\int_M \mathbb{S}$ is G -equivariantly homotopy equivalent to the sphere spectrum, but $\int_M^{nu} \mathbb{S}$ is G -equivariantly homotopy equivalent to $\Sigma_+^\infty \text{Ran}(M)$ for the Ran space of M (the space of finite non-empty subsets of M). Non-equivariantly, the Ran space is contractible and this is a model of \mathbb{S} , but equivariantly, when we take $M = G$ (as for the norm above) and we take $H < G$ a positive dimensional closed subgroup, the geometric fixed points satisfy

$$(\Sigma_+^\infty \text{Ran}(G))^{\Phi H} \cong \Sigma_+^\infty (\text{Ran}(G)^H) = *.$$

This is the wrong answer because factorization homology should take smash products to smash products in the X variable, and so the empty smash product \mathbb{S} should go to the empty smash product \mathbb{S} . But this essentially the only thing that goes wrong: technology of [BHM1] appears sufficient to prove that (under mild hypotheses on X , e.g., the inclusion of the unit $\mathbb{S} \rightarrow X$ is a Hurewicz cofibration) the unital construction fits into a homotopy pushout square

$$\begin{array}{ccc} \int_M^{nu} \mathbb{S} & \longrightarrow & \int_M^{nu} X \\ \downarrow & & \downarrow \\ \mathbb{S} \simeq \int_M \mathbb{S} & \longrightarrow & \int_M X. \end{array}$$

Ayala and Francis [3, 2.1.4] define a filtration on non-unital factorization homology that they call the “cardinality filtration” and they identify the homotopy type of the cofiber in filtration level q as

$$Filt^q \int_M^{nu} X / Filt^{q-1} \int_M^{nu} X \simeq C^{nu}(q, M)^+ \wedge_{\Sigma_q} X^{(q)}$$

(in the case of a compact parallelized manifold M), where C^{nu} denotes the subspace of components of the configuration space where at least one point of the configuration lies in each component of M . Here $(-)^+$ denotes one-point compactification,

and $(-)^{(q)}$ denotes q th smash power. In [CFH], the authors construct a point-set version of this map with enough naturality in the map and the homotopies¹ that in the case of $N_e^G X$, we get an equivariant homotopy equivalence

$$Fil^q \int_G^{nu} X / Fil^{q-1} \int_G^{nu} X \simeq C^{mu}(q, G)^+ \wedge_{\Sigma_q} X^{(q)}.$$

The point-set change of universe functor preserves G -homotopy equivalences. Moreover, as indicated in the introduction, in the case when the underlying orthogonal spectrum of X is cofibrant, the genuine G -equivariant homotopy type of

$$I_{UG}^U(C(q, G)^+ \wedge_{\Sigma_q} X^{(q)})$$

is invariant under weak equivalences in X ; this happens in particular in the case when X is cofibrant as a \mathcal{D}^d -algebra and the result holds also in the case when X is a cofibrant commutative ring orthogonal spectrum. This gives the following result.

Theorem 2.2. *Let X' be a \mathcal{D}^d -algebra whose underlying orthogonal spectrum is cofibrant (e.g., cofibrant \mathcal{D}^d -algebras) and let X be either a \mathcal{D}^d -algebra whose underlying orthogonal spectrum is cofibrant or a cofibrant commutative ring orthogonal spectrum. For the point-set construction of \int in [CFH], a weak equivalence $X' \rightarrow X$ induces a weak equivalence $N_e^G X' \rightarrow N_e^G X$.*

Corollary 2.3. *The left derived functor of N_e^G exists. Moreover, the composite with the derived functor of the forgetful functor $\iota: \text{Com} \rightarrow \mathcal{S}[\mathcal{D}^d]$ from commutative ring orthogonal spectra to \mathcal{D}^d -algebras is the derived functor of the composite $N_e^G \iota$.*

In the case when $X = R$ is a cofibrant commutative algebra, we have another interpretation of the homotopy type of $N_e^G R$, which aligns with the finite theory. To start, the category of commutative ring orthogonal spectra admits indexed colimits over topological spaces; in particular this means that for a space M , we have a commutative orthogonal spectrum $R \otimes M$ with the universal property that the space of maps from $R \otimes M$ to a commutative ring orthogonal spectrum A is homeomorphic to the space of maps from M to the space of maps from R to A

$$\mathcal{T}(M, \mathcal{S}[\text{Com}](R, A)) \cong \mathcal{S}[\text{Com}](R \otimes M, A).$$

In the case $M = G$, this means that $(-) \otimes G$ is the free functor from the category of commutative ring orthogonal spectra to the category of left G -objects in commutative ring orthogonal spectra. The point-set change of universe I_{UG}^U is the free functor from the category of left G -objects in commutative ring orthogonal spectra to the category of commutative ring orthogonal G -spectra indexed on U . Thus, $I_{UG}^U(- \otimes G)$ is the free functor from the category of commutative ring orthogonal spectra to commutative ring orthogonal G -spectra indexed on U (and as such it is clearly a Quillen left adjoint, and in particular preserves weak equivalences between cofibrant objects). In the case when G is a finite group, the tensor $R \otimes G$ is just the smash power of R indexed on the elements of G and the change of universe $I_{UG}^U(R \otimes G)$ is precisely the norm $N_e^G R$ [2, 2.17–18], [6, A.52, A.56].

The tensor $R \otimes G$ also has a cardinality filtration, but in this case the associated graded cofibers look like

$$Fil^q(R \otimes G) / Fil^{q-1}(R \otimes G) \simeq C(q, G)^+ \wedge_{\Sigma_q} (R/\mathbb{S})^{(q)}.$$

¹Specifically, we can make naturality work for a compact Lie group of automorphisms, and we claim no more generality than that.

If the unit map $\mathbb{S} \rightarrow R$ is a Hurewicz cofibration of the underlying orthogonal spectra, then the maps in the filtration are Hurewicz cofibrations and the display above is a G -equivariant homotopy equivalence. The construction of f in [CFH] admits a corresponding filtration and a filtration preserving map $\int_G R \rightarrow R \otimes G$, which induces a homotopy equivalence on associated graded cofibers; the constructions are natural enough that the comparison is G -equivariant, but in this case we have less sharp results on the covariance of the homotopies and do not know that we can make the homotopies G -equivariant. This is not good enough to directly conclude that the map after point-set change of universe I_{UG}^U is a weak equivalence. Nevertheless, the goal of [PMI] is to study geometric fixed points of constructions of the form

$$I_{UG}^U(C(q, G)^+ \wedge_{\Sigma_q} X^{(q)}).$$

Let $K < G$ be a closed subgroup. We note that $C(q, G)^K$ is empty unless K is a finite group whose cardinality divides q , in which case we have a canonical identification

$$C(q, X)^K \cong C(q/\#K, K \backslash G)$$

(where $K \backslash G$ is the left quotient orbit space). We therefore expect that for reasonable X , we will have

$$(I_{UG}^U(C(q, G)^+ \wedge_{\Sigma_q} X^{(q)}))^{\Phi K} \simeq *$$

if K is positive dimensional or K is finite and $\#K$ does not divide q , and

$$(I_{UG}^U(C(q, G)^+ \wedge_{\Sigma_q} X^{(q)}))^{\Phi K} \simeq C(p, G)^+ \wedge_{\Sigma_p} X^{(p)}$$

when $p = q/\#K$ is an integer. A careful study of the comparison map $N_e^G R \rightarrow I_{UG}^U(R \otimes G)$ should prove that it is a homotopy equivalence on each geometric fixed point spectrum, which would then prove the following as a result.

Conjecture 2.4. *Let R be a cofibrant commutative algebra. Then there is a natural weak equivalence of genuine G -spectra*

$$N_e^G R \simeq I_{UG}^U(R \otimes G)$$

where $I_{UG}^U(- \otimes G)$ is the left adjoint of the forgetful functor from commutative ring orthogonal G -spectra indexed on U to commutative ring spectra.

We can use the same ideas as in the discussion preceding the previous conjecture to study the geometric fixed points of the construction N_e^G for more general \mathcal{D}^d -algebras. In the case when $K < G$ is positive dimensional, the following should hold.

Conjecture 2.5. *Let $K < G$ be a closed subgroup of positive dimension. If X is a \mathcal{D}^d -algebra whose underlying orthogonal spectrum is cofibrant or X is a cofibrant commutative ring orthogonal spectrum, then the inclusion of the unit $\mathbb{S} \rightarrow X$ induces a weak equivalence of derived geometric K -fixed point spectra*

$$(N_e^G X)^{\Phi K} \simeq \mathbb{S}.$$

The previous result gives further justification for the identification of N_e^G as a norm, as its analogue is known to hold for the functor $I_{UG}^U(R \otimes G)$.

Theorem 2.6 (Hill-Hopkins [7]). *Let R be a cofibrant commutative ring orthogonal spectrum, and let $K < G$ be a closed subgroup of positive dimension. Then the*

inclusion of the unit \mathbb{S} in R induces a weak equivalence of derived geometric K -fixed point spectra

$$(I_{UG}^U(R \otimes G))^{\Phi K} \simeq \mathbb{S}$$

Proof. (Compare [7, 8.5].) For notational convenience, let $A = I_{UG}^U(R \otimes G)$. Without loss of generality, we can assume that R is a cell commutative orthogonal spectrum; then A is a cell commutative G -spectrum built from attaching a commutative ring cell of the form

$$\mathbb{P}_G(G_+ \wedge \Sigma_{\mathbb{R}^m}^\infty(B^n, \partial B^n)_+)$$

for each cell of the form

$$\mathbb{P}(\Sigma_{\mathbb{R}^m}^\infty(B^n, \partial B^n)_+)$$

building R (where \mathbb{P} and \mathbb{P}_G are the free functors from orthogonal spectra to commutative ring orthogonal spectra and from orthogonal G -spectra to commutative ring orthogonal G -spectra, respectively, and $\Sigma_{\mathbb{R}^m}^\infty$ denotes the left adjoint to the \mathbb{R}^m space functor in either category). The underlying orthogonal G -spectrum of A is then built from \mathbb{S} by attaching orthogonal spectrum cells of the form

$$C = G^q \wedge_H \Sigma_V^\infty(B(W), \partial B(W))_+$$

where $q > 0$, H is a subgroup of Σ_q , V and W are finite dimensional inner product spaces with orthogonal H -actions, and $B(W)$ denotes the unit ball (or more naturally, $B(W)$ is a polydisk $B^{n_1} \times \cdots \times B^{n_q}$ with H acting by permuting the factors). Precisely, for $Z \subset U$ a finite dimensional G -stable subspace, the Z space pair of the orthogonal G -spectrum pair C is

$$C(Z) = G_+^q \wedge_H (\mathcal{J}(V, Z) \wedge (B(W), \partial B(W))_+)$$

(where \mathcal{J} is the Thom space of [12, II.4.1]). Because H is finite and K is not, neither space in the pair $C(Z)$ can have any K -fixed points other than the base point, and so both genuine G -spectra in the pair C have trivial derived geometric K fixed point spectra, for example, by [12, V.4.8.(ii), V.4.8.12]. It follows that cell attachment by C does not change derived geometric K -fixed points, and that the map $\mathbb{S} \rightarrow A$ induces an equivalence on derived geometric K -fixed points. \square

Given the discussion of geometric fixed points above, we would expect that for K finite,

$$(R \otimes G)^{\Phi K} \simeq R \otimes (K \backslash G)$$

as an NK/K -spectrum. The analogous formula

$$(N_e^G X)^{\Phi K} \simeq \int_{K \backslash G} X$$

does not always make sense because $K \backslash G$ is not always parallelizable: $K \backslash G$ inherits a parallelization from G if and only if the Ad action of K on the tangent space of G at the identity $T_e G$ is trivial. In this case, we make the following conjecture.

Conjecture 2.7. *Let $K < G$ be a finite subgroup and assume the Ad action of K on $T_e G$ is trivial. For composite of derived functors $\Phi^K N_e^G$, there is a natural weak equivalence in the non-equivariant stable category*

$$(N_e^G X)^{\Phi K} \simeq \int_{K \backslash G} X.$$

If K is normal, then the Ad action is automatically trivial, and we can identify the righthand side as $N_e^{G/K} X$; in this case, we further conjecture that the above weak equivalence refines to a weak equivalence in the stable category of genuine G/K -spectra. When $K \setminus G$ is not parallelizable, a \mathcal{D}^d -algebra does not have enough structure to make sense of $\int_{K \setminus G}$. Instead, the weak equivalence should take the following form.

Conjecture 2.8. *Let $K < G$ be finite. There is a natural non-equivariant weak equivalence*

$$(N_e^G X)^{\Phi K} \simeq \int_{N \setminus G} \int_{K \setminus N \times T_e(N \setminus G)} X$$

where N is the normalizer of K in G .

Interpreting the formula in the preceding conjecture takes some work. The manifold $N \setminus G$ is not generally parallelizable but its tangent bundle admits a canonical reduction of structure to N for the N -representation given by the action on $T_e(N \setminus G)$; factorization homology $\int_{N \setminus G}$ admits as input N -framed little $T_e(N \setminus G)$ -disk algebras (see Definition 3.10). The inner factorization homology $\int_{K \setminus N \times T_e(N \setminus G)} X$ comes with a canonical such structure. In the case when K acts trivially on $T_e G$, N contains the identity component G_e , and $\int_{N \setminus G}$ is a finite smash power, so the two conjectures agree.

3. FACTORIZATION HOMOLOGY OF H -FRAMED MANIFOLDS

The construction of relative norms N_H^G requires factorization homology for non-parallelizable manifolds; we use this section to review framings of smooth manifolds and how the framing fits into factorization homology. We take a somewhat different approach from most other treatments of factorization homology in that we work in terms of a reduction of structure *group* for the tangent bundle rather than working with reduction of structure on the classifying space level. The work in this section is fundamentally non-equivariant, though it intrinsically involves a structure group H for the framings. We construct a point-set factorization homology functor in this context, which we use in the next section to define relative norms.

Before starting, we fix the following convention. The first condition ensures that the categories we consider are small. The second condition, while undesirable in some contexts, is convenient in the context of factorization homology constructions.

Convention 3.1. When we use the term *manifold*, we will always understand that its underlying topological space is a subspace of \mathbb{R}^∞ and that it has finitely many components.

We begin by fixing some terminology and notation. For M a smooth d -manifold, let $FM \rightarrow M$ denote the frame bundle, a principal $GL(d)$ -bundle: a point consists of an element m of M and a basis for the tangent space $T_m M$. The frame bundle used \mathbb{R}^d as the standard model vector space, but in our most important examples, G and G/H , using the tangent space at the identity $T_e G$ or at the identity coset $T_e H G/H$ is more natural (choice-free) and so we formulate framings in terms of an arbitrary d -dimensional vector space V . A smooth d -manifold has a V -frame bundle $F_V M \rightarrow M$, where a point in $F_V M$ consists of an element m of M and a linear isomorphism from V to $T_m M$. The V -frame bundle has the canonical

structure of a principal $\mathrm{GL}(V)$ -bundle and there is a tautological isomorphism of $\mathrm{GL}(V)$ -bundles

$$F_V M \cong FM \times_{\mathrm{GL}(d)} \mathrm{Iso}(V, \mathbb{R}^d),$$

where $\mathrm{Iso}(V, \mathbb{R}^d)$ denotes the space of linear isomorphisms from V to \mathbb{R}^d .

Terminology 3.2. Let H be a topological group with a given linear action on V , i.e., a homomorphism $\rho: H \rightarrow \mathrm{GL}(V)$. A *tangential H, V -structure* on a smooth d -manifold M (or *tangential H -structure*, when V, ρ is understood) consists of a principal H -bundle $P \rightarrow M$ and an isomorphism of principal $\mathrm{GL}(V)$ -bundles

$$\phi: P \times_H \mathrm{GL}(V) \longrightarrow F_V M.$$

An *H -framed manifold* is a smooth d -manifold together with a choice of tangential H -structure; we use the notation $F_H M \rightarrow M$ for its structural principal H -bundle written $P \rightarrow M$ above and ϕ_M for the structural isomorphism written ϕ above.

When H is the trivial group and $V = \mathbb{R}^d$, we use *parallelized* as a synonym for H -framed.

There is an obvious definition of maps of H -framed manifold in terms of lifts of derivatives that we call H -framed local isometry, but it is too constrained for many purposes. The looser definition of H -framed immersion is the right one for factorization homology.

Definition 3.3. Let L and M be H -framed manifolds (for fixed $\rho: H \rightarrow \mathrm{GL}(V)$). An H -framed immersion $L \rightarrow M$ consists of a smooth immersion (i.e., local diffeomorphism) $f: L \rightarrow M$, a map of principal H -bundles

$$Ff: F_H L \longrightarrow f^* F_H M,$$

and a principal $\mathrm{GL}(V)$ -bundle homotopy

$$If: F_V L \times I \longrightarrow f^* F_V M$$

such that If ends at the derivative viewed as a map of frame bundles $F_V L \rightarrow f^* F_V M$ and begins at the map

$$F_V L \xrightarrow{\phi_L^{-1}} F_H L \times_H \mathrm{GL}(V) \xrightarrow{Ff} f^* F_H M \times_H \mathrm{GL}(V) \xrightarrow{f^* \phi_M} f^* FM.$$

induced by Ff . An H -framed embedding is an H -framed immersion whose underlying map is an open embedding (i.e., diffeomorphism onto an open subset). An H -framed diffeomorphism is a surjective H -framed embedding, or equivalently, an H -framed embedding that has an inverse, which is also an H -framed embedding. An H -framed local isometry is an H -framed immersion where the homotopy If is constant, i.e., the map of principal H -bundles Ff is a lift of the derivative.

Notation 3.4. Let \mathcal{E}_H (or $\mathcal{E}_{H,V}$ when specifying V is needed for clarity) denote the category whose objects are the H -framed manifolds and whose maps are the H -framed embeddings. We topologize the mapping spaces using the (k-ification of) the smooth compact open topology on the space of smooth embeddings.

When V admits an H -invariant inner product, H -framed local isometries are true local isometries for the resulting Riemannian structure; if not, the terminology fits less well.

Example 3.5. Let $V = \mathbb{R}^d$, $H = O(d)$, with ρ the standard inclusion. An H -framing on a d -manifold M then consists of a continuous (in $m \in M$) choice of the orthogonal frames in $T_m M$, and so is equivalent to the choice of a continuous Riemannian metric on M . Given H -framed manifolds L, M , an H -framed local isometry from L to M consists of a smooth map $f: L \rightarrow M$ such that the derivative at each point sends orthogonal frames to orthogonal frames, i.e., it is precisely a local isometry in the usual sense. An H -framed immersion consists of a smooth immersion $f: L \rightarrow M$ and a $\mathrm{GL}(d)$ -equivariant homotopy $I'f_x: FL_x \rightarrow FM_{f(x)}$, continuous in $x \in L$, that starts at a map that takes orthogonal frames to orthogonal frames and ends at Df_x . Any other such homotopy $I''f_x$ can be obtained by pointwise multiplication by a path in $\mathrm{GL}(d)$ that starts at an element of $O(d)$ and ends at the identity, that is to say, an element of the homotopy fiber of ρ . It follows that for fixed f , the space of lifts of f to an H -framed immersion is the space of sections of a locally trivial principal $\mathrm{hofib}(\rho)$ -bundle. Since $\mathrm{hofib}(\rho)$ is contractible, so is this space of sections.

The work of the previous section implicitly used the following example.

Example 3.6. A Lie group G has two canonical H -framings for H the trivial group and $V = T_e G$ the tangent space at the identity: one given by left-invariant vector fields and the other given by right-invariant vector fields. Choosing an isomorphism $\mathbb{R}^d \cong T_e G$ gives G a parallelization. Consider the left-invariant framing or parallelization. For this framing, left multiplication by an element of G is an H -framed local isometry; moreover, it is easy to see that when G is connected, all H -framed local isometries are of this form. The inclusion of G into the space of H -framed local isometries and into the space of H -framed embeddings is continuous.

We will always use the convention of left invariant vector fields. The following generalization will form the basis for the relative norm.

Example 3.7. Let G be a Lie group and $H < G$ a closed subgroup. The orbit space G/H has a canonical $H, T_{eH}G/H$ -tangential structure with H -frame bundle the quotient map $G \rightarrow G/H$: given $g \in G$, the derivative $DL_g|_{eH}$ of left multiplication by g gives an isomorphism of $T_{eH}(G/H)$ with the tangent space of gH . Left multiplication by elements of G give H -framed local isometries.

Another important example is the vector space V itself.

Example 3.8. The vector space V viewed as an d -manifold has the canonical structure of an H -framed manifold: the canonical identification of V with the tangent space $T_v V$ at every point $v \in V$ induces a splitting of the V -frame bundle

$$F_V V \cong V \times \mathrm{GL}(V)$$

and we use the split H -frame bundle

$$F_H V := V \times H$$

with the map $F_H V \rightarrow F_V V$ induced by $\rho: H \rightarrow \mathrm{GL}(V)$. For an H -framed manifold M and an open embedding $f: V \rightarrow M$, a map of H -principal bundles $F_H V \rightarrow f^* F_H M$ is determined by the section

$$V \cong V \times \{e\} \subset V \times H = F_H V \longrightarrow f^* F_H M$$

and an arbitrary section $V \rightarrow f^* F_H M$ induces a map of H -principal bundles $F_H V \rightarrow f^* F_H M$. Similar observations apply to homotopies of maps of principal

$\mathrm{GL}(V)$ -bundles $F_V V \rightarrow f^* F_V M$. It follows that a lift of f to an H -framed embedding determines and is determined by a section s of $f^* F_H M$ and a homotopy over V starting from the composite of s with the map $f^* F_H M \rightarrow F_V M$ induced by ρ and ending at the derivative, viewed as a map $V \rightarrow f^* F_H M$. Similar remarks apply to maps out of any d -manifold that has a given splitting of its V -frame bundle.

Convention 3.9. For the rest of the section, we now fix the vector space V , the topological group H , which we assume to be a Lie group, the homomorphism $\rho: H \rightarrow \mathrm{GL}(V)$, and an H -invariant inner product structure on V , which we assume exists. (This in particular factors ρ through $O(V)$, the linear isometries of V .)

In this context, factorization homology is built from the space of H -framed embeddings of copies of V in H -framed manifolds. In the case where the target is V itself, the H -framed little disks operad gives a small rigid model.

Definition 3.10. Write D for the open unit disk in V . The H -framed little V -disk operad \mathcal{D}_H^V has n th space defined as follows: $\mathcal{D}_H^V(0)$ consists of a single point. An element of $\mathcal{D}_H^V(1)$ consists of an ordered pair (λ, h) where $h \in H$ and λ is a affine transformation $\lambda: V \rightarrow V$ of the form

$$\lambda(v) = v_0 + rhv$$

for some $v_0 \in D$, $r \in (0, 1]$ (and the given element $h \in H$) that takes D into D ; (λ, h) is then completely determined by v_0 , r , h , and we topologize $\mathcal{D}_H^V(1)$ as a subspace of $D \times (0, 1] \times H$. For $n > 1$, $\mathcal{D}_H^V(n)$ is the subspace of $\mathcal{D}_H^V(1) \times \cdots \times \mathcal{D}_H^V(1)$ where the images of D under the affine transformations are disjoint. The identity affine transformation and identity element of H give an identity element in $\mathcal{D}_H^V(1)$. We have a right action of the symmetric group Σ_n on $\mathcal{D}(n)$ by permuting the copies of $\mathcal{D}_H^V(1)$, and we have an operadic composition defined by composing maps and multiplying group elements: the composition

$$\mathcal{D}_H^V(n) \times \mathcal{D}_H^V(j_1) \times \cdots \times \mathcal{D}_H^V(j_n) \longrightarrow \mathcal{D}_H^V(j)$$

(for $j = j_1 + \cdots + j_n$) is defined by

$$\left(\begin{array}{c} ((\lambda_1, h_1), \dots, (\lambda_n, h_n)), \\ ((\mu_{1,1}, g_{1,1}), \dots, (\mu_{1,j_1}, g_{1,j_1})), \\ \dots, \\ ((\mu_{n,1}, g_{n,1}), \dots, (\mu_{n,j_n}, g_{n,j_n})) \end{array} \right) \mapsto \left(\begin{array}{c} (\lambda_1 \circ \mu_{1,1}, h_1 g_{1,1}), \dots, \\ (\lambda_1 \circ \mu_{1,j_1}, h_1 g_{1,j_1}), \\ \dots, \\ (\lambda_n \circ \mu_{n,1}, h_n g_{n,1}), \dots, \\ (\lambda_n \circ \mu_{n,j_n}, h_n g_{n,j_n}) \end{array} \right).$$

We emphasize that the operad \mathcal{D}_H^V depends on the action ρ of H on V and not just the abstract topological group H and vector space V .

Remark 3.11. When the kernel of $\rho: H \rightarrow O(V)$ is trivial, the element h of H in the ordered pair in the definition of $\mathcal{D}_H^V(1)$ (and $\mathcal{D}_H^V(n)$) is completely determined by the affine transformation λ and we can more concisely define $\mathcal{D}_H^V(1)$ as the space of affine transformations $\lambda: V \rightarrow V$ of the form $x \mapsto v_0 + rhv$ that send the unit disk into the unit disk (\mathcal{D}_H^V remains the subspace of $\mathcal{D}_H^V(1)^n$ where the images of the D under the affine transformations are disjoint).

We contrast the H -framed little V -disk operad with the following H -equivariant little V -disk operad often used in equivariant factorization homology.

Definition 3.12. The H -equivariant little V -disk operad \mathcal{D}_V has n th space $\mathcal{D}_V(n)$ the space of those ordered n -tuples of affine transformations of the form

$$v \mapsto v_0 + rv$$

that send the closed unit disk into the closed unit disk where (for $n > 1$) the images overlap only possibly on the boundary. The identity element is the identity map in $\mathcal{D}_V(1)$ and the operadic composition is induced by composing affine transformations. The action of H on V induces an action of H on the embedding spaces by conjugation: $(h \cdot \lambda)(v) = h\lambda(h^{-1}v)$. The identity element $\text{id} \in \mathcal{D}_V(1)$ is fixed under this action, and the operadic composition maps

$$\mathcal{D}_V(n) \times \mathcal{D}_V(j_1) \times \cdots \times \mathcal{D}_V(j_n) \longrightarrow \mathcal{D}_V(j)$$

(for $j = j_1 + \cdots + j_n$) are H -equivariant, making \mathcal{D}_V an operad in the category of H -spaces.

Non-equivariantly \mathcal{D}_V is \mathcal{D}_e^V where e denotes the trivial group, but \mathcal{D}_V and \mathcal{D}_H^V are related equivariantly as follows. Let \mathcal{H} be the operad with $\mathcal{H}(n) = H^n$, where composition is induced by diagonal maps and the group multiplication:

$$\begin{pmatrix} (h_1, \dots, h_n), \\ (k_{1,1}, \dots, k_{1,j_1}), \\ \dots, \\ (k_{n,1}, \dots, k_{n,j_n}) \end{pmatrix} \mapsto \begin{pmatrix} h_1 k_{1,1}, \dots, h_1 k_{1,j_1} \\ \dots, \\ h_n k_{n,1}, \dots, h_n k_{n,j_n} \end{pmatrix}.$$

The H -framed little V -disks operad \mathcal{D}_H^V is then isomorphic to the semidirect product $\mathcal{D}_V \rtimes \mathcal{H}$ of the H -equivariant little V -disks operad \mathcal{D}_V and the operad $\mathcal{H}(n) = H^n$, with composition on the \mathcal{D}_V factor twisted by the action of H on $\mathcal{D}(j_i)$: the isomorphism $\mathcal{D}_V \rtimes \mathcal{H} \rightarrow \mathcal{D}_H^V$ is given on the n -ary part by the formula

$$(3.13) \quad ((\lambda_1, \dots, \lambda_n), (h_1, \dots, h_n)) \in \mathcal{D}_V(n) \times H^n = (\mathcal{D}_V \rtimes \mathcal{H})(n) \\ \mapsto ((\lambda_1 \circ h_1, h_1), \dots, (\lambda_n \circ h_n, h_n)) \in \mathcal{D}_H^V(n).$$

The relationship between the operads is given by the following proposition, which is purely formal and holds in any suitable topological category (see, for example, [15, 2.3]). In it, we understand an H -equivariant \mathcal{D}_V -algebra to be an object X with an action of both the operad \mathcal{D}_V and the topological group H such that the algebra structure maps are H -equivariant.

Proposition 3.14. *The point-set category of H -equivariant \mathcal{D}_V -algebras is isomorphic to the point-set category of \mathcal{D}_H^V -algebras.*

The difference between the two categories is then purely structural or philosophical. We use \mathcal{D}_H^V when we need to work in a non-equivariant context and \mathcal{D}_V when we need to work in an equivariant context. In terms of homotopy categories or ∞ -categories, viewing \mathcal{D}_H^V merely as an operad, the natural notion of weak equivalence on \mathcal{D}_H^V -algebras would correspond to Borel equivalence of H -equivariant \mathcal{D}_V -algebras, which is never what we want here. As a category of H -equivariant orthogonal spectra with extra structure, the category of H -equivariant \mathcal{D}_V -algebras, we have U -universe homotopy theories for all H -universes.

As indicated above its definition, the H -framed little V -disk operad models H -framed embeddings of copies of V into V . To make this precise, we first note that the open disk D is H -framed diffeomorphic to V where we used the canonical

splitting of the V -frame bundle of D to define the H -framed structure. Choosing and fixing an H -framed diffeomorphism, it suffices to discuss H -framed embeddings of copies of D . We have a continuous map

$$\mathcal{D}_H^V(n) \longrightarrow \mathcal{E}_H(D \times \{1, \dots, n\}, D)$$

defined as follows. We can specify the H -framed structure on a smooth map as in Example 3.8. For an element (λ, h) of $\mathcal{D}_H^V(1)$, the underlying smooth map $D \rightarrow D$ is λ , the element h , viewed as a constant section

$$D \longrightarrow f^* F_H D \cong D \times H$$

induces the lift of frame bundles, and we use $t \mapsto r^t \rho(h)$ as the homotopy over D

$$D \times I \longrightarrow f^* F_V D \cong D \times \mathrm{GL}(V)$$

from the image of the lift to the derivative (where r is as in the notation of Definition 3.10, $\lambda(v) = v_0 + rhv$). This specifies a continuous map $\mathcal{D}_H^V(1) \rightarrow \mathcal{E}_H(D \times \{1\}, D)$, and for $n > 1$, the element $((\lambda_1, h_1), \dots, (\lambda_n, h_n))$ goes to the H -framed map $D \times \{1, \dots, n\} \rightarrow D$ that does the lift of (λ_i, h_i) just described on the i th copy. Taken together, the collection $\mathcal{E}_H(D \times \{1, \dots, n\}, D)$ (as n varies) forms an operad, a version of the endomorphism operad $\mathcal{E}nd_{\Pi}(D)$ in \mathcal{E}_H (for the symmetric monoidal product given by disjoint union). The following observation is clear by construction.

Proposition 3.15. *The map $\mathcal{D}_H^V(n) \rightarrow \mathcal{E}_H(D \times \{1, \dots, n\}, D)$ is compatible with the symmetric group action and composition, giving a map of operads $\mathcal{D}_H^V \rightarrow \mathcal{E}nd_{\Pi}(D)$.*

It well-known and well-documented in the literature that the map $\mathcal{D}_H^V(n) \rightarrow \mathcal{E}_H(D \times \{1, \dots, n\}, D)$ is a homotopy equivalence. In fact, we can say more than this. There is an obvious inclusion of the wreath product group $\Sigma_n \wr H$ in the H -framed self-diffeomorphisms of $D \times \{1, \dots, n\}$, where Σ_n permutes the factors and elements of H act by the H -isometries on each summand (an element h of H has underlying smooth map $\rho(h)$, lift specified by the constant section h , and the homotopy over D also constant). This induces a natural action of $\Sigma_n \wr H$ on $\mathcal{E}_H(D \times \{1, \dots, n\}, M)$ (for any H -framed manifold M), and for $M = D$, it restricts to a compatible action on $\mathcal{D}_H^V(n)$. In [CFH], we prove the following theorem about this $\Sigma_n \wr H$ -action.

Theorem 3.16.

- (i) *For any H -framed manifold M , the $\Sigma_n \wr H$ -space $\mathcal{E}_H(D \times \{1, \dots, n\}, M)$ is equivariantly homotopy equivalent to a free $\Sigma_n \wr H$ -CW complex.*
- (ii) *The map $\mathcal{D}_H^V(n) \rightarrow \mathcal{E}_H(D \times \{1, \dots, n\}, D)$ is a $\Sigma_n \wr H$ -equivariant homotopy equivalence.*

We construct factorization homology as a homotopy coend for the (left) action of the operad \mathcal{D}_H^V on an spectrum and the right action of \mathcal{D}_H^V on the following embedding spaces.

Notation 3.17. For an H -framed manifold M , let $\mathcal{E}_M(n) = \mathcal{E}_H(D \times \{1, \dots, n\}, M)$.

We have a map

$$\mathcal{E}_M(n) \times (\mathcal{D}_H^V(j_1, 1) \times \dots \times \mathcal{D}_H^V(j_n, 1)) \longrightarrow \mathcal{E}_M(j)$$

(for $j = j_1 + \dots + j_n$) obtained by composing H -framed embeddings. The collection $\mathcal{E}_M(n)$ forms a symmetric sequence (which is just to say that each $\mathcal{E}_M(n)$ comes with an action of Σ_n), and the map above can be re-interpreted in the category of symmetric sequences as a right action of \mathcal{D}_H^V on \mathcal{E}_M for the plethysm product. This is simpler to explain in terms of associated functors: consider the endofunctors \mathbb{E}_M and \mathbb{D} on orthogonal spectra defined by

$$\mathbb{E}_M X = \bigvee_{n \geq 0} \mathcal{E}_M(n)_+ \wedge_{\Sigma_n} X^{(n)}, \quad \mathbb{D} X = \bigvee_{n \geq 0} \mathcal{D}_H^V(n)_+ \wedge_{\Sigma_n} X^{(n)}.$$

Then \mathbb{D} is the monad associated to the operad \mathcal{D}_H^V , and the composition maps above define a right action of \mathbb{D} on \mathbb{E}_M

$$\mathbb{E}_M \circ \mathbb{D} \longrightarrow \mathbb{E}_M$$

in the category of endofunctors of orthogonal spectra (in terms of composition). In this setting, we have the monadic bar construction of May [13, §9]:

Construction 3.18. Let M be an H -framed manifold and let A be a \mathcal{D}_H^V -algebra in the category of orthogonal spectra. Define the simplicial object $B_\bullet(M; A)$ to be the monadic bar construction $B_\bullet(\mathbb{E}_M, \mathbb{D}, A)$:

$$B_q(M; A) := B_q(\mathbb{E}_M, \mathbb{D}, A) = \mathbb{E}_M \underbrace{\mathbb{D} \cdots \mathbb{D}}_{n \text{ factors}} A$$

where the face maps d_i are induced by the monadic composition $\mathbb{D}\mathbb{D} \rightarrow \mathbb{D}$ (for $0 < i < q$), the action of \mathcal{D}_H^V on A , $\mathbb{D}A \rightarrow A$ (for $i = 0$), and the right action of \mathcal{D}_H^V on \mathcal{E}_M , $\mathbb{E}_M\mathbb{D} \rightarrow \mathbb{E}_M$ (for $i = q$). The degeneracy maps s_i are induced by the monadic unit maps $\text{Id} \rightarrow \mathbb{D}$. We write $B(M; A)$ for the geometric realization, or $B_H(M; A)$ when it is necessary to denote or emphasize the structure group H

The previous construction $B(M; A)$ is a standard formulation of factorization homology $\int_M A$ in the context of H -framed manifolds, at least under some cofibrancy hypotheses on A ; see, for example, [1, IX.1.5], [11, 5.5.2.6], [14, §2.3], [9, Def. 35], [17, 3.14]. Indeed, because $\mathcal{E}_M(n)$ and $\mathcal{D}_H^V(n)$ are Σ_n -equivariantly homotopic to free Σ_n -CW complexes, $B(M; -)$ preserves weak equivalences in \mathcal{D}_H^V -algebras A whose underlying orthogonal spectra are “flat” in the sense that the point-set smash product functor $A \wedge (-)$ preserves weak equivalences. It even suffices for the underlying orthogonal spectra just to have the right smash powers in the sense that the map in the stable category from the derived smash power to the point-set smash power is a weak equivalence. $B(M; A)$ correctly computes $\int_M A$ just under this kind of minimal hypothesis on A .

When a topological group G acts on M through H -framed diffeomorphisms, $B(M; A)$ obtains a natural G -action. When H is the trivial group, this gives the point-set construction of factorization homology from [CFH] with the properties we asserted in the previous section. When H is non-trivial, the construction is too flabby to have the correct G -equivariant homotopy type. We correct this with the following “compressed” bar construction.

Construction 3.19. Let $\bar{\mathbb{D}}$ be the free H -equivariant \mathcal{D}_V -algebra monad on H -equivariant orthogonal spectra

$$\bar{\mathbb{D}} X := \bigvee_{n \geq 0} \mathcal{D}_V(n) \times_{\Sigma_n} X^{(n)}$$

and for an H -framed manifold M , let \bar{E}_M denote the functor from H -equivariant orthogonal spectra to orthogonal spectra defined by

$$\bar{E}_M X := \bigvee_{n \geq 0} \mathcal{E}_M(n) \times_{\Sigma_n \wr H} X^{(n)}.$$

For a \mathcal{D}_H^V -algebra A (viewed as an H -equivariant \mathcal{D}_V -algebra), define the simplicial object $\bar{B}_\bullet(M; A)$ to be the monadic bar construction $B_\bullet(\bar{E}_M, \bar{\mathbb{D}}, A)$:

$$\bar{B}_q(M; A) := B_q(\bar{E}_M, \bar{\mathbb{D}}, A) = \bar{E}_M \underbrace{\bar{\mathbb{D}} \cdots \bar{\mathbb{D}}}_n A$$

with the usual face and degeneracy maps (see Construction 3.18). Write $\bar{B}(M; A)$ for the geometric realization, or $\bar{B}_H(M; A)$ when it is necessary to denote or emphasize the structure group H .

In terms of the H -framed little V -disk operad, the monad $\bar{\mathbb{D}}$ is naturally isomorphic to the monad (on H -equivariant orthogonal spectra)

$$\bigvee_{n \geq 0} \mathcal{D}_H^V(n)_+ \wedge_{\Sigma_n \wr H} X^{(n)}$$

where the resulting H -action is the H -action from the \mathcal{D}_H^V -algebra structure. In concrete terms, the H -action on the n th summand is induced by the left H -action on $\mathcal{D}_H^V(n)$ coming from the H -action on D in the category of H -framed manifolds. (Specifically, $h \in H$ sends $((\lambda_1, h_1), \dots, (\lambda_n, h_n))$ to $((h \circ \lambda_1, hh_1), \dots, (h \circ \lambda_n, hh_n))$.) Under the isomorphism $\mathcal{D}_H^V(n) \cong \mathcal{D}^V(n) \times H^n$ of (3.13), this action corresponds to the diagonal action on

$$\mathcal{D}_V(n)_+ \wedge_{\Sigma_n} X^{(n)}$$

used in Construction 3.19. This makes clear the relationship between $B(M; A)$ and $\bar{B}(M; A)$: expanding out the definitions in terms of the spaces $\mathcal{E}_M(n)$ and $\mathcal{D}_H^V(n)$, $B(M; A)$ is formed from products of these smashed with smash powers of A by coequalizing symmetric group actions and $\bar{B}(M; A)$ is formed by the same products and smash powers by coequalizing the action of the corresponding wreath product with H .

The quotient map

$$B(M; A) \longrightarrow \bar{B}(M; A)$$

is natural in both the H -framed manifold M and the \mathcal{D}_H^V -algebra A , and a straightforward ‘‘Quillen Theorem A’’ argument (plus Theorem 3.16.(i)) proves that it is always a homotopy equivalence. In particular, this is a natural weak equivalence, but the homotopy inverse and homotopy data cannot be made natural in M .

4. THE RELATIVE CASE

We now consider the case when $H < G$ is a closed subgroup of a positive dimensional compact Lie group G and discuss the relative norm N_H^G . In the case when $H < G$ is finite index, we already know how to construct this norm as an equivariant smash power [7, 8.1]; the more interesting case is when $H < G$ is positive codimension. In the finite index case, the relative norm makes sense for any genuine H -spectrum; in the positive codimension case, we need additional structure of precisely the kind introduced in the previous section.

Let G be a compact Lie group and $H < G$ a closed subgroup. Let $V = T_e H G/H$ with its natural H -action. We then have a canonical tangential H, V structure (q.v. Terminology 3.2 and Example 3.7) on G/H and the left multiplication action of G on G/H is an action in the category of H -framed manifolds. Our setup in the previous section assumed an H -invariant inner product on V ; as the space of such inner products is contractible and our constructions are continuous, we can choose one arbitrarily (but a uniform way to choose the inner product for all $H < G$ at once is to choose a G -invariant inner product on $T_e G$). As in general for norms, we work with complete universes: we fix a complete G universe U , which we assume (wlog) contains \mathbb{R}^∞ and also a complete H -universe U_H , which contains \mathbb{R}^∞ . (We can as in Section 2 also assume that $U^G = \mathbb{R}^\infty$ and $(U_H)^H = \mathbb{R}^\infty$ if we so choose, but this is not necessary, and without this requirement, a uniform way to choose U_H for all $H < G$ at once is to use U with action restricted to H .)

Definition 4.1. For X an H -equivariant \mathcal{D}_V -algebra indexed on U_H , we define the relative norm $N_H^G X$ as the point-set functor

$$N_H^G X = I_{\mathbb{R}^\infty}^U \bar{B}(G/H; I_{U_H}^{\mathbb{R}^\infty} X)$$

using the construction \bar{B} of 3.19 and point set change of universe functors I . Here the G -action on $\bar{B}(G/H; -)$ comes from topological functoriality of \bar{B} and the action of G on G/H in the category of H -framed manifolds.

Just as in the absolute case, the relative norm comes with a cardinality filtration from the Ayala-Francis cardinality filtration on (non-unital) factorization homology. In this case, the associated graded cofiber at filtration level q looks (non-equivariantly) like

$$C_H^{nu}(q, G/H)^+ \wedge_{\Sigma_q, H} X^{(q)}$$

where $C_H(q, M)$ denotes the H -framed version of the configuration space: an element consists of a q -tuple of elements of $F_H M$ whose image in M is a configuration (and C_H^{nu} is the subspace where the configuration in M is surjective on π_0). Equivariantly, we expect this piece to be weakly equivalent to

$$I_{\mathbb{R}^\infty}^U (C_H^{nu}(q, G/H)^+ \wedge_{\Sigma_q, H} (I_{U_H}^{\mathbb{R}^\infty} X)^{(q)})$$

at least under the hypothesis that the underlying H -equivariant orthogonal spectrum of X is cofibrant in the U_H -universe model structure or X is an H -equivariant commutative ring orthogonal spectrum and cofibrant in the commutative ring U_H -universe model structure. If this is the case, expected results from [PMI] would then imply that the following conjecture holds.

Conjecture 4.2. *Let X' and X satisfy the Cofibrancy Hypothesis 4.3 below. For the point-set construction of Definition 4.1, a U_H -universe weak equivalence $X' \rightarrow X$ induces a U -universe weak equivalence $N_H^G X' \rightarrow N_H^G X$.*

Hypothesis 4.3. For the purposes of this section, we say that an H -equivariant \mathcal{D}_V -algebra satisfies the ‘‘Cofibrancy Hypothesis’’ if one of the following holds:

- Its underlying H -equivariant orthogonal spectrum is cofibrant in the U_H -universe model structure
- It inherits its H -equivariant \mathcal{D}_V -algebra by virtue of being an H -equivariant commutative ring orthogonal spectrum, and it is cofibrant in the U_H -universe model structure on commutative ring orthogonal spectra.

Restricting to the second case in the hypothesis, the following is an immediate corollary of the conjecture.

Corollary 4.4 (Conjectural). *The left derived functor of N_H^G exists. Moreover, the composite with the derived functor (for the U_H -universe homotopy categories) of the forgetful functor ι from H -equivariant commutative ring orthogonal spectra to H -equivariant \mathcal{D}_V -algebras is the derived functor of the composite $N_H^G \iota$.*

When X is an H -equivariant commutative ring orthogonal spectrum, we would like to compare $N_H^G X$ to the left adjoint functor of the forgetful functor from G -equivariant commutative ring orthogonal spectra to H -equivariant commutative ring orthogonal spectra. Denote this left adjoint as $(-)\otimes_H G$. The point-set model, up to isomorphism, does not depend on the indexing universe, and indexing on \mathbb{R}^∞ , we can identify $(I_{U_H}^{\mathbb{R}^\infty} X)\otimes_H G$ as a quotient of the free G -equivariant commutative ring orthogonal spectrum on $(I_{U_H}^{\mathbb{R}^\infty} X)\wedge_H G_+$. Filtering this with the q th level the image of $((I_{U_H}^{\mathbb{R}^\infty} X)\wedge_H G_+)^{(q)}/\Sigma_q$, the associated graded point-set quotients are then given by the orthogonal G -spectra

$$C_H(q, G/H)_+ \wedge_{\Sigma_q \wr H} (I_{U_H}^{\mathbb{R}^\infty} (X/\mathbb{S}))^{(q)}$$

which we can re-index to U using the point-set change of universe $I_{\mathbb{R}^\infty}^U$. When the inclusion of \mathbb{S} in X is a Hurewicz cofibration of orthogonal H -spectra, the maps in the filtration are Hurewicz cofibrations of orthogonal G -spectra, and we can use this filtration to analyze $X\otimes_H G$ homotopically. Just as in the absolute case, factorization homology of unital algebras has a unital version of the cardinality filtration, with the q -level associated graded cofiber expected (under suitable cofibrancy hypotheses) to be U -universe weakly equivalent to

$$I_{\mathbb{R}^\infty}^U (C_H(q, G/H)^+ \wedge_{\Sigma_q \wr H} (I_{U_H}^{\mathbb{R}^\infty} (X/\mathbb{S}))^{(q)}),$$

that is, the U -re-indexed associated graded quotient above. If this all works, it would then establish the following conjecture.

Conjecture 4.5. *Let R be a cofibrant H -equivariant commutative ring orthogonal spectrum in the U_H -universe model structure. Then there is a natural U -universe weak equivalence*

$$N_H^G R \simeq R \otimes_H G$$

where $(-)\otimes_H G$ is the left adjoint of the (point-set) forgetful functor from G -equivariant commutative ring orthogonal spectra to H -equivariant commutative ring orthogonal spectra.

Analyzing the cardinality filtration gives conjectures for the geometric fixed points:

Conjecture 4.6. *Let X be an H -equivariant \mathcal{D}_V -algebra satisfying the Cofibrancy Hypothesis 4.3 above. If K is a normal subgroup of G and H is finite index in HK , then $H\cap K$ acts trivially on V , the map of H -equivariant inner product spaces $V = T_{e_H}G/H \rightarrow T_{e_{HK}}G/HK$ is an isomorphism, and there exists a natural U^K -universe G/K -equivariant weak equivalence*

$$(N_H^G X)^{\Phi K} \simeq N_{(G/K)/(HK/K)}^{G/K} (X^{\Phi(H\cap K)}).$$

If K is a normal subgroup of G and H is not finite index in HK , then the unit map induces a U^K -universe G/K -equivariant weak equivalence $\mathbb{S} \rightarrow (N_H^G X)^{\Phi K}$.

Having formulated a relative construction, it is now possible to try to iterate norms. For $K < H < G$, we expect an equivalence between $N_K^G X$ and an iterated construction along the lines of $N_H^G N_K^H X$. The first issue that arises is that the inputs for N_K^G and N_K^H do not match: the former wants a K -equivariant little $T_{eK}(G/K)$ -disk algebra whereas the latter wants a K -equivariant little $T_{eK}(H/K)$ -disk algebra. This is minor because (having chosen a G -invariant inner product on $T_e G$), we have a decomposition of K -equivariant inner product spaces

$$(4.7) \quad T_{eK}(G/K) \cong T_{eK}(H/K) \oplus T_{eH}(G/H),$$

and this gives a forgetful functor from the input for N_K^G to the input for N_K^H at least on the level of homotopy categories. A more serious issue is that even when we use a K -equivariant little $T_{eK}(G/K)$ -disk algebra X as the input, the output of N_K^H does not obviously have the structure of a H -equivariant little $T_{eH}G/H$ -disk algebra, which is what is needed as the input to N_H^G . On the other hand, when X is a K -equivariant little $T_{eK}(G/K)$ -disk algebra, we can make sense of the factorization homology

$$\bar{B}(H/K \times D(T_{eH}(G/H)); X)$$

where D denotes the open unit disk and we understand $H/K \times D(T_{eH}(G/H))$ as a $K, T_{eK}(G/K)$ -framed manifold using the isomorphism (4.7) again. We expect the following to hold:

Conjecture 4.8. *Let $K < H < G$ and let X be a K -equivariant little $T_{eK}(G/K)$ -disk algebra. There is a natural zigzag of H -equivariant homotopy equivalence*

$$\bar{B}(H/K \times D(T_{eH}(G/H)); X) \simeq B(H/K, i^* X)$$

where i^* denotes a functor to K -equivariant little $T_{eK}(G/K)$ -disk algebras modeling the homotopical forgetful functor for the decomposition in (4.7).

In particular, $I_{\mathbb{R}^\infty}^{U_H} \bar{B}(H/K \times D(T_{eH}(G/H)); I_{U_K}^{\mathbb{R}^\infty} X)$ should be an acceptable stand-in for $N_K^H(i^* X)$.

The advantage of $\bar{B}(H/K \times D(T_{eH}(G/H)); X)$ is that the K -framed manifold $H/K \times D(T_{eH}(G/H))$ comes with the structure of a $\mathcal{D}_H^{T_{eH}(G/H)}$ -algebra in the category \mathcal{E}_K , as we now explain. The key observation is that the (diagonal) (left) multiplication by H on $H/K \times D(T_{eH}G/H)$ is a K -framed local isometry. The tangential $K, T_{eK}(G/K)$ -structure on $H/K \times D(T_{eH}G/H)$ has K -frame bundle $H \times D(T_{eH}G/H)$ with K acting on the right of H (acting trivially on $D(T_{eH}G/H)$). The identification of

$$(H \times D(T_{eH}G/H)) \times_K T_{eK}(G/K)$$

with the tangent bundle of $H/K \times D(T_{eH}G/H)$ sends an element

$$\begin{aligned} ((h, u), v \oplus w) &\in (H \times D(T_{eH}G/H)) \times_K (T_{eK}(H/K) \oplus T_{eH}(G/H)) \\ &\cong (H \times D(T_{eH}G/H)) \times_K T_{eK}(G/K) \end{aligned}$$

to $((h, u), DL_h|_{eK} v \oplus h \cdot w)$, where $DL_h|_{eK}$ denotes the derivative of left multiplication by h on H/K . For an element $g \in H$, multiplication by g on $H/K \times D(T_{eH}G/H)$ then clearly sends K -frames to K -frames. Given an element

$$((\lambda, h_1), \dots, (\lambda_n, h_n)) \in \mathcal{D}_H^{T_{eH}(G/H)}(n)$$

of the H -framed little $T_{eK}(G/H)$ -disk operad, we then get a K -framed embedding

$$(H/K \times D(T_{eH}G/H)) \amalg \cdots \amalg (H/K \times D(T_{eH}G/H)) \longrightarrow (H/K \times D(T_{eH}G/H))$$

using the group elements h_i diagonally and the affine transformations λ on the disks $D(T_{eH}G/H)$.

A fundamental property of factorization homology is that it is symmetric monoidal in both variables. In particular, our construction \bar{B} is strong symmetric monoidal on the point-set level in the manifold variable: it takes disjoint unions of manifolds to smash products of orthogonal spectra up to coherent natural isomorphism. It follows that an operadic algebra structure on the manifold M (for disjoint union in the category \mathcal{E}_K) induces the same kind of operadic algebra structure on $\bar{B}(M; X)$ for fixed X . In the present context, this discussion proves the following observation.

Proposition 4.9. *Let $K < H < G$ and let X be a K -equivariant little $T_{eK}(G/K)$ -disk algebra. The orthogonal spectrum $\bar{B}(H/K \times D(T_{eH}(G/H)); X)$ has the canonical structure of a $\mathcal{D}_H^{T_{eH}(G/H)}$ -algebra.*

This allows us to make sense of the iterated norm construction. We conjecture the following relationship.

Conjecture 4.10. *Let $K < H < G$ and let X be a K -equivariant little $T_{eK}(G/K)$ -disk algebra whose underlying K -equivariant orthogonal spectrum is cofibrant in the U_K -universe. There is a natural U -universe G -equivariant weak equivalence*

$$N_K^G X \simeq N_H^G I_{\mathbb{R}^\infty}^{U_H} \bar{B}(H/K \times D(T_{eH}(G/H))); I_{U_K}^{\mathbb{R}^\infty} X.$$

As mentioned above, factorization homology is symmetric monoidal in both variables. Symmetric monoidality in the algebra variable should imply this expected property of norms.

Conjecture 4.11. *Let X and Y be H -equivariant \mathcal{D}_V -algebras whose underlying H -equivariant orthogonal spectra are cofibrant in the U_H -universe model structure. There is a natural U -universe G -equivariant weak equivalence*

$$N_H^G X \wedge N_H^G Y \simeq N_H^G (X \wedge Y).$$

5. THE CASE OF THE CIRCLE GROUP

The paper [2] defines norms and relative norms for the circle group S^1 in terms of cyclic bar constructions. In this section, we compare the point-set construction of the norms in [2, 1.1, 8.2] to the point-set construction of the norms in the previous section in the case $G = S^1$; see Theorem 5.1 for a precise statement. This section is independent of the work of [CFH] and [PMI], and depends on the rest of this paper only in its use of the notation, terminology, and definitions.

We fix the positive integer m and consider the subgroup $C_m < S^1$. To avoid notational confusion in what follows, we consistently write Z_n for the cyclic subgroup of Σ_n generated by the cyclic permutation $(1 \cdots n)$.

Let U denote a complete S^1 -universe; we write U_{C_m} for U regarded as a complete C_m -universe. The point-set [2, 8.2] norm is a functor from C_m -equivariant associative ring spectra indexed on U_{C_m} to S^1 -equivariant orthogonal spectra indexed on U built as a composite

$$I_{\mathbb{R}^\infty}^U N_{\wedge}^{cyc, C_m} I_{U_{C_m}}^{\mathbb{R}^\infty}$$

using a C_m relative cyclic bar construction (which we review starting after Theorem 5.14 below) and point-set change of universe functors. For the point-set norm of Definition 4.1, we note that the action of C_m on the tangent space $T_{eC_m} S^1/C_m$ is trivial, and (to simplify notation) we identify this tangent space with \mathbb{R} using the standard metric and orientation on S^1 (with total length 2π , giving S^1/C_m length $2\pi/m$). The norm

$$I_{\mathbb{R}^\infty}^U \bar{B}_{C_m}(S^1/C, I_{U_{C_m}}^{\mathbb{R}^\infty}(-))$$

is then a functor from C_m -equivariant $\mathcal{D}_{\mathbb{R}}$ -algebras (little 1-disk algebras) in orthogonal spectra indexed on U_{C_m} to S^1 -equivariant orthogonal spectra indexed on U . The inputs for these functors differ, but we have a forgetful functor from C_m -equivariant associative ring orthogonal spectra to C_m -equivariant $\mathcal{D}_{\mathbb{R}}$ -algebras (which is an equivalence on homotopy categories), so we state the comparison in the associative case. The following is the main result of this section.

Theorem 5.1. *Let R be a C_m -equivariant associative ring orthogonal spectrum indexed on \mathbb{R}^∞ . There is a natural zigzag of natural S^1 -equivariant homotopy equivalences*

$$\bar{B}_{C_m}(S^1/C_m, R) \longleftarrow B(\bar{\mathbb{E}}, \bar{\mathbb{D}}, R) \longrightarrow B(\bar{\mathbb{E}}^c, \bar{\mathbb{D}}^c, R) \longleftarrow B(\bar{\mathbb{C}}^c, \mathbb{T}, R)$$

and a natural isomorphism

$$B(\bar{\mathbb{C}}^c, \mathbb{T}, R) \cong N_{\wedge}^{cyc, C_m} B(\mathbb{T}, \mathbb{T}, R)$$

of S^1 -equivariant orthogonal spectra indexed on \mathbb{R}^∞ .

The functors of the form $B(-, -, R)$ are all monadic bar constructions, which we explain in detail below. In the first display, all the maps are geometric realizations of maps that are natural S^1 -equivariant homotopy equivalences on each simplicial level (the homotopy inverses and homotopy data is natural in R). See Propositions 5.6, 5.11, 5.12 below. In both displays, \mathbb{T} denotes the free associative algebra monad (in the category of C_m -equivariant orthogonal spectra) and in the second display, $B(\mathbb{T}, \mathbb{T}, R)$ is the geometric realization of the “standard construction” or the two-sided bar construction. The C_m -equivariant associative ring spectrum $B(\mathbb{T}, \mathbb{T}, R)$ comes with a natural map in the category of C_m -equivariant associative ring spectra

$$B(\mathbb{T}, \mathbb{T}, R) \longrightarrow R$$

which is a natural C_m -equivariant homotopy equivalence of the underlying orthogonal spectra. Because point-set change of universe functors are topologically enriched and therefore preserve homotopy equivalences, we get the following norm comparison as a corollary (applying the previous theorem to $I_{C_m}^{\mathbb{R}^\infty} A$ and applying $I_{\mathbb{R}^\infty}^U$ to the resulting zigzag).

Corollary 5.2. *Let A be a C_m -equivariant associative ring orthogonal spectrum indexed on U_{C_m} . The relative norm $N_{C_m}^{S^1} A$ of Definition 4.1 is naturally S^1 -equivariantly homotopy equivalent to the relative norm $N_{C_m}^{S^1} B(\mathbb{T}, \mathbb{T}, A)$ of [2, 8.2].*

The relative norm of [2, 8.2] is known to preserve the weak equivalences used there (the “ \mathcal{F} -equivalences”) under mild hypotheses on its input. For example, it is good enough if the underlying C_m -equivariant orthogonal spectrum indexed on U_{C_m} is cofibrant, and we note that if A satisfies this, then so does $B(\mathbb{T}, \mathbb{T}, A)$. In this case then, we get a weak equivalence between both relative norms on A .

We note that in the case $m = 1$, the theorem and corollary above give an equivariant comparison between factorization homology and THH .

We now begin the proof of the theorem. For the first display, the argument amounts to little more than defining terms. The functor \bar{E} in the statement is a simplification of \bar{E}_{S^1/C_m} along the lines that the little disk operad is a simplification of the disk embedding spaces.

Construction 5.3. Let $E(1)$ be the set of ordered pairs (ζ, r) with $\zeta \in S^1$ and $r \in (0, \pi/m]$. Such an ordered pair specifies an element of $\mathcal{E}_{S^1/C_m}(1)$ where

- The embedding $f: D \rightarrow S^1/C_m$ is the map $t \mapsto [e^{irt}\zeta]$ (writing $[\alpha]$ for the image in S^1/C_m of an element $\alpha \in S^1$, and thinking of S^1 as the unit complex numbers).
- The map of C_m -frame bundles $F_{C_m}D \rightarrow f^*F_{C_m}S^1/C_m$ is determined by the section $t \mapsto e^{irt}\zeta$.
- The map of C_m -frame bundles lies over the map of frame bundles that is the identity (under the identification of the tangent space of S^1/C_m with \mathbb{R} given above) and the derivative is multiplication by r in the fiber of each point $t \in D$; we use the homotopy $s \mapsto r^s$.

We let $E(n)$ be the subspace of $E(1)^n$ where the images of the embeddings do not overlap. We then get an inclusion $E(n) \rightarrow \mathcal{E}_{S^1/C_m}(n)$. We write E for the collection $E(n)$, $n \geq 0$.

The following is clear from construction.

Proposition 5.4. *The right $\mathcal{D}_{C_m}^{\mathbb{R}}$ -action on \mathcal{E}_{S^1/C_m} restricts to define a right $\mathcal{D}_{C_m}^{\mathbb{R}}$ -action on E .*

The map $E(n) \rightarrow \mathcal{E}_{S^1/C_m}(n)$ is $S^1 \times (\Sigma_n \wr C_m)^{\text{op}}$ -equivariant (equivariant for both the left action of S^1 and the right action of $\Sigma_n \wr C_m$). Since $C_n \rightarrow \text{GL}(\mathbb{R})$ is the trivial map, all embeddings in $\mathcal{E}_{S^1/C_m}(n)$ are oriented. The exponential map from \mathbb{R} to the oriented transformations in $\text{GL}(\mathbb{R})$ is an isomorphism, and the following proposition is then easy using linear homotopies.

Proposition 5.5. *For each n , the inclusion of $E(n)$ in $\mathcal{E}_{S^1/C_m}(n)$ is a $S^1 \times (\Sigma_n \wr C_m)^{\text{op}}$ -equivariant homotopy equivalence.*

We define the functor \bar{E} from C_m -equivariant orthogonal spectra to S^1 -equivariant orthogonal spectra as in Construction 3.19: let

$$\bar{E}X = \bigvee_{n \geq 0} E(n)_+ \wedge_{\Sigma_n \wr C_m} X^{(n)}.$$

We then get a monadic bar construction $B(\bar{E}, \bar{\mathbb{D}}, -)$ with input C_m -equivariant $\mathcal{D}_{\mathbb{R}}$ -algebras in orthogonal spectra and output S^1 -equivariant orthogonal spectra. The map of right $\mathcal{D}_{C_m}^{\mathbb{R}}$ -spaces $E \rightarrow \mathcal{E}_{S^1/C_m}$ induces a map of bar constructions, and Proposition 5.5 then implies the following proposition.

Proposition 5.6. *For any C_m -equivariant $\mathcal{D}_{\mathbb{R}}$ -algebra X , the map of monadic bar constructions*

$$B_{\bullet}(\bar{E}, \bar{\mathbb{D}}, X) \longrightarrow B_{\bullet}(\mathcal{E}_{S^1/C_m}, \bar{\mathbb{D}}, X)$$

is on each level a natural S^1 -equivariant homotopy equivalence.

The idea for \bar{E}^c and \bar{D}^c is to expand E and $\mathcal{D}_{\mathbb{R}}$ to allow the radius of the disk images to go to zero, while retaining the correct overall homotopy type for these embedding spaces. This is easiest to first define in the non-symmetric context and then put the symmetries back in. To do this, let $u\mathcal{D}_{\mathbb{R}}(n)$ be the subspace of $\mathcal{D}_{\mathbb{R}}(n)$ consisting of those n -tuples $(\lambda_1, \dots, \lambda_n)$ such that

$$\lambda_1(0) < \dots < \lambda_n(0);$$

then $\mathcal{D}_{\mathbb{R}}(n) \cong u\mathcal{D}_{\mathbb{R}} \times \Sigma_n$. The operadic composition map preserves these components (making $u\mathcal{D}_{\mathbb{R}}$ a “non- Σ operad”) and the isomorphism $\mathcal{D}_{\mathbb{R}}(-) \cong u\mathcal{D}_{\mathbb{R}}(-) \times \Sigma_-$ is an isomorphism of operads where the operadic composition (and permutation action) on the symmetric groups is the standard one, defining the operad for associative monoids \mathcal{A} . We extend this operad as follows.

Construction 5.7. Let $u\mathcal{D}^c(1)$ be the subspace of points $(v, r) \in \mathbb{R} \times [0, 1]$ such that the affine transformation

$$\lambda: t \mapsto v + rt$$

sends $D = [-1, 1] \subset \mathbb{R}$ into D . Let $u\mathcal{D}^c(n)$ be the subset of $(u\mathcal{D}^c(1))^n$ consisting of those n -tuples $((v_1, r_1), \dots, (v_n, r_n))$ that satisfy

$$v_1 \leq \dots \leq v_n$$

and whenever $v_j + r_j > v_{j+1} - r_{j+1}$ we have $r_j = r_{j+1} = 0$; in terms of the affine transformations, whenever the images of two overlap, they are both constant to the same point. Let $\mathcal{D}^c(n) = u\mathcal{D}^c(n) \times \Sigma_n$, with the operadic multiplication induced by diagonal composition and block sum of permutations: composition takes the element corresponding to

$$(((\lambda_1, \dots, \lambda_n), \sigma), ((\mu_{1,1}, \dots), \tau_1), \dots, ((\dots, \mu_{n,j_n}), \tau_n))$$

in $\mathcal{D}^c(n) \times \mathcal{D}^c(j_1) \times \dots \times \mathcal{D}^c(j_n)$ to the element corresponding to

$$((\lambda_1 \circ \mu_{\sigma^{-1}(1),1}, \dots, \lambda_1 \circ \mu_{\sigma^{-1}(1),j_{\sigma^{-1}(1)}}), \sigma \circ (\tau_1 \oplus \dots \oplus \tau_n))$$

in $\mathcal{D}^c(j)$.

The inclusion of $\mathcal{D}_{\mathbb{R}}$ in \mathcal{D}^c is a map of operads, and we also get a map of operads $\mathcal{A} \rightarrow \mathcal{D}^c$ from the operad \mathcal{A} for associative monoids: the map

$$\mathcal{A}(n) = \Sigma_n \longrightarrow u\mathcal{D}^c(n) \times \Sigma_n = \mathcal{D}^c(n)$$

sends $\sigma \in \Sigma_n$ to $((0, 0), \dots, (0, 0), \sigma)$. Because $u\mathcal{D}_{\mathbb{R}}(n)$ and $u\mathcal{D}^c(n)$ are contractible for all n , we get the following proposition.

Proposition 5.8. *The maps of operads $\mathcal{D}_{\mathbb{R}} \rightarrow \mathcal{D}^c \leftarrow \mathcal{A}$ induce Σ_n -equivariant homotopy equivalences*

$$\mathcal{D}_{\mathbb{R}}(n) \longrightarrow \mathcal{D}^c(n) \longleftarrow \mathcal{A}(n)$$

for all n .

We can do something similar for $E(n)$: let $uE(n)$ be the subspace of $E(n)$ of those n -tuples $((\zeta_1, r_1), \dots, (\zeta_n, r_n))$ such that for each i , the counterclockwise path from $[\zeta_j]$ to $[\zeta_{j+1}]$ does not pass through any of the other $[\zeta_k]$, that is, starting with $[\zeta_1]$, the points $[\zeta_j]$ are numbered in strictly counterclockwise order. This has a

canonical action of $Z_n < \Sigma_n$ (for $Z_n = \langle (1 \cdots n) \rangle$) and $uE(n)$ is an $S^1 \times (Z_n \wr C_m)^{\text{op}}$ -subspace of $E(n)$. Moreover, $E(n)$ is isomorphic as an $S^1 \times (\Sigma_n \wr C_m)^{\text{op}}$ -space to $uE(n) \times_{C_n} \Sigma_n$. The right action of $\mathcal{D}_{C_m}^{\mathbb{R}}$ on E restricts to give maps of the form

$$uE(n) \times uD_{\mathbb{R}}(j_1) \times \cdots \times uD_{\mathbb{R}}(j_n) \longrightarrow uE(j)$$

which are $S^1 \times (C_m^n)^{\text{op}}$ -equivariant where C_m^n acts by diagonal blocks on the right. Writing $uD_{C_m}^{\mathbb{R}}$ for the non- Σ version of $\mathcal{D}_{C_m}^{\mathbb{R}}$, $uD_{\mathbb{R}} \rtimes C_m$, the corresponding action map

$$uE(n) \times uD_{C_m}^{\mathbb{R}}(j_1) \times \cdots \times uD_{C_m}^{\mathbb{R}}(j_n) \longrightarrow uE(j)$$

is $S^1 \times (C_m^j)^{\text{op}}$ -equivariant. We have a further cyclic Z_n invariance of the following form: for

$$(f, g_1, \dots, g_n) \in uE(n) \times uD_{\mathbb{R}}(j_1) \times \cdots \times uD_{\mathbb{R}}(j_n) \longrightarrow uE(j),$$

$\alpha \in Z_n$, and $\alpha_{j_1, \dots, j_n} \in \Sigma_j$ the permutation that cycles the blocks j_1, \dots, j_n by α , the following compositions are equal:

$$(f\alpha) \circ (g_1, \dots, g_n) = (f \circ (g_{\alpha^{-1}(1)}, \dots, g_{\alpha^{-1}(n)}))\alpha_{j_1, \dots, j_n} \in uE(j)$$

We note that $\alpha_{j_1, \dots, j_n} \in Z_j < \Sigma_j$; for example, if α is the cycle $(1 \cdots n)$ then α_{j_1, \dots, j_n} is $(1 \cdots j)^{j^n}$. Adding the full symmetric group symmetries back in, these action maps induce the $\mathcal{D}_{C_m}^{\mathbb{R}}$ -action maps.

The Z_n -action on $uE(n)$ adds an extra complication to constructing the extension $uE^c(n)$. Since Z_1 is the trivial group, no issues arise at the 1-ary level, and we can take $uE^c(1) = E^c(1)$ to be the set of ordered pairs (ζ, r) with $\zeta \in S^1$ and $r \in [0, \pi/m]$. The problem arises at the 2-level: consider the elements

$$((1, r), (e^{2ir}, r)), ((e^{2ir}, r), (1, r)) \in uE(2).$$

As r goes to zero, these need to converge to different elements of $uE^c(2)$, and so we cannot just take $uE^c(2)$ to be the obvious subspace of $uE^c(1)^2$. Instead, we note that when r is small, the center points 1 and e^{2ir} are close together and we can interpret the point 1 as being “first” in the counter-clockwise order; we can identify it as first because traveling only counter-clockwise, most of the circle has to be traversed to reach it from the other point. We use this idea to redefine $uE(n)$ in a way that we extends to allow the size of the disk images to be zero.

Define $\theta_j: C(n, S^1/C_m) \rightarrow (0, 2\pi/m)$ to be the (continuous) function that takes a configuration (x_1, \dots, x_n) to the length of the counter-clockwise arc from x_j to x_{j+1} (for $j < n$) or from x_n to x_1 (for $j = n$). We define θ_j on $E(n)$ and $uE(n)$ using the *center point map* $E(n) \rightarrow C(n, S^1/C_m)$ that takes an element $((\zeta_1, r_1), \dots, (\zeta_n, r_n))$ to the configuration given by the center points of the disk images $([\zeta_1], \dots, [\zeta_n])$. Since for elements of $uE(n)$, the center points occur cyclically in the counter-clockwise direction, we have that the sum of the lengths always adds up to the circumference of the circle,

$$\theta_1|_{uE(n)} + \cdots + \theta_n|_{uE(n)} \equiv 2\pi/m.$$

Now let $uE'(n)$ be the subspace of $E(1)^n \times (0, 2\pi/m)^n$ consisting of the points

$$((\zeta_1, r_1), \dots, (\zeta_n, r_n)), (\phi_1, \dots, \phi_n)$$

such that

- Starting at $[\zeta_1]$, the points $[\zeta_1], \dots, [\zeta_n]$ occur in counter-clockwise order;
- The intervals $t \mapsto [e^{\pi i r_j t} \zeta_j]$, $t \in (-1, 1)$ do not overlap;
- $[\zeta_{j+1}] = [e^{\pi i \phi_j} \zeta_j]$ for $j = 0, \dots, n-1$ where $\zeta_0 := \zeta_n$ and $\phi_0 := \phi_n$; and

- $\phi_1 + \cdots + \phi_n = 2\pi/m$.

Then the projection $uE'(n) \rightarrow uE(n)$ and the map $uE(n) \rightarrow uE'(n)$ given by the inclusion and $\theta_1, \dots, \theta_n$ are inverse homeomorphisms. Moreover, if we let Z_n act on $(0, 2\pi/m)^n$ by permuting coordinates (and let S^1 and C_m act trivially on $(0, 2\pi/m)^n$), then these homeomorphisms are $S^1 \times (Z_n \wr C_m)^{\text{op}}$ -equivariant. The effect of the action of $u\mathcal{D}_{\mathbb{R}}$ on the new ϕ coordinates is straight-forward but tedious to describe; to avoid unnecessary redundancy, we just write it out for the extension uE^c below.

Construction 5.9. Let $uE^c(0)$ be a point and $uE^c(1)$ the set of ordered pairs (ζ, r) with $\zeta \in S^1$ and $r \in [0, \pi/m]$. For $n > 1$, let $uE'(n)$ be the subspace of $E^c(1)^n \times [0, 2\pi/m]^n$ consisting of the points

$$((\zeta_1, r_1), \dots, (\zeta_n, r_n)), (\phi_1, \dots, \phi_n)$$

such that

- Starting at $[\zeta_1]$, the points $[\zeta_1], \dots, [\zeta_n]$ occur in counter-clockwise order;
- If for some $j < k \in \{1, \dots, n\}$ the intervals $t \mapsto [e^{\pi i r_j t} \zeta_j]$ and $t \mapsto [e^{\pi i r_k t} \zeta_k]$, $t \in (-1, 1)$ overlap, then $r_j = r_k = 0$;
- $[\zeta_{j+1}] = [e^{\pi i \phi_j} \zeta_j]$ for $j = 0, \dots, n-1$ where $\zeta_0 := \zeta_n$ and $\phi_0 := \phi_n$; and
- $\phi_1 + \cdots + \phi_n = 2\pi/m$.

We have an $S^1 \times (Z_n \wr C_m)^{\text{op}}$ -action with the left S^1 action diagonally on the ζ_j , the right C_m actions individually on the ζ_j , and the Z_n action permuting the indexes on the ζ_j, r_j , and ϕ_j .

We define the action map

$$uE^c(n) \times u\mathcal{D}^c(j_1) \times \cdots \times u\mathcal{D}^c(j_n) \longrightarrow uE^c(j)$$

($j = j_1 + \cdots + j_n$) as follows. For

$$\begin{aligned} ((\zeta_1, r_1), \dots, (\zeta_n, r_n), (\phi_1, \dots, \phi_n)) &\in uE^c, \\ ((v_{i,1}, s_{i,1}), \dots, (v_{i,j_i}, s_{i,j_i})) &\in u\mathcal{D}^c(j_i), \end{aligned}$$

the resulting element of $uE^c(j)$,

$$((\xi_1, t_1), \dots, (\xi_j, t_j), (\psi_1, \dots, \psi_j))$$

is given as follows. For $\ell \in \{1, \dots, j\}$ define $j(\ell)$ to be the smallest integer such that

$$j_1 + \cdots + j_{j(\ell)} \geq \ell$$

(so $j(\ell) = 1$ for $\ell \in \{1, \dots, j_1\}$, $j(\ell) = 2$ for $\ell \in \{j_1 + 1, \dots, j_1 + j_2\}$, etc.), and define

$$k(\ell) = \ell - (j_1 + \cdots + j_{j(\ell)-1})$$

(where we understand the parenthetical sum as 0 when $j(\ell) = 1$). We have made this definition so that the ℓ index in the codomain corresponds to the $j(\ell), k(\ell)$ index in the domain, and we take

$$\begin{aligned} \xi_\ell &= e^{i r_{j(\ell)} v_{j(\ell), k(\ell)}} \zeta_{j(\ell)} \\ t_\ell &= r_{j(\ell)} s_{j(\ell), k(\ell)}; \end{aligned}$$

in other words, in terms of the corresponding maps of the interval, we do the usual diagonal composition. For the parameters ψ_ℓ , we take

$$\psi_\ell = \begin{cases} r_{j(\ell)}(v_{j(\ell),k(\ell)+1} - v_{j(\ell),k(\ell)}) & \text{if } \ell < j \text{ and } j(\ell+1) = j(\ell) \\ \phi_{j(\ell)} - r_{j(\ell)}v_{j(\ell),k(\ell)} + r_{j(\ell)+1}v_{j(\ell)+1,1} & \text{if } \ell < j \text{ and } j(\ell+1) = j(\ell) + 1 \\ \phi_n - r_n v_{n,j_n} + r_1 v_{1,1} & \text{if } \ell = j; \end{cases}$$

an easy check shows that this defines an element of $uE^c(j)$.

We define $E^c(n)$ to be the $S^1 \times (\Sigma_n \wr C_m)^{\text{op}}$ -space $E^c(n) \times_{Z_n} \Sigma_n$. The cyclic permutation action on the $uE^c(n)$ has the same compatibility with composition as on the $uE(n)$, and we get a right action of \mathcal{D}^c on E^c generalizing the formulas above for the right action of $\mathcal{D}_{\mathbb{R}}$ on E .

As discussed above, the functions θ_j on $uE(n)$ fill in parameters ϕ_j to define a map $uE(n) \rightarrow uE^c(n)$ and hence a map $E(n) \rightarrow E^c(n)$, which are easily seen to be inclusions. The latter map is $S^1 \times (\Sigma_n \wr C_m)^{\text{op}}$ -equivariant; we show it is an $S^1 \times (\Sigma_n \wr C_m)^{\text{op}}$ -equivariant homotopy equivalence.

Proposition 5.10. *The map $E(n) \rightarrow E^c(n)$ is an $S^1 \times (\Sigma_n \wr C_m)^{\text{op}}$ -equivariant homotopy equivalence.*

Proof. There is nothing to show in the case $n = 0$ and the case $n = 1$ is clear. For $n \geq 2$, we can identify $E(n)$ with its homeomorphic image, which consists of the elements where (in our usual notation) none of the r_j 's are zero. Let X denote the subspace of $E^c(n)$ where all the r_j 's are zero. We have an obvious equivariant deformation retraction of $E(n)$ on to X , which induces an equivariant homotopy equivalence between $E(n)$ and the subspace X_0 of X^c where the $[\zeta_j]$ are all distinct elements of S^1/C_m . (The center point map gives an equivariant homeomorphism from X to the C_m -framed configuration space $C_{C_m}(n, S^1/C_m)$ described in the paragraph following Definition 4.1, with the functions θ above inducing the inverse.) We can equivalently describe X_0 as the subspace where all the ϕ_j are positive; let $X_k \subset X$ be the subspace where at most k of the ϕ_j are zero. Then $X = X_n$. Consider the equivariant self-homotopy of X that at time t sends the element represented by

$$(((((\zeta_1, 0), \dots, (\zeta_n, 0)), (\phi_1, \dots, \phi_n)), \sigma)$$

to the element represented by

$$(((((e^{it\phi_1/2}\zeta_1, 0), \dots, (e^{it\phi_n/2}\zeta_n, 0)), (\phi_1(1-t/2) + \phi_2 t/2, \dots, \phi_n(1-t/2) + \phi_1 t/2)), \sigma).$$

This starts at the identity and ends at an endomorphism f of X . The endomorphism f sends X_k into X_{k-1} for $k > 0$. The n th iterate then sends X into X_0 and is evidently an equivariant homotopy inverse to the inclusion. \square

Let \bar{E}^c be the functor from C_m -equivariant orthogonal spectra to S^1 -equivariant orthogonal spectra defined by

$$\bar{E}^c X = \bigvee_{n \geq 0} E^c(n)_+ \wedge_{\Sigma_n \wr C_m} X^{(n)} \cong \bigvee_{n \geq 0} uE^c(n)_+ \wedge_{Z_n \wr C_m} X^{(n)}$$

and let $\bar{\mathbb{D}}^c$ be the monad in C_n -equivariant orthogonal spectra associated to the C_n -equivariant operad \mathcal{D}^c ,

$$\bar{\mathbb{D}}^c X = \bigvee_{n \geq 0} \mathcal{D}^c(n)_+ \wedge X^{(n)}.$$

The right action of \mathcal{D}^c on E^c descends to give a right action of the monad \mathbb{D}^c on \bar{E}^c , and the inclusion of E in E^c is compatible with the actions in the sense that the diagram

$$\begin{array}{ccc} \bar{E}\bar{\mathbb{D}} & \longrightarrow & \bar{E} \\ \downarrow & & \downarrow \\ \bar{E}^c\bar{\mathbb{D}}^c & \longrightarrow & \bar{E}^c \end{array}$$

of functors from C_m -equivariant orthogonal spectra to S^1 -equivariant orthogonal spectra precisely commutes. This induces a map of monadic bar constructions

$$B(\bar{E}, \bar{\mathbb{D}}, -) \longrightarrow B(\bar{E}^c, \bar{\mathbb{D}}^c, -).$$

Propositions 5.8 and 5.10 now prove the following proposition.

Proposition 5.11. *For any C_m -equivariant $\mathcal{D}_{\mathbb{R}}$ -algebra X , the map of monadic bar constructions*

$$B_{\bullet}(\bar{E}, \bar{\mathbb{D}}, X) \longrightarrow B_{\bullet}(\bar{E}^c, \bar{\mathbb{D}}^c, X)$$

is on each level a natural S^1 -equivariant homotopy equivalence.

For the last zigzag, let $uC^c(n)$ be the subspace of $uE^c(n)$ where the r_j 's are all zero, and let $C^c(n) = uC^c \times_{Z_n} \Sigma_n$. Viewing $C^c(n)$ as a subspace of $E^c(n)$, it inherits a $S^1 \times (\Sigma_n \wr C_m)^{\text{op}}$ -action and the inclusion of $C^c(n)$ in $E^c(n)$ is an equivariant homotopy equivalence. The right \mathcal{D}^c -action on E^c restricts to C^c , and using the map of operads $\mathcal{A} \rightarrow \mathcal{D}^c$, we get a right action of \mathcal{A} on C^c , making the following diagram precisely commute

$$\begin{array}{ccc} \bar{C}^c\mathbb{T} & \longrightarrow & \bar{C}^c \\ \downarrow & & \downarrow \\ \bar{E}^c\bar{\mathbb{D}}^c & \longrightarrow & \bar{E}^c \end{array}$$

where \mathbb{T} is the free associative algebra monad (the monad associated to the operad \mathcal{A}) and \bar{C}^c denotes the functor from C_m -equivariant orthogonal spectra to S^1 -equivariant orthogonal spectra

$$\bar{C}^c X = \bigvee_{n \geq 0} C^c(n)_+ \wedge_{\Sigma_n \wr C_m} X^{(n)} \cong \bigvee_{n \geq 0} uC^c(n)_+ \wedge_{Z_n \wr C_m} X^{(n)}$$

We get a corresponding map of bar constructions and the following proposition is now clear.

Proposition 5.12. *For any C_m -equivariant associative ring orthogonal spectrum R , the map of monadic bar constructions*

$$B_{\bullet}(\bar{C}^c, \mathbb{T}, R) \longrightarrow B_{\bullet}(\bar{E}^c, \bar{\mathbb{D}}^c, R)$$

is on each level a natural S^1 -equivariant homotopy equivalence.

This completes the proof of the assertion about the first display in Theorem 5.1. For the isomorphism in the second display, we use the following construction.

Construction 5.13. For R a C_m -equivariant associative ring orthogonal spectrum. Define the S^1 -equivariant orthogonal spectrum $\bar{C}^c \otimes_{\mathbb{T}} X$ to be the (point-set) coequalizer

$$\bar{C}^c\mathbb{T}R \rightrightarrows \bar{C}^c R \longrightarrow \bar{C}^c \otimes_{\mathbb{T}} R$$

with one map $\bar{C}^c\mathbb{T}R \rightarrow \bar{C}^c R$ induced by the right \mathbb{T} -action on \bar{C}^c and the other induced by the left \mathbb{T} -action on R .

Since the functors \bar{C}^c and \mathbb{T} commute with geometric realization, we have a natural isomorphism

$$\bar{C}^c \otimes_{\mathbb{T}} B(\mathbb{T}, \mathbb{T}, R) \cong B(\bar{C}^c, \mathbb{T}, R).$$

The proof of Theorem 5.1 is therefore completed by the verification of the following theorem.

Theorem 5.14. *There is a natural isomorphism*

$$\bar{C}^c \otimes_{\mathbb{T}} (-) \cong N_{\wedge}^{cyc, C_m}(-)$$

of functors from C_m -equivariant associative ring orthogonal spectra to S^1 -equivariant orthogonal spectra.

Before explaining the isomorphism, we begin with a brief review of the functor N_{\wedge}^{cyc, C_m} from C_m -equivariant associative ring orthogonal spectra to S^1 -equivariant orthogonal spectra. Non-equivariantly, N_{\wedge}^{cyc, C_m} is a variant of the cyclic bar construction: it is the geometric realization of the simplicial object with q th object the $(q+1)$ th smash power

$$N_{\wedge, q}^{cyc, C_m} R = R^{(q+1)}$$

with degeneracy s_i induced by the inclusion of the identity in the $(i+1)$ th position and face map d_i for $i = 0, \dots, q-1$, the multiplication in positions $i+1$ and $i+2$. The last face map d_{q+1} cycles the last position around to the front acts on it by the generator $e^{2\pi i/m} \in C_m < S^1 < \mathbb{C}^\times$ and then multiplies the (new) 1st and 2nd positions. Then $d_{q+1} = d_0 \circ \tau_q$ where τ_q is the operation on $R^{(q+1)}$ that cycles the factors and then applies $e^{2\pi i/m}$ in the first factor. The face, degeneracy, and τ operators satisfy the relations

$$(5.15) \quad \begin{aligned} \tau_q^{m(q+1)} &= \text{id} \\ d_0 \tau_q &= d_q & s_0 \tau_q &= \tau_{q+1}^2 s_q \\ d_i \tau_q &= \tau_{q-1} d_{i-1} & s_i \tau_q &= \tau_{q+1} s_{i-1} \quad (1 \leq i \leq q) \end{aligned}$$

(in addition to the simplicial identities relating just the faces and degeneracies). As in [4, §1] (but with slightly different indexing conventions), this implicitly defines a category $\mathbf{\Lambda}_m^{\text{op}}$, generalizing Connes' cyclic category, such that $N_{\wedge, \bullet}^{cyc, C_m}$ is a functor from $\mathbf{\Lambda}_m^{\text{op}}$ to orthogonal spectra. We use the terminology *m-cyclic orthogonal spectrum* (or *m-cyclic space*) for a functor from $\mathbf{\Lambda}_m^{\text{op}}$ to orthogonal spectra (or spaces).

Let $\Lambda_m[q]$ denote the geometric realization of the representable object

$$\Lambda_m[q]_{\bullet} = \Lambda_m(\bullet, q).$$

As q varies $\Lambda_m[q]$ is a functor from $\mathbf{\Lambda}_m$ to spaces. Then (as in [4, 1.8]), for any $\mathbf{\Lambda}_m^{\text{op}}$ -object X_{\bullet} (in spaces or orthogonal spectra), the inclusion of the simplex category $\mathbf{\Delta}$ in $\mathbf{\Lambda}_m$ induces an isomorphism from usual geometric realization of X_{\bullet} to an m -cyclic realization given as the coend over $\mathbf{\Lambda}_m$ of $X_{\bullet} \wedge \Lambda_m[\bullet]_+$ (in the case of orthogonal spectra). The spaces $\Lambda_m[q]$ have a natural (in $q \in \mathbf{\Lambda}_m^{\text{op}}$) action of the circle S^1 ; see [4, 1.6]. This gives the geometric realization of any m -cyclic orthogonal spectrum (or space) a natural S^1 -action.

To be precise, $\Lambda_m[q]$ is isomorphic to the space

$$\mathbb{R}/m\mathbb{Z} \times \Delta[q].$$

Writing an element as $(r + m\mathbb{Z}, t_0, \dots, t_q)$ where $r \in \mathbb{R}$ and $t_i \geq 0$, $t_0 + \dots + t_q = 1$, the circle acts by

$$e^{i\theta} \cdot (r + m\mathbb{Z}, t_0, \dots, t_q) = (r + \theta/(2\pi/m) + m\mathbb{Z}, t_0, \dots, t_q).$$

As a functor of $\mathbf{\Lambda}_m$, the face and degeneracy maps act in the usual manner on the simplices and the twist τ_q acts by

$$\tau_q(r + m\mathbb{Z}, t_0, \dots, t_q) = (r - t_q + m\mathbb{Z}, t_q, t_0, \dots, t_{q-1}).$$

(This is [4, 1.6] adjusted for our indexing convention.)

The spaces $\Lambda_m[q]$ are closely related to the spaces $uC^c(n)$. To simplify notation, we write a typical element of $uC^c(n)$ as

$$(\zeta_1, \dots, \zeta_n, \phi_1, \dots, \phi_n)$$

for $\zeta_i \in S^1$, $\phi_i \in [0, 2\pi/m]$ (dropping the $r_j = 0$ from the notation we used above and flattening parentheses). Let $v_{m,n} \in Z_n \wr C_m$ denote the element

$$((1 \cdots n); 1, \dots, 1, e^{-2\pi i/m}) \in Z_n \times C_m^n = Z_n \wr C_m.$$

Then $v_{m,n}$ generates a cyclic subgroup of order mn in $Z_n \wr C_m$ that acts on $X^{(n)}$ (for a C_m -equivariant orthogonal spectrum X) by acting by $e^{-2\pi i/m}$ on the last factor and then cycling it to the first position. The precise relationship between the spaces $\Lambda_m[q]$ and $uC^c(n)$ is as follows.

Proposition 5.16. *The map $\Lambda_m[q] \rightarrow uC^c(q+1)$ that sends $(r + m\mathbb{Z}, t_0, \dots, t_q)$ to*

$$(e^{(2\pi/m)ir}, e^{(2\pi/m)i(r+t_0)}, \dots, e^{(2\pi/m)i(r+t_0+\dots+t_{q-1})}, (2\pi/m)t_0, \dots, (2\pi/m)t_q)$$

induces a $S^1 \times (Z_{q+1} \wr C_m)^{\text{op}}$ -equivariant isomorphism

$$\Lambda_m[q] \times_{C_{m(q+1)}} (Z_{q+1} \wr C_m) \longrightarrow uC^c(q+1)$$

(where on the left, we are using the isomorphism $C_{m(q+1)} \cong \langle u_{m,q+1} \rangle \subset Z_{q+1} \wr C_m$ sending the generator $e^{2\pi i/(m(q+1))}$ to $v_{m,q+1}$).

Proof. The displayed formula for the map $\Lambda_m[q] \rightarrow uC^c(q+1)$ is clearly well-defined and S^1 -equivariant; moreover, it is equivariant for the right action of $C_{m(q+1)}$ on $\Lambda_m[q]$ and the $\langle v_{m,q+1} \rangle$ action on $uC^c(q+1)$ under the given isomorphism since

$$e^{-2\pi i/m} e^{(2\pi/m)i(r+t_0+\dots+t_{q-1})} = e^{(2\pi/m)i(r-t_q)}$$

$(-1 + t_0 + \dots + t_{q-1} = -t_q)$. The map $\Lambda_m[q] \times_{C_{m(q+1)}} (Z_{q+1} \wr C_m) \rightarrow uC^c(q+1)$ is therefore well-defined and $S^1 \times (Z_{q+1} \wr C_m)^{\text{op}}$ -equivariant. It is a continuous bijection of compact Hausdorff spaces and therefore an isomorphism. \square

Using the isomorphism above, we get a well defined map

$$(5.17) \quad \begin{aligned} \bar{C}^c R &\cong \bigvee_{n \geq 0} uC^c(n)_+ \wedge_{Z_n \wr C_m} R^{(n)} \\ &\longrightarrow N_{\wedge}^{\text{cyc}, C_m}(R) \cong \left(\bigvee_{q \geq 0} \Lambda_m[q]_+ \wedge R^{(q+1)} \right) / \sim \end{aligned}$$

sending the $n = 0$ summand $uC^c(0)_+ \wedge R^{(0)} \cong \mathbb{S}$ by the inclusion of $\{1\}_+ \wedge \mathbb{S}$ in $\Lambda_m[0] \wedge R^{(0)} \cong S_+^1 \wedge \mathbb{S}$, and for $n > 0$, sending the n th summand through the $q = n - 1$ summand using the isomorphism

$$uC^c(q+1)_+ \wedge_{Z_{q+1} \wr C_m} R^{(q+1)} \cong \Lambda_m[q]_+ \wedge_{C_m(q+1)} R^{(q+1)}$$

implied by the previous proposition. This is well defined because on the right hand side, the $C_m(q+1)$ -actions on $\Lambda_m[q]_+ \wedge R^{(q+1)}$ are coequalized as part of the coend. It is obvious that this map is S^1 -equivariant on the n th summand for $n > 0$, but it is also S^1 -equivariant on the 0th summand (as a consequence of the fact that on the image of \mathbb{S} in $N_{\wedge, 0}^{cyc, C_m}$, $\tau_1 s_0 = s_0$).

For the coequalizer forming $\bar{C}^c \otimes_{\mathbb{T}} R$ from $\bar{C}^c R$,

$$\bar{C}^c \otimes_{\mathbb{T}} R \cong \left(\bigvee_{n \geq 0} uC^c(n)_+ \wedge_{Z_n \wr C_m} R^{(n)} \right) / \sim,$$

the equivalence relation induced by the action of \mathcal{A} on C^c and R can be written in terms of faces, degeneracies, and twists in the m -cyclic object $N_{\wedge, \bullet}^{cyc, C_m}$ and relations involving the the unit $\mathbb{S} \rightarrow R$. As a consequence, the map $\bar{C}^c R \rightarrow N_{\wedge}^{cyc, C_m} R$ induces a map $\bar{C}^c \otimes_{\mathbb{T}} R \rightarrow N_{\wedge}^{cyc, C_m} R$ and we get the following proposition.

Proposition 5.18. *The map of (5.17) induces a natural transformation*

$$\bar{C}^c \otimes_{\mathbb{T}} (-) \longrightarrow N_{\wedge}^{cyc, C_m}(-)$$

of functors from C_m -equivariant associative ring orthogonal spectra to S^1 -equivariant orthogonal spectra.

A more careful analysis of the equivalence relation forming $\bar{C}^c \otimes_{\mathbb{T}} R$ from $\bar{C}^c R$ should show that the natural transformation is an isomorphism, but we take a different approach. Consider the case when $R = \mathbb{T}X$ for some C_m -equivariant orthogonal spectrum X . Then the inclusion of X in $\mathbb{T}X$ induces an isomorphism

$$\bar{C}^c X \xrightarrow{\cong} \bar{C}^c \otimes_{\mathbb{T}} \mathbb{T}X.$$

In this case, the m -cyclic object $N_{\wedge, \bullet}^{cyc, C_m}(\mathbb{T}X)$ breaks up into a wedge sum of m -cyclic objects

$$N_{\wedge, \bullet}^{cyc, C_m}(\mathbb{T}X) = N(0)_{\bullet} \vee N(1)_{\bullet} \vee \dots$$

where $N(n)_{\bullet}$ consists of the $X^{(n)}$ summands in $N_{\wedge, \bullet}^{cyc, C_m}(\mathbb{T}X)$. Then $N(0)$ is the constant m -cyclic object on \mathbb{S} , and for $n > 0$, the inclusion of $X^{(n)}$ in $(\mathbb{T}X)^{(n)} = N_{\wedge, n-1}^{cyc, C_m}(\mathbb{T}X)$ induces an isomorphism of C_m -cyclic objects

$$N(n)_{\bullet} \xleftarrow{\cong} (\Lambda_m[n-1]_{\bullet})_+ \wedge_{C_m n} X^{(n)}.$$

The map

$$\bar{C}^c X \longrightarrow \bar{C}^c \otimes_{\mathbb{T}} \mathbb{T}X \longrightarrow N_{\wedge}^{cyc, C_m}(\mathbb{T}X)$$

respects homogeneous degree and induces an isomorphism

$$uC^c(n)_+ \wedge_{Z_n \wr C_m} X^{(n)} \longrightarrow \Lambda_m[n-1]_+ \wedge_{C_m n} X^{(n)} = N(n)$$

for each n . This proves the following proposition.

Proposition 5.19. *Let $R = \mathbb{T}X$ for X a C_m -equivariant orthogonal spectrum. The natural S^1 -equivariant map $\bar{C}^c \otimes_{\mathbb{T}} \mathbb{T}X \rightarrow N_{\wedge}^{cyc, C_m}(\mathbb{T}X)$ of Proposition 5.18 is an isomorphism.*

We can now prove Theorem 5.14 (which completed the proof of Theorem 5.1).

Proof of Theorem 5.14. Proposition 5.18 constructs the natural transformation. Because smash powers preserve reflexive coequalizers, both $\bar{C}^c \otimes_{\mathbb{T}}(-)$ and $N_{\wedge}^{cyc, C_m}(-)$ preserve reflexive coequalizers. Any C_m -equivariant associative ring orthogonal spectrum is the reflexive coequalizer

$$\mathbb{T}TR \rightrightarrows TR \longrightarrow R$$

where the maps $\mathbb{T}TR \rightarrow TR$ are given by the monadic product $\mathbb{T}\mathbb{T} \rightarrow \mathbb{T}$ and \mathbb{T} applied to the action map $TR \rightarrow R$. The previous proposition implies that the natural transformation is an isomorphism for $\mathbb{T}TR$ and TR , and so we conclude that it is an isomorphism for R . \square

6. FACTORIZATION HOMOLOGY OF V -FRAMED G -MANIFOLDS (EQUIVARIANT THEORY)

In this section, we show how to obtain a version of G -equivariant factorization homology of V -framed G -manifolds as a special case of the factorization homology discussed in Section 3. We start with a review of genuine equivariant factorization homology for V -framed G -manifolds. We then compare the category of V -framed G -manifolds to a category of (rigid) G -objects in a category of G -framed embeddings as defined in Section 3. Finally, we compare the bar construction defining factorization homology of V -framed G -manifolds with the bar construction of Construction 3.19. Convention 3.1 is in effect in this section.

We begin by reviewing the definition of V -framed G -manifolds and their factorization homology as defined by Horev [8] and Zou [17] in the case of a finite group G . We follow the latter precisely, but work in the context of G -equivariant orthogonal spectra rather than G -spaces (see also Remark 6.4 below); we expect that this agrees with the genuine equivariant factorization homology of the former (which has an axiomatic characterization) but make no justification for that here.

Definition 6.1. Let V be finite dimensional vector space with linear G -action and G -invariant inner product. A V -framed G -manifold M is a smooth manifold with smooth G -action together with an isomorphism of G -equivariant vector bundles

$$\theta_M: TM \cong M \times V.$$

We write $\theta_{M,V}: TM \rightarrow V$ for the composite of θ_M with the projection to V . Given V -framed G -manifolds L and M , a V -framed embedding $L \rightarrow M$ consists of a smooth embedding $f: L \rightarrow M$, a map

$$\alpha: TL \times [0, \infty) \longrightarrow V$$

that is a linear isomorphism $T_x L \rightarrow V$ on each fiber, and a locally constant function $\ell: L \rightarrow [0, \infty)$ such that

- for all $\xi \in TL$, $\alpha(\xi, 0) = \theta_{L,V}(\xi)$; and
- for all $\xi \in TL_x$ and $t \geq \ell(x)$, $\alpha(\xi, t) = \theta_{M,V}(Df(\xi))$.

We write $\mathcal{E}mb_V$ for the G -space of such maps with its intrinsic topology and conjugation G -action. The composition of V -framed embeddings

$$\mathcal{E}mb_V(L, N) \times \mathcal{E}mb_V(M, N) \longrightarrow \mathcal{E}mb_V(L, M)$$

is defined by treating ℓ as the length for Moore path composition: given $(f, \alpha, \ell) \in \mathcal{E}mb_V(L, M)$ and $(f', \alpha', \ell') \in \mathcal{E}mb_V(M, N)$, the composite map $(f'', \alpha'', \ell'') \in \mathcal{E}mb_V(L, N)$ is defined by

$$f'' = f' \circ f, \quad \ell'' = \ell' + \ell$$

$$\alpha''(\xi, t) = \begin{cases} \alpha(\xi, t) & 0 \leq t \leq \ell(x) \\ \alpha'(Df(\xi), t - \ell(x)) & t \geq \ell(x) \end{cases}$$

(where $\xi \in T_x L$). This makes $\mathcal{E}mb_V$ a G -topological category (morphisms are G -spaces, composition is G -equivariant, identity maps are G -fixed points) with the identity map in $\mathcal{E}mb_V(M, M)$ given by $(\text{id}, \alpha, 0)$ with $\alpha(\xi, t) = \theta_{M, V}(\xi)$.

The topological category agrees with the Definition 3.6 of [17] (with the minor correction to ℓ to make Remark 3.9, *ibid.* work). For $n \geq 0$, let $V(n) = V \times \{1, \dots, n\}$ with $V(0)$ the empty set. We note that $V(n)$ has a canonical V -framed G -manifold structure with θ the usual identification $TV \cong V \times V$.

Notation 6.2. For $n \geq 0$, let $\mathcal{R}_M(n) = \mathcal{E}mb_V(V(n), M)$, and let $\bar{\mathcal{R}}_M$ denote the functor from G -equivariant orthogonal spectra to G -equivariant orthogonal spectra defined by

$$\bar{\mathcal{R}}_M X = \bigvee_{n \geq 0} \mathcal{R}_M(n)_+ \wedge_{\Sigma_n} X^{(n)}$$

with the diagonal G -action.

In the case when $M = V$, composition gives \mathcal{R}_V the structure of an operad in G -spaces, and $\bar{\mathcal{R}}_V$ is the free \mathcal{R}_V -algebra monad on G -equivariant orthogonal spectra. We then get a right $\bar{\mathcal{R}}_V$ action on $\bar{\mathcal{R}}_M$, and we can form the monadic bar construction.

Construction 6.3. For a V -framed G -manifold M and a \mathcal{R}_V -algebra in G -equivariant orthogonal spectra X , define

$$\bar{B}_V(M; X) := B(\bar{\mathcal{R}}_M, \bar{\mathcal{R}}_V, X).$$

For X an \mathcal{R}_V -algebra in G -equivariant orthogonal spectra indexed on a complete universe U , define G -equivariant factorization homology by

$$\int_M X := I_{\mathbb{R}^\infty}^U \bar{B}_V(M; I_U^{\mathbb{R}^\infty} X).$$

Remark 6.4. The analogue for spaces obviously does not use point-set change of universe, and should take

$$\bar{\mathcal{R}}_M X = \prod_{n \geq 0} \mathcal{R}_M(n) \times_{\Sigma_n} X^{(n)}.$$

Zou [17, 3.14] uses the reduced constructions that glue along the inclusion of units $X^n \rightarrow X^{n+1}$ and the operad degeneracies (operadic composition with $\mathcal{R}_V(0)$). A straight-forward Quillen A argument shows that the bar construction for the reduced and the unreduced functors are G -equivariantly homotopy equivalent.

To compare this to the theory of Section 3, we note that Definition 6.1 uses path composition of homotopies whereas Definition 3.3 uses pointwise multiplication of homotopies. This is easy to fix by enlarging $\mathcal{E}mb_V$ to use both.

Definition 6.5. Let $\mathcal{E}mb_V^\square$ be the G -topological category whose objects are the V -framed G -manifolds and where a map from L to M consists of a smooth embedding $f: L \rightarrow M$, a map

$$\alpha: TL \times [0, \infty) \times [0, 1] \longrightarrow V$$

that is a linear isomorphism $T_x L \rightarrow V$ in each fiber, and a locally constant function $\ell: L \rightarrow [0, \infty)$ such that

- for all $\xi \in TL$, $\alpha(\xi, 0, 0) = \theta_{L,V}(\xi)$;
- for all $\xi \in TL_x$, $\alpha(\xi, \ell(x), 1) = \theta_{M,V}(Df(\xi))$;
- for all $\xi \in TL_x$, $s \geq \ell(x)$, and $t \in [0, 1]$, $\alpha(\xi, s, t) = \alpha(\xi, \ell(x), t)$.

We topologize $\mathcal{E}mb_V^\square$ with its intrinsic topology and use conjugation for the G -action. The composition

$$\mathcal{E}mb_V^\square(M, N) \times \mathcal{E}mb_V^\square(L, M) \longrightarrow \mathcal{E}mb_V^\square(L, N)$$

is defined by treating ℓ as the length for Moore path composition and doing pointwise multiplication (of linear isomorphisms) on the overlap: given $(f, \alpha, \ell) \in \mathcal{E}mb_V^\square(L, M)$ and $(f', \alpha', \ell') \in \mathcal{E}mb_V^\square(M, N)$, the composite map $(f'', \alpha'', \ell'') \in \mathcal{E}mb_V^\square(L, N)$ is given by

$$\begin{aligned} f'' &= f' \circ f, & \ell'' &= \ell' + \ell \\ \alpha''(\xi, s, t) &= \begin{cases} \alpha'(\tilde{\alpha}(\xi, s, t), 0, t) & 0 \leq s \leq \ell(x) \\ \alpha'(\tilde{\alpha}(\xi, s, t), s - \ell(x), t) & s \geq \ell(x) \end{cases} \end{aligned}$$

where $\xi \in TL_x$ and $\tilde{\alpha}: TL \times [0, 1] \times [0, \infty) \rightarrow TM$ is defined by

$$\theta_M(\tilde{\alpha}(\xi, s, t)) = (f(x), \alpha(\xi, s, t)) \in M \times V.$$

With this definition $\mathcal{E}mb_V$ includes (isomorphically) as the subcategory $\mathcal{E}mb_V^-$ of maps where α is constant in the $[0, 1]$ direction. The inclusion of $\mathcal{E}mb_V^-(L, M)$ in $\mathcal{E}mb_V^\square(L, M)$ is always a G -equivariant homotopy equivalence. We can define an analogue of \mathcal{R}_M^\square of \mathcal{R}_M using $\mathcal{E}mb_V^\square$ in place of $\mathcal{E}mb_V$. With this, we get a corresponding monadic bar construction $B_V^\square(M; -)$, and for any G -equivariant \mathcal{R}_V^\square -algebra X , we get a map

$$(6.6) \quad \bar{B}_V(M; X) \longrightarrow \bar{B}_V^\square(M; X)$$

which is evidently a natural G -equivariant homotopy equivalence. Let $\mathcal{E}mb_V^\downarrow$ be the subcategory where $\ell = 0$; the inclusion of $\mathcal{E}mb_V^\downarrow(L, M)$ in $\mathcal{E}mb_V^\square(L, M)$ is always a G -equivariant homotopy equivalence. We get an analogue \mathcal{R}_M^\downarrow of \mathcal{R}_M^\square and a corresponding monadic bar construction $B_V^\downarrow(M; -)$; for any G -equivariant \mathcal{R}_V^\square -algebra X , the induced map

$$(6.7) \quad \bar{B}_V^\downarrow(M; X) \longrightarrow \bar{B}_V^\square(M; X)$$

is a natural G -equivariant homotopy equivalence. It is therefore harmless to use $\mathcal{E}mb_V^\downarrow$ in place of $\mathcal{E}mb_V$.

In the context of Section 3, we will take $H = G$, but the group G will play two different roles here, and will retain the notation H for G as the structure group as in that section when needed to avoid confusion between the different roles. (The following definition still makes sense for $H \neq G$ provided it is H that acts on V .) We are then considering the category $\mathcal{E}_{H,V}$ of H -framed embeddings of manifolds

with tangential H, V -structure as in Definition 3.3 and we define a G -equivariant H -framed G -manifold to consist of an H -framed manifold M together with an action of G on M in $\mathcal{E}_{H,V}$ where each element g of G acts by an H -framed local isometry. We view this as a G -topological category with G -space of maps given by the space of maps in $\mathcal{E}_{H,V}$ with G acting by conjugation; we denote this category as $\mathcal{E}_{H,V;G}$.

A V -framed G -manifold M gives an object in $\mathcal{E}_{G,V;G}$ as follows: we take the $G = H$ -frame bundle to be the projection map $M \times G \rightarrow M$, and we then have an isomorphism $F_G M \times_G V \rightarrow TM$

$$F_G M \times_G V = (M \times G) \times_G V \cong M \times V \xrightarrow{\theta_M^{-1}} TM$$

which induces an isomorphism of $\mathrm{GL}(V)$ -principal bundles from $F_G M \times_G \mathrm{GL}(V)$ to the V -frame bundle of M . (More concretely, the adjoint to θ_M^{-1} gives a section of M into the V -frame bundle, and the G -action on V then gives the reduction of structure to $M \times G$.) This defines the tangential G, V -structure. Since for $g \in G$, the diagram

$$\begin{array}{ccc} TM & \xrightarrow{Dg} & TM \\ \theta_M \downarrow & & \downarrow \theta_M \\ M \times V & \xrightarrow{(g,g)} & M \times V \end{array}$$

commutes, the diagonal action of G on $M \times G$ gives a map of $G = H$ -frame bundles lifting the derivative on the V -frame bundle. In other words, lifting g to the diagonal g -action on $F_G M$ endows g with the structure of a $G = H$ -framed local isometry.

The resulting object of $\mathcal{E}_{G,V;G}$ comes with the extra structure of a section s_M of the G -frame bundle, namely the section at the identity element of G . This section is compatible with the group action on M (in $\mathcal{E}_{G,V;G}$) and on V in the sense that for all $x \in M$ and all $g \in G$, the equation

$$g \cdot s_M(x) = s_M(gx) \cdot g \in (F_G M)_{gx},$$

where the action on the lefthand side denotes the action of g on the H -frame bundle of M as an H -framed local isometry and the action on the righthand side denotes the right action of $g \in G$ on the $G = H$ -principal bundle. (Using the section to identify the G -frames at each point as G , this equation reads $g \cdot e = e \cdot g \in G$.) For an arbitrary object M of $\mathcal{E}_{G,V;G}$, we call a section s_M of the G -frame bundle satisfying the equation above G, V -compatible. We can now state the precise relationship between the categories $\mathcal{E}mb_V^{\perp}$ and $\mathcal{E}_{G,V;G}$.

Theorem 6.8. *The G -topological category $\mathcal{E}mb_V^{\perp}$ is equivalent to the G -topological category \mathcal{P} where an object is an ordered pair (M, s_M) with M an object of $\mathcal{E}_{G,V;G}$ and s_M is a G, V -compatible section of its G -frame bundle and where the G -space of maps $(L, s_L) \rightarrow (M, s_M)$ is the subspace of $\mathcal{E}_{G,V}(L, M)$ of maps whose induced map of G -frame bundles takes s_L to the pullback of s_M .*

Proof. We have already described the functor in the forward direction on objects. On maps, it takes $(f, \alpha, 0) \in \mathcal{E}mb_V^{\perp}(L, M)$ as above to the map $(f, Ff, If) \in \mathcal{E}_{G,V}(L, M)$ defined as follows. The map

$$Ff: F_G L = L \times G \longrightarrow f^* F_G M \cong L \times G$$

is $f \times \text{id}: L \times G \rightarrow L \times G$, which takes the section s_L of $F_G L$ to the section $f^* s_M$ of $f^* F_G M$ as required. We note that for $x \in L$, the element $s_L(x) \in (F_G L)_x$ maps to the V -frame $\theta_{L,x}^{-1}$ in $(F_V L)_x = \text{Iso}(V, T_x L)$, where $\theta_{L,x}^{-1}: V \cong T_x L$ denotes the restriction of the bundle map $\theta_L^{-1}: L \times V \cong TL$ to the point x . The $\text{GL}(V)$ -principal bundle homotopy If needs to start at the $\text{GL}(V)$ -principal bundle map that sends $\theta_{L,x}^{-1}$ to $\theta_{M,f(x)}^{-1}$ and end at the map that takes $\theta_{L,x}^{-1}$ to $Df_x \circ \theta_{L,x}^{-1}$. Define If fiberwise by

$$\begin{aligned} If_x: \text{Iso}(V, T_x L) \times [0, 1] &\longrightarrow \text{Iso}(V, T_{f(x)} M) \\ If_x(\phi, t) &= \theta_{M,f(x)}^{-1} \circ \alpha((x, -), 0, t) \circ \phi, \end{aligned}$$

where we understand $\alpha((x, -), 0, t)$ as a linear isomorphism $T_x L \rightarrow V$ in the composition formula. The formula is clearly right $\text{GL}(V)$ -equivariant and specifies a homotopy of maps of $\text{GL}(V)$ -principal bundles. Then since $\alpha((x, -), 0, 0) = \theta_{L,x}$, we have

$$If_x(\theta_{L,x}^{-1}, 0) = \theta_{M,f(x)}^{-1} \circ \theta_{L,x} \circ \theta_{L,x}^{-1} = \theta_{M,f(x)}^{-1},$$

and since $\alpha((x, -), 0, 1) = \theta_{M,x} \circ Df_x$, we have

$$If_x(\theta_{L,x}^{-1}, 1) = \theta_{M,f(x)}^{-1} \circ \theta_{M,x} \circ Df_x \circ \theta_{L,x}^{-1} = Df_x \circ \theta_{L,x}^{-1},$$

as required.

Now given M, s_M , we get a V -framed G -manifold structure on M using the given G action on M and the isomorphism $\theta_M^{-1}: M \times V \cong TM$ adjoint to the section s_M ; the G, V -compatibility precisely implies that this isomorphism is G -equivariant. Given (L, s_L) , (M, s_M) , and a map $(f, Ff, If) \in \mathcal{E}_{G,V}(L, M)$ that on G -frame bundles sends s_L to $f^* s_M$, we produce a map $(f, \alpha, 0) \in \mathcal{E}mb_V^{\downarrow}(L, M)$, by defining

$$\alpha((x, -), 0, t) = \theta_{M,f(x)} \circ If_x(\theta_{L,x}^{-1}, t) \circ \theta_{L,x}.$$

Since by hypothesis Ff sends s_L to $f^* s_M$, we have that $If_x(\theta_{L,x}^{-1}, 0) = \theta_{M,f(x)}^{-1}$, and we see that $\alpha((x, -), 0, 0) = \theta_{L,x}$, as required. Since by definition $If_x(\theta_{L,x}^{-1}, 1) = Df_x \circ \theta_{L,x}^{-1}$, we see that $\alpha((x, -), 0, 1) = \theta_{M,f(x)} Df_x$, as also required.

It is straight-forward to check that these formulas define functors, that the composite functor on $\mathcal{E}mb_V^{\downarrow}$ is the identity, and that the composite functor on the pair category is naturally isomorphic to the identity. \square

We note that both $\mathcal{E}mb_V^{\downarrow}$ and $\mathcal{E}_{G,V;G}$ have a coproduct, given on the underlying G -manifolds by disjoint union, and that the functor in the previous definition preserves the coproduct.

For a V -framed G -manifolds, the main difference (philosophically) between maps in $\mathcal{E}mb_V^{\downarrow}$ and maps in $\mathcal{E}_{G,V;G}$ is that for $g \in G$, the self-map $g: M \rightarrow M$ in $\mathcal{E}_{G,V;G}$ is never in the image of the maps in $\mathcal{E}mb_V^{\downarrow}$ unless $g = e$ (even if G acts trivially on both the vector space V and the underlying smooth manifold of M). We can put this action back in with the following extension of $\mathcal{E}mb_V^{\downarrow}$.

Definition 6.9. Let $\mathcal{E}mb_V^{\downarrow \times G}$ be the topological category where the objects are the V -framed G -manifolds and where the maps are defined by

$$\mathcal{E}mb_V^{\downarrow \times G}(L, M) = \mathcal{E}mb_V^{\downarrow}(L, M) \times G^{\pi_0 L},$$

where $G^{\pi_0 L}$ denotes the space of locally constant maps from L to G . Using the observation about coproducts above, writing $L = L_1 \amalg \cdots \amalg L_p$ for the components of L , we have

$$\mathcal{E}mb_V^{\times G}(L, M) \cong \mathcal{E}mb_V^{\times G}(L_1, M) \times \cdots \times \mathcal{E}mb_V^{\times G}(L_p, M);$$

moreover, writing $M = M_1 \amalg \cdots \amalg M_q$ for the components of M , we also have

$$\mathcal{E}mb_V^{\times G}(L_i, M) \cong \mathcal{E}mb_V^{\times G}(L_i, M_1) \amalg \cdots \amalg \mathcal{E}mb_V^{\times G}(L_i, M_q),$$

and so it suffices to describe compositions

$$\mathcal{E}mb_V^{\times G}(M, N) \times \mathcal{E}mb_V^{\times G}(L, M) \longrightarrow \mathcal{E}mb_V^{\times G}(L, N)$$

when all of L , M , and N are connected. For

$$\begin{aligned} (\phi, g) &\in \mathcal{E}mb_V^{\times G}(L, M) \times G = \mathcal{E}mb_V^{\times G}(L, M), \\ (\phi', g') &\in \mathcal{E}mb_V^{\times G}(M, N) \times G = \mathcal{E}mb_V^{\times G}(L, M), \end{aligned}$$

the composite in $\mathcal{E}mb_V^{\times G}(L, N)$ is given by $(\phi' \circ g' \phi, g'g)$ where the superscript g' denotes the G -action on $\mathcal{E}mb_V^{\times G}(L, M)$.

We then have a functor $\mathcal{E}mb_V^{\times G} \rightarrow \mathcal{E}_{G, V; G}$ which sends $(\phi, g) \in \mathcal{E}mb_V^{\times G}(L, M)$ to the composite of the self-map of L given by g and the image of the map ϕ under the functor of Theorem 6.8. Composition in $\mathcal{E}mb_V^{\times G}$ is defined precisely to make this functorial. For V -framed G -manifolds L, M , every map in $\mathcal{E}_{G, V; G}(L, M)$ comes with an associated continuous map $L \rightarrow G$ given by the inherent map on G -frame bundles and canonical sections; when G is finite the map is locally constant and decomposes the element of $\mathcal{E}_{G, V; G}(L, M)$ as an element of $\mathcal{E}mb_V^{\times G}$. This proves the following, which we regard as a corollary of Theorem 6.8.

Corollary 6.10. *If G is a finite group, then the topologically enriched functor $\mathcal{E}mb_V^{\times G} \rightarrow \mathcal{E}_{G, V; G}$ is full and faithful: for all L, M in $\mathcal{E}mb_V^{\times G}$, the map*

$$\mathcal{E}mb_V^{\times G}(L, M) \longrightarrow \mathcal{E}_{G, V; G}(L, M)$$

is a homeomorphism.

The relationship is not so tight when G is a positive dimensional compact Lie group; however, for the purposes of factorization homology, it suffices to understand the relationship for the disjoint union of copies of V . In that case we can study $\mathcal{E}_{G, V; G}(V(n), M)$ in terms of configurations; looking at the center point of the disk (in M and in the G -frame bundle map), we get a commutative diagram

$$\begin{array}{ccc} \mathcal{E}mb_V^{\times G}(V(n), M) & \longrightarrow & \mathcal{E}_{G, V; G}(V(n), M) \\ & \searrow & \swarrow \\ & C(n, M) \times G^n & \end{array}$$

and well-known arguments show that the downward maps are homotopy equivalences. Keeping track of equivariance, we have a left G -action on M and a left $\Sigma_n \wr G$ action on $V(n)$ that the mapping spaces converts to a right $\Sigma_n \wr G$ -action, which is compatible with the right $\Sigma_n \wr G$ action on G^n . An equivariant elaboration of the usual argument shows that the maps are in fact equivariant homotopy equivalences:

Proposition 6.11. *For an arbitrary compact Lie group G , and a V -framed G -manifold M , the map*

$$\mathcal{E}mb_V^{\times G}(V(n), M) \longrightarrow \mathcal{E}_{G, V; G}(V(n), M)$$

is a $G \times (\Sigma_n \wr G)^{\text{op}}$ -equivariant homotopy equivalence.

Unlike in the category $\mathcal{E}mb_V$, in the category $\mathcal{E}mb_V^{\downarrow}$, the vector space V and its unit disk D are isomorphic: choosing a smooth diffeomorphism $\psi: [0, 1) \rightarrow [0, \infty)$ that is the identity near 0, we can use ψ radially to get a diffeomorphism of G -manifolds $\Psi: D \rightarrow V$. We take α to be

$$\alpha((x, v), 0, t) = (\psi'(|x|))^t v_x + (\psi(|x|)/|x|)^t v_{\perp x}$$

where v_x denotes orthogonal projection in the x direction and $v_{\perp x}$ denotes the orthogonal complement; for fixed x, v , this is a homotopy from the identity to the derivative of Ψ . (For points x near 0, we have $\psi'(|x|) = 1$ and $\psi(|x|)/|x| = 1$, so α is the constant homotopy, and we understand the formula this way also at the point $x = 0$.)

Writing $\mathcal{R}_M^{\downarrow D}(n) = \mathcal{E}mb_V^{\downarrow}(D(n), M)$ for $D(n) = D \times \{1, \dots, n\}$ and

$$\bar{\mathcal{R}}_M^{\downarrow D} X = \bigvee_{n \geq 0} \mathcal{R}_M^{\downarrow D}(n)_+ \wedge_{\Sigma_n} X^{(n)}$$

(with the diagonal G -action), we then get a monadic bar construction

$$\bar{B}_V^{\downarrow D}(M; -) := B(\bar{\mathcal{R}}_M^{\downarrow D}, \bar{\mathcal{R}}_D^{\downarrow D}, -),$$

and the isomorphism Ψ above induces a G -equivariant isomorphism

$$(6.12) \quad \bar{B}_V^{\downarrow D}(M; -) \cong \bar{B}_V^{\downarrow}(M; -)$$

where we also use Ψ to translate between the inputs.

The little V -disk operad \mathcal{D}_V admits an obvious map of G -equivariant operads $\mathcal{D}_V \rightarrow \mathcal{R}_D^{\downarrow D}$, where we interpret an affine transformation $\lambda(v) = v_0 + rv$ as the map in $\mathcal{E}mb_V^{\downarrow}$ given by λ and the homotopy $\alpha((x, v), 0, t) = r^t v$. Looking at configuration spaces, we see that the maps

$$\mathcal{D}_V(n) \longrightarrow \mathcal{R}_D^{\downarrow D}(n)$$

are $G \times \Sigma_n^{\text{op}}$ -equivariant homotopy equivalences, and so the induced map on monadic bar constructions

$$(6.13) \quad B(\bar{\mathcal{R}}_M^{\downarrow D}, \bar{\mathbb{D}}, -) \longrightarrow B(\bar{\mathcal{R}}_M^{\downarrow D}, \bar{\mathcal{R}}_D^{\downarrow D}, -) = \bar{B}_V^{\downarrow D}(M; -)$$

is a natural G -equivariant homotopy equivalence.

To compare the monadic bar construction $B(\bar{\mathcal{R}}_M^{\downarrow D}, \bar{\mathbb{D}}, -)$ to the monadic bar construction

$$\bar{B}(M; -) = B(\bar{\mathbb{E}}_M, \bar{\mathbb{D}}, -)$$

of Section 3, we observe that as functors from G -equivariant orthogonal spectra to itself, the functor

$$\bar{\mathcal{R}}_M^{\downarrow D} = \bigvee_{n \geq 0} \mathcal{E}mb_V^{\downarrow}(D(n), M)_+ \wedge_{\Sigma_n} X^{(n)}$$

with the diagonal G -action is naturally isomorphic to the functor

$$\bigvee_{n \geq 0} \mathcal{E}mb_V^{\times G}(D(n), M)_+ \wedge_{\Sigma_n \wr G} X^{(n)}$$

with the G -action given by the left G -action on $\mathcal{E}mb_V^{\times G}(D(n), M)$. The functor in Corollary 6.10 then gives a natural transformation

$$\bar{R}_M^{|D}(X) \longrightarrow \bar{E}_M(X) = \bigvee_{n \geq 0} \mathcal{E}_{G,V}(D(n), M)_+ \wedge_{\Sigma_n \wr G} X^{(n)},$$

which is a natural G -equivariant homotopy equivalence (an isomorphism when G is finite). Moreover, the natural transformation $\bar{R}_M^{|D} \rightarrow \bar{E}_M$ is a map of right $\bar{\mathbb{D}}$ -functors, and so we get an induced map on bar constructions that is also a natural G -equivariant homotopy equivalence:

Proposition 6.14. *For any G -equivariant \mathcal{D}_V -algebra X , the natural map*

$$B(\bar{R}_M^{|D}, \bar{\mathbb{D}}, X) \longrightarrow B(\bar{E}_M, \bar{\mathbb{D}}, X) = \bar{B}(M; X)$$

is a natural G -equivariant homotopy equivalence.

All together, (6.6), (6.7), (6.12), (6.13), and Proposition 6.14 give a zigzag of natural G -equivariant homotopy equivalences between genuine equivariant factorization homology for V -framed G -manifolds as defined in Definition 6.3 (inspired by Horev [8] and Zou [17]) and the theory

$$I_{\mathbb{R}^\infty}^U \bar{B}(M; I_U^{\mathbb{R}^\infty} X)$$

of Section 3.

MISSING PIECES

[CFH] Andrew J. Blumberg and Michael A. Mandell. Configuration spaces and factorization homology. In preparation.

[PMI] Andrew J. Blumberg, Michael A. Hill, and Michael A. Mandell. Parametrized multiplicative induction in equivariant stable homotopy theory. In progress.

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