ON THE SLICE SPECTRAL SEQUENCE FOR QUOTIENTS OF NORMS OF REAL BORDISM

AGNÈS BEAUDRY, MICHAEL A. HILL, TYLER LAWSON, XIAOLIN DANNY SHI, AND MINGCONG ZENG

Abstract. In this paper, we study equivariant quotients of the multiplicative norm $MU^{(\!(C_{2^n})\!)}$ of the Real bordism spectrum by permutation summands, a concept defined here. These quotients are interesting because of their relationship to the so-called "higher real K-theories". We provide new tools for computing the equivariant homotopy groups of such quotients of $MU^{(\!(C_{2^n})\!)}$ and quotients of the closely related spectrum $BP^{(\!(C_{2^n})\!)}$.

As a new example, we study spectra denoted by $BP^{(\!(C_{2^n})\!)}\langle m,m\rangle$, which have non-trivial chromatic localizations only at heights equal to rm where $0 \leqslant r \leqslant 2^{n-1}$. These spectra are natural equivariant generalizations of integral Morava-K-theories. For σ the real sign representation of $C_{2^{n-1}}$, we give a complete computation of the a_σ-localized slice spectral sequence of $i^*_{C_{2^{n-1}}} BP^{(\!(C_{2^n})\!)}\langle m,m\rangle$. We do this by establishing a correspondence between this localized slice spectral sequence and the $H\mathbb{F}_2$-based Adams spectral sequence in the category of $H\mathbb{F}_2 \wedge H\mathbb{F}_2$-modules. We also give a complete computation of the a_λ-localized slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2,2\rangle$ for λ a rotation of $\mathbb{R}^2$ by an angle of $\pi/2$. The non-localized slice spectral sequences can be recovered completely from these localizations.

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This material is based upon work supported by the National Science Foundation under Grant No. DMS-1906227, DMS-2105019 and DMS-2104844.

1. Introduction

1.1. **Motivation.** Let $E(k,\Gamma)$ be the Lubin–Tate spectrum associated to a non-additive formal group Γ of height h over a finite field k of characteristic 2. The Goerss–Hopkins–Miller theorem says that $E(k,\Gamma)$ is a commutative ring spectrum and that we have an action of $\mathrm{Aut}(\Gamma)$ on $E(k,\Gamma)$ by commutative ring maps. For a finite subgroup G of $\mathrm{Aut}(\Gamma)$, we can therefore view $E(k,\Gamma)$ as a G-equivariant commutative ring spectrum via the cofree functor and define a theory $EO_{(k,\Gamma)}(G)$ by

$$EO_{(k,\Gamma)}(G) \simeq E(k,\Gamma)^{hG}.$$

These are the Hopkins–Miller higher real K-theory spectra, so named because when the height is 1 and G is C_2, these are a form of 2-complete real K-theory.

Up to an étale extension, these spectra do not depend on (k,Γ) and we will suppress this choice by letting

$$E_h = E(k,\Gamma) \quad \text{and} \quad EO_h(G) = EO_{(k,\Gamma)}(G).$$

The spectra $EO_h(G)$ play a central role in chromatic homotopy theory. Reasons of their importance include:

(1) The fact that they detect interesting elements in the homotopy groups of spheres. For example, Hill–Hopkins–Ravenel's work on manifolds of Kervaire invariant one [19] grew out of work of Ravenel [40] which can be reinterpreted in terms of the Hurewicz image of the $EO_{p-1}(C_p)$. More recently Li–Shi–Wang–Xu study the Hurewicz image in $EO_h(C_2)$ [32].
(2) Their role as fundamental building blocks for the $K(h)$-local sphere via the theory of finite resolutions. The theory of finite resolutions was developed by Goerss–Henn–Mahowald–Rezk [14] and expanded by work of Henn [17], followed by work of Bobkova–Goerss [10].
(3) The relative computability of their homotopy groups in contrast with the homotopy groups of spheres and its chromatic localizations. For example, the homotopy groups of $EO_{p-1}(C_p)$ were computed by Hopkins–Miller, see Nave [38], and those of $EO_{k(p-1)}(C_p)$ for $k>1$ by Hill–Hopkins–Ravenel (unpublished). Computations for $p=2$ are discussed below.

Few computations have been done of homotopy groups of $EO_h(G)$ when $p=2$ for chromatic heights $h>2$. At height $h=1$, these computation are well understood via the relationship with complex and real K-theory. At $h=2$, computations are done using the close relationship of the higher real K-theories with the spectrum tmf of topological modular forms and its analogues with level structures. Computations for tmf are due to Hopkins–Mahowald (see [11, 4]), and are related to $EO_2(G)$ where G is the binary tetrahedral group. Topological modular forms with level 5 structures is related to $EO_2(C_4)$ and those computations are due to Behrens–Ormsby [9], while tmf with level 3 structures is related to $EO_2(C_6)$ and its homotopy was first computed by Mahowald–Rezk [33]. See also work of Hill–Meier [25] and Hill–Hopkins–Ravenel [24].

But at chromatic heights $h > 2$, such computations have been out of reach for a long time, one reason being the lack of nice geometric "global models" for the higher real K-theories such as ko and tmf.

However, the work of Hill–Hopkins–Ravenel [19] has made such computations more achievable. This is the approach we take in this paper, as we now describe.

By work of Hewett, when $p = 2$ a finite 2-subgroup of $\mathrm{Aut}(\Gamma)$ is either a cyclic 2-group of order 2^n whenever $h = 2^{n-1}m$ for a natural number m, or, when $h = 2m$ for m odd, G may be the quaternions [18]. In this paper, we restrict to the case of cyclic 2-groups. Our reason for this restriction is that, in this case, we can use the slice filtration and related machinery developed in [19] for cyclic groups. So, moving forward, we assume that that $G = C_{2^n}$ and that $h = 2^{n-1}m$ for m any natural number.

The case of $G = C_2$. The first case to consider is when $G = C_2$. On the one hand, there is a central subgroup C_2 in $\mathrm{Aut}(\Gamma)$ coming from "formal inversion" as an automorphism of any formal group and we obtain a spectrum $EO_h(C_2)$. On the other hand, complex conjugation gives an action of C_2 on the spaces of MU, which can be used to construct the Real bordism $MU_{\mathbb{R}}$, a C_2-spectrum first considered by Fujii [13] and Landweber [31]. Hahn–Shi [15] have produced a Real orientation for any of the Lubin–Tate spectra at the prime 2:

$$MU_{\mathbb{R}} \to i^*_{C_2} E_h.$$

This allows us to combine the two C_2-actions under one perspective, and to construct E_h as localizations of quotients of $MU_{\mathbb{R}}$.

We first discuss two families of C_2-spectra that refine classical spectra of interest in chromatic homotopy theory and are constructed as quotients of $MU_{\mathbb{R}}$. More specifically, after localization at 2, $MU_{\mathbb{R}}$ splits as a wedge of suspensions of the Real Brown–Peterson spectrum $BP_{\mathbb{R}}$ and the spectra we discuss are quotients of the latter.

Let ρ_2 be the real regular representation. By work of Araki [2] and Landweber [31], we have

$$\pi^{C_2}_{*\rho_2} BP_{\mathbb{R}} \cong \mathbb{Z}_{(2)}[\bar{v}_1, \bar{v}_2, \bar{v}_3, \ldots]$$

for generators $\bar{v}_r \in \pi^{C_2}_{(2^r-1)\rho_2} BP_{\mathbb{R}}$ whose underlying homotopy classes give generators $v_i \in \pi_{(2^i-1)2} BP$ for BP_*.

(1) First, we have Real truncated Brown–Peterson spectra

$$BP_{\mathbb{R}}\langle h\rangle = BP_{\mathbb{R}}/(\bar{v}_j \mid j > h)$$

The underlying non-equivariant spectrum is a classical truncated Brown–Peterson spectrum $BP\langle h\rangle$, which has chromatic height $\leqslant h$ in the sense that $L_{K(r)} BP\langle h\rangle$ is trivial if and only if $r > h$.

The $K(h)$-localization of $i^*_e BP_{\mathbb{R}}\langle h\rangle$ gives, up to periodization, a model of Lubin–Tate theory E_h with its canonical C_2-action obtained through Goerss–Hopkins–Miller theory. The fixed points of $BP_{\mathbb{R}}\langle h\rangle$ are thus global models for $EO_h(C_2)$.

These equivariant spectra and their $\bar{v}_h$-localizations were first studied by Hu–Kriz [29] and Kitchloo–Wilson [30]. They give a chromatic filtration of the fixed point spectrum $(BP_{\mathbb{R}})^G$ via the tower

$$\ldots \to BP_{\mathbb{R}}\langle h\rangle^{C_2} \to \ldots \to BP_{\mathbb{R}}\langle 1\rangle^{C_2} \to BP_{\mathbb{R}}\langle 0\rangle^{C_2} \simeq H\mathbb{Z}^{C_2}.$$

(2) Another example are the Real integral Morava K-theories, which we denote by

$$BP_{\mathbb{R}}\langle h, h\rangle := BP_{\mathbb{R}}/(\bar{v}_i \mid i \neq 0, h).$$

The underlying spectra are classical integral Morava K-theories, which have chromatic height concentrated at $r = 0, h$ in the sense that $L_{K(r)}BP\langle h, h\rangle$ is non-trivial in exactly these two cases. These are closely related to the classical Morava K-theories $K(h)$. Their equivariant generalizations introduced below are among the main examples we study in this paper.

Nonequivariant versions of these integral Morava K-theories have been previously studied, e.g. in [37].

Larger cyclic 2-groups. In this paper, we study generalizations of the above examples to larger cyclic groups $G = C_{2^n}$. In particular, we will study the generalizations of the integral Morava K-theories $BP_{\mathbb{R}}\langle h, h\rangle$ in great computational depth.

Let G be a finite subgroup of $\mathrm{Aut}(\Gamma)$ which contains C_2. Since E_h is an equivariant commutative ring spectrum, the norm-forgetful adjunction produces a G-equivariant orientation map

$$MU^{(\!(G)\!)} := N_{C_2}^G MU_{\mathbb{R}} \to E_h.$$

Again, since we are working 2-locally, using Quillen's idempotent we can replace $MU_{\mathbb{R}}$ with $BP_{\mathbb{R}}$ and get a map

$$BP^{(\!(G)\!)} := N_{C_2}^G BP_{\mathbb{R}} \to E_h$$

This allows us to view

$$\left(BP^{(\!(G)\!)}\right)^G$$

as a global approximation to any $EO_h(G)$.

Thus, the spectrum $\left(BP^{(\!(G)\!)}\right)^G$ contains transchromatic information for all heights h where $\mathrm{Aut}(\Gamma)$ contains G as a subgroup.

Remark 1.1. In fact, to capture more global phenomena from the $\left(BP^{(\!(G)\!)}\right)^G$-perspective, we can assemble these fixed point spectra as G varies into a "detection tower" of ring spectra

$$\cdots \to \left(BP^{(\!(C_{2^n})\!)}\right)^{C_{2^n}} \to \left(BP^{(\!(C_{2^{n-1}})\!)}\right)^{C_{2^{n-1}}} \to \cdots \to \left(BP^{(\!(C_2)\!)}\right)^{C_2} \to BP,$$

where the maps are constructed using the unit of the restriction-coinduction adjunction. The Hurewicz image for any $\left(BP^{(\!(C_{2^n})\!)}\right)^{C_{2^n}}$ is therefore bounded by that of the limit

$$\varprojlim \left(BP^{(\!(C_{2^n})\!)}\right)^{C_{2^n}}.$$

We continue to focus on $G = C_{2^n}$ and $h = 2^{n-1}m$. To compute the homotopy groups of $\left(BP^{(\!(G)\!)}\right)^G$, Hill, Hopkins, and Ravenel introduced the equivariant slice spectral sequence [19, 22]. However, due to the complexity of the equivariant computations, besides $\left(BP^{(\!(C_2)\!)}\right)^{C_2}$, not much is known about the homotopy groups of $\left(BP^{(\!(G)\!)}\right)^G$.

The current approach has been to pass to fixed points of quotients of $BP^{(\!(G)\!)}$ that are generalizations of the $BP_{\mathbb{R}}\langle h\rangle$. To define these quotients, note that as in [19], one can compute the C_2-equivariant homotopy groups of $BP^{(\!(G)\!)}$ in degrees an integer multiple of ρ_2 to be

$$\pi^{C_2}_{*\rho_2} BP^{(\!(G)\!)} \cong \mathbb{Z}_{(2)}[G\cdot \bar{v}_1^G, G\cdot \bar{v}_2^G, \ldots]$$

where $\bar{v}_r^G \in \pi^{C_2}_{(2^r-1)\rho_2}$. Here, $G\cdot\bar{v}_r^G$ denotes a set of 2^{n-1} elements

$$\bar{v}_r^G, \gamma\bar{v}_r^G, \ldots, \gamma^{2^{n-1}-1}\bar{v}_r^G,$$

where γ represents a generator of G and the Weyl action of G is made obvious by the notation except for $\gamma^{2^{n-1}}\bar{v}_r^G = -\bar{v}_r^G$. The method of twisted monoid rings of Hill–Hopkins–Ravenel [19] then allows one to form quotients of $BP^{(\!(G)\!)}$ by collections of *permutation summands* of the form $G\cdot\bar{v}_r^G$. This is the first thing we explain and explore in Section 2. Such quotients by permutation summands are our main objects of study.

The generalizations of the quotients of $BP_{\mathbb{R}}$ mentioned above are the following quotients by permutation summands:

(1) The spectra

$$BP^{(\!(G)\!)}\langle m\rangle := BP^{(\!(G)\!)}/(G\cdot\bar{v}^G_{m+1}, G\cdot\bar{v}^G_{m+2}, \ldots)$$

generalize the spectrum $BP_{\mathbb{R}}\langle h\rangle$ in the case when $G = C_2$. In [7, Theorem 7.5], we showed that this is a spectrum of height $\leqslant h$, where $h = 2^{n-1}m$. Up to periodization, the $K(h)$-localization of the spectrum underlying $BP^{(\!(G)\!)}\langle m\rangle$ gives a model for a height h Lubin–Tate theory equipped with a G-action coming from the Morava stabilizer group. So, the fixed points of the $BP^{(\!(G)\!)}\langle m\rangle$ are global models for $EO_h(G)$.

These quotients give a chromatic filtration of $(BP^{(\!(G)\!)})^G$ via the tower

$$\ldots \to (BP^{(\!(G)\!)}\langle m\rangle)^G \to (BP^{(\!(G)\!)}\langle m-1\rangle)^G \to \ldots \to (BP^{(\!(G)\!)}\langle 1\rangle)^G \to H\mathbb{Z}^G.$$

The height increases by 2^{n-1} at each step of this tower.

Computations of the slice spectral sequence of these quotients are still few, limited to the case of $BP_{\mathbb{R}}\langle m\rangle$ [29], of $BP^{(\!(C_4)\!)}\langle 1\rangle$ [24], and of $BP^{(\!(C_4)\!)}\langle 2\rangle$ [26].

We do not extend these computations in this paper, but rather consider the homotopy groups of the next quotient on our list, the generalization of the integral Morava K-theories, as a preliminary step to understanding $BP^{(\!(G)\!)}\langle m\rangle$ in more generality.

(2) The spectra

$$BP^{(\!(G)\!)}\langle m,m\rangle := BP^{(\!(G)\!)}\langle m\rangle/(G\cdot\bar{v}^G_1, \ldots, G\cdot\bar{v}^G_{m-1})$$

have chromatic height concentrated in degrees rm for $0 \leqslant r \leqslant 2^{n-1}$, as shown in Theorem 3.1 below. When $n=1$, these are the $BP_{\mathbb{R}}\langle h,h\rangle$.

These spectra, to our knowledge, have not been considered before and are the main new example of a quotient by permutation summands studied in this paper. They serve as a computable approximation of equivariant truncated Brown–Peterson spectra $BP^{(\!(G)\!)}\langle m\rangle$. A key point is that their slice differentials are easier to compute than those of the $BP^{(\!(G)\!)}\langle m\rangle$.

1.2. **Main results.**

We now describe the content of this paper and state our results.

Section 2. The first part of this paper is dedicated to the definition of the various equivariant quotients by permutation summands and the study of their slices. The results of this section, while not stated elsewhere, are straightforward generalizations of the work of Hill–Hopkins–Ravenel in [19].

A typical example of a permutation summand for $BP^{(\!(G)\!)}$ is a collection of elements of the form $G \cdot \bar{v}_j^G$ (see Definition 2.1). The main point is that the slice associated graded of quotients by permutation summands is easy to describe:

Theorem (*cf.* Proposition 2.5)**.** *The slice associated graded of the quotient*

$$BP^{(\!(G)\!)}/(G \cdot \bar{v}_j^G \mid j \in J)$$

where J is a set of natural numbers, is the generalized Eilenberg–Mac Lane spectrum

$$H\underline{\mathbb{Z}}[G \cdot \bar{v}_i^G \mid i \notin J].$$

Example. The slice associated graded for $BP^{(\!(G)\!)}\langle m, m\rangle$ is $H\underline{\mathbb{Z}}[G \cdot \bar{v}_m^G]$.

In fact, the results we prove do not depend on the choice of generators of the permutation summand: We can replace $\bar{v}_m$ by any element $\bar{s}_{2^m-1}$ in $\pi^{C_2}_{(2^m-1)\rho_2} BP^{(\!(G)\!)}$ that generates a permutation summand. For $\bar{S} = \{\bar{s}_{2^j-1} \mid j \in J\}$, we will write

$$G \cdot \bar{S} = \{G \cdot \bar{s}_{2^j-1} \mid j \in J\}.$$

Section 3. We determine the chromatic height of the generalized integral Morava K-theory spectra $BP^{(\!(G)\!)}\langle m, m\rangle$. Namely, we show:

Theorem (Theorem 3.1)**.** *The spectrum underlying $BP^{(\!(G)\!)}\langle m, m\rangle$ has non-trivial chromatic localizations only at heights equal to rm where $0 \leqslant r \leqslant 2^{n-1}$. That is,*

1. *for $r = km$ where $0 \leqslant k \leqslant 2^{n-1}$, $L_{K(r)} i_e^* BP^{(\!(G)\!)}\langle m, m\rangle \not\simeq *$, and*
2. *for all other $r \geqslant 0$, $L_{K(r)} i_e^* BP^{(\!(G)\!)}\langle m, m\rangle \simeq *$.*

Theorem 3.1 states that the underlying spectrum of $BP^{(\!(G)\!)}\langle m, m\rangle$ captures chromatic information at heights 0, m, $2m$, $3m$, $\ldots$, $2^{n-1}m$.

Section 4. In this expository section, we discuss the tools needed for the computations of the rest of the paper.

To compute the homotopy groups of quotients of the form $BP^{(\!(G)\!)}/G \cdot \bar{S}$, we use a tool called the *localized slice spectral sequence*. This spectral sequence is developed in [36]. We briefly recall some of its highlights here, but Section 4 is dedicated to a more detailed discussion of these ideas.

Let X be a G-spectrum and let $\{P^\bullet\}$ denote the slice tower for X. For $H \subset G$ a normal subgroup, the spectral sequence associated to the tower

$$\{Q^\bullet\} := \{\tilde{E}\mathcal{F}[H] \wedge P^\bullet\}$$

is the *localized slice spectral sequence* of $\tilde{E}\mathcal{F}[H] \wedge X$, where $\mathcal{F}[H]$ is a certain family described in Definition 4.3. By [36], it converges strongly to the G-equivariant homotopy groups of $\tilde{E}\mathcal{F}[H] \wedge X$, which is also equal to the homotopy groups of $\Phi^H(X)^{G/H}$ (see Theorem 4.5).

Example. When $G = H = C_{2^n}$, the localized slice spectral sequence is the a_σ-localization of the slice spectral sequence, for σ the sign representation. If $G = C_{2^n}$ and $H = C_{2^i}$ for $i < n$, then the localized slice spectral sequence is the $a_{\lambda_{i-1}}$-localization of the slice spectral sequence for λ_{i-1} a rotation of the plane by an angle of $\pi/2^{n-i}$.

For $G = C_{2^n}$, the slice spectral sequence of X is also divided into different regions, separated by the lines through the origin of slopes $(2^i - 1)$, $0 \leqslant i \leqslant n$. In [35] (forthcoming), the authors prove a result which they called the *Slice Recovery Theorem*, which we state in Theorem 4.6. It implies that for X a (-1)-connected G-spectrum, the map from the slice spectral sequence of X to the localized slice spectral sequence of $\tilde{E}\mathcal{F}[C_{2^i}] \wedge X$ induces an isomorphism between all the differentials on or above the line of slope $(2^{i-1} - 1)$ for all $1 \leqslant i \leqslant n$.

Therefore, the localized slice spectral sequence serves as a bridge between the fixed points X^G and the residual fixed points $(\Phi^{C_{2^i}} X)^{G/C_{2^i}}$. More precisely, even though the localized slice spectral sequence only computes the geometric fixed points, its E_2-page and differentials are closely related to the original slice spectral sequence, which computes the fixed points.

Section 5. Equipped with the tools reviewed above, we completely compute the a_σ-localized slice spectral sequence of $a_\sigma^{-1} BP^{(\!(G)\!)}/G \cdot \bar{S}$. This is the same as the localized slice spectral sequence for $\tilde{E}\mathcal{F}[G] \wedge BP^{(\!(G)\!)}/G \cdot \bar{S}$. This spectral sequence converges to the homotopy groups of the G-geometric fixed points of $BP^{(\!(G)\!)}/G \cdot \bar{S}$.

This computation, which is the content of Theorem 5.9, is done by an application of the Slice Differential Theorem of Hill–Hopkins–Ravenel [19]. The quotient map

$$BP^{(\!(G)\!)} \longrightarrow BP^{(\!(G)\!)}/G \cdot \bar{S}$$

induces a map of the corresponding localized slice spectral sequences. The Slice Differential Theorem produces all the differentials in the a_σ-localized slice spectral sequence of $a_\sigma^{-1} BP^{(\!(G)\!)}$. By using the module structure and naturality, we deduce all the differentials in the a_σ-localized slice spectral sequence of $a_\sigma^{-1} BP^{(\!(G)\!)}/G \cdot \bar{S}$.

Using the Slice Recovery Theorem, the computation of the a_σ-localized slice spectral sequence of $a_\sigma^{-1} BP^{(\!(G)\!)}/G \cdot \bar{S}$ gives us a complete understanding of all the differentials above the line of slope $2^{n-1} - 1$ in the original slice spectral sequence of $BP^{(\!(G)\!)}/G \cdot \bar{S}$.

Remark 1.2. When specialized to the case $G = C_2$, the a_σ-localized slice spectral sequence of Section 5 produces all the differentials in the slice spectral sequence of $BP_{\mathbb{R}}/\bar{S}$. This is explained in Corollary 5.12.

The results of this section are also used to show that, in stark contrast to the non-equivariant setting, most of the quotients of $BP^{(\!(G)\!)}$ by permutation summands do not admit a ring structure:

Proposition (Proposition 5.16)**.** Let $J \subseteq \mathbb{N}$ and $\bar{S} = \{\bar{s}_{2^j-1} \mid j \in J\}$ be a set of generators for permutation summands. If there is a $k \in J$ such that $(k+1) \notin J$, then $BP^{(\!(G)\!)}/G \cdot \bar{S}$ does not have a ring structure in the homotopy category.

Section 6. We analyze the next region of the slice spectral sequence of $BP^{(\!(G)\!)}\langle m, m\rangle$, that is, the region between the lines of slopes $(2^{n-2} - 1)$ and $(2^{n-1} - 1)$. Letting $G' = C_{2^{n-1}}$ be the subgroup of index two in $G = C_{2^n}$, the differentials in this region can be obtained by computing the localized slice spectral sequence of

$$\tilde{E}\mathcal{F}[G'] \wedge BP^{(\!(G)\!)}\langle m, m\rangle \simeq a_{\lambda_{n-2}}^{-1} BP^{(\!(G)\!)}\langle m, m\rangle,$$

which computes the homotopy groups of the G/G'-fixed points of the spectrum $\Phi^{G'}(BP^{(\!(G)\!)}\langle m, m\rangle)$.

A useful input to computing this spectral sequence is its restriction to the group G', which computes the homotopy groups of $\Phi^{G'} i^*_{G'} BP^{(\!(G)\!)}\langle m, m\rangle$. Then the Mackey functor structure allows us to deduce information about the G-equivariant spectral sequence from the simpler G'-equivariant spectral sequence.

This is where yet another spectral sequence comes into play, namely, the $H\mathbb{F}_2$-based Adams spectral sequence in the category of A-module spectra (as in Baker–Lazarev [3]), where

$$A := H\mathbb{F}_2 \wedge H\mathbb{F}_2.$$

Here, $H\mathbb{F}_2$ is given an A-module structure via the multiplication map $A \to H\mathbb{F}_2$. We call this spectral sequence the *relative Adams spectral sequence*.

The link between this more classical spectral sequence and this work is that, as non-equivariant spectra, there is an equivalence

$$\Phi^{G'}(i^*_{G'} BP^{(\!(G)\!)}\langle m, m\rangle) \simeq A/(\xi_i, \zeta_i : i \neq m)$$

for ξ_i and ζ_i the usual Milnor generators and their conjugates. Furthermore, there is an intimate connection between the relative Adams spectral sequence with the G'-equivariant localized slice spectral sequence given by the following result, which we prove in Section 6.2:

Theorem (Theorem 6.7 and Corollary 6.10)**.** *After a reindexing of filtrations, the G'-equivariant localized slice spectral sequence of $\tilde{E}\mathcal{F}[G'] \wedge BP^{(\!(G)\!)}\langle m, m\rangle$ is isomorphic to the relative Adams spectral sequence of $A/(\xi_i, \zeta_i : i \neq m)$.*

In Section 6.3, we compute the relative Adams spectral sequence of $A/(\xi_i, \zeta_i : i \neq m)$, extending our computations from [6].

Combining the above theorem relating the two spectral sequences with this computation gives us the G'-equivariant localized slice spectral sequence of $BP^{(\!(G)\!)}\langle m, m\rangle$.

As an example, in Section 6.4 we compute C_2-equivariant localized slice spectral sequence of

$$\tilde{E}\mathcal{F}[C_2] \wedge i^*_{C_2} BP^{(\!(C_4)\!)}\langle 2, 2\rangle$$

by using the comparison of spectral sequences of Section 6.2 and the computations of Section 6.3.

Section 7. In the final section the paper we specialize to the case $m = 2$ and $G = C_4$, letting $a_{\lambda_0} = a_\lambda$. We compute the localized slice spectral sequence of $a_\lambda^{-1} BP^{(\!(C_4)\!)}\langle 2, 2\rangle$, converging to the C_4-equivariant homotopy groups of

$$a_\lambda^{-1} BP^{(\!(C_4)\!)}\langle 2, 2\rangle \simeq \tilde{E}\mathcal{F}[C_4] \wedge BP^{(\!(C_4)\!)}\langle 2, 2\rangle.$$

Remark 1.3. Although we do not do this in details in this paper, the Slice Recovery Theorem 4.6 can be used to recover all the differentials in the C_4-equivariant slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2, 2\rangle$. By Theorem 3.1, the theory $BP^{(\!(C_4)\!)}\langle 2, 2\rangle$ is of heights 0, 2 and 4. This computation is thus a height-4 computation of a spectrum that is closely related to $BP^{(\!(C_4)\!)}\langle 2\rangle$, the spectrum studied in [26].

Remark 1.4. The slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2, 2\rangle$ is easier to compute than that of $BP^{(\!(C_4)\!)}\langle 2\rangle$ because there are less classes in the spectral sequence. We suspect these computations can be generalized to higher heights, to compute the homotopy of $BP^{(\!(C_4)\!)}\langle m, m\rangle$ for any $m \geqslant 1$. This is a C_4-equivariant theory of heights 0, m and $2m$. The computation of the equivariant homotopy groups of $BP^{(\!(C_4)\!)}\langle m, m\rangle$ for all $m \geqslant 1$ would be a significant addition to the bank of computations at higher chromatic heights.

Remark 1.5. The computation of the C_4-equivariant slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2,2\rangle$, combined with the Slice Recovery Theorem of [35] (forthcoming) determines the slice spectral sequences of $BP^{(\!(G)\!)}\langle 2,2\rangle$ for any $G = C_{2^n}$ above the line of slope $2^{n-2}-1$.

1.3. **Acknowledgements.** The authors would like to thank Michael J. Hopkins and Lennart Meier for helpful conversations.

2. Quotient modules of $MU^{(\!(G)\!)}$

2.1. **Slices for some $MU^{(\!(G)\!)}$ modules.** For a graded ring R with augmentation ideal I, let Q_nR denote the degree n elements in I/I^2. One of the key computations in [19] was a convenient choice of algebra generators for the $*\rho_2$-graded homotopy groups of $MU^{(\!(G)\!)}$. In particular, we have an isomorphism of graded $\mathbb{Z}[G]$-modules

$$Q_{*\rho_2}\big(\pi^{C_2}_{*\rho_2}MU^{(\!(G)\!)}\big) \cong \bigoplus_{k\geqslant 1}\Sigma^{k\rho_2}\operatorname{Ind}^G_{C_2}(\mathbb{Z}_-^{\otimes k}),$$

where $\mathbb{Z}_-$ is the integral sign representation.

Definition 2.1. Let $J \subseteq \mathbb{N}$ and

$$\bar{S} = \{\bar{s}_j \in \pi^{C_2}_{j\rho_2}MU^{(\!(G)\!)} \mid j \in J\}$$

be a collection of elements. Associated to $\bar{S}$, we have a C_2-equivariant map

$$f_{\bar{S}}\colon \bigoplus_{j\in J}\Sigma^{j\rho_2}\mathbb{Z}_-^{\otimes j} \to \bigoplus_{j\in J}Q_{j\rho_2}\big(MU^{(\!(G)\!)}\big).$$

We say that $\bar{S}$ *generates a permutation summand* if the adjoint G-equivariant map

$$\tilde{f}_{\bar{S}}\colon \bigoplus_{j\in J}\Sigma^{j\rho_2}\operatorname{Ind}^G_{C_2}(\mathbb{Z}_-^{\otimes j}) \to \bigoplus_{j\in J}Q_{j\rho_2}\big(MU^{(\!(G)\!)}\big)$$

is an isomorphism.

Given any element in the $RO(C_2)$-graded homotopy of $MU^{(\!(G)\!)}$, we can use the method of twisted monoid algebras from [19]. For each $j \in J$, we have a free associative algebra

$$\mathbb{S}^0[\bar{s}_j] := \bigvee_{i=0}^{\infty} S^{ij\rho_2},$$

and we have a canonical associative algebra map

$$\mathbb{S}^0[\bar{s}_j] \to i^*_{C_2}MU^{(\!(G)\!)}$$

adjoint to the map defining $\bar{s}_j$. Using the norm maps on $MU^{(\!(G)\!)}$ and the multiplication, we get an associative algebra map

$$\mathbb{S}^0[G\cdot\bar{S}] = \bigwedge_{j\in J}N^G_{C_2}\mathbb{S}^0[\bar{s}_j] \to MU^{(\!(G)\!)}.$$

Definition 2.2. For $J \subseteq \mathbb{N}$, let

$$MU^{(\!(G)\!)}/G\cdot\bar{S} := MU^{(\!(G)\!)}/\big(G\cdot\bar{s}_j \mid j\in J\big) = MU^{(\!(G)\!)}\underset{\mathbb{S}^0[G\cdot\bar{S}]}{\wedge}\mathbb{S}^0.$$

We call $MU^{(\!(G)\!)}/G\cdot\bar{S}$ a *quotient by permutation summands.*

The following is a slight generalization of the Slice Theorem of Hill–Hopkins–Ravenel [19, Theorem 6.1]. Recall from [19] that

$$\pi^{C_2}_{*\rho_2} MU^{(\!(G)\!)} \cong \mathbb{Z}_{(2)}[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots]$$

for generators $\bar{r}_i$ in $\pi^{C_2}_{i\rho_2} MU^{(\!(G)\!)}$ introduced in (5.39) of [19]. In fact, the conditions on the classes $\bar{s}_j$ guarantee that we can use them instead of the $\bar{r}_j$ for $j \in J$. We then extend the set $\bar{S}$ to form a set of equivariant algebra generators, as in [19, Section 5]. Any set

$$\bar{S}' := \{\bar{s}_j \mid j \notin J\}$$

of elements $\bar{s}_j \in \pi^{C_2}_{j\rho_2} MU^{(\!(G)\!)}$ which generate a permutation summand will do to extend $\bar{S}$ to a set $\bar{S} \cup \bar{S}'$ of equivariant generators for $\pi^{C_2}_{*\rho_2} MU^{(\!(G)\!)}$.

Definition 2.3. Let $J \subseteq \mathbb{N}$ and $\bar{S}$ and $\bar{S}'$ be as above. Define

$$H\underline{\mathbb{Z}}[G \cdot \bar{s}_1, G \cdot \bar{s}_2, \dots] := H\underline{\mathbb{Z}} \underset{\mathbb{S}^0}{\wedge} \mathbb{S}^0[G \cdot \bar{s}_1, \dots]$$

and

$$H\underline{\mathbb{Z}}[G \cdot \bar{s}_1, G \cdot \bar{s}_2, \dots]/G \cdot \bar{S} := H\underline{\mathbb{Z}}[G \cdot \bar{s}_1, \dots] \underset{\mathbb{S}^0[G \cdot \bar{S}]}{\wedge} \mathbb{S}^0 .$$

Remark 2.4. The spectrum $H\underline{\mathbb{Z}}[G \cdot \bar{s}_1, G \cdot \bar{s}_2, \dots]/G \cdot \bar{S}$ is very simple. In fact, it is equivalent to $H\underline{\mathbb{Z}}[G \cdot \bar{S}']$, which itself is the smash product over $j \notin J$ of the norms, in the category of $H\underline{\mathbb{Z}}$-modules, of $H\underline{\mathbb{Z}}[\bar{s}_j] \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{ij\rho_2}$.

Proposition 2.5. *The slice associated graded of $MU^{(\!(G)\!)}/G \cdot \bar{S}$ is the generalized Eilenberg–Mac Lane spectrum*

$$H\underline{\mathbb{Z}}[G \cdot \bar{s}_1, G \cdot \bar{s}_2, \dots]/(G \cdot \bar{S}).$$

Proof. We have a natural equivalence

$$MU^{(\!(G)\!)}/G \cdot \bar{S} \simeq MU^{(\!(G)\!)} \underset{\mathbb{S}^0[G \cdot \bar{s}_1, \dots]}{\wedge} \mathbb{S}^0[G \cdot \bar{S}'].$$

The result now follows exactly as [19, Slice Theorem 6.1], using the natural degree filtration on $\mathbb{S}^0[G \cdot \bar{S}']$. □

Letting $\bar{S}$ be the generators killed by the Quillen idempotent, this recovers the usual form of the slice associated graded for $BP^{(\!(G)\!)}$. We could moreover always append this to any collection $\bar{S}$ we consider, which allows us to deduce all of the analogous results for $BP^{(\!(G)\!)}$. We will do so without comment moving forward.

Remark 2.6. The left action of $MU^{(\!(G)\!)}$ on itself always endows $MU^{(\!(G)\!)}/G \cdot \bar{S}$ with a canonical $MU^{(\!(G)\!)}$-module structure, and the same is true with $BP^{(\!(G)\!)}$ instead.

Notation 2.7. In the homotopy of the spectrum $BP^{(\!(G)\!)}$, let

$$\bar{v}_k^G := \bar{t}_k^G \in \pi^{C_2}_{(2^k-1)\rho_2} BP^{(\!(G)\!)},$$

as defined and considered in [7].

Definition 2.8. For each $m \geqslant 0$, let $J_m = \{k \mid k > m\}$. Let

$$\bar{S}_m = \left\{\bar{s}_{2^j-1} \in \pi^{C_2}_{(2^j-1)\rho_2} BP^{(\!(G)\!)} \mid j \in J_m\right\}$$

generate a permutation summand. When for each $j \in J_m$, $\bar{s}_{2^j-1} = \bar{v}_j^G$, we name the quotient

$$BP^{(\!(G)\!)}/G \cdot \bar{S}_m = BP^{(\!(G)\!)}\langle m \rangle.$$

More generally, we say that the $BP^{(\!(G)\!)}$-module

$$BP^{(\!(G)\!)}/G \cdot \bar{S}_m$$

is a *form of* $BP^{(\!(G)\!)}\langle m\rangle$.

Notation 2.9. Let v_k^G be the restriction to the trivial group of $\bar{v}_k^G$.

Remark 2.10. Just as in [20], we note that since the underlying rings are all polynomial rings, the map

$$\pi_*^{\{e\}} BP^{(\!(G)\!)} \to \pi_*^{\{e\}} BP^{(\!(G)\!)}\langle m\rangle = \mathbb{Z}_{(2)}[G \cdot v_1^G, \dots, G \cdot v_m^G]$$

has a section.

A form of $BP^{(\!(G)\!)}\langle m\rangle$ is a quotient module M with the property that for any section, the composite

$$\mathbb{Z}_{(2)}[G \cdot v_1^G, \dots, G \cdot v_m^G] \to \pi_*^{\{e\}} BP^{(\!(G)\!)} \to \pi_*^{\{e\}} M$$

is an isomorphism. The difference between the forms lies in the $BP^{(\!(G)\!)}$-module structure, not in the underlying homotopy groups.

Corollary 2.11. *The slice associated graded for any form of* $BP^{(\!(G)\!)}\langle m\rangle$ *is*

$$H\underline{\mathbb{Z}}[G \cdot \bar{v}_1^G, \dots, G \cdot \bar{v}_m^G].$$

Definition 2.12. Let k and m be natural numbers with $1 \leqslant k \leqslant m$. Let

$$\bar{S}_{k,m} = \big\{\bar{v}_j^G \mid 0 < j < k \text{ or } j > m\big\},$$

and let

$$BP^{(\!(G)\!)}\langle k, m\rangle = BP^{(\!(G)\!)}/G \cdot \bar{S}_{k,m}.$$

Remark 2.13. As in Definition 2.8, we also define forms of $BP^{(\!(G)\!)}\langle k, m\rangle$ as quotients by elements $\bar{s}_{2^j-1}$, for $0 < j < k$ or $j > m$ that generate permutation summands.

Corollary 2.14. *The slice associated graded for* $BP^{(\!(G)\!)}\langle k, m\rangle$ *(or for any form) is*

$$H\underline{\mathbb{Z}}[G \cdot \bar{v}_k^G, \dots, G \cdot \bar{v}_m^G].$$

One of the main examples we will analyze is

$$BP^{(\!(G)\!)}\langle m, m\rangle = BP^{(\!(G)\!)}/\big(G \cdot \bar{v}_1^G, \dots, G \cdot \bar{v}_{m-1}^G, G \cdot \bar{v}_{m+1}^G, \dots\big)$$

where $m \geqslant 1$.

The slice associated graded for $BP^{(\!(G)\!)}\langle m, m\rangle$ is very simple, given by

$$H\underline{\mathbb{Z}}[G \cdot \bar{v}_m^G].$$

3. Chromatic Height of $BP^{(\!(G)\!)}\langle m, m\rangle$

In this section, we study the underlying chromatic height of the spectra $BP^{(\!(G)\!)}\langle m, m\rangle$.

Theorem 3.1.

(1) For $r = km$ where $0 \leqslant k \leqslant 2^{n-1}$, $L_{K(r)} i_e^* BP^{(\!(G)\!)}\langle m, m\rangle \not\simeq *$;

(2) For all other $r \geqslant 0$, $L_{K(r)} i_e^* BP^{(\!(G)\!)}\langle m, m\rangle \simeq *$.

Proof. Our proof will be similar to that of Proposition 7.4 and Theorem 7.5 in [7]. In this proof, let $X = i_e^* BP^{(\!(G)\!)}\langle m, m\rangle$. For any r, there is a cofinal sequence $J(i) = (j_0, j_1, \ldots, j_{r-1})$ of positive integers and generalized Moore spectra

$$M_{J(i)} = S^0/(v_0^{j_0}, \ldots, v_{r-1}^{j_{r-1}})$$

with maps $M_{J(i+1)} \to M_{J(i)}$ such that

$$L_{K(r)}X \simeq \operatorname{holim}_i \left(L_r X \wedge M_{J(i)}\right).$$

See [28, Prop. 7.10].

Since X is a BP-module, it follows from [27, Cor. 1.10] that the natural map $L_r^f X \to L_r X$ is an equivalence (since it is a BP-equivalence between BP-local spectra). Therefore,

$$L_r X \wedge M_{J(i)} \simeq L_r^f X \wedge M_{J(i)} \simeq X \wedge L_r^f M_{J(i)} \simeq X \wedge v_r^{-1} M_{J(i)} \simeq v_r^{-1} X \wedge M_{J(i)}.$$

Here, we have used the fact that since $M_{J(i)}$ is a type r spectrum, its finite localization is the telescope [34, Prop. 3.2]. We have also used the fact that L_r^f is smashing.

To prove (1), we assume r is of the form km with $0 \leqslant k \leqslant 2^{n-1}$. We will first show that under the map

$$BP \longrightarrow v_r^{-1} X \wedge M_{J(i)},$$

the image of $v_r \in \pi_* BP$ is nonzero. Note that

$$v_r^{-1} X \wedge M_{J(i)} = v_r^{-1} X \wedge_{MU} MU/(v_0^{j_0}, \ldots, v_{r-1}^{j_{r-1}}) = v_r^{-1} X/(v_0^{j_0}, \ldots, v_{r-1}^{j_{r-1}}).$$

By an iterative application of the formula

$$v_r^{C_{2^{n-1}}} \equiv v_r^{C_{2^n}} + \gamma_n v_r^{C_{2^n}} + \sum_{j=1}^{r-1} \gamma_n v_j^{C_{2^n}} (v_{r-j}^{C_{2^n}})^{2^j} \pmod{I_r}$$

(where $I_r = (2, v_1, \ldots, v_{r-1})$) in [7, Theorem 1.1], the images of $v_r^j = (v_r^{C_2})^j$ in $(\pi_* X)/(v_0, \ldots, v_{r-1})$ are all nonzero for $j \geqslant 1$. This implies that their images are also nonzero in $\pi_* \left(X/(v_0, \ldots, v_{r-1})\right)$. Therefore, the image of v_r in $\pi_*(v_r^{-1} X/(v_0^{j_0}, \ldots, v_{r-1}^{j_{r-1}}))$ is nonzero. After taking the homotopy limit, the image of v_r under the map $\pi_* BP \to \pi_* L_{K(r)} X$ will also be nonzero. It follows that $\pi_* L_{K(r)} X \not\simeq *$.

To prove (2), we will consider two cases, based on the divisibility of r by m. If r is not divisible by m, then the degree of v_r, $2(2^r - 1)$, is not divisible by $2(2^m - 1)$. However, the homotopy groups of X are concentrated in degrees that are divisible by $2(2^m - 1)$. This implies that the multiplication by v_r map

$$\Sigma^{|v_r|} X \longrightarrow X$$

induces the zero map on homotopy, and

$$\pi_* v_r^{-1} X \cong v_r^{-1} \pi_* X = 0.$$

It follows that $v_r^{-1} X \simeq *$ and therefore $L_{K(r)} X \simeq *$.

Now, suppose m divides r. Let $r = km$ for some $k > 2^{n-1}$. The result of [7, Proposition 7.3] implies that $v_r \in (2, v_1, \ldots, v_{r-1})$ so that $v_r^q \in (v_0^{j_0}, \ldots, v_{r-1}^{j_{r-1}})$ for some $q > 0$. Now,

$$X \wedge M_{J(i)} = X \wedge_{MU} MU/(v_0^{j_0}, \ldots, v_{r-1}^{j_{r-1}})$$

and there is a Künneth spectral sequence [12, Theorem IV.4.1]

$$E_2^{s,t} = \mathrm{Tor}_{-s,t}^{MU_*}(\pi_* X, \pi_* MU/(v_0^{j_0}, \ldots, v_{r-1}^{j_{r-1}})) \Longrightarrow \pi_{t-s}\left(X \wedge M_{J(i)}\right).$$

This is a cohomologically graded lower half-plane spectral sequence. As in the proof of Theorem 7.5(2) in [7], the fact that for some q multiplication by v_r^q raises filtration implies that every element in the homotopy groups of $X \wedge M_{J(i)}$ is killed by some finite power of v_r. It follows that $L_{K(r)}X = \mathrm{holim}_i \left(v_r^{-1} X \wedge M_{J(i)}\right) \simeq *$. □

4. Localized spectral sequences

In our computations below, we will make use of various localizations of the slice spectral sequence of quotients of $MU^{((G))}$. In this section, we recall results from [21] and [36] that we will use here. As a reminder, we continue to let $G = C_{2^n}$.

4.1. **Some notation.** Here, we introduce some notation. We refer the reader to [23] for more details.

Consider 2-local homotopy equivalence classes of representation spheres S^V where V is a finite dimensional orthogonal representation. This is a semi-group with respect to the smash product. Let $JO(G)$ be the group completion.

Definition 4.1. Define $\lambda_j = \lambda_j(G)$ to be the 2-dimensional irreducible real representation of G for which the generator $\gamma \in G$ acts on $\mathbb{R}^2$ by a rotation by $2\pi/2^{n-j}$. We also have the one-dimensional sign representation $\sigma_n = \sigma(G)$, for which the generator acts by multiplication by -1.

Note that $\lambda_{n-1} = 2\sigma_n$. There is an isomorphism of underlying abelian groups

$$JO(G) \cong \mathbb{Z}\{1, \sigma_n, \lambda_0, \ldots, \lambda_{n-2}\}$$

where the equivalence sends S^V to V.

Definition 4.2. For each representation V, there is a homotopy class

$$a_V : S^0 \to S^V$$

which corresponds to the inclusion of $S^0 = \{0, \infty\}$. We call this the *Euler class*.

If V is an orientable representation of dimension d, we also get classes

$$u_V \in \pi_{d-V} H\underline{\mathbb{Z}}.$$

We call these *orientation classes*.

We have commutative diagrams

$$\begin{tikzcd} S^0 \arrow[r, "a_{\lambda_i}"] \arrow[dr, "a_{\lambda_{i+1}}"'] & S^{\lambda_i} \arrow[d, "(-)^2"] \\ & S^{\lambda_{i+1}} \end{tikzcd}$$

where the vertical arrow is a double cover. Therefore, a_{λ_i} divides $a_{\lambda_{i+1}}$ for each $0 \leqslant i \leqslant n-2$.

4.2. **Localizations and isotropy separation.**

Definition 4.3. For each $0 \leqslant i \leqslant n-1$, we have families

$$\mathcal{F}_i = \mathcal{F}[C_{2^{i+1}}] = \{H \mid H \subsetneq C_{2^{i+1}}\}$$

and $\mathcal{F}_n = \mathcal{A}ll$ be the family of all subgroups of G.

These families interpolate between $\mathcal{F}_0 = \{e\}$ and $\mathcal{F}_n = \mathcal{A}ll$.

The universal and couniversal spaces for the family $\mathcal{F}_i$ can be written in very algebraic terms.

Proposition 4.4. *For $0 \leqslant i \leqslant n-1$, we have*

$$E\mathcal{F}_i \simeq \lim_{\rightarrow} S(k\lambda_i) = S(\infty\lambda_i)$$

and

$$\tilde{E}\mathcal{F}_i \simeq S^{\infty\lambda_i} = S^0[a_{\lambda_i}^{-1}].$$

Proof. The representation λ_i has kernel exactly $C_{2^i} \subset G$, and the residual action of G/C_{2^i} is faithful. The result follows. □

We now state two results of Meier–Shi–Zeng that we will use later.

Theorem 4.5 (Meier–Shi–Zeng [36])**.** *For X a G-spectrum with regular slice tower $P^\bullet X$, the spectral sequence associated to the tower $\tilde{E}\mathcal{F}_i \wedge P^\bullet X$, which corresponds to the a_{λ_i}-localized spectral sequence*

$$a_{\lambda_i}^{-1} E_2^{s,t+\alpha} = a_{\lambda_i}^{-1} \pi_{t-s+\alpha}^G P_t^t X \Longrightarrow \pi_{t-s+\alpha}^G(a_{\lambda_i}^{-1} X), \qquad \alpha \in JO(G)$$

converges strongly.

Theorem 4.6 (Meier–Shi–Zeng [35])**.** *Let X be a (-1)-connected G-spectrum. Let L_i be the line of slope $(2^i - 1)$ through the origin. The following statements hold:*

(1) On the integer graded page, the map from the slice spectral sequence of X to the a_{λ_i}-localized slice spectral sequence of X induces an isomorphism on the E_2-page for the classes above L_i, and a surjection for the classes that are on L_i.

(2) The map of spectral sequences above induces an isomorphism between differentials that originate from classes that are on or above L_i.

In what follows, we will compute heavily with localized slice spectral sequences. The following remark explains the advantages of this approach.

Remark 4.7. It follows from Theorem 4.6 that all the differentials in the slice spectral sequence of X that are on or above L_i can be immediately recovered from the a_{λ_i}-localized slice spectral sequence of X by truncating off the latter spectral sequence below L_i. In particular, all the differentials in the slice spectral sequence of X can be recovered by truncating off the a_{λ_0}-inverted slice spectral sequence below the horizontal line $s = 0$.

In addition, the a_{λ_i}-localized slice spectral sequence are individually easier to compute that the non-localized spectral sequences. Of course, by localizing, we loose the information below the line L_i, but the approach is to work inductively, starting with the $a_{\lambda_{n-1}}$-localization (which is the same as the a_{σ_n}-localization) and ending with the a_{λ_0}-localization. As we explained above, all differentials can be recovered from the a_{λ_0} localization so that at that stage, we have not actually lost any information at all.

One final remark on the advantage of computing with the a_{λ_0}-localized spectral sequence is that it actually records information about the slice spectral sequences

$$E_2^{s,t+\star} = \pi^G_{t-s+\star} P^t_t X \Rightarrow \pi^G_{t-s+\star} X$$

for any $\star = k\lambda_0$ where $k \in \mathbb{Z}$. Indeed, we can recover all the differentials in the $* + k\lambda_0$-graded spectral sequence by truncating the a_{λ_0}-localized spectral sequence below the horizontal line $s = -2k$. So, the localized spectral sequence contains much more information than simply the integer graded spectral sequence.

5. C_{2^n}-geometric fixed points and quotients of $BP^{(\!(G)\!)}$

As a proof-of-concept and for later computations, in this section, we will compute the homotopy of the geometric fixed points

$$\Phi^G(BP^{(\!(G)\!)}/G \cdot \bar{S}) \simeq (\tilde{E}\mathcal{P} \wedge BP^{(\!(G)\!)}/G \cdot \bar{S})^G \simeq (a_\sigma^{-1} BP^{(\!(G)\!)}/G \cdot \bar{S})^G$$

of quotients by permutation summands via the a_σ-localized slice spectral sequence.

On the one hand, we know the answer, since we know the homotopy type of the geometric fixed points.

Proposition 5.1. *We have a weak-equivalence of $H\mathbb{F}_2$-modules*

$$\Phi^G\big(BP^{(\!(G)\!)}/G \cdot \bar{S}\big) \simeq H\mathbb{F}_2 \wedge \Sigma^\infty_+ \left(\prod_{j \in J} S^{2^j}\right).$$

Proof. The geometric fixed points functor is strong symmetric monoidal, and we have

$$\Phi^G(BP^{(\!(G)\!)}) = H\mathbb{F}_2. \qquad \square$$

On the other hand, the a_σ-localization map of the slice spectral has a particular simple target, and this will tell us a great deal about any of the slice spectral sequences for these quotients.

5.1. **General quotients of $BP^{(\!(G)\!)}$.**

We now consider the a_σ-localized slice spectral sequence

$$a_\sigma^{-1} E_2^{s,t} := a_\sigma^{-1} \pi^G_{t-s} P^t_t (BP^{(\!(G)\!)}/G \cdot \bar{S}) \Longrightarrow \pi^G_{t-s} a_\sigma^{-1}(BP^{(\!(G)\!)}/G \cdot \bar{S}).$$

Inverting a_σ has the effect of killing the transfer from any proper subgroups. This means that the E_2-page of the a_σ-localizaed slice spectral sequence has a particular simple form:

$$a_\sigma^{-1} E_2^{*,\star} = \mathbb{F}_2[N^G_{C_2} \bar{s}_i \mid i \notin J][u_{2\sigma}, a_\sigma^{\pm 1}, a_{\lambda_0}^{\pm 1}, \ldots, a_{\lambda_{n-2}}^{\pm 1}]\ .$$

Definition 5.2. For each $j \in \mathbb{N}$, let

$$\bar{f}_j = a_{\bar{\rho}}^{2^j - 1} N^G_{C_2} \bar{s}_j.$$

This definition *a priori* depends heavily on the choices of the $\bar{s}_i$. However, from the point of view of differentials, these choices will not matter, due to a small lemma.

Lemma 5.3. *Let $\bar{s}_{2^m-1}$ be any element in degree $(2^m - 1)\rho_2$ that generates a permutation summand. We have*

$$N^G_{C_2} \bar{s}_{2^m-1} \equiv N^G_{C_2} \bar{v}^G_m \mod (N^G_{C_2} \bar{v}^G_1, \ldots, N^G_{C_2} \bar{v}^G_{m-1}) + Im(tr),$$

where $Im(tr)$ denotes the image of the transfer.

In particular, $\bar{f}_i$ is independent of the choice of $\bar{s}_i$, modulo the lower $\bar{v}_j$.

Proof. We have
$$\pi^G_{*\rho_G}\big(BP^{(\!(G)\!)}\big)/Im(tr) \cong \mathbb{F}_2\big[N^G_{C_2}\bar{v}^G_1, N^G_{C_2}\bar{v}^G_2, \dots\big],$$
and the map
$$x \mapsto N^G_{C_2} x$$
gives a ring homomorphism
$$\pi^{C_2}_{*\rho_2} BP^{(\!(G)\!)} \to \pi^G_{*\rho_G}\big(BP^{(\!(G)\!)}\big)/Im(tr).$$
Additionally, Weyl equivariance of the norm shows that for any $\gamma \in G$,
$$N^G_{C_2}(\gamma x) = \gamma N^G_{C_2}(x) \equiv N^G_{C_2}(x) \mod Im(tr).$$

The lemma can be restated as saying that for any generator
$$\bar{s}_{2^m-1} \in Q_{(2^m-1)\rho_2}\big(\pi^{C_2}_{*\rho_2} BP^{(\!(G)\!)}\big),$$
we have
$$N^G_{C_2}\bar{s}_{2^m-1} = N^G_{C_2}\bar{v}^G_m \in Q_{(2^m-1)\rho_G}\big(\mathbb{F}_2[N^G_{C_2}\bar{v}^G_1, \dots]\big).$$
The above argument shows that the norm is a Weyl-equivariant ring homomorphism, and hence it induces a linear map
$$\Big(Q_{(2^m-1)\rho_2}\big(\pi^{C_2}_{*\rho_2} BP^{(\!(G)\!)}\big)\Big)_G \to Q_{(2^m-1)\rho_G}\big(\mathbb{F}_2\big[N^G_{C_2}\bar{v}^G_1, \dots\big]\big).$$
Both the source and target are isomorphic to $\mathbb{F}_2$, and choosing $\bar{v}^G_m$ as the generator of the source shows the map to be non-zero. It is therefore non-zero on any generator for the source. □

Corollary 5.4. *The E_2-term for the a_σ-localized slice spectral sequence for $BP^{(\!(G)\!)}$ is given by*
$$\mathbb{F}_2[a_\sigma^{\pm 1}, a_{\lambda_{n-2}}^{\pm 1}, \dots, a_{\lambda_0}^{\pm 1}][b, \bar{f}_1, \dots],$$
where the bidegree of $\bar{f}_i$ is $\big(2^i-1, (2^n-1)(2^i-1)\big)$ and where the bidegree of
$$b = u_{2\sigma}/a_\sigma^2$$
is $(2,-2)$.

Corollary 5.5. *For any $\bar{S}$, the a_σ-localized slice spectral sequence for $BP^{(\!(G)\!)}/G\cdot\bar{S}$ is a module over that for $BP^{(\!(G)\!)}$, and the E_2-term is the quotient*
$$\mathbb{F}_2[a_\sigma^{\pm 1}, a_{\lambda_{n-2}}^{\pm 1}, \dots, a_{\lambda_0}^{\pm 1}][b, \bar{f}_1, \dots]/\big(\bar{f}_j \mid j \in J\big)$$

We start with examining the a_σ-localized slice spectral sequence for $BP^{(\!(G)\!)}$, since all other cases are modules over this.

Proposition 5.6. *Let $\bar{s}_i$ for $i \in \mathbb{N}$ be any choice of permutation summand generators for $\pi^{C_2}_{*\rho_2} BP^{(\!(G)\!)}$.*

Then in the a_σ-localized slice spectral sequence for $BP^{(\!(G)\!)}$, the differentials are determined by
$$d_{\ell(k)}(b^{2^k}) = \bar{f}_{k+1}, \quad \ell(k) := 2^n(2^{k+1}-1)+1, \quad k \geqslant 0. \tag{5.1}$$

Proof. Lemma 5.3 shows that
$$\bar{f}_j \equiv a_{\bar{\rho}}^{(2^j-1)} N^G_{C_2}\bar{v}^G_j$$
modulo the earlier generators $\bar{f}_i$ with $i < j$, so the Slice Differentials Theorem of [19] implies that we have the differentials (5.1). □

Remark 5.7. The $\bar{f}_i$ all lie on the line of slope $2^n - 1$ through the origin in the $(t-s, s)$-plane. This is a vanishing line for both the spectral sequence of $a_\sigma^{-1} BP^{(\!(G)\!)}$ and that of quotient by permutation summands, so the differential in (5.1) is the last possible on b^{2^n}.

We next use this result to study the a_σ-localized slice spectral sequence of other quotients.

Definition 5.8. Let $A(J)$ be the set of non-negative integers r such that the dyadic expansion $r = \varepsilon_0 + \varepsilon_1 \cdot 2 + \varepsilon_2 \cdot 4 + \dots$ satisfies $\varepsilon_i = 0$ if $i + 1 \notin J$.

Theorem 5.9. *The a_σ-localized slice spectral sequence of $a_\sigma^{-1}(BP^{(\!(G)\!)}/G \cdot \bar{S})$ can be completely described as follows:*

1. *the only non-trivial differentials are of lengths $\ell(k) = 2^n(2^{k+1} - 1) + 1$ for some $k \geqslant 0$.*
2. *The $E_{\ell(k)}$-page is the module over*
$$\mathbb{F}_2[\bar{f}_i \mid i \geqslant k+1 \text{ and } i \notin J][b^{2^k}]$$
generated by the set of permanent cycles b^r where $0 \leqslant r < 2^k$ and $r \in A(J)$.
3. *If $k+1 \notin J$, then there are non-trivial differentials are multiples of*
$$d_{\ell(k)}(b^{2^k}) = \bar{f}_{k+1}$$
by the $d_{\ell(k)}$-cycles
$$\mathbb{F}_2[\bar{f}_i \mid i \geqslant k+1 \text{ and } i \notin J][b^{2^{k+1}}]\{b^r \mid r < 2^k \text{ and } r \in A(J)\}.$$
There are no other differentials of that length.
4. *If $k+1 \in J$, then $E_{\ell(k)} = E_{\ell(k+1)}$.*
5. *Consequently, $E_\infty \cong \mathbb{F}_2\{b^r \mid r \in A(J)\}$.*

Proof. We will prove the statements by induction on k. For the base case when $k = 0$, we have $\ell(0) = 2^n + 1$. The first possible non-trivial differential by sparseness is $d_{\ell(0)}(b) = \bar{f}_1$. Therefore, $E_{\ell(0)} = E_2$ and the class b^0 is a permanent cycle. The claims hold.

Now, suppose that the $E_{\ell(k)}$-page is as claimed. If $k+1 \in J$, then b^{2^k} is a $d_{\ell(k)}$-cycle, and hence a permanent cycle. Any element on the $E_{\ell(k)}$-page of the form b^r with $r < 2^{k+1}$ is of the form $r' + \varepsilon_k \cdot 2^k$ for $r' < 2^k$ with $r' \in A(J)$ and $\varepsilon_k \in \{0, 1\}$. Since $k+1 \in J$, $r \in A(J)$. Using the module structure over the a_σ-localized spectral sequence of $a_\sigma^{-1} BP^{(\!(G)\!)}$, the elements b^r are also $d_{\ell(k)}$-cycles. By sparseness of the $E_{\ell(k)}$-page, it is impossible for b^r to support a differential of length longer than $\ell(k)$ because there are no possible targets. Therefore, the elements b^r are permanent cycles. This implies that $E_{\ell(k)} = E_{\ell(k+1)}$ which proves our claims when $k+1 \in J$.

On the other hand, if $k+1 \notin J$, we have a non-trivial differential $d_{\ell(k)}(b^{2^k}) = \bar{f}_{k+1}$. For $\alpha \geqslant 1$, consider the element $\bar{f}\bar{f}_{k+1}^{\alpha} b^{r+2^k t}$ with $0 \leqslant r < 2^k$, $t \geqslant 0$ even, and $\bar{f}$ a monomial in the $\bar{f}_i$'s for $i > k+1$ and $i \notin J$. Such an element is the target of the $d_{\ell(k)}$-differential on $\bar{f}\bar{f}_{k+1}^{\alpha-1} b^{r+2^k(t+1)}$. It follows that the $E_{\ell(k)+1}$-page is a polynomial algebra over
$$\mathbb{F}_2[\bar{f}_i \mid i \geqslant k+2 \text{ and } i \notin J][b^{2^{k+1}}]$$
generated by the already established permanent cycles b^r, $r < 2^k$. Since $k+1 \notin J$, the set $\{r \in A(J) \mid r < 2^{k+1}\}$ is equal to $\{r \in A(J) \mid r < 2^k\}$. This completes the induction step. $\square$

As an immediate consequence, using Theorem 4.6, we have:

Theorem 5.10. *In the integer graded slice spectral sequence*

$$E_2^{s,t} = \pi_{t-s}^G P_t^t BP^{(\!(G)\!)}/G \cdot \bar{S} \Rightarrow \pi_{t-s} BP^{(\!(G)\!)}/G \cdot \bar{S},$$

we have:

1. *Above the line of slope $2^{n-1}-1$, the E_2-page is isomorphic to the integer graded part of the E_2-page of the a_σ-localization of the spectral sequence, as described in Corollary 5.5.*
2. *The only non-trivial differentials whose sources lie on or above the line of slope $2^{n-1}-1$ are in one-to-one correspondence with the non-trivial differentials of the integer graded a_σ-localized slice spectral sequence above this line. They are of lengths $\ell(k)$ for $k+1 \notin J$, and are generated under the module structure of the spectral sequence for $BP^{(\!(G)\!)}$ by the differentials*
$$d_{\ell(k)}(u_{2\sigma}^{2^k}) = a_\sigma^{2^{k+1}} a_{\bar{\rho}}^{2^{k+1}-1} N_{C_2}^G \bar{s}_{2^{k+1}-1}$$
for $k+1 \notin J$.

We will now give some examples to illustrate the results above. We start with example for the group $G = C_2$. Since $\bar{v}_j \equiv \bar{v}_{k+1}$ modulo the previous $\bar{v}_j$'s, we write $\bar{v}_{k+1} = \bar{s}_{2^{k+1}-1}$, but any choice of permutation summand generators gives the same results.

Example 5.11. Consider $BP_{\mathbb{R}}/\bar{S}$ for $\bar{S} = \{\bar{s}_{2^j-1} \mid j \in J\}$. The E_2-page are the a_σ-localized slice spectral sequence is

$$a_\sigma^{-1} E_2^{*,\star} = \mathbb{F}_2[\bar{v}_i \mid i \notin J][u_{2\sigma}, a_\sigma^{\pm 1}],$$

the non-trivial differentials are generated by $d_{2^{k+2}-1}(b^{2^k}) = \bar{f}_{k+1}$ and

$$E_\infty^{*,\star} \cong \mathbb{F}_2[a_\sigma^{\pm 1}]\{b^r \mid r \in A(J)\}.$$

Figure 1 shows the example for $BP_{\mathbb{R}}\langle 2\rangle$.

In this case, using Theorem 4.6 as in Theorem 5.10 together with the fact that the a_σ-localized spectral sequence records information about many $JO(C_2)$ degrees of the slices spectral sequence (as noted in Remark 4.7), we can easily describe a large part of the (unlocalized) $JO(C_2)$-graded slice spectral sequence of $BP_{\mathbb{R}}/\bar{S}$.

Corollary 5.12. *Let $JO(C_2)^+ \subseteq JO(C_2)$ be the elements of the form $a + b\sigma$ with $a - b \geqslant 0$. The $JO(C_2)^+$ graded slice E_2-page of $BP_{\mathbb{R}}/\bar{S}$ is*

$$E_2^{*,\star} \cong \mathbb{Z}_{(2)}[\bar{v}_i \mid i \notin J][u_{2\sigma}, a_\sigma]/(2a_\sigma), \quad \star \in JO(C_2)^+ \ . \tag{5.2}$$

1. *The only non-trivial differentials are of lengths $(2^{k+2}-1)$ for some $k \geqslant 0$.*
2. *If $k+1 \notin J$, then the nontrivial-differentials are multiples of*
$$d_{2^{k+2}-1}(u_{2\sigma}^{2^k+r}) = \bar{v}_{k+1} u_{2\sigma}^r a_\sigma^{2^{k+2}-1}, \ \ r \in A(J)$$
by the $d_{2^{k+2}-1}$-cycles $\mathbb{F}_2[\bar{v}_i, a_\sigma \,|\, i \geqslant k+1$ and $i \notin J]$.
3. *If $k+1 \in J$, then $E_{2^{k+2}-1} = E_{2^{k+3}-1}$.*
4. *Consequently, the E_∞-page is*
$$E_\infty^{*,\star} \cong \mathbb{Z}_{(2)}[\bar{v}_i \mid i \notin J][a_\sigma]/(2a_\sigma, \bar{v}_i a_\sigma^{2^{i+1}-1})\{u_{2\sigma}^r, 2u_{2\sigma}^s \mid r \in A(J), s \notin A(J)\}.$$

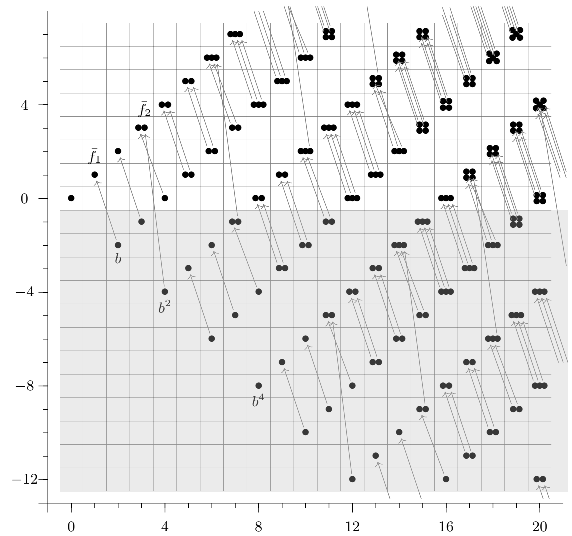

FIGURE 1. The a_σ-localized slice spectral sequence of $a_\sigma^{-1}BP_{\mathbb{R}}\langle 2\rangle$ in integer degrees. The slice spectral sequence of $BP_{\mathbb{R}}\langle 2\rangle$ is obtained by removing the region below the horizontal line $s = 0$ and replacing $\bullet = \mathbb{Z}/2$ by copies of $\mathbb{Z}$, which reintroduces the transfers.

As an explicit example, we show the computation of the slice spectral sequences of $BP_{\mathbb{R}}\langle 2, 2\rangle$, deduced from that of $a_\sigma^{-1}BP_{\mathbb{R}}\langle 2, 2\rangle$. The computation is illustrated in Figure 2.

Remark 5.13. On the E_∞-page of $BP_{\mathbb{R}}\langle 2, 2\rangle$ (the third picture of Figure 2), there is an exotic extension $\eta = \bar{v}_1 a_\sigma$, as shown by the dashed line. This extension follows from the work in [8]. More precisely, letting $(n, k, b) = (2, 1, 0)$ in Corollary 3.11 of [8] gives the exotic $\pi_\star MU_{\mathbb{R}}$-multiplication $\bar{v}_1 u_{2\sigma} = \bar{v}_2 a_\sigma^4$. It follows that

$$(\bar{v}_2 a_\sigma u_{2\sigma}) \cdot (\bar{v}_1 a_\sigma) = (\bar{v}_2 a_{2\sigma}) \cdot (\bar{v}_1 u_{2\sigma}) = (\bar{v}_2 a_{2\sigma}) \cdot (\bar{v}_2 a_{4\sigma}) = \bar{v}_2^2 a_\sigma^6.$$

For the next two examples, the group $G = C_4$ with generators $\bar{s}_{2^i-1} = \bar{v}_i^G$, but any choice of permutation summand generators gives the same results.

Example 5.14. Consider $BP^{((C_4))}\langle 2\rangle$, so that $J = \{j \in \mathbb{N} \mid j > 2\}$. The E_2-page are the a_σ-localized slice spectral sequence is

$$a_\sigma^{-1}E_2^{*,*} \cong \mathbb{F}_2[\bar{f}_1, \bar{f}_2, b] \ .$$

The spectral sequence has two types of differentials, namely

$$d_5(b) = \bar{f}_1, \quad \text{and} \quad d_{13}(b^2) = \bar{f}_2 \ .$$

The class b^4 is a permanent cycle, and we have

$$\pi_*^{C_4} a_\sigma^{-1} BP^{((C_4))}\langle 2\rangle \cong \mathbb{F}_2[b^4].$$

The computation is illustrated in Figure 3.

Example 5.15. Consider $BP^{((C_4))}\langle 2, 2\rangle$. In this case, the E_2-page are the a_σ-localized slice spectral sequence is

$$a_\sigma^{-1}E_2^{*,*} = \mathbb{F}_2[\bar{f}_2, b] \ .$$

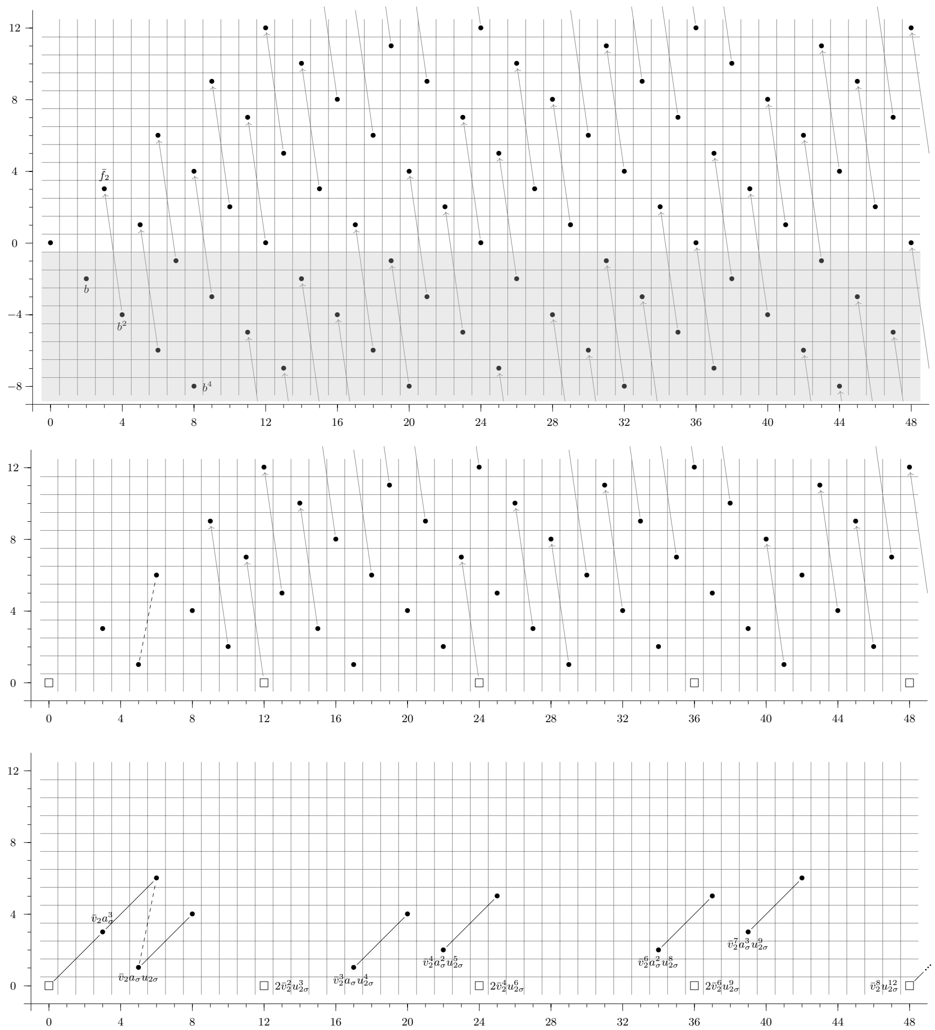

FIGURE 2. The a_σ-localized slice spectral sequences of $a_\sigma^{-1}BP_{\mathbb{R}}\langle 2,2\rangle$ (top). The middle figure is the slice spectral sequence of $BP_{\mathbb{R}}\langle 2,2\rangle$ and the bottom is its E_∞-page. A $\square$ denotes $\mathbb{Z}_{(2)}$, a $\bullet$ denotes $\mathbb{Z}/2$.

There is only one family of differentials, generated by

$$d_{13}(b^2) = \bar{f}_2$$

and the answer is

$$\pi_*^{C_4} a_\sigma^{-1} BP^{((C_4))}\langle 2,2\rangle \cong \mathbb{F}_2[b^4]\{1, b\} \ .$$

The computation is illustrated in Figure 4.

5.2. **Application: Multiplicative structure.** While the left action of $MU^{((G))}$ on itself always endows $MU^{((G))}/G\cdot\bar{S}$ with a canonical $MU^{((G))}$-module structure, and the same is true with $BP^{((G))}$ instead, much less is known for ring structures.

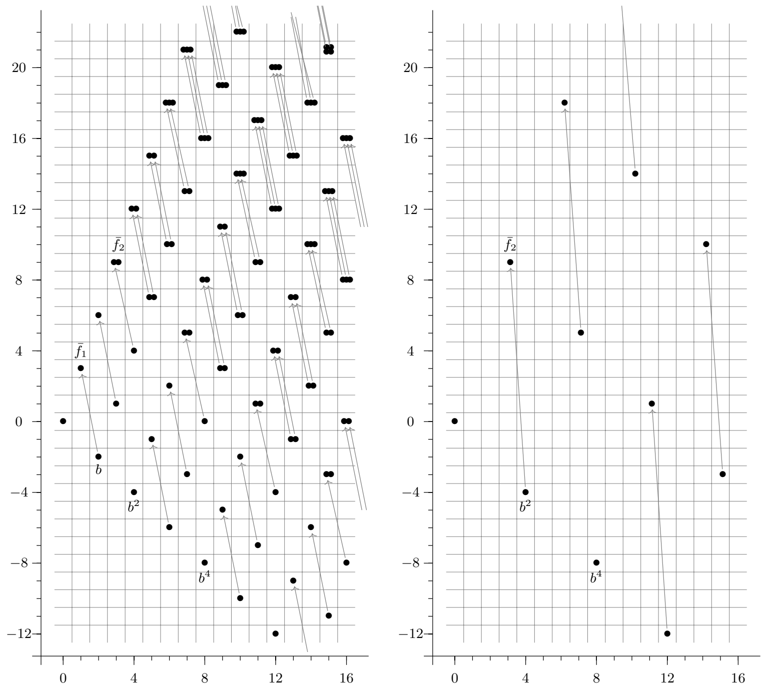

FIGURE 3. The a_σ-localized slice spectral sequence of $a_\sigma^{-1}BP^{(\!(C_4)\!)}\langle 2\rangle$.

We do have the following restrictive condition on quotients as a straightforward consequence of the techniques introduced above.

Proposition 5.16. *Let $J \subseteq \mathbb{N}$ and $\bar{S} = \{\bar{s}_{2^j-1} \mid j \in J\}$ be a set of generators for permutation summands. If there is a $k \in J$ such that $(k+1) \notin J$, then $BP^{(\!(G)\!)}/G \cdot \bar{S}$ does not have a ring structure in the homotopy category.*

Proof. If there is a ring structure on $BP^{(\!(G)\!)}/G \cdot \bar{S}$, then the map $BP^{(\!(G)\!)}/G \cdot \bar{S} \to H\underline{\mathbb{Z}}$ to the zero slice induces a map of multiplicative spectral sequences. This remains true after inverting a_σ. Since $\pi_*^G(a_\sigma^{-1}H\underline{\mathbb{Z}}) \cong \mathbb{F}_2[b]$ and the map from $\pi_*^G(a_\sigma^{-1}BP^{(\!(G)\!)}/G \cdot \bar{S})$ to $\pi_*^G(a_\sigma^{-1}H\underline{\mathbb{Z}})$ is the natural inclusion, the former is a subring of the latter. However, if $k \in J$ and $k+1 \notin J$, then $b^{2^{k-1}}$ is nonzero in $\pi_*^G(a_\sigma^{-1}BP^{(\!(G)\!)}/G \cdot \bar{S})$, but its square b^{2^k} is zero. This is a contradiction. $\square$

Put another way, Proposition 5.16 says that the only possible $BP^{(\!(G)\!)}$-module quotients $BP^{(\!(G)\!)}/G \cdot \bar{S}$ by permutation summands which could be rings are the forms of $BP^{(\!(G)\!)}\langle m\rangle$. Even here, we know very little.

Example 5.17. For $G = C_2$, $BP_{\mathbb{R}}\langle 1\rangle = k_{\mathbb{R}}$ and $tmf_1(3)$ is a form of $BP_{\mathbb{R}}\langle 2\rangle$. Both admit commutative ring structures. For $m > 2$ we do not know if $BP_{\mathbb{R}}\langle m\rangle$ admits an associative ring structure.

For $G = C_4$, $tmf_1(5)$ is a form of $BP^{(\!(C_4)\!)}\langle 1\rangle$. For $m > 1$, we do not know if $BP^{(\!(C_4)\!)}\langle m\rangle$ admits even an associative ring structure.

If we instead look only at the underlying spectrum, then work of Angeltveit and of Robinson shows that we have nice ring structures [1, 41]. This has been

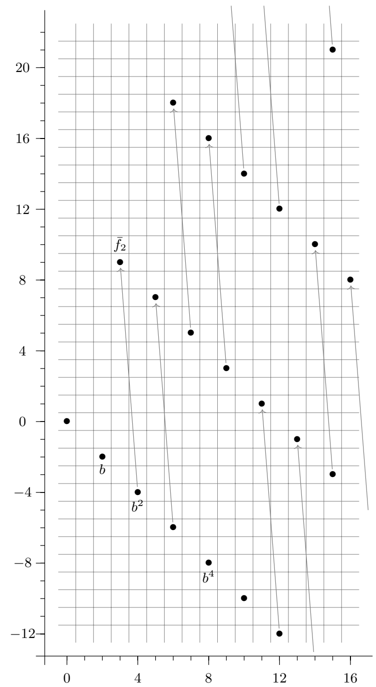

FIGURE 4. The a_σ-localized slice spectral sequence of $a_\sigma^{-1}BP^{(\!(C_4)\!)}\langle 2,2\rangle$.

refined by Hahn–Wilson to show that this is still true in the category of $MU^{(\!(G)\!)}$ or $BP^{(\!(G)\!)}$-modules [16].

Proposition 5.18. *For any $J \subseteq \mathbb{N}$ and for any $\bar{S}$, the spectrum*

$$i_e^* BP^{(\!(G)\!)}/G \cdot \bar{S}$$

is an associative $i_e^* BP^{(\!(G)\!)}$-algebra spectrum.

Proof. The assumptions on $\bar{S}$ ensure that the sequence

$$\left(\gamma^i \bar{s}_j \mid 0 \leqslant i \leqslant 2^{n-1}-1, j \in J\right)$$

forms a regular sequence in the homotopy groups of the even spectrum $i_e^* BP^{(\!(G)\!)}$. The result follows from [16, Theorem A]. □

Remark 5.19. The Hahn–Shi Real orientation shows the restriction to C_2 of the spectrum $BP^{(\!(G)\!)}/G \cdot \bar{S}$ always admits an E_σ-algebra structure [15].

6. The $C_{2^{n-1}}$-geometric fixed points

Let $G' = C_{2^{n-1}}$, the subgroup of index two in $G = C_{2^n}$. We extend the results of the previous section slightly, considering the $a_{\lambda_{n-2}}$-localized slice spectral sequence for permutation quotients. This is again a spectral sequence of Mackey functors, now essentially for $C_4 \cong G/C_{2^{n-2}}$. In this section, we study the $C_2 \cong G'/C_{2^{n-2}}$-equivariant level, since we can tell an increasingly complete story here. The C_4-fixed points are more subtle, as we will see in Section 7.

Note that since

$$i_{G'}^{*}\lambda_{n-2} = \lambda_{(n-1)-1} = 2\sigma,$$

the restriction to G' of the $a_{\lambda_{n-2}}$-localized slice spectral sequence for $BP^{(\!(G)\!)}/G\cdot\bar{S}$ is the a_σ-localized slice spectral sequence for

$$i_{G'}^{*}BP^{(\!(G)\!)}/G\cdot\bar{S}.$$

Just as for the G-geometric fixed points, we start by identifying the homotopy type. In this case, since

$$i_{G'}^{*}BP^{(\!(G)\!)} \simeq BP^{((G'))} \wedge BP^{((G'))},$$

we have

$$\Phi^{G'}BP^{(\!(G)\!)} \simeq H\mathbb{F}_2 \wedge H\mathbb{F}_2,$$

and all of the geometric fixed points we consider will take place in the category of modules over

$$A = H\mathbb{F}_2 \wedge H\mathbb{F}_2.$$

Composing with the localization map

$$i_{G'}^{*}BP^{(\!(G)\!)} \to \tilde{E}\mathcal{P} \wedge i_{G'}^{*}BP^{(\!(G)\!)},$$

the element $N_{C_2}^{G'}\bar{s}_i$ gives us a polynomial in the dual Steenrod algebra.

Definition 6.1. Let $g_i \in \pi_i A$ be the image of $N_{C_2}^{G'}\bar{s}_i$.

Note that the residual $C_2 \cong G'/C_{2^{n-2}}$-action interchanges

$$N_{C_2}^{G'}\bar{s}_i \text{ and } \gamma N_{C_2}^{G'}\bar{s}_i,$$

while acting as the conjugation in the dual Steenrod algebra.

Lemma 6.2. *The G'-geometric fixed points of $BP^{(\!(G)\!)}/G\cdot\bar{S}$ are the A-module*

$$A/(g_i, \bar{g}_i \mid i \in J).$$

In general, the homotopy type of this module very heavily depends on the choices of generators. We have several cases where we can explicitly identify the images, however. Using [19, Proposition 2.57] and [36, Proposition 6.2], we see that for Hill–Hopkins–Ravenel generators $\overline{v}_i^G$ of $BP^{(\!(G)\!)}$, the G'-geometric fixed points of their norms satisfies

$$\begin{aligned}\xi_i &= \Phi^{G'}N_{C_2}^{G}\overline{v}_i^G\\ \zeta_i &= \Phi^{G'}N_{C_2}^{G}\gamma\overline{v}_i^G,\end{aligned}$$

where ξ_i are the Milnor generators of the mod 2 dual Steenrod algebra, and ζ_i are their dual.

6.1. **Forms of $BP^{(\!(G)\!)}\langle k,m\rangle$.** We can get much more explicit answers for the geometric fixed points with certain forms of $BP^{(\!(G)\!)}\langle k,m\rangle$, since here we can identify the geometric fixed points of the norms exactly.

Corollary 6.3. *The G'-geometric fixed points of $BP^{(\!(G)\!)}\langle k,m\rangle$ are given by the A-module*

$$A/\big(\xi_i,\zeta_i \mid i<k \text{ or } i>m\big) \simeq A/\big(\xi_i,\zeta_i \mid i<k\big) \underset{A}{\wedge} A/\big(\xi_j,\zeta_j \mid j>m\big).$$

Writing this module in several ways makes working with this easier, as we can connect this with a series of modules and computations studied in [6].

Definition 6.4. For any subset I of the natural numbers, let

$$M_I = \bigwedge_{i\in I} A/(\xi_i) \text{ and } \overline{M}_I = \bigwedge_{i\in I} A/(\zeta_i).$$

Let

$$R_k = \mathrm{End}_A\left(M_{\{1,\dots,k-1\}}\right)$$

and let

$$A\langle m\rangle = M_{\{m+1,m+2,\dots\}}.$$

As the endomorphisms of a module, R_k is always an associative algebra. By [6,], for any m, $A\langle m\rangle$ and $\overline{A}\langle m\rangle$ are associative algebras as well. More surprisingly, by [6,], we have

$$R_k \simeq \Sigma M_{\{1,\dots,k-1\}} \underset{A}{\wedge} \overline{M}_{\{1,\dots,k-1\}},$$

which allows us to rewrite $\Phi^{G'} BP^{(\!(G)\!)}\langle k, m\rangle$.

Corollary 6.5. *For any $k \leqslant m$, we have*

$$\Phi^{G'} BP^{(\!(G)\!)}\langle k, m\rangle \simeq \Sigma R_k \underset{A}{\wedge} A\langle m\rangle \underset{A}{\wedge} \overline{A}\langle m\rangle,$$

the suspension of an associative A-algebra.

The extreme case of this is $BP^{(\!(G)\!)}\langle m, m\rangle$.

Corollary 6.6. *The G'-geometric fixed points of $BP^{(\!(G)\!)}\langle m, m\rangle$ are given by the A-module*

$$\Sigma R_m \underset{A}{\wedge} A\langle m\rangle \underset{A}{\wedge} \overline{A}\langle m\rangle.$$

The homotopy of this A-module is more complicated than one might have initially expected. These kinds of modules were studied by the authors [6], where we used a Baker–Lazarev style Adams spectral sequence based on $H\mathbb{F}_2$-homology, but in the category of A-modules [3]. A remarkable feature of the case of $BP^{(\!(G)\!)}\langle m, m\rangle$ is that this relative Adams spectral sequence completely determines the a_σ-localized slice spectral sequence.

6.2. **A comparison of spectral sequences.** Let $P_\bullet = P_\bullet BP^{(\!(G)\!)}\langle m, m\rangle$ be the slice covering tower of $BP^{(\!(G)\!)}\langle m, m\rangle$. That is, P_t is the homotopy fibre of the cannonical map

$$BP^{(\!(G)\!)}\langle m, m\rangle \to P^{t-1} BP^{(\!(G)\!)}\langle m, m\rangle$$

where $P^\bullet = P^\bullet BP^{(\!(G)\!)}\langle m, m\rangle$ is the regular slice tower.

The slices $P_t^t BP^{(\!(G)\!)}\langle m, m\rangle$ are non-trivial only in dimensions of the form $t = 2i(2^m - 1)$. Therefore we can "speed-up" the slice tower without losing any information. Define

$$\tilde{P}_t = P_{2t(2^m-1)}\ .$$

This re-indexes the slice tower, so that

$$\tilde{P}_t^t = P_{2t(2^m-1)}^{2t(2^m-1)} BP^{(\!(G)\!)}\langle m, m\rangle.$$

Since there is an equivalence

$$\Phi^{G'} BP^{(\!(G)\!)} \simeq H\mathbb{F}_2 \wedge H\mathbb{F}_2 = A,$$

$\Phi^{G'}\tilde{P}_\bullet$ is a covering tower converging to $\Phi^{G'} BP^{(\!(G)\!)}\langle m, m\rangle$ in the category of A-modules.

Theorem 6.7. *The tower $\Phi^{G'}\tilde{P}_\bullet BP^{(\!(G)\!)}\langle m,m\rangle$ is an $H\mathbb{F}_2$-Adams resolution of $\Phi^{G'}BP^{(\!(G)\!)}\langle m,m\rangle$ in the category of A-modules.*

Proof. Let $Q_\bullet = \Phi^{G'}\tilde{P}_\bullet BP^{(\!(G)\!)}\langle m,m\rangle$ for convenience. Then $Q_\bullet$ is an $H\mathbb{F}_2$-Adams resolution of $Q_0 = \Phi^{G'}BP^{(\!(G)\!)}\langle m,m\rangle$ in A-modules if the following conditions are met for each $i \geqslant 0$ [39, Def. 2.2.1.3]:

(1) Q_i^i is a wedge of suspensions of $H\mathbb{F}_2$'s, and
(2) the map $Q_i \to Q_i^i$ is monomorphic in $H\mathbb{F}_2$-homology.

We now verify the first condition. By definition,

$$Q_0^0 = \Phi^{G'}P_0^0 BP^{(\!(G)\!)}\langle m,m\rangle = \Phi^{G'}H\underline{\mathbb{Z}} = H\mathbb{F}_2[b],$$

with A-module structure defined by the geometric fixed points of the reduction map $BP^{(\!(G)\!)} \to H\underline{\mathbb{Z}}$. By [19, Prop. 7.6], for each i, $\overline{v}_i^G$ and its conjugate $\gamma\overline{v}_i^G$ act trivially on $H\underline{\mathbb{Z}}$, thus the geometric fixed points of $N_{C_2}^{G'}\overline{v}_i^G$ and $N_{C_2}^{G'}\gamma\overline{v}_i^G$, which are ξ_i and ζ_i, act trivially on $H\mathbb{F}_2[b]$. Therefore, as an A-module, $Q_0^0 \simeq \bigvee_{j=0}^{\infty}\Sigma^{2j}H\mathbb{F}_2$. The Slice Theorem [19, Thm. 6.1] implies that for $i > 0$, Q_i^i is a wedge of suspensions of Q_0^0, thus the first condition is met.

We verify the second condition by an alternative construction of the slice covering tower of $BP^{(\!(G)\!)}\langle m,m\rangle$. As in [19, §6], let $R = \mathbb{S}^0[G\cdot \overline{v}_m^G]$ be the homotopy refinement of $BP^{(\!(G)\!)}\langle m,m\rangle$, and M_i be the subcomplex of R consisting of spheres of dimension $\geqslant 2i(2^m-1)$. The arguments in [19, §6.1] tell us that

$$\tilde{P}_i \simeq BP^{(\!(G)\!)}\langle m,m\rangle \wedge_R M_i.$$

Notice that G'-equivariantly, $M_{i+1} \subset M_i$ is the sub R-module $(\overline{v}_m^G, \gamma\overline{v}_m^G)M_i$, thus the quotient M_i/M_{i+1} is equivalent to $M_i/(\overline{v}_m^G, \gamma\overline{v}_m^G)M_i$. Taking the G'-geometric fixed points on the cofibration

$$BP^{(\!(G)\!)}\langle m,m\rangle \wedge_R M_{i+1} \to BP^{(\!(G)\!)}\langle m,m\rangle \wedge_R M_i \to BP^{(\!(G)\!)}\langle m,m\rangle \wedge_R M_i/M_{i+1},$$

we obtain the cofibration

$$Q_{i+1} \to Q_i \to Q_i^i \simeq Q_i/(\xi_m, \zeta_m)Q_i$$

because $\Phi^{G'}N_{C_2}^{G'}\overline{v}_m^G = \xi_m$ and $\Phi^{G'}N_{C_2}^{G'}\gamma\overline{v}_m^G = \zeta_m$. Since ξ_m and ζ_m have trivial image under $A \to H\mathbb{F}_2$, the map $Q_i \to Q_i^i$ induces a monomorphism in $H\mathbb{F}_2$-homology. $\square$

Corollary 6.8. *We have an isomorphism of spectral sequences between the relative Adams spectral sequence for $A/(\xi_i, \zeta_i \mid i \neq m)$ and the speeded-up a_σ-localized slice spectral sequence for $i_{G'}^* BP^{(\!(G)\!)}\langle m,m\rangle$.*

The dictionary here can be a little confusing, due to the scaling in the slice filtration. We record the un-scaled version here:

Remark 6.9. A relative Adams d_r corresponds to an ordinary a_σ-localized slice differential $d_{2(2^m-1)r+1}$.

Corollary 6.10. *The integer graded $E_{2^{m+1}}$-page of the G'-equivariant a_σ-localized slice spectral sequence of $i_{G'}^* BP^{(\!(G)\!)}\langle m,m\rangle$ computing $\pi_*^{G'}a_\sigma^{-1}i_{G'}^*BP^{(\!(G)\!)}\langle m,m\rangle$ is isomorphic to the E_2-page of the relative Adams spectral sequence of the spectrum $\Phi^{G'}i_{G'}^*BP^{(\!(G)\!)}\langle m,m\rangle$.*

6.3. **The Relative Adams spectral sequence for $\Phi^{G'} i^*_{G'} BP^{(\!(G)\!)}\langle m, m\rangle$.** By Corollary 6.6, the G'-geometric fixed points of $BP^{(\!(G)\!)}\langle m, m\rangle$ are a suspension of an associative ring spectrum. Because of this, we instead work with the associative algebra, since then the spectral sequence will be one of associative algebras.

One of the most surprising results from [6] was a decomposition of $A\langle m\rangle \underset{A}{\wedge} \overline{A}\langle m\rangle$, and hence of further quotients, into simpler, finite pieces. This makes our computations even easier.

Definition 6.11. For each $m \geqslant 1$, let

$$\tilde{A}\langle m\rangle = A\langle m\rangle \underset{A}{\wedge} \overline{M}_{\{m+1,\ldots,2m\}}.$$

Proposition 6.12 ([6, Corollary 5.6])**.** *For each m, we have a decomposition of A-modules*

$$A\langle m\rangle \underset{A}{\wedge} \overline{A}\langle m\rangle \simeq \bigvee_{k\geqslant 0} \Sigma^{2^{2m+1}k} \tilde{A}\langle m\rangle.$$

Proposition 6.13 ([6, Theorem 5.9])**.** *There is an associative algebra structure on $\tilde{A}\langle m\rangle$ such that the projection map*

$$A\langle m\rangle \underset{A}{\wedge} \overline{A}\langle m\rangle \to \tilde{A}\langle m\rangle$$

is a map of associative algebras.

This reduces the study of modules of the form

$$M \underset{A}{\wedge} A\langle m\rangle \underset{A}{\wedge} \overline{A}\langle m\rangle$$

to the study of $\tilde{A}\langle m\rangle$-modules of the form

$$M \underset{A}{\wedge} \tilde{A}\langle m\rangle.$$

We apply this in the case of $M = R_m$, where we again have an associative algebra structure.

Definition 6.14. Let

$$R\langle m\rangle = R_m \underset{A}{\wedge} \tilde{A}\langle m\rangle.$$

By [6], the E_2-page of the relative Adams spectral sequence of $R\langle m\rangle$ is given by

$$\mathbb{F}_2[\xi_1, \ldots, \xi_m, e]/e^{2^m} \otimes E(\beta_{\text{-}2}, \beta_{\text{-}4}, \ldots, \beta_{\text{-}2^{m-1}})/\big(\xi_i + \xi_{i+1}\beta_{\text{-}2^i} \mid 1 \leqslant i \leqslant m-1\big),$$

where the bidegrees are given by:

$$|e| = (2^{m+1}, 0)$$
$$|\xi_m| = (2^m - 1, 1)$$
$$|\beta_{\text{-}k}| = (-k, 0).$$

Since for $1 \leqslant i < m$, we have the relation

$$\xi_i = \xi_{i+1}\beta_{\text{-}2^i},$$

this simplifies as an algebra to

$$\mathbb{F}_2[\xi_m] \otimes E(\beta_{\text{-}2}, \beta_{\text{-}4}, \ldots, \beta_{\text{-}2^{m-1}}) \otimes \mathbb{F}_2[e_{2^{m+1}}]/(e_{2^{m+1}}^{2^m}).$$

Notation 6.15. We will use the following convenient notation:

$$\beta(2\epsilon_1 + 4\epsilon_2 + \ldots + 2^{m-1}\epsilon_{m-1}) := \beta_{-2}^{\epsilon_1}\beta_{-4}^{\epsilon_2}\cdots\beta_{-2^{m-1}}^{\epsilon_{m-1}} .$$

We get elements,

$$\beta(0), \beta(2), \ldots, \beta(2^m - 2)$$

where $\beta(\ell)$ has degree $-\ell$. Note that, as element on the E_2-page for $R\langle m\rangle$, for $0 \leqslant \ell \leqslant \ell'$

$$\beta(\ell)\beta(\ell') = \begin{cases} \beta(\ell') & \ell = 0 \\ \left(\binom{\ell'}{\ell} - 1\right)\beta(\ell+\ell') & \ell > 0 . \end{cases} \tag{6.1}$$

In [6], we determined a number of differentials in these kinds of relative Adams spectral sequences.

Proposition 6.16 ([6, Corollary 7.5])**.** *In the relative Adams spectral sequence for $\widetilde{A}\langle m\rangle$, for each $1 \leqslant i \leqslant m$ and $n \geqslant 0$, we have differentials*

$$d_{1+2^{i+1}}(e^{2^i+2^{i+1}n}) = \xi_m^{2^{i+1}}\xi_{i+1} \cdot e^{2^{i+1}n}.$$

The spectrum $R\langle m\rangle$ is an $\widetilde{A}\langle m\rangle$-algebra. Therefore, by naturality and the relations on the E_2-page, in the relative Adams spectral sequence of $R\langle m\rangle$ there are differentials

$$d_{1+2^{i+1}}(e^{2^i+2^{i+1}n}) = \xi_m^{1+2^{i+1}}e^{2^{i+1}n}\beta(2^m - 2^{i+1}) \tag{6.2}$$

for $0 \leqslant i \leqslant m-1$ and $0 \leqslant n < 2^{m-(i+1)}$, provided that the target survives to the $E_{1+2^{i+1}}$-page. We will see that all other differentials will be determined by these and the multiplicative structure of the spectral sequence.

We start with two useful lemmas.

Lemma 6.17. *If $d_{1+2r}(\beta(\ell)e^k)$ is non-zero, then*

$$d_{1+2r}(\beta(\ell)e^k) = \xi_m^{1+2r}\beta(\ell + 2^m - 2r)e^{k-r}$$

for some $1 \leqslant r \leqslant k$.

Proof. Let

$$d_{1+2r}(\beta(\ell)e^k) = \xi_m^{1+2r}\beta(\ell')e^{k'}.$$

Then $k' \leqslant k$ so we let s be a number such that $k' = k - s$. Note that $0 \leqslant s < 2^m$ and $0 < r < 2^m$. Computing degrees, we obtain the equation

$$2^{m+1}k - \ell - 1 = (2^m - 1)(1 + 2r) + 2^{m+1}(k - s) - \ell' \tag{6.3}$$

This simplifies to

$$-2^m + (\ell' - \ell) + 2r = 2^{m+1}(r - s).$$

Since $0 \leqslant \ell, \ell' \leqslant 2^m - 2$, we have

$$2 - 2^m \leqslant \ell' - \ell \leqslant 2^m - 2.$$

Furthermore, since $0 < r < 2^m$, $0 < 2r < 2^{m+1}$. Therefore the absolute value of $-2^m + (\ell' - \ell) + 2r$ is less than 2^{m+1}. The equation above implies that this quantity is divisible by 2^{m+1}. This implies that both sides of the equation must be zero. It follows that $r = s$ and $\ell' = \ell + 2^m - 2r$. $\square$

The next lemma is a straightforward but annoying exercise analyzing inequalities and we do not include the proof here.

Lemma 6.18. *Consider pairs (ℓ, k), where ℓ is even and $0 \leqslant \ell, k \leqslant 2^m - 1$. Define subsets of such pairs by*

$$S = \{(\ell, k) : k \leqslant \ell\}, \quad \text{and} \quad S' = \{(\ell, k) : \ell < k\}$$

as follows. Let $k \leqslant \ell$. Set

$$j = \max \left\{ 0 \leqslant r \leqslant m-1 : \binom{\ell}{2^r} \equiv 0 \mod 2 \right\},$$
$$i = \min \left\{ j \leqslant r \leqslant m-1 : \binom{k}{2^r} \equiv 0 \mod 2 \right\}.$$

Then letting

$$\phi(\ell, k) = (\ell - (2^m - 2^{i+1}), k + 2^i)$$

gives a bijection $\phi \colon S \to S'$.

Remark 6.19. If $\ell < k$ and we fix $i \geqslant 0$, then for $2^i \leqslant \kappa < 2^{i+1}$ and $\kappa - 2^i \leqslant \ell < \kappa$, if k can be written in the form $k = \kappa + 2^{i+1}n$, then $\phi^{-1}(\ell, k) = (\ell + 2^m - 2^{i+1}, k - 2^i)$. This formulation of the above bijection will be useful for proving the next result.

Theorem 6.20. *In the relative Adams spectral sequence of $R\langle m \rangle$, for*

$$0 \leqslant k, \ell \leqslant 2^m - 1$$

with ℓ even, we have the following:

1. *the class $\beta(\ell)e^k$ is a permanent cycle if and only if $k \leqslant \ell$;*
2. *if $\ell < k$, the class $\beta(\ell)e^k$ supports a non-trivial differential, determined as follows. Fix $i \geqslant 0$. For $2^i \leqslant \kappa < 2^{i+1}$ and $\kappa - 2^i \leqslant \ell < \kappa$, if $k = \kappa + 2^{i+1}n$, then there is a differential*

$$d_{1+2^{i+1}}(\beta(\ell)e^k) = \xi_m^{1+2^{i+1}} \beta(\ell + 2^m - 2^{i+1}) e^{k-2^i} .$$

These are the only non-trivial differentials.

Proof. This is a multiplicative spectral sequence. At E_2-page, there is a vanishing line of slope $1/(2^m - 1)$ with intercept on the $(t-s)$-axis at $2^m - 2$ (the vanishing line is formed by the ξ_m-multiples on $\beta(2^m - 2)$). Furthermore, looking at the map of spectral sequences from $\widetilde{A}\langle n \rangle$, we see that the class e^{2^i} survives to the $E_{1+2^{i+1}}$-page for $1 \leqslant i \leqslant m-1$. Therefore, the differentials d_r are e^{2^i}-linear for $r < 2^{i+1} + 1$. The first non-zero class in positive filtration is $\xi_m \beta(2^m - 2)$ which has topological degree $(2^m - 1) - (2^m - 2) = 1$. Therefore, every element of $E(\beta)$ is a permanent cycle and the spectral sequence is one of modules over this exterior algebra.

We will prove the following statements inductively on $0 \leqslant j \leqslant m - 1$:

1. For $2^j \leqslant k < 2^{j+1}$, if $k - 2^j \leqslant \ell < k$, then

$$d_{1+2^{j+1}}(\beta(\ell)e^{k+2^{j+1}n}) = \xi_m^{1+2^{j+1}} \beta(\ell + 2^m - 2^{j+1}) e^{k-2^j+2^{j+1}n} .$$

2. For $2^j \leqslant k < 2^{j+1}$ and $k \leqslant \ell$, the class $\beta(\ell)e^k$ is a permanent cycle.
3. There are no other non-trivial differentials until the $E_{1+2^{j+2}}$-page.

We note that (1) and (2) inductively imply that any class $\beta(\ell)e^k$ with $k < 2^{j+1}$ either supports a differential d_r for $r \leqslant 1 + 2^{j+1}$, or is a permanent cycle.

To prove the inductive claim, we start with $j = 0$, so that $k = 1$ in (1). Using that ℓ is even, in (1), the range forces $\ell = 0$. The first possible non-trivial differential for degree reasons is on e, and this differential is forced by the d_3-differential

$$d_3(e) = \xi_m^3 \beta(2^m - 2)$$

in $\widetilde{A}\langle n\rangle$. All d_3-s are then determined by e^2-linearity and given by

$$d_3(e^{1+2n}) = \xi_m^3\beta(2^m-2)e^{2n}.$$

Here, we have used the fact that the differentials are linear over $\mathbb{F}_2[\xi_m]\otimes E(\beta)$. For degree reasons, the classes $\beta(\ell)e$ for $\ell \geqslant 2$ are permanent cycles, proving (2). The differentials are e^2-linear and all other classes that could support a d_3 are the product of e^2 with permanent cycles. So they survive to the E_5-page.

Let $i > 0$ and assume that (1), (2), (3) hold for smaller values of $0 \leqslant j < i$. As noted above, the differentials in the spectral sequence of $\widetilde{A}\langle m\rangle$ implies the differentials

$$d_{1+2^{i+1}}(e^{2^i+2^{i+1}n}) = \xi_m^{1+2^{i+1}}\beta(2^m-2^{i+1})e^{2^{i+1}n},$$

provided that the targets survive to the $E_{1+2^{i+1}}$-page. By the induction hypothesis and Lemma 6.18, this is the case. This proves the differential of (1) for $k = 2^i$ and $\ell = 0$.

Now, choose k and ℓ so that $2^i \leqslant k < 2^{i+1}$ and $k - 2^i \leqslant \ell < k$ as in (1). In particular, $\ell > 0$. The class $[\beta(\ell)e^{k-2^i}]$ is a permanent cycle by the induction hypothesis. Therefore,

$$\begin{aligned} d_{1+2^{i+1}}(\beta(\ell)e^{k+2^{i+1}n}) &= d_{1+2^{i+1}}([\beta(\ell)e^{k-2^i}]e^{2^i+2^{i+1}n}) \\ &= [\beta(\ell)e^{k-2^i}]d_{1+2^{i+1}}(e^{2^i})e^{2^{i+1}n} \\ &= [\beta(\ell)e^{k-2^i}]\xi_m^{1+2^{i+1}}\beta(2^m-2^{i+1})e^{2^{i+1}n} \\ &= \xi_m^{1+2^{i+1}}\beta(\ell)\beta(2^m-2^{i+1})e^{k-2^i+2^{i+1}n}\ . \end{aligned}$$

The binomial expansion of $2^m - 2^{i+1}$ is

$$2^{m-1} + \ldots + 2^{i+2} + 2^{i+1}.$$

The bounds on ℓ give $0 < \ell < 2^{i+1}$, which guarantees that $\binom{2^m-2^{i+1}}{\ell} = 0$ since $\ell > 0$. So, by (6.1)

$$\beta(\ell)\beta(2^m-2^{i+1}) = \beta(\ell+2^m-2^{i+1}).$$

We get a non-trivial differential as long as the target is non-zero, which is the case by the induction hypothesis and Lemma 6.18. This proves (1).

We next show that the classes $\beta(\ell)e^k$ for $k \leqslant \ell$ and $2^i \leqslant k < 2^{i+1}$ are permanent cycles. Suppose that for $2^i \leqslant r \leqslant k$,

$$d_{1+2r}(\beta(\ell)e^k) = \xi_m^{1+2r}\beta(\ell+2^m-2r)e^{k-r}.$$

The form of the differential comes from Lemma 6.17. Note that

$$k-r \leqslant \ell - r \leqslant \ell - r + 2^m - r = \ell + 2^m - 2r.$$

This shows that the target is a permanent cycle by the induction hypothesis. We will now show that this target is actually killed by a shorter differential.

Since

$$\ell + 2^m - 2r - (2^m - 2^{i+1}) = 2^{i+1} + \ell - 2r \geqslant 2^{i+1} + k - 2k = 2^{i+1} - k > 0.$$

Therefore, $\ell + 2^m - 2r > 2^m - 2^{i+1}$ and so we can write

$$\begin{aligned} \ell + 2^m - 2r &= 2^m - 2^{j+1} + \bar{\ell}, & \bar{\ell} < 2^j,\ 0 \leqslant j \leqslant i \\ k - r &= \bar{k} + 2^h - 2^j + 2^{h+1}n & j \leqslant h,\ \bar{k} < 2^j,\ 0 \leqslant n \end{aligned}$$

Let
$$\ell' = \ell + 2^m - 2r - (2^n - 2^{h+1}) = \bar{\ell} + 2^{h+1} - 2^{j+1}$$
and
$$\kappa' = k - r - 2^{h+1}n + 2^h = \bar{k} + 2^{h+1} - 2^j.$$
Then, $\kappa' - 2^h \leqslant \ell' < \kappa'$ and $2^h \leqslant \kappa' < 2^{h+1}$. So by the induction hypothesis,
$$\begin{aligned} d_{1+2^{h+1}}(\xi_m^{2r-2^{h+1}}\beta(\ell')e^{\kappa'+2^{h+1}n}) &= \xi_m^{2r-2^{h+1}}\left(\xi_m^{1+2^{h+1}}\beta(\ell'+2^m-2^{h+1})e^{\kappa'-2^h+2^{h+1}n}\right) \\ &= \xi_m^{1+2r}\beta(\ell+2^m-2r)e^{k-r}\ . \end{aligned}$$
Therefore, the target was killed by a shorter differential. This proves (2).

Finally, (3) holds by the linearity of the differentials with respect to the $d_{2^{i+2}}$-cycle $e^{2^{i+1}}$. □

Finally, the correspondence between the a_σ-localized slice spectral sequence of $i_*^{G'}BP^{(\!(G)\!)}\langle m,m\rangle$ and the relative Adams spectral sequence is as follows:

Summary 6.21. The E_2-page of the relative Adams spectral sequence of the geometric fixed points $\Phi^{G'}BP^{(\!(G)\!)}\langle m,m\rangle$ is, additively,
$$E_2^{*,*} \cong \Sigma^{2^m-2}\mathbb{F}_2[\xi_m]\otimes E(\beta)\otimes\mathbb{F}_2[e_{2^{m+1}}]$$
where the shift preserves the filtration and adds 2^m-2 to the topological degree.

The correspondence between the slice spectral sequence and the relative Adams spectral sequence is as follows:

(1) The elements b^k for $0 \leqslant k \leqslant 2^{m-1}-1$ correspond to $\beta(2^m-2(k+1))$. Note that $b^{2^{m-1}-1}$ corresponds to $\beta(0)=1$, the multiplicative unit in the relative Adams spectral sequence of $R\langle m\rangle$.

(2) For $\ell = 2^{n-1}(2^m-1)+1$, $0 \leqslant k < 2^{m-1}$ and $r \geqslant 0$, the element $b^{k+2^{m-1}}$ in the localized sliced spectral sequence supports the differential $d_\ell(b^{k+2^{m-1}}) = b^k(\bar{f}_m + \gamma\bar{f}_m)$, forced by the slice differential
$$d_\ell(u_{2\sigma}^{2^{m-1}}) = a_\sigma^{2^{m+1}-1}(\overline{v}_m^G + \gamma\overline{v}_m^G).$$
These differentials are b^{2^m}-linear, leaving behind b^k where the dyadic expansion of $k = \varepsilon_0 + \varepsilon_1\cdot 2 + \ldots$ satisfies $\varepsilon_{m-1}=0$. This family of differentials identifies $\overline{f}_m$ with $\gamma\overline{f}_m$ in the whole slice spectral sequence.

(3) After the slice differentials above, multiplication by either $\bar{f}_m$ or $\gamma\overline{f}_m$ corresponds to multiplication by ξ_m in the relative Adams spectral sequence.

(4) The remaining elements b^k are in one-to-one correspondence with the elements in filtration $s=0$ in the relative Adams spectral sequence, shifted by a degree 2^m-2. In particular, the element $b^{2^m+2^{m-1}-1}$ corresponds to $e_{2^{m+1}}$.

6.4. **The C_2-slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2,2\rangle$.** As an application, we illustrate the above correspondence of spectral sequences by computing the C_2-slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2,2\rangle$.

The C_2-slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2,2\rangle$ is determined by the relative Adams spectral sequence for
$$R\langle 2\rangle = \widetilde{A}\langle 2\rangle/(\zeta_5,\ldots)\wedge_A \mathrm{End}_A(M_1),$$

whose computation follows from Section 6.3 above. The essential features were also completely computed in Section 7.3 of [6]. The E_2-page is

$$\mathbb{F}_2[\xi_2] \otimes E(\beta_{\text{-}2}) \otimes \mathbb{F}_2[e_8].$$

There are only d_3- and d_5-differentials. The d_3-differentials are generated by

$$d_3(e_8^{1+2*}) = e_8^{2*}\xi_2^3\beta_{-2},$$

and the d_5-differentials are generated by

$$d_5(e_8^{2+4*}) = e_8^{4*}\xi_2^5,$$
$$d_5(e_8^{3+4*}\beta_{\text{-}2}) = e_8^{1+4*}\xi_2^5\beta_{\text{-}2}.$$

The E_2-term of the a_σ-localized slice spectral sequence for $a_\sigma^{-1}i^*_{C_2}BP^{(\!(C_4)\!)}\langle 2,2\rangle$ is

$$\mathbb{F}_2[u_{2\sigma}, a_\sigma^{\pm 1}, \bar{t}_2, \gamma\bar{t}_2].$$

As before, let $b = u_{2\sigma}/a_{2\sigma}$. The shortest differentials in this spectral sequence are the d_7-differentials, whose effects are to identify $\bar{t}_2$ with $\gamma\bar{t}_2$. The d_7-differentials are generated by

$$d_7(b^{2+4*}) = b^{4*}(\bar{t}_2 + \gamma\bar{t}_2)a_\sigma^3,$$
$$d_7(b^{3+4*}) = b^{1+4*}(\bar{t}_2 + \gamma\bar{t}_2)a_\sigma^3.$$

After the d_7-differentials, we can then import the differentials from the relative Adams spectral sequence as explained in Summary 6.21. As a result, the Adams d_3-differentials become the slice d_{19}-differentials, generated by

$$d_{19}(b^{5+8*}) = b^{1+8*}\bar{t}_2^3a_\sigma^9.$$

The Adams d_5-differentials become the slice d_{31}-differentials, generated by

$$d_{31}(b^{9+16*}) = b^{1+16*}\bar{t}_2^5a_\sigma^{15},$$
$$d_{31}(b^{12+16*}) = b^{4+16*}\bar{t}_2^5a_\sigma^{15}.$$

Figure 5 shows the a_σ-localized slice spectral sequence for $a_\sigma^{-1}i^*_{C_2}BP^{(\!(C_4)\!)}\langle 2,2\rangle$. At E_2, the classes $\bullet$ denote families of monomials formed by the classes $a_\sigma^3\bar{t}_2$ and $a_\sigma^3\gamma\bar{t}_2\}$ on the various powers of b. At E_8, a $\bullet$ depicted as the target of a d_7-differential becomes a copy of $\mathbb{Z}/2$, represented by a class of the form $a_\sigma^{3*}\bar{t}_2^*b^k$ for $k \neq 2,3 \mod 4$. Each $\bullet$ depicted as the source of a d_7-differential is completely gone as the d_7-differentials are injective.

As in Theorem 4.6, to obtain the differentials in the C_2-slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2,2\rangle$, we truncate at the horizontal line of filtration $s = 0$ and remove the region below this line.

7. The C_4-localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}\langle 2,2\rangle$

In this section, we compute the integer-graded C_4-localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}\langle 2,2\rangle$, using all the tools that we have developed in the previous sections. This computation serves to demonstrate the robustness of our techniques as well as providing insights to higher differentials phenomena when generalized to higher heights.

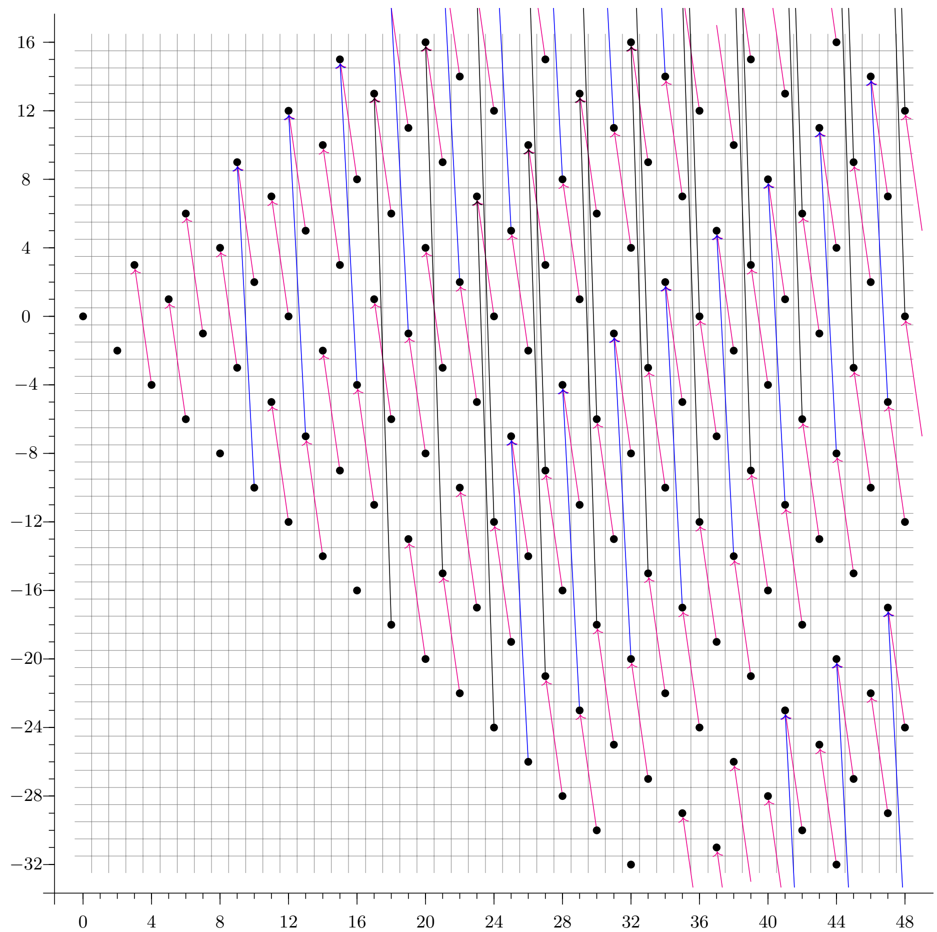

FIGURE 5. The a_σ-localized slice spectral sequence of $a_\sigma^{-1} i_{C_2}^* BP^{(\!(C_4)\!)}\langle 2,2\rangle$.

Remark 7.1. Just like the integer-graded spectral sequence, the full $RO(C_4)$-graded spectral sequence can be computed by the exact same method. We have opted to only compute the integer-graded slice spectral sequence because it is more convenient to present its diagrams.

Remark 7.2. By the discussion in Section 4, the slice spectral sequence of the unlocalized spectrum $BP^{(\!(C_4)\!)}\langle 2,2\rangle$ is completely determined by the a_λ-localized slice spectral sequence by truncating away the region below the line of filtration $s = 0$.

Remark 7.3. The following facts are good to keep in mind while doing the computation:

(1) The differentials with source on or above the line of slope 1 in the C_4-localized slice spectral sequence of $a_\lambda^{-1} BP^{(\!(C_4)\!)}\langle 2,2\rangle$ are determined by the C_4-localized slice spectral sequence of $a_\sigma^{-1} BP^{(\!(C_4)\!)}\langle 2,2\rangle$ computed in Example 5.15. These are all d_{13} differentials.

(2) Many of the differentials $d_{\leqslant 31}$ are determined using the C_2-slice differentials computed in Section 6.4 and the Mackey functor structure (i.e. restriction and transfer).
(3) The C_4-localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}\langle 2,2\rangle$ is a module over the spectral sequence of $a_\lambda^{-1}MU^{(\!(C_4)\!)}$, but very little of that structure is needed for the computation (see Section 7.3). Multiplication with respect to two key classes gives rise to periodicity of differentials and a vanishing line (Theorem 7.20). These phenomena determine all higher differentials ($d_{>31}$).

7.1. **Organization of the slice associated graded.** For the rest of this section, we let

$$\bar{t}_1 = \bar{v}_1^{C_4} \quad \text{and} \quad \bar{t}_2 = \bar{v}_2^{C_4}.$$

From Corollary 2.14, the slice associated graded for $BP^{(\!(C_4)\!)}\langle 2,2\rangle$ is $H\underline{\mathbb{Z}}[\bar{t}_2, \gamma\bar{t}_2]$, so the E_2-page of our spectral sequence is obtained by a_λ-localization of the homotopy of this slice associated graded.

We organize the slice cells by powers of

$$\bar{\mathfrak{d}}_{\bar{t}_2} := N_{C_2}^{C_4}(\bar{t}_2).$$

Remark 7.4. This mirrors the approach taken by Hill–Hopkins–Ravenel in [24] to compute the slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 1\rangle = BP^{(\!(C_4)\!)}\langle 1,1\rangle$, where they organized the slice cells by powers of $\bar{\mathfrak{d}}_{\bar{t}_1} := N_{C_2}^{C_4}(\bar{t}_1)$.

The slice cells are organized according to the following matrix:

$$\begin{pmatrix} \bar{\mathfrak{d}}_{\bar{t}_2}^0 & \bar{\mathfrak{d}}_{\bar{t}_2}^1 & \bar{\mathfrak{d}}_{\bar{t}_2}^2 & \cdots \\ \bar{\mathfrak{d}}_{\bar{t}_2}^0(\bar{t}_2, \gamma\bar{t}_2) & \bar{\mathfrak{d}}_{\bar{t}_2}^1(\bar{t}_2, \gamma\bar{t}_2) & \bar{\mathfrak{d}}_{\bar{t}_2}^2(\bar{t}_2, \gamma\bar{t}_2) & \cdots \\ \bar{\mathfrak{d}}_{\bar{t}_2}^0(\bar{t}_2^2, \gamma\bar{t}_2^2) & \bar{\mathfrak{d}}_{\bar{t}_2}^1(\bar{t}_2^2, \gamma\bar{t}_2^2) & \bar{\mathfrak{d}}_{\bar{t}_2}^2(\bar{t}_2^2, \gamma\bar{t}_2^2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{7.1}$$

To read this, note that $i_{C_2}^*\bar{\mathfrak{d}}_{\bar{t}_2} = \bar{t}_2\gamma\bar{t}_2$ so that every monomial in $\bar{t}_2, \gamma\bar{t}_2$ appears in some entry of the matrix.

Entries $\bar{\mathfrak{d}}_{\bar{t}_2}^j$ in the matrix represent slice cells of the form

$$H\underline{\mathbb{Z}} \wedge S^{3j\rho_4},$$

where ρ_4 is the regular representation of C_4. These are the regular (or non-induced) cells. Entries of the form $\bar{\mathfrak{d}}_{\bar{t}_2}^j(\bar{t}_2^i, \gamma\bar{t}_2^i)$ represent slice cells of the form

$$H\underline{\mathbb{Z}} \wedge (C_{4+} \wedge_{C_2} S^{i_{C_2}^*(3j\rho_4+3i\rho_2)}) \simeq H\underline{\mathbb{Z}} \wedge (C_{4+} \wedge_{C_2} S^{(6j+3i)\rho_2}).$$

These are the induced cells.

For the non-induced cells, the homotopy groups of $H\underline{\mathbb{Z}} \wedge S^{*\rho_4}$ are computed in [24] and depicted in Figure 3 of that reference. For the induced cells, we get the induced Mackey functors

$$H\underline{\mathbb{Z}}_*(C_{4+} \wedge S^{*\rho_2}) \cong \uparrow_{C_2}^{C_4} H\underline{\mathbb{Z}}_*^{C_2}(S^{*\rho_2}),$$

(see Definition 2.6 of [24]) whose values are also known. After inverting a_λ, we obtain the E_2-page depicted in Figure 6.

We use the same notation as in Table 1 of [24] for the Mackey functors. Blue Mackey functors are supported on induced cells and represent multiple copies of

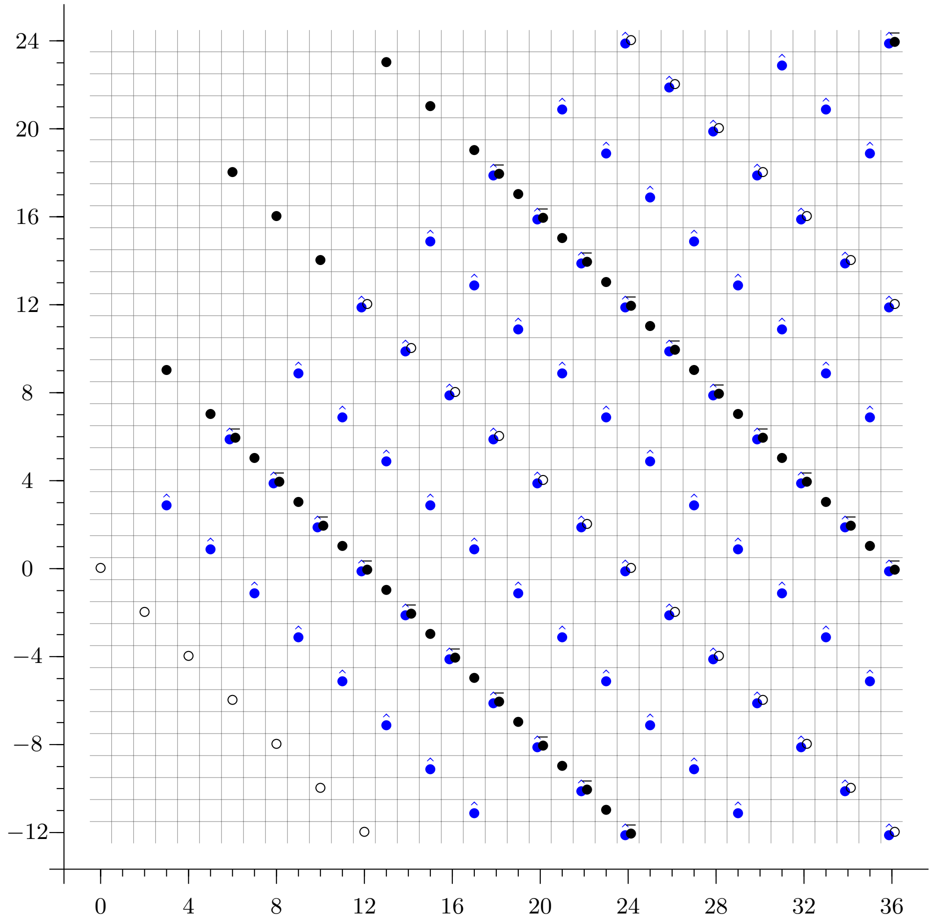

FIGURE 6. The E_2-page of the Mackey functor valued a_λ-localized slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2,2\rangle$.

the Mackey functor $\hat{\bullet}$ of [24], supported on the various monomials in the matrix (7.1) that are not in the first row.

We name the classes that do not come from induced cells. First, there are classes of order four (which have non-trivial restrictions):

- $\bar{\mathfrak{d}}_{\bar{t}_2}^{2i} u_{(6i-j)\lambda} u_{6i\sigma} a_{j\lambda}$ in degree $(24i-2j, 2j)$ for $0 \leqslant i$ and $j \leqslant 6i$.

Next, the classes of order two which do not come from induced cells are:

- $\bar{\mathfrak{d}}_{\bar{t}_2}^{i} u_{2k\sigma} a_{3i\lambda} a_{(3i-2k)\sigma}$ in degree $(3i+2k, 9i-2k)$ for $0 \leqslant i$ and $0 \leqslant 2k < 3i$. These are above the line of slope one.
- $\bar{\mathfrak{d}}_{\bar{t}_2}^{i} u_{j\lambda} u_{(3i-1)\sigma} a_{(3i-j)\lambda} a_\sigma$ in degree $(6i+2j-1, 6i-2j+1)$ for i odd, $0 \leqslant i$ and $0 \leqslant j$.

The induced cells are named by treating them as images of the transfer map from the corresponding classes in the C_2-slice spectral sequence.

7.2. **The d_7-differentials.** The first differentials are the d_7-differentials. They occur between classes supported on slice cells that are in the same column of the matrix (7.1). The d_7-differentials are all proven by restricting to the a_σ-localized slice spectral sequence of $a_\sigma^{-1} i^*_{C_2} BP^{(\!(C_4)\!)}\langle 2,2\rangle$. More specifically, the restriction of certain classes to the C_2-spectral sequence support the d_7-differentials discussed in Section 6.4, and therefore by naturality and degree reasons their preimages must also support d_7-differentials in the C_4-spectral sequence. By naturality and degree reasons, these are all the d_7-differentials that can occur, after which we obtain the E_8-page.

Let $b = \frac{u_\lambda}{a_\lambda}$. The restriction of this class is the class $b = \frac{u_{2\sigma}}{a_{2\sigma}}$ in the C_2-spectral sequence. We have d_7-differentials

$$d_7\left(b^{2+4*}\right) = tr_{C_2}^{C_4}(\bar{t}_2 a_{3\sigma_2}) b^{4*}$$
$$d_7\left(b^{3+4*}\right) = tr_{C_2}^{C_4}(\bar{t}_2 a_{3\sigma_2}) b^{1+4*}.$$

Figure 7 depicts the E_7-page. The $\circ$ classes are $\mathbb{Z}/4$'s coming from non-induced cells. The black $\bullet$ classes are $\mathbb{Z}/2$'s coming from the non-induced cells, while the blue $\bullet$ classes are direct sums of $\mathbb{Z}/2$'s coming from the induced cells.

Figure 8 depicts the E_8-page. In that figure, all dots represent a copy of $\mathbb{Z}/2$, with the exception of the white $\circ$ classes which represent $\mathbb{Z}/4$'s. The pink denotes a degree where, on a previous page, there was a $\mathbb{Z}/4$ but the generator supported a non-zero differential and is no longer present. As before, blue classes come from induced cells.

7.3. **Strategy for computing higher differentials.** Before computing the higher differentials (those of lengths $\geqslant 13$), we describe our strategy. There are two classes that will be very important for our computation. They are the classes

$$\alpha = \bar{\mathfrak{d}}_{\bar{t}_2}^8 u_{24\sigma} a_{24\lambda} \quad \text{at } (48, 48),$$
$$b^{32} = u_{32\lambda}/a_{32\lambda} \quad \text{at } (64, -64).$$

We will use the following crucial facts about the spectral sequence of $a_\lambda^{-1} MU^{(\!(C_4)\!)}$. These results are the only significant computational inputs from the localized slice spectral sequence for $a_\lambda^{-1} MU^{(\!(C_4)\!)}$ and the much harder computations of [26].

Proposition 7.5. *The class $\alpha = \bar{\mathfrak{d}}_{\bar{t}_2}^8 u_{24\sigma} a_{24\lambda}$ is a permanent cycle in the localized slice spectral sequence for $a_\lambda^{-1} MU^{(\!(C_4)\!)}$. Consequently, the differentials are linear with respect to multiplication by α.*

Proof. This is a direct consequence of the Slice Differential Theorem of Hill–Hopkins–Ravenel [19, Theorem 9.9]. More precisely, all the differentials in the region on or above the line of slope 1 in the C_4 localized slice spectral sequence for $a_\lambda^{-1} MU^{(\!(C_4)\!)}$ can be completely computed. $\square$

Proposition 7.6. *In the localized slice spectral sequence for $a_\lambda^{-1} MU^{(\!(C_4)\!)}$, we have the following facts:*

(1) The class $b^8 = u_{8\lambda}/a_{8\lambda}$ supports a d_{15}-differential.
(2) The class $b^{16} = u_{16\lambda}/a_{16\lambda}$ supports a d_{31}-differential.
(3) The class $b^{32} = u_{32\lambda}/a_{32\lambda}$ is a d_{61}-cycle.

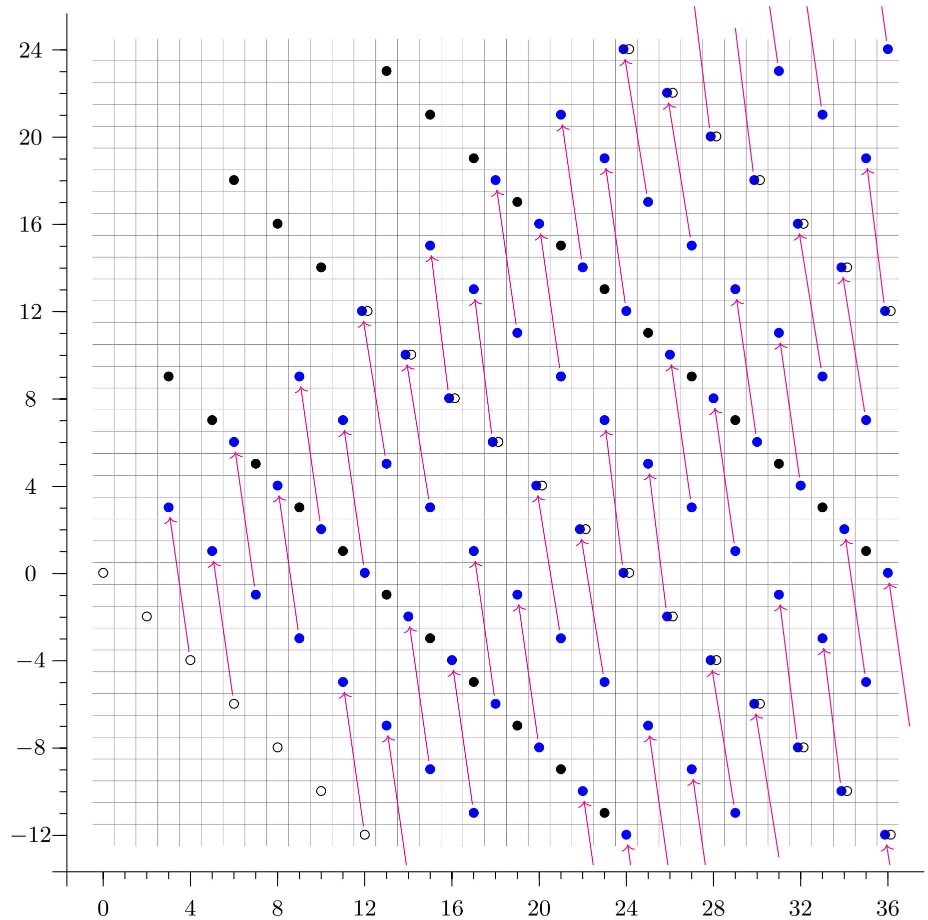

FIGURE 7. The E_7-page of the a_λ-localized slice spectral sequence of $a_\lambda^{-1} BP^{(\!(C_4)\!)}\langle 2, 2\rangle$.

Proof. All the claims are direct consequences of the computations for the C_4 slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2\rangle$ (see Table 1 in [26]). We will elaborate on each of them below.

For (1), the restriction of the class $u_{8\lambda}$ is $u_{16\sigma_2}$, which supports the d_{31}-differential

$$d_{31}(u_{16\sigma_2}) = \bar{v}_4^{C_2} a_{31\sigma_2}$$

in the C_2-spectral sequence of $MU^{(\!(C_4)\!)}$. Therefore by naturality of the restriction map, the class $u_{8\lambda}$ must support a differential of length at most 31. By a stem-wise computation, it is impossible for the class $u_{8\lambda}$ to support a differential of length $\leqslant 13$, as there are no possible targets. We will not carry out the details of this computation here, as [26, Section 7.3] already contains the arguments and results to show that $u_{8\lambda}$ supports a d_{15}-differential in the slice spectral sequence of $MU^{(\!(C_4)\!)}$. It follows that the class $\frac{u_{8\lambda}}{a_{8\lambda}}$ must also support a d_{15}-differential in the a_λ-localized slice spectral sequence of $a_\lambda^{-1} MU^{(\!(C_4)\!)}$. This proves (1).

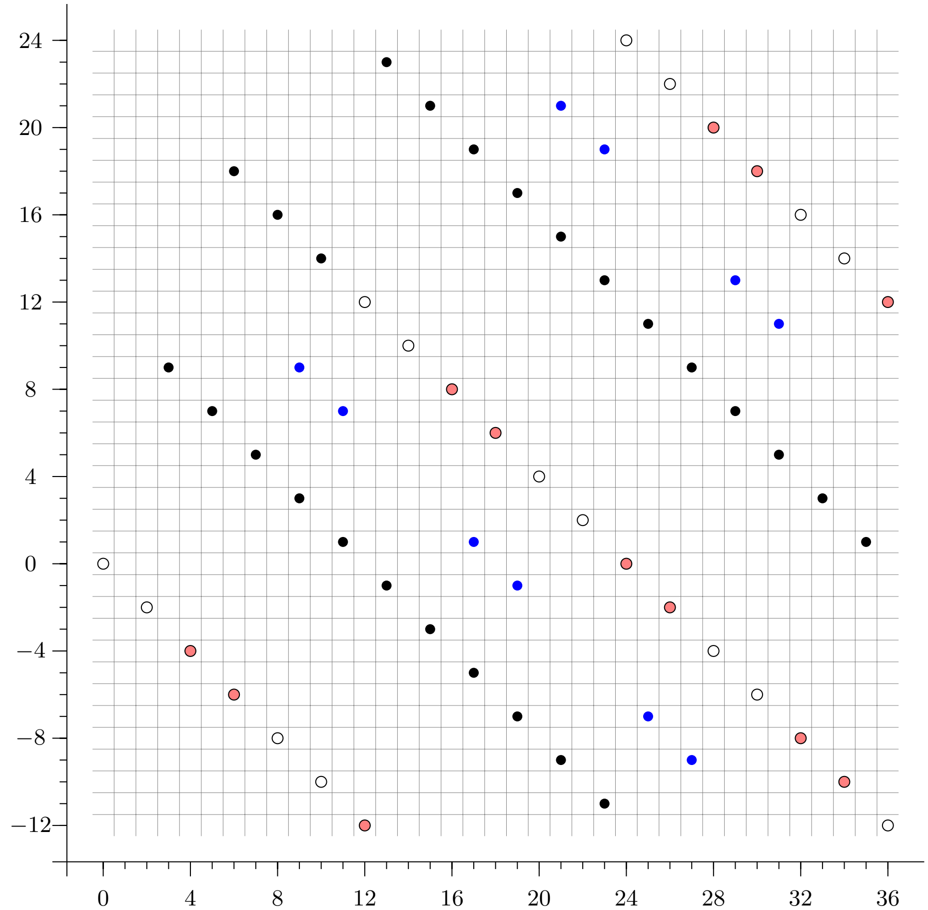

FIGURE 8. The E_8-page of the a_λ-localized slice spectral sequence of $a_\lambda^{-1} BP^{(\!(C_4)\!)}\langle 2,2\rangle$.

For (2), the restriction of the class $u_{16\lambda}$ is $u_{32\sigma_2}$, which supports the d_{63}-differential

$$d_{63}(u_{32\sigma_2}) = \bar{v}_5^{C_2} a_{63\sigma_2}$$

in the C_2-spectral sequence of $MU^{(\!(C_4)\!)}$. Therefore by naturality of the restriction map, the class $u_{16\lambda}$ must support a differential of length at most 63. By a stem-wise computation, it is impossible for the class $u_{16\lambda}$ to support a differential of length $\leqslant 19$, as there are no possible targets. Again, we do not write down the details here, as the computation in [26] already shows that $u_{16\lambda}$ supports a d_{31}-differential in the slice spectral sequence of $MU^{(\!(C_4)\!)}$ (see the discussion in Section 11.2 and the chart in Section 13 of [26]). It follows that the class $\frac{u_{16\lambda}}{a_{16\lambda}}$ must also support a d_{31}-differential in the a_λ-localized slice spectral sequence of $a_\lambda^{-1} MU^{(\!(C_4)\!)}$. This proves (2).

For (3), it is a consequence of the computation of $a_\lambda^{-1} BP^{(\!(C_4)\!)}\langle 2,2\rangle$ that $u_{32\lambda}$ is a d_{61}-cycle in the slice spectral sequence of $MU^{(\!(C_4)\!)}$ (in fact, it can be shown that it supports the d_{63}-differential $d_{63}(u_{32\lambda}) = N(\bar{v}_5 a_{63\sigma_2})u_{16\lambda}$, but we will not prove it here). $\square$

Multiplication by the classes α and b^{32} give the spectral sequence a large amount of structure which we will exploit in our computation. We describe this below, focusing on each class at a time.

7.3.1. *Multiplication by α.* The class α is extremely important for this computation. A key consequence of the behavior we describe here is that this allows us to compute differentials out of order, flipping back and forth between different pages of the spectral sequence without loosing the thread of its story. We make this precise now, starting with the following straightforward lemma.

Lemma 7.7. *In the C_4-localized slice spectral sequence of $a_\lambda^{-1} BP^{(\!(C_4)\!)}\langle 2,2\rangle$, on the E_{13}-page, we have:*

(1) Multiplication by α is injective.
(2) Let x be a class in bidegree $(t-s, s)$. If

$$s - 48 \geqslant -(t-s-48) \quad \textit{and} \quad s - 48 \leqslant 3(t-s-48)$$

then x is α divisible.

See Figure 9.

The key result is then the following:

Proposition 7.8. *Let $r \geqslant 13$. Suppose that $d_r(x) = y$ is a non-trivial differential on the E_r-page of the localized slice spectral sequence of $a_\lambda^{-1} BP^{(\!(C_4)\!)}\langle 2,2\rangle$. Then y is α-free in the sense that no multiple $\alpha^i y$ is zero at E_r. Consequently, x is also α-free.*

Proof. First, note that if y is α-free, the linearity of the differentials with respect to multiplication by α implies that x must also be α-free.

We prove that y is α-free by induction on r. If $r = 13$, the claim follows immediately from the fact that all classes are α-free at E_{13} (Lemma 7.7). Suppose that the claim holds for all $r' < r$, that $d_r(x) = y$ is a non-trivial differential and that y is α-torsion. Then there exists $i > 0$, $r' < r$ and z such that

$$d_{r'}(z) = \alpha^i y.$$

Choose a minimal such i, so that $\alpha^{i-1} y$ is non-zero at E_r. A comparison of degrees then implies that the bidegree of z satisfies the conditions of Lemma 7.7 (2), so that z is α-divisible. It cannot be the case that $d_{r'}(z/\alpha) = \alpha^{i-1} y$ since this contradicts the minimality of i. So we must have that $d_{r''}(z/\alpha) = v \neq 0$ for some $r'' < r' < r$. But by the induction hypothesis, v is then α-free which means that $d_{r''}(z) = \alpha v \neq 0$, which is also a contradiction. $\square$

Remark 7.9. We will show in Section 7.4.3 that α is killed by a d_{61}-differential. This will imply that for any permanent cycle x, the class αx must be hit by a differential of length at most 61.

We now explain the upshot of Proposition 7.8. Given any class y at E_{13}, there is a unique class x which is not α-divisible (so is in the complement of the region of Lemma 7.7 (2)) and such that $y = \alpha^i x$ for some $i \geqslant 0$. We say that x generates an α-free family, where the family is the collection of classes $\{\alpha^i x \mid i \geqslant 0\}$.

Now, Proposition 7.8 implies that α-free families come in pairs: one family in the pair, generated by x say, supports differentials which truncate the second family in the pair, generated by y say. (In fact, by Remark 7.9, the differentials must be of the form $d_r(\alpha^i x) = \alpha^{i+1} y$.) All classes in the α-free family generated by x are then gone at the E_{r+1}-page, having supported a no-trivial differential. The class y is now α-torsion, and by Proposition 7.8, it cannot support any further differential. This allows us to discard y from the rest of the computation, making the spectral sequence effectively sparser. Furthermore, we may now run differentials out of order if we find a unique possibility for pairing α-free families, even if this is through very long differentials.

Remark 7.10. This is in fact a common behavior for spectral sequences. For example, what we have here is very similar to the situation explained in a certain elliptic spectral sequence [5, Section 6], where there the role of α is played by the class $\bar{\kappa}$.

7.3.2. *Multiplication by b^{32}.* Multiplication by the permanent cycle b^{32} acts as a periodicity generator for most of the spectral sequence. More precisely, we have:

Lemma 7.11. *Let $r \geqslant 13$. Multiplication by b^{32} is injective on the E_r-page for classes on or below the line of slope 1 through the origin.*

It follows that if a differential has both source and target on or below the line of slope 1 through the origin, then $d_r(x) = y$ occurs if and only if $d_r(b^{32}x) = b^{32}y$ occurs. Differentials whose source and target are above the line of slope 1 through the origin are determined by the a_σ-localized spectral sequence. Some differentials, fall in neither category in the sense that they cross the line of slope one. That is, the source is on or below the line of slope one and the target is above. For these differentials, the target may be b^{32}-torsion while the source is not.

As one does the computation however, one sees that the target of such differentials have bidegree $(t-s, s)$ such that

$$s \leqslant (t-s) + 14.$$

This can be seen from the d_{13}-differentials that are obtained from the a_σ-localized spectral sequence and the structure of the E_{14}-page. Since the longest differential is a d_{61} and classes are concentrated in degrees with $t-s$ even, classes strictly below the line

$$s = (t-s) - 60$$

cannot support differentials that cross the line of slope 1 through the origin. Therefore, to completely determine the differentials of the spectral sequence using b^{32}-linearity and α-linearity, it is sufficient to determine:

- The d_{13}'s with source on or above the line of slope 1 through the origin, all obtained from the a_σ-localized spectral sequence.
- The differentials on classes with source of bidegree $(t-s, s)$ where is in the rectangular region:

$$s \leqslant (t-s) \qquad\qquad s \geqslant -(t-s)$$
$$s \geqslant (t-s) - 188 \qquad\qquad s < -(t-s) + 96$$

This region is larger than what is needed in practice, but the goal of this discussion is simply to illustrate the strategy and make a rough estimate on what differentials need to be determined. As we go through the computation, we learn that the region that determines all differentials is in fact smaller but, a priori, this is not clear.

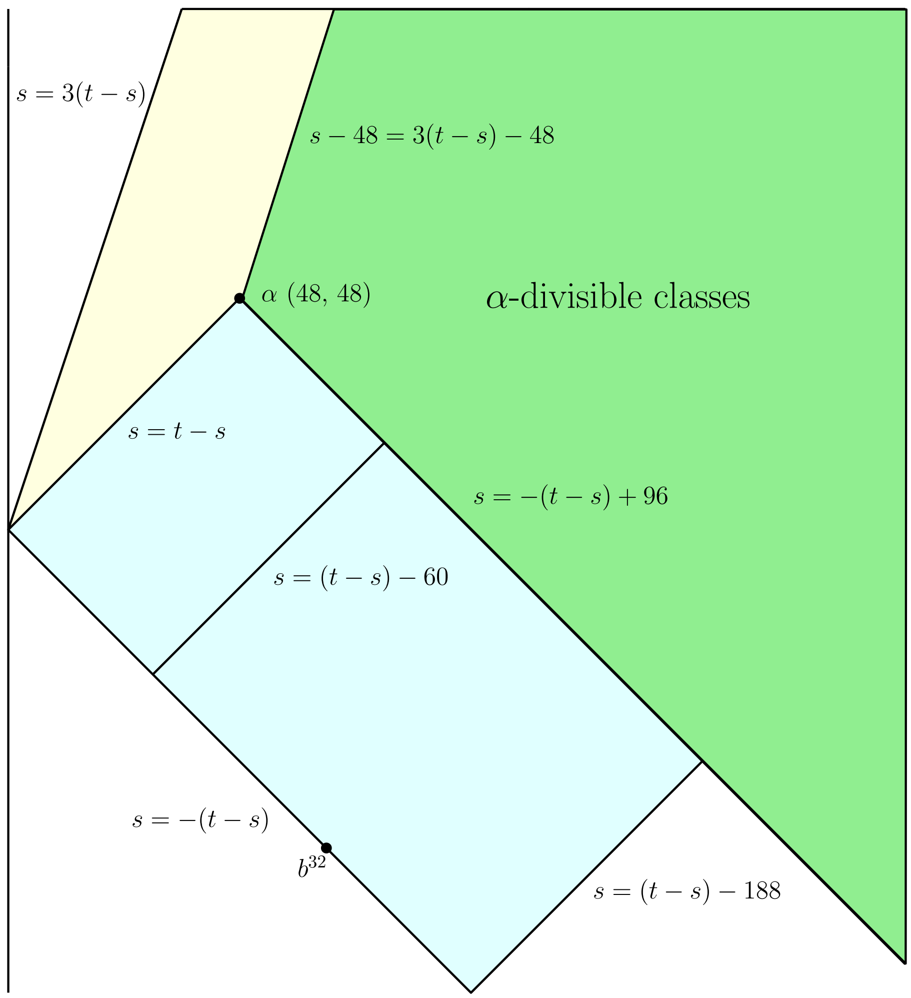

FIGURE 9. Features of the a_λ-localized slice spectral sequence of $a_\lambda^{-1} BP^{(\!(C_4)\!)}\langle 2, 2\rangle$

7.3.3. *Summary.* To summarize, we just have to focus on the classes in the shaded rectangular region of Figure 9 which is the union of a cone and a rectangle. Once we have figured out the fate of all the classes in this region, we can propagate by the classes α and b^{32} to obtain the rest of the differentials. Furthermore, once an α-multiple of a class gets truncated by a differential, that class can no longer support differentials and can be disregarded from future arguments.

7.4. **Differentials of length at most 13.**

7.4.1. d_{13}-*differentials.* By degree reasons, the next possible differentials are the d_{13}-differentials.

The differentials on or above the line of slope 1 are all obtained by computing the a_σ-localized spectral sequence, as explained in Section 5.1. This spectral sequence is depicted in Figure 4. The differentials are summarized in the following proposition.

Proposition 7.12. *The d_{13}-differentials that are on or above the line of slope 1 are generated by*

(1) $d_{13}\left((\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^{i+4k}\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^3 u_{2\sigma} a_{9\lambda} a_{7\sigma} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^{i-1+4k}$
$i = 1, 2,\ k \geqslant 0.$

(2) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{4\sigma} a_{6\lambda} a_{2\sigma} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^{i+4k}\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^3 a_{9\lambda} a_{9\sigma} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^{i+4k}$
$i = 0, 3,\ k \geqslant 0.$

(3) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2}^5 u_{14\sigma} a_{15\lambda} a_\sigma \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^{i+4k}\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^6 u_{10\sigma} a_{18\lambda} a_{8\sigma} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^{i+4k}$
$i = 0, 1,\ k \geqslant 0.$

To prove the d_{13}-differentials that are under the line of slope 1, we would like to first point out that the class $\bar{\mathfrak{d}}_{\bar{t}_2} u_\lambda u_{2\sigma} a_{2\lambda} a_\sigma$ in bidegree $(7, 5)$ is a permanent-cycle by degree reasons.

The class $\bar{\mathfrak{d}}_{\bar{t}_2}^4 u_{12\sigma} a_{12\lambda}$ in bidegree $(24, 24)$ will also be important. By the Hill–Hopkins–Ravenel Slice differential theorem [19, Theorem 9.9]. This class supports the d_{13}-differential

$$d_{13}(\bar{\mathfrak{d}}_{\bar{t}_2}^4 u_{12\sigma} a_{12\lambda}) = \bar{\mathfrak{d}}_{\bar{t}_2}^5 u_{8\sigma} a_{15\lambda} a_{7\sigma}$$

in the slice spectral sequence of $BP^{(\!(C_4)\!)}$. By naturality, this differential also appears in the slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2, 2\rangle$. When applying the Leibniz rule, the class $\bar{\mathfrak{d}}_{\bar{t}_2}^4 u_{12\sigma} a_{12\lambda}$ $(24, 24)$ acts as if it is a d_{13}-cycle for differentials whose sources are below the line of slope 1. More specifically, the target of the d_{13}-differential on this class multiplied with the source of another d_{13}-differential below the line of slope 1 is always 0.

Proposition 7.13. *We have the following d_{13}-differentials:*

(1) $d_{13}\left(b^4\right) = \bar{\mathfrak{d}}_{\bar{t}_2} u_\lambda u_{2\sigma} a_{2\lambda} a_\sigma$
(2) $d_{13}\left(b^5\right) = \bar{\mathfrak{d}}_{\bar{t}_2} u_{2\lambda} u_{2\sigma} a_\lambda a_\sigma$
(3) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2} u_{4\lambda} u_{2\sigma} a_{-\lambda} a_\sigma\right) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda}$
(4) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2} u_{4\lambda} u_{2\sigma} a_{-\lambda} a_\sigma \cdot b\right) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda} \cdot b$
(5) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2} u_{4\lambda} u_{2\sigma} a_{-\lambda} a_\sigma \cdot b^2\right) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda} \cdot b^2$
(6) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2} u_{4\lambda} u_{2\sigma} a_{-\lambda} a_\sigma \cdot b^3\right) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda} \cdot b^3$

Proof. To prove (1), we will first prove the differential

$$d_{13}\left(b^{12}\right) = \bar{\mathfrak{d}}_{\bar{t}_2} u_{9\lambda} u_{2\sigma} a_{-6\lambda} a_\sigma \ \ (d_{13}(24, -24) = (23, -11))$$

The source of this differential restricts to a class in the C_2-spectral sequence that supports a d_{31}-differential. By naturality and degree reasons, we must have the d_{13}-differential claimed above. Applying Leibniz with the class b^8 in degree $(16, -16)$ proves (1).

The source of (2) restricts to a class that supports a d_{19}-differential in the C_2-spectral sequence. Therefore the source class must support either a d_{13}- or a d_{19}-differential. By naturality, it cannot be a d_{19}-differential because the target does not restrict to the target of the d_{19}-differential in the C_2-spectral sequence.

The targets of (3) is in the image of the transfer. The preimage is killed by a d_{19}-differential. Therefore by naturality and degree reasons, we have the d_{13}-differential claimed in (3).

Differentials (4) and (5) are obtained by applying the Leibniz rule using the class $\bar{\mathfrak{d}}_{\bar{t}_2} u_\lambda u_{2\sigma} a_{2\lambda} a_\sigma$ $(7,5)$ with differentials (1) and (2) (and also using the gold relation).

It remains to prove differential (6). Consider the class $\bar{\mathfrak{d}}_{\bar{t}_2} u_{7\lambda} u_{2\sigma} a_{-4\lambda}$. The restriction of this class is $\bar{t}_2 \gamma \bar{t}_2 u_{14\sigma_2} a_{-8\sigma_2}$, which supports a d_7-differential in the C_2-spectral sequence. Therefore, the class $\bar{\mathfrak{d}}_{\bar{t}_2} u_{7\lambda} u_{2\sigma} a_{-4\lambda}$ also supports a d_7-differential in the C_4-spectral sequence. The existence of this d_7-differential shows that there is an exotic restriction of filtration 6 for the class $\bar{\mathfrak{d}}_{\bar{t}_2} u_{7\lambda} u_{2\sigma} a_{-4\lambda} a_\sigma$ $(19,-7)$. It must have nonzero restriction, restricting to the class $\bar{t}_2^3 u_{10\sigma_2} a_{-\sigma_2}$ $(19,-1)$ after the E_7-page.

Since the class $\bar{t}_2^3 u_{10\sigma_2} a_{-\sigma_2}$ $(19,-1)$ supports a d_{19}-differential in the C_2-spectral sequence, the class $\bar{\mathfrak{d}}_{\bar{t}_2} u_{7\lambda} u_{2\sigma} a_{-4\lambda} a_\sigma$ $(19,-7)$ cannot survive past the E_{25}-page. The only possibility is for it to support the d_{13}-differential claimed by (6). $\square$

The same proof for differentials (1)–(6) can be used to prove six more differentials that are obtained by multiplying both the source and the target of each differential by $(12,12)$: $\bar{\mathfrak{d}}_{\bar{t}_2} u_{6\sigma} a_{6\lambda}$ (note that we can't just directly propagate by this class using the Leibniz rule because it supports a d_5-differential in $\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$).

Proposition 7.14. *The following classes are d_{13}-cycles:*

(1) $2b^6$ $(12,-12)$*;*
(2) $2\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\lambda} u_{6\sigma}$ $(24,0)$*;*
(3) $\bar{\mathfrak{d}}_{\bar{t}_2}^3 u_{8\lambda} u_{8\sigma} a_\lambda a_\sigma$ $(33,3)$.

Proof. For (1), the class $2\bar{\mathfrak{d}}_{\bar{t}_2}^4 u_{6\lambda} u_{12\sigma} a_{6\lambda}$ $(36,12)$ is a d_{13}-cycle by using the class $\bar{\mathfrak{d}}_{\bar{t}_2} u_\lambda u_{2\sigma} a_{2\lambda} a_\sigma$ $(7,5)$ to apply the Leibniz rule to the d_{13}-differential

$$d_{13}(\bar{\mathfrak{d}}_{\bar{t}_2}^3 u_{6\lambda} u_{8\sigma} a_{3\lambda} a_\sigma) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^4 u_{2\lambda} u_{12\sigma} a_{10\lambda} \quad (d_{13}(29,7) = (28,20)).$$

Therefore, by Leibniz with the class $\bar{\mathfrak{d}}_{\bar{t}_2}^4 u_{12\sigma} a_{12\lambda}$ $(24,24)$, the class $2b^6$ $(12,-12)$ is also a d_{13}-cycle.

(2) is proven by the exact same method, by using the class $\bar{\mathfrak{d}}_{\bar{t}_2} u_\lambda u_{2\sigma} a_{2\lambda} a_\sigma$ $(7,5)$ to apply the Leibniz rule to the d_{13}-differential

$$d_{13}(\bar{\mathfrak{d}}_{\bar{t}_2} u_{6\lambda} u_{2\sigma} a_{-3\lambda} a_\sigma) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{2\lambda} u_{6\sigma} a_{4\lambda} \quad (d_{13}(17,-5) = (16,8)).$$

For (3), the class $\bar{\mathfrak{d}}_{\bar{t}_2} u_{8\lambda} u_{8\sigma} a_\lambda a_\sigma$ is a d_{13}-cycle in $\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$. Therefore by naturality it is a d_{13}-cycle in $\mathrm{SliceSS}(BP^{(\!(C_4)\!)}\langle 2,2\rangle)$. $\square$

Now, we can propagate all the differentials by the classes $\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{12\sigma} a_{12\lambda}$ $(24,24)$ and b^8 $(16,-16)$. The d_{13}-differentials under the line of slope 1 are summarized in the following proposition.

Proposition 7.15. *The d_{13}-differentials that are under the line of slope 1 are*

(1) $d_{13}\left(b^{4+i+8j} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^k\right) = \bar{\mathfrak{d}}_{\bar{t}_2} u_\lambda u_{2\sigma} a_{2\lambda} a_\sigma \cdot b^{i+8j} (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^k$,
$0 \leqslant i \leqslant 1,\ j,k \geqslant 0$.
(2) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2} u_{4\lambda} u_{2\sigma} a_{-\lambda} a_\sigma \cdot b^{i+8j} (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^k\right) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda} \cdot b^{i+8j} (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^k$,
$0 \leqslant i \leqslant 3,\ j,k \geqslant 0$.

They are shown in Figure 10.

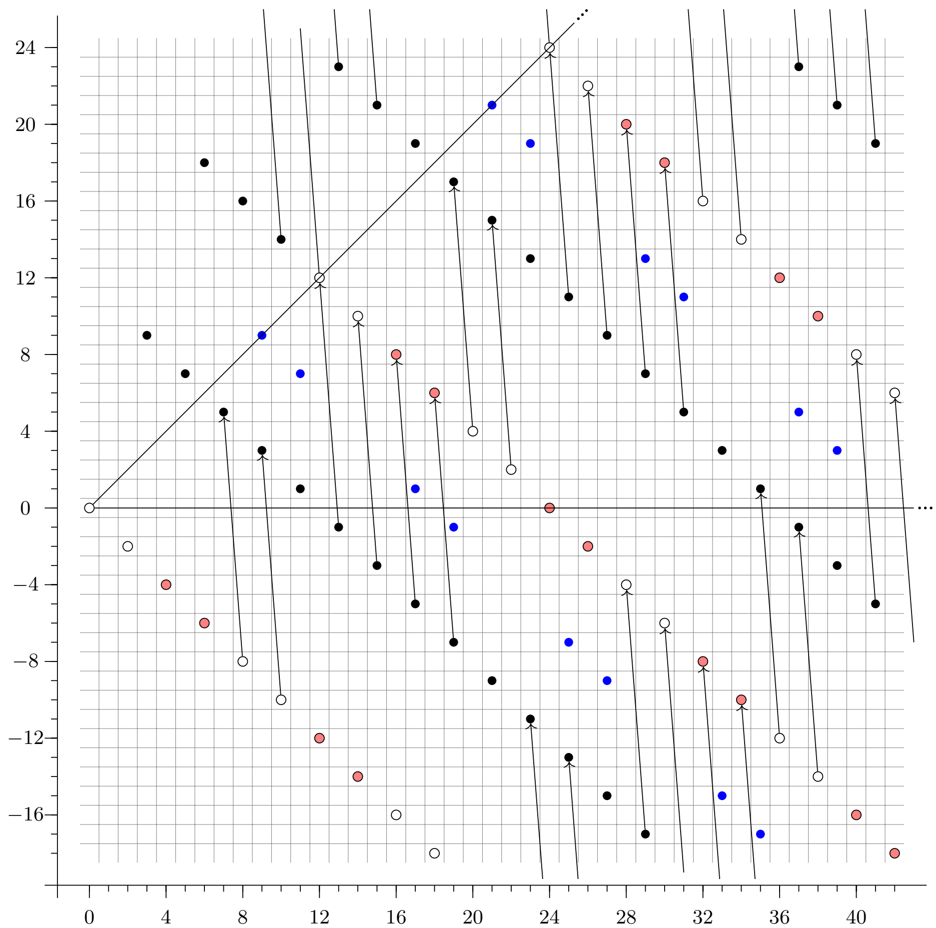

FIGURE 10. The E_{13}-page of the a_λ-localized slice spectral sequence of $a_\lambda^{-1} BP^{(\!(C_4)\!)}\langle 2,2\rangle$.

Remark 7.16. We will see that these are the last non-trivial d_{13} differentials. However, at this point in the computation, there are possibilities for other non-trivial d_{13} differentials. Later, (in Lemmas 7.21 and 7.24) we will show that these do not occur.

7.4.2. d_{19}*-differentials.*

Proposition 7.17. *The following d_{19}-differentials exist:*

(1) $d_{19}\left(2b^5\right) = tr(\bar{t}_2^3 a_{9\sigma_2})$ $(d_{19}(10,-10) = (9,9))$
(2) $d_{19}\left(2b^6\right) = tr(\bar{t}_2^3 u_{2\sigma_2} a_{7\sigma_2})$ $(d_{19}(12,-12) = (11,7))$
(3) $d_{19}\left(b^9\right) = tr(\bar{t}_2^3 u_{8\sigma_2} a_{\sigma_2})$ $(d_{19}(18,-18) = (17,1))$
(4) $d_{19}\left(2b^{13}\right) = tr(\bar{t}_2^3 u_{16\sigma_2} a_{-7\sigma_2})$ $(d_{19}(26,-26) = (25,-7))$

Proof. Differential (1) is obtained by applying transfer to the d_{19}-differential

$$d_{19}\left(\frac{u_{10\sigma_2}}{a_{10\sigma_2}}\right) = \bar{t}_2^3 a_{9\sigma_2}$$

in the C_2-spectral sequence.

For differentials (2) and (3), the classes $\bar{t}_2^3 u_{2\sigma_2} a_{7\sigma_2}$ and $\bar{t}_2^3 u_{8\sigma_2} a_{\sigma_2}$ are killed by d_{31}-differentials in the C_2-spectral sequence. Therefore their images under the transfer map must also be killed by differentials of lengths at most 31. The only possibilities are the differentials claimed.

Differential (4) is obtained by applying the transfer to the d_{19}-differential

$$d_{19}\left(\frac{u_{26\sigma_2}}{a_{26\sigma_2}}\right) = \bar{t}_2^3 u_{16\sigma_2} a_{-7\sigma_2}$$

in the C_2-spectral sequence. □

The same arguments in the proof above can be used to prove twelve more d_{19}-differentials, obtained by multiplying the four d_{19}-differentials in Proposition 7.17 by $\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\lambda} a_{6\lambda}$ $(12, 12)$, $\bar{\mathfrak{d}}_{\bar{t}_2}^4 u_{12\lambda} a_{12\lambda}$ $(24, 24)$, and $\bar{\mathfrak{d}}_{\bar{t}_2}^6 u_{18\lambda} a_{18\lambda}$ $(36, 36)$.

Proposition 7.18. *The following d_{19}-differentials exist:*

(1) $d_{19}(2b^{14}) = tr(\bar{t}_2^3 u_{18\sigma_2} a_{-9\sigma_2})$ $(d_{19}(28, -28) = (27, -9))$

(2) $d_{19}\left(2b^{14} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})\right) = tr(\bar{t}_2^7 u_{18\sigma_2} a_{3\sigma_2})$ $(d_{19}(40, -16) = (39, 3))$

(3) $d_{19}\left(2b^{14} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^2\right) = tr(\bar{t}_2^{11} u_{18\sigma_2} a_{15\sigma_2})$ $(d_{19}(52, -4) = (51, 15))$

(4) $d_{19}\left(2b^{14} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma} a_{6\lambda})^3\right) = tr(\bar{t}_2^{15} u_{18\sigma_2} a_{27\sigma_2})$ $(d_{19}(64, 8) = (63, 27))$

Proof. We will prove differential (1) first. Consider the class $tr(\bar{t}_2^{19} u_{18\sigma_2} a_{39\sigma_2})$ $(75, 39)$. This class must die on or before the E_{61}-page. There are three possible ways for this class to die. It can support a d_{31}-differential hitting the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{12} u_\lambda u_{36\sigma} a_{35\lambda}$ $(74, 70)$; it can be the target of a d_{19}-differential from the class $2\bar{\mathfrak{d}}_{\bar{t}_2}^8 u_{14\lambda} u_{24\sigma} a_{10\lambda}$ $(76, 20)$; or it can be the target of a d_{43}-differential from the class $2\bar{\mathfrak{d}}_{\bar{t}_2}^6 u_{20\lambda} u_{18\sigma} a_{-2\lambda}$ $(76, -4)$.

It is impossible for this class to support a d_{31}-differential because it is the transfer of a class that supports a d_{31}-differential in the C_2-slice spectral sequence, and the target does not transfer to the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{12} u_\lambda u_{36\sigma} a_{35\lambda}$ $(74, 70)$.

The d_{43}-differential also cannot happen because the class $2\bar{\mathfrak{d}}_{\bar{t}_2}^6 u_{20\lambda} u_{18\sigma} a_{-2\lambda}$ $(76, -4)$ is the transfer of $\bar{t}_2^{12} u_{40\sigma_2} a_{-4\sigma_2}$, which is the target of a d_{31}-differential in the C_2-spectral sequence. Therefore it must be killed by a differential of length at most 31.

It follows that the d_{19}-differential

$$d_{19}(2\bar{\mathfrak{d}}_{\bar{t}_2}^8 u_{14\lambda} u_{24\sigma} a_{10\lambda}) = tr(\bar{t}_2^{19} u_{18\sigma_2} a_{39\sigma_2}) \ \ (d_{19}(76, 20) = (75, 39))$$

exists. Applying Leibniz with respect to the class $\bar{\mathfrak{d}}_{\bar{t}_2}^4 u_{24\sigma} a_{24\lambda}$ $(48, 48)$ proves (1).

Differentials (2), (3), (4) are proven by the exact same method. □

Now, we can propagate the d_{19}-differentials that we have proven by the classes b^{16} $(32, -32)$ and $\bar{\mathfrak{d}}_{\bar{t}_2}^4 u_{24\sigma} a_{24\lambda}$ $(48, 48)$ to obtain the rest of the d_{19}-differentials.

Proposition 7.19. *The d_{19}-differentials are*

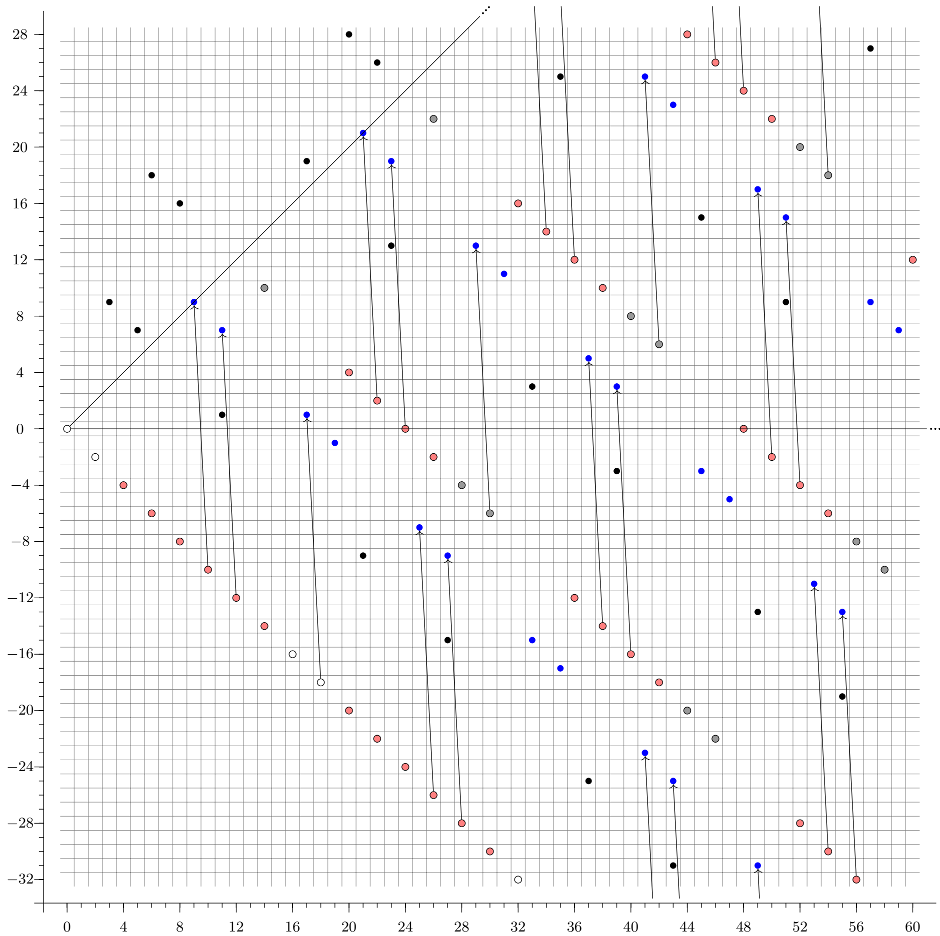

FIGURE 11. The E_{19}-page of the a_λ-localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}\langle 2,2\rangle$.

(1) $d_{19}\left(2b^{i+8j}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^k\right) = tr(\bar{t}_2^3 a_{9\sigma_2})\cdot b^{i-5+8j}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^k$
$i=5,6,\ j,k\geqslant 0$

(2) $d_{19}\left(b^{9+16i}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^k\right) = tr(\bar{t}_2^3 u_{8\sigma_2}a_{\sigma_2})\cdot b^{16i}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^k$
$i,k\geqslant 0$.

They are shown in Figure 11.

7.4.3. *The vanishing theorem.*

Theorem 7.20 (Vanishing Theorem)**.** *In the a_λ-localized slice spectral sequence for $a_\lambda^{-1}BP^{(\!(C_4)\!)}\langle 2,2\rangle$, we have the d_{61}-differential*

$$d_{61}(\bar{\mathfrak{d}}_{\bar{t}_2}^3 u_{16\lambda}u_{8\sigma}a_{-7\lambda}a_\sigma) = \bar{\mathfrak{d}}_{\bar{t}_2}^8 u_{24\sigma}a_{24\lambda}\ \ (d_{61}(49,-13)=(48,48)).$$

Furthermore, any class of the form $(\bar{\mathfrak{d}}_{\bar{t}_2}^8 u_{24\sigma}a_{24\lambda})\cdot x$ must die on or before the E_{61}-page.

Proof. In the a_λ-localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}$, the class $N(\bar{v}_4)u_{16\sigma}a_{31\lambda}$ must die on or before the E_{61}-page because it is the target of the predicted d_{61}-differential

$$d_{61}\left(u_{16\lambda}a_\sigma\right) = N(\bar{v}_4)u_{16\sigma}a_{31\lambda},$$

obtained by norming up the d_{31}-differential $d_{31}(u_{16\sigma_2}) = \bar{v}_4 a_{31\sigma_2}$ in the C_2-spectral sequence. Therefore, if we multiply the target by $\bar{\mathfrak{d}}_{\bar{t}_2}^{11}u_{32\sigma}a_{32\lambda}$, the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{11}N(\bar{v}_4)u_{48\sigma}u_{48\lambda}$ $(96,96)$ must die on or before the E_{61}-page.

Under the map

$$a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{(\!(C_4)\!)}) \longrightarrow a_\lambda^{-1}\,\mathrm{SliceSS}(BP^{(\!(C_4)\!)}\langle 2,2\rangle),$$

the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{11}N(\bar{v}_4)u_{48\sigma}u_{48\lambda}$ is sent to $\bar{\mathfrak{d}}_{\bar{t}_2}^{16}u_{48\sigma}u_{48\lambda}$ $(96,96)$. By naturality and degree reasons, the only possibility that this class can die on or before the E_{61}-page is for it to be killed by a d_{61}-differential. This implies that the original class must also be killed by a d_{61}-differential in the a_λ-localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}$. Furthermore, by the module structure, any class in the a_λ-localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}\langle 2,2\rangle$ of the form $(\bar{\mathfrak{d}}_{\bar{t}_2}^{16}u_{48\sigma}u_{48\lambda})\cdot x$ must die on or before the E_{61}-page.

The class $\bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_{24\sigma}a_{24\lambda}$ $(48,48)$ is also the target of a d_{61}-differential because after multiplying it by $\bar{\mathfrak{d}}_{\bar{t}_2}^{16}u_{48\sigma}a_{48\lambda}$ $(96,96)$, it must die on or before the E_{61}-page. By degree reasons, the only possibility is for it to be killed by a d_{61}-differential. Since multiplication by $\bar{\mathfrak{d}}_{\bar{t}_2}^{16}u_{48\sigma}a_{48\lambda}$ $(96,96)$ induces an injection on the E_2-page, and all the classes above the line of slope (-1) with this class as the origin are all divisible by it, the claimed d_{61}-differential must occur.

Similarly, for any class of the form $(\bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_{24\sigma}a_{24\lambda})\cdot x$, we can multiply it by $\bar{\mathfrak{d}}_{\bar{t}_2}^{16}u_{48\sigma}a_{48\lambda}$ $(96,96)$ to deduce that the product must die on or before the E_{61}-page. It follows from the same reasoning as the previous paragraph that the original class must also die on or before the E_{61}-page. $\square$

7.4.4. *d_{31}-differentials.* To prove the d_{31}-differentials, we will first prove the nonexistence of certain d_{13}-differentials.

Lemma 7.21. *At the E_{13}-page, we have*

(1) $d_{13}(\bar{\mathfrak{d}}_{\bar{t}_2}u_{8\lambda}u_{2\sigma}a_{-5\lambda}a_\sigma) \neq 2\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{4\lambda}u_{6\sigma}a_{2\lambda}$ $(d_{13}(21,-9)\neq(20,4))$.

(2) $d_{13}(\bar{\mathfrak{d}}_{\bar{t}_2}^{3}u_{8\lambda}u_{8\sigma}a_{\lambda}a_\sigma) \neq 2\bar{\mathfrak{d}}_{\bar{t}_2}^{4}u_{4\lambda}u_{12\sigma}a_{8\lambda}$ $(d_{13}(33,3)\neq(32,16))$.

Proof. Suppose (1) exists. By applying the Leibniz rule with respect to the classes $\bar{\mathfrak{d}}_{\bar{t}_2}^{4}u_{12\sigma}a_{12\lambda}$ $(24,24)$ and b^8 $(16,-16)$, the d_{13}-differential

$$d_{13}(\bar{\mathfrak{d}}_{\bar{t}_2}^{5}u_{16\lambda}u_{14\sigma}a_{-\lambda}a_\sigma) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^{6}u_{12\lambda}u_{18\sigma}a_{6\lambda} \quad (d_{13}(61,-1)=(60,12))$$

must also exist. Consider the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{9}u_{3\lambda}u_{26\sigma}a_{24\lambda}a_\sigma$ in $(59,49)$. By Theorem 7.20, this class must die on or before the E_{61}-page. However, with the class $2\bar{\mathfrak{d}}_{\bar{t}_2}^{6}u_{12\lambda}u_{18\sigma}a_{6\lambda}$ $(60,12)$ gone, there are no classes that could kill it or be killed by this class on or before the E_{61}-page. Contradiction.

Now, suppose (2) exists. By applying the Leibniz rule with respect to the classes $\bar{\mathfrak{d}}_{\bar{t}_2}^{4}u_{12\sigma}a_{12\lambda}$ $(24,24)$ and b^8 $(16,-16)$, the d_{13}-differential

$$d_{13}(\bar{\mathfrak{d}}_{\bar{t}_2}^{7}u_{16\lambda}u_{20\sigma}a_{5\lambda}a_\sigma) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_{12\lambda}u_{24\sigma}a_{12\lambda} \ \ (d_{13}(73,11)=(72,24))$$

must also exist. Consider the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{11}u_{3\lambda}u_{32\sigma}a_{30\lambda}a_\sigma$ $(71,61)$. By Theorem 7.20, this class must die on or before the E_{61}-page. Just like the previous case, there is no class that could kill it or be killed by it on or before the E_{61}-page. Contradiction. $\square$

Proposition 7.22. *We have the following d_{31}-differentials:*

(1) $d_{31}\left(tr(\bar{t}_2^{11}u_{10\sigma_2}a_{23\sigma_2})\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_{18\sigma}a_{24\lambda}a_{6\sigma}$;

(2) $d_{31}\left(tr(\bar{t}_2^{11}u_{10\sigma_2}a_{23\sigma_2})\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_{18\sigma}a_{24\lambda}a_{6\sigma}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})$;

(3) $d_{31}\left(tr(\bar{t}_2^{11}u_{24\sigma_2}a_{9\sigma_2})\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^i\right) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_{4\lambda}u_{24\sigma}a_{20\lambda}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^i$, $0 \leqslant i \leqslant 3$;

(4) $d_{31}\left(tr(\bar{t}_2^{11}u_{24\sigma_2}a_{9\sigma_2})\cdot b^{16}(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^i\right) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_{4\lambda}u_{24\sigma}a_{20\lambda}\cdot b^{16}(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^i$, $0 \leqslant i \leqslant 3$.

Proof. To prove (1), first multiply the predicted target, $\bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_{18\sigma}a_{24\lambda}a_{6\sigma}$ $(42, 54)$, by $\bar{\mathfrak{d}}_{\bar{t}_2}^{16}u_{48\sigma}a_{48\lambda}$ $(96, 96)$. By Theorem 7.20 and degree reasons, the product must be killed by a differential of length 61. It follows that (1) must hold.

By Theorem 7.20 and degree reasons, the target of (2) must be killed by a differential of length at most 61. The only possible differential is the ones claimed.

To prove (3), note that in the a_{σ_2}-localized slice spectral sequence of $i^*_{C_2}BP^{(\!(C_4)\!)}\langle 2,2\rangle$, we have the differential

$$d_{31}(\bar{t}_2^{11}u_{24\sigma_2}a_{9\sigma_2}) = \bar{t}_2^{16}u_{8\sigma_2}a_{40\sigma_2} \;\; (d_{31}(57,9) = (56,40)).$$

Applying transfer to the target shows that the image of the target under the transfer map must be killed on or before the E_{31}-page. There are only two possibilities. Either the claimed d_{31}-differential exists, or it is killed by a d_{13}-differential from $\bar{\mathfrak{d}}_{\bar{t}_2}^{7}u_{8\lambda}u_{20\sigma}a_{13\lambda}a_{\sigma}$ $(57, 27)$. By Lemma 7.21, the d_{13}-differential does not exist. Therefore the claimed d_{31}-differential happens for $i = 0$. The rest of the differentials in (3) and all the differentials in (4) are proven by the same method. $\square$

We can propagate the differentials in Proposition 7.22 with respect to the classes $\bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_{24\sigma}a_{24\lambda}$ $(48, 48)$ and b^{32} $(64, -64)$ to obtain the rest of the d_{31}-differentials.

Proposition 7.23. *The d_{31}-differentials are*

(1) $d_{31}\left(tr(\bar{t}_2^{11}u_{10\sigma_2}a_{23\sigma_2})\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^{i+4j}\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_{18\sigma}a_{24\lambda}a_{6\sigma}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^{i+4j}$, $i = 0, 1,\ j \geqslant 0$;

(2) $d_{31}\left(tr(\bar{t}_2^{3}u_{24\sigma_2}a_{-15\sigma_2})\cdot b^{16i}(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^j\right) = 2\bar{\mathfrak{d}}_{\bar{t}_2}^{4}u_{4\lambda}u_{12\sigma}a_{8\lambda}\cdot b^{16i}(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^j$, $i, j \geqslant 0$.

They are shown in Figure 12.

7.4.5. *d_{37}-differentials.* To prove the d_{37}-differentials, we will first prove the nonexistence of certain d_{13}-differentials.

Lemma 7.24. *At the E_{13}-page, we have*

(1) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2}^{12}u_{\lambda}u_{36\sigma}a_{35\lambda}\right) \neq \bar{\mathfrak{d}}_{\bar{t}_2}^{13}u_{34\sigma}a_{39\lambda}a_{5\sigma}$;

(2) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2}^{12}u_{\lambda}u_{36\sigma}a_{35\lambda}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})\right) \neq \bar{\mathfrak{d}}_{\bar{t}_2}^{13}u_{34\sigma}a_{39\lambda}a_{5\sigma}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})$;

(3) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2}^{11}u_{3\lambda}u_{32\sigma}a_{30\lambda}a_{\sigma}\right) \neq \bar{\mathfrak{d}}_{\bar{t}_2}^{12}u_{34\sigma}a_{36\lambda}a_{2\sigma}$;

(4) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2}^{11}u_{3\lambda}u_{32\sigma}a_{30\lambda}a_{\sigma}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})\right) \neq \bar{\mathfrak{d}}_{\bar{t}_2}^{12}u_{34\sigma}a_{36\lambda}a_{2\sigma}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})$;

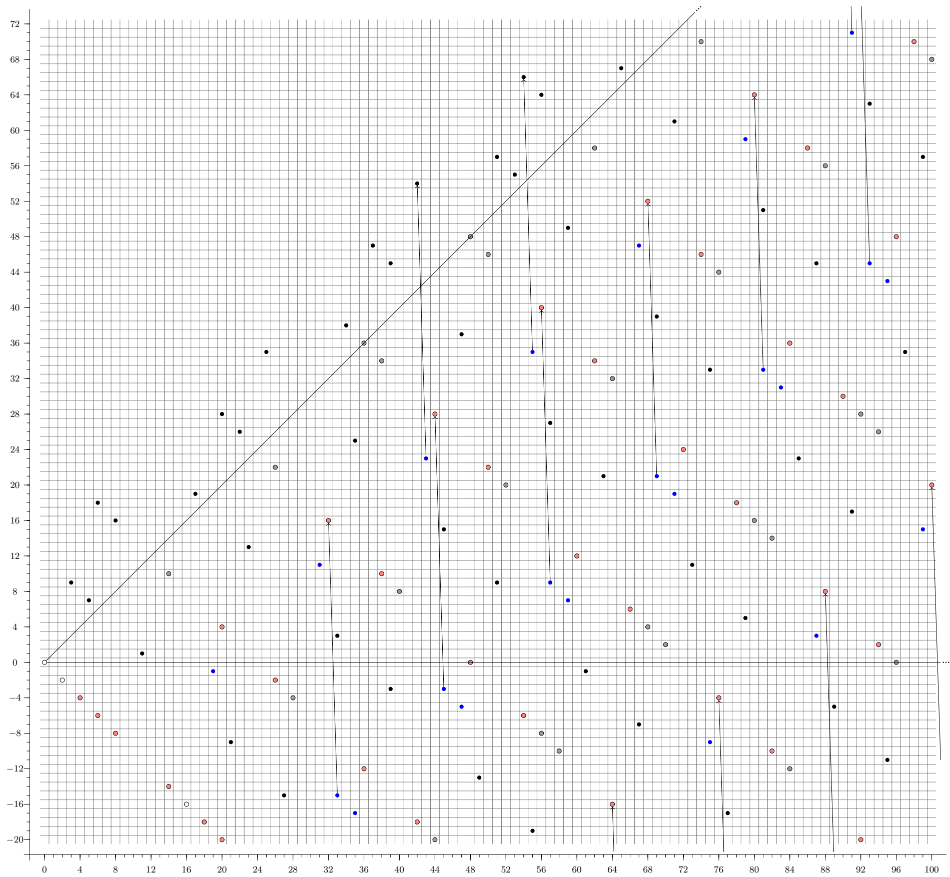

FIGURE 12. The E_{31}-page of the a_λ-localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}\langle 2,2\rangle$.

(5) $d_{13}\left(\bar{\mathfrak{d}}_{\bar{t}_2}^{9}u_{19\lambda}u_{26\sigma}a_{8\lambda}a_\sigma\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^i\right)\neq 2\bar{\mathfrak{d}}_{\bar{t}_2}^{10}u_{15\lambda}u_{30\sigma}a_{15\lambda}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^i$, $0\leqslant i\leqslant 3$.

Proof. To prove (1), note that if the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{12}u_\lambda u_{36\sigma}a_{35\lambda}$ $(74,70)$ supports a d_{13}-differential, then applying the Leibniz rule with respect to the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{4}u_{12\sigma}a_{12\lambda}$ $(24,24)$ would show that the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_\lambda u_{24\sigma}a_{23\lambda}$ $(50,46)$ must support a differential of length at most 13. This is a contradiction because there are no possible targets. The nonexistence of differentials (2), (3), and (4) can be proven by the same method. The differentials in (5) follows from (1)-(4) by applying the Leibniz rule with respect to $\bar{\mathfrak{d}}_{\bar{t}_2}^{4}u_{12\sigma}a_{12\lambda}$ $(24,24)$ and b^{16} $(32,-32)$. □

Proposition 7.25. *We have the following d_{37}-differentials for $i=0,1$:*

$$d_{37}(\bar{\mathfrak{d}}_{\bar{t}_2}^{5}u_{27\lambda}u_{14\sigma}a_{-12\lambda}a_\sigma\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^i)=\bar{\mathfrak{d}}_{\bar{t}_2}^{8}u_{17\lambda}u_{24\sigma}a_{7\lambda}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^{2}u_{6\sigma}a_{6\lambda})^i.$$

Proof. To prove the differential when $i=0$, we will show that the d_{37}-differential

$$d_{37}(\bar{\mathfrak{d}}_{\bar{t}_2}^{13}u_{27\lambda}u_{38\sigma}a_{12\lambda}a_\sigma)=\bar{\mathfrak{d}}_{\bar{t}_2}^{16}u_{17\lambda}u_{48\sigma}a_{31\lambda}\ \ (d_{37}(131,25)=(130,62))$$

exists. Propagating with respect to the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{8} u_{24\sigma} a_{24\lambda}$ $(48, 48)$ would then prove the desired differential. Note that by Theorem 7.20, the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{16} u_{17\lambda} u_{48\sigma} a_{31\lambda}$ $(130, 62)$ must die on or before the E_{61}-page. There are two possibilities: either it supports a d_{37}-differential hitting $\bar{\mathfrak{d}}_{\bar{t}_2}^{19} u_{8\lambda} u_{56\sigma} a_{49\lambda} a_\sigma$ $(129, 99)$, or the claimed differential exists. Suppose the first case happens, then we claim there is no possibility for the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{13} u_{27\lambda} u_{38\sigma} a_{12\lambda} a_\sigma$ $(131, 25)$ to die on or before the E_{61}-page. This is because if the class does die, then the only possibility is for it to support a d_{13}-differential hitting $2\bar{\mathfrak{d}}_{\bar{t}_2}^{14} u_{23\lambda} u_{42\sigma} a_{19\lambda}$ $(130, 38)$. However, if this d_{13}-differential exists, then by applying the Leibniz rule with respect to the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{4} u_{12\sigma} a_{12\lambda}$ $(24, 24)$, the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{9} u_{27\lambda} u_{26\sigma} a_\sigma$ $(107, 1)$ must also support a differential of length at most 13. This is a contradiction because we must have the d_{37}-differential

$$d_{37}(\bar{\mathfrak{d}}_{\bar{t}_2}^{9} u_{27\lambda} u_{26\sigma} a_\sigma) = \bar{\mathfrak{d}}_{\bar{t}_2}^{12} u_{17\lambda} u_{36\sigma} a_{19\lambda} \ \ (d_{37}(107, 1) = (106, 38))$$

by the Vanishing Theorem and degree reasons (Theorem 7.20). It follows that the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{9} u_{27\lambda} u_{26\sigma} a_\sigma$ $(107, 1)$ supports a d_{37}-differential.

The second differential, when $i = 1$, is proven by the same method. $\square$

Proposition 7.26. *The d_{37}-differentials are*

(1) $d_{37}\left(\bar{\mathfrak{d}}_{\bar{t}_2}^{4} u_{8\lambda} u_{12\sigma} a_{4\lambda} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{i+4j}\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^{7} u_{18\sigma} a_{21\lambda} a_{3\sigma} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{i+4j}$, $i = 0, 1,\ j \geqslant 0$;

(2) $d_{37}\left(\bar{\mathfrak{d}}_{\bar{t}_2} u_{8\lambda} u_{2\sigma} a_{-5\lambda} a_\sigma \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{i+4j}\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^{4} u_{8\sigma} a_{12\lambda} a_{4\sigma} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{i+4j}$, $i = 0, 3,\ j \geqslant 0$;

(3) $d_{37}\left(2\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{7\lambda} u_{6\sigma} a_{-\lambda} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{i+4j}\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^{5} u_{10\sigma} a_{15\lambda} a_{5\sigma} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{i+4j}$, $i = 0, 1,\ j \geqslant 0$;

(4) $d_{37}\left(2b^{12+16i} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{j}\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^{3} u_{3\lambda} u_{8\sigma} a_{6\lambda} a_\sigma \cdot b^{16i} (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{j}$, $i, j \geqslant 0$;

(5) $d_{37}\left(\bar{\mathfrak{d}}_{\bar{t}_2} u_{11\lambda} u_{2\sigma} a_{-8\lambda} a_\sigma \cdot b^{16i} (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{j}\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^{4} u_{\lambda} u_{12\sigma} a_{11\lambda} \cdot b^{16i} (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{j}$, $i, j \geqslant 0$.

They are shown in Figure 13.

Proof. All the differentials can be proven immediately from the Vanishing Theorem and degree reasons (Theorem 7.20 and Lemma 7.24), Proposition 7.25, and propagation with respect to the classes $\bar{\mathfrak{d}}_{\bar{t}_2}^{8} u_{24\sigma} a_{24\lambda}$ $(48, 48)$ and b^{32} $(64, -64)$. $\square$

7.4.6. *d_{43}-differentials.*

Proposition 7.27. *The following d_{43}-differentials exist for $i = 0, 1$:*

$$d_{43}(\bar{\mathfrak{d}}_{\bar{t}_2}^{12} u_{40\lambda} u_{36\sigma} a_{-4\lambda} \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{i}) = tr(\bar{t}_2^{31} u_{58\sigma_2} a_{35\sigma_2}) \cdot (\bar{\mathfrak{d}}_{\bar{t}_2}^{2} u_{6\sigma} a_{6\lambda})^{i}.$$

Proof. When $i = 0$, note that by the Vanishing Theorem (Theorem 7.20), the class $tr(\bar{t}_2^{31} u_{58\sigma_2} a_{35\sigma_2})$ $(151, 35)$ must die on or before the E_{61}-page. There are two possibilities. Either the claimed differential occurs, or it supports a d_{55}-differential hitting $2\bar{\mathfrak{d}}_{\bar{t}_2}^{20} u_{15\lambda} u_{60\sigma} a_{45\lambda}$ $(150, 90)$. The second case does not occur because the class $2\bar{\mathfrak{d}}_{\bar{t}_2}^{20} u_{15\lambda} u_{60\sigma} a_{45\lambda}$ $(150, 90)$ needs to support a d_{61}-differential killing the class $\bar{\mathfrak{d}}_{\bar{t}_2}^{25} u_{74\sigma} a_{75\lambda} a_\sigma$ $(149, 151)$, or else no class would be able to kill $\bar{\mathfrak{d}}_{\bar{t}_2}^{25} u_{74\sigma} a_{75\lambda} a_\sigma$

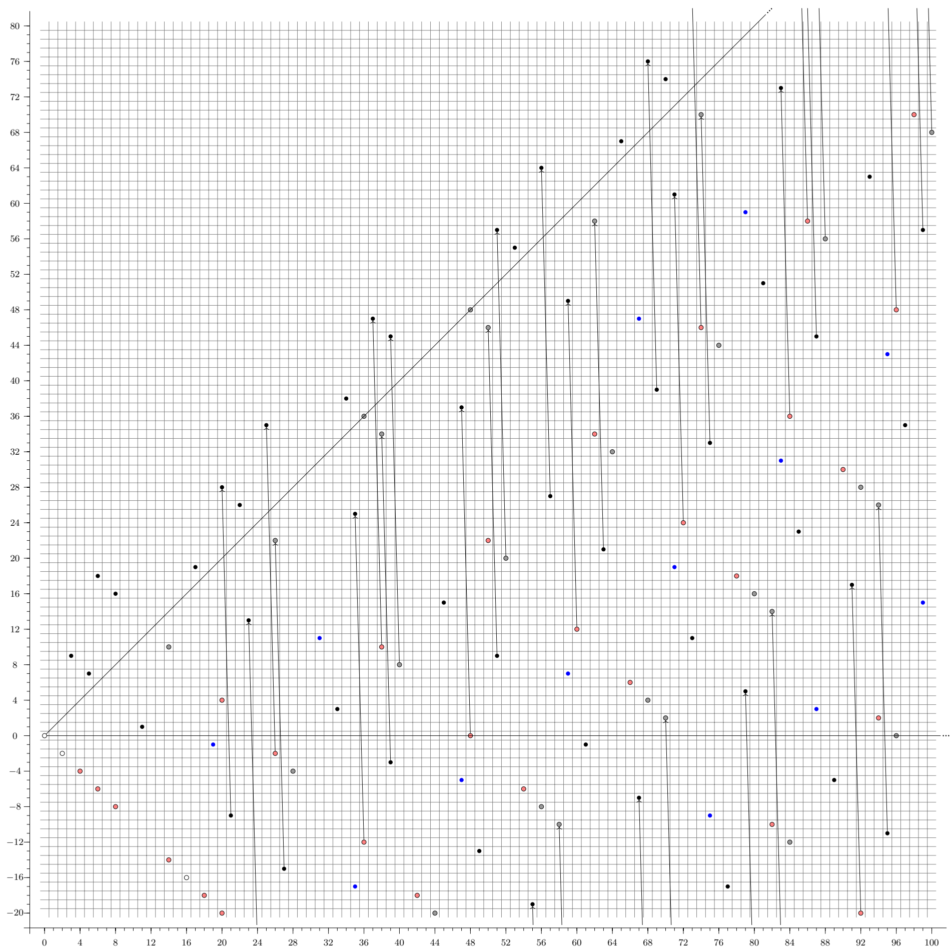

FIGURE 13. The E_{37}-page of the a_λ-localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}\langle 2,2\rangle$.

$(149, 151)$ on or before the E_{61}-page and we would reach a contradiction with Theorem 7.20.

The second differential is proven by the same method. $\square$

Proposition 7.28. *The d_{43}-differentials are*

(1) $d_{43}\left(b^{16+32i}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}\right) = tr(\bar{t}_2^7 u_{10\sigma_2}a_{11\sigma_2})\cdot b^{32i}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}$, $i,k\geqslant 0,\ j=0,3$;

(2) $d_{43}\left(b^{24+32i}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{8\lambda}u_{6\sigma}a_{-2\lambda})^{\ell}\right)$
$= tr(\bar{t}_2^7 u_{26\sigma_2}a_{-5\sigma_2})\cdot b^{32i}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{8\lambda}u_{6\sigma}a_{-2\lambda})^{\ell}$, $i,k\geqslant 0,\ j=0,1,\ \ell=0,1,2.$

They are shown in Figure 14.

Proof. All the differentials can be proven immediately from the Vanishing Theorem and degree reasons (Theorem 7.20), Proposition 7.27, and propagation with respect to the classes $\bar{\mathfrak{d}}_{\bar{t}_2}^8 u_{24\sigma}a_{24\lambda}$ $(48, 48)$ and b^{32} $(64, -64)$. □

7.4.7. d_{55}*-differentials.*

Proposition 7.29. *The d_{55}-differentials are*

(1) $d_{55}\left(tr(\bar{t}_2^3 u_{26\sigma_2}a_{-17\sigma_2})\cdot b^{32i}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}\right) = \bar{\mathfrak{d}}_{\bar{t}_2}^6 u_{16\sigma}a_{18\lambda}a_{2\sigma}\cdot b^{32i}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}$, $i, k \geqslant 0$, $j = 0, 3$;

(2) $d_{55}\left(tr(\bar{t}_2^3 u_{26\sigma_2}a_{-17\sigma_2})\cdot b^{8+32i}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{8\lambda}u_{6\sigma}a_{-2\lambda})^{\ell}\right)$ $= \bar{\mathfrak{d}}_{\bar{t}_2}^6 u_{16\sigma}a_{18\lambda}a_{2\sigma}\cdot b^{8+32i}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{8\lambda}u_{6\sigma}a_{-2\lambda})^{\ell}$, $i, k \geqslant 0$, $j = 0, 1$, $\ell = 0, 1, 2$.

They are shown in Figure 14.

Proof. All the differentials can be deduced from the Vanishing Theorem and degree reasons (Theorem 7.20), and propagation with respect to the classes $\bar{\mathfrak{d}}_{\bar{t}_2}^8 u_{24\sigma}a_{24\lambda}$ $(48, 48)$ and b^{32} $(64, -64)$. □

7.4.8. d_{61}*-differentials.*

Proposition 7.30. *We have the following d_{61}-differentials:*

(1) $d_{61}\left(\bar{\mathfrak{d}}_{\bar{t}_2} u_{16\lambda}u_{2\sigma}a_{-13\lambda}a_{\sigma}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{8\lambda}u_{6\sigma}a_{-2\lambda})^i(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}b^{32\ell}\right)$ $= \bar{\mathfrak{d}}_{\bar{t}_2}^6 u_{18\sigma}a_{18\lambda}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{8\lambda}u_{6\sigma}a_{-2\lambda})^i(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}b^{32\ell}$, $(i, j) = (0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1), (3, -3), (3, 0)$, $k, \ell \geqslant 0$;

(2) $d_{61}\left(2\bar{\mathfrak{d}}_{\bar{t}_2}^4 u_{15\lambda}u_{12\sigma}a_{-3\lambda}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{8\lambda}u_{6\sigma}a_{-2\lambda})^i(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}b^{32\ell}\right)$ $= \bar{\mathfrak{d}}_{\bar{t}_2}^9 u_{26\sigma}a_{27\lambda}a_{\sigma}\cdot(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{8\lambda}u_{6\sigma}a_{-2\lambda})^i(\bar{\mathfrak{d}}_{\bar{t}_2}^2 u_{6\sigma}a_{6\lambda})^{j+4k}b^{32\ell}$, $(i, j) = (0, 0), (0, 1), (1, -3), (1, 0), (2, -4), (2, -3), (3, -4), (3, -3)$, $k, \ell \geqslant 0$.

They are shown in Figure 14.

Proof. All the differentials can be deduced from the Vanishing Theorem and degree reasons (Theorem 7.20), and propagation with respect to the classes $\bar{\mathfrak{d}}_{\bar{t}_2}^8 u_{24\sigma}a_{24\lambda}$ $(48, 48)$ and b^{32} $(64, -64)$. □

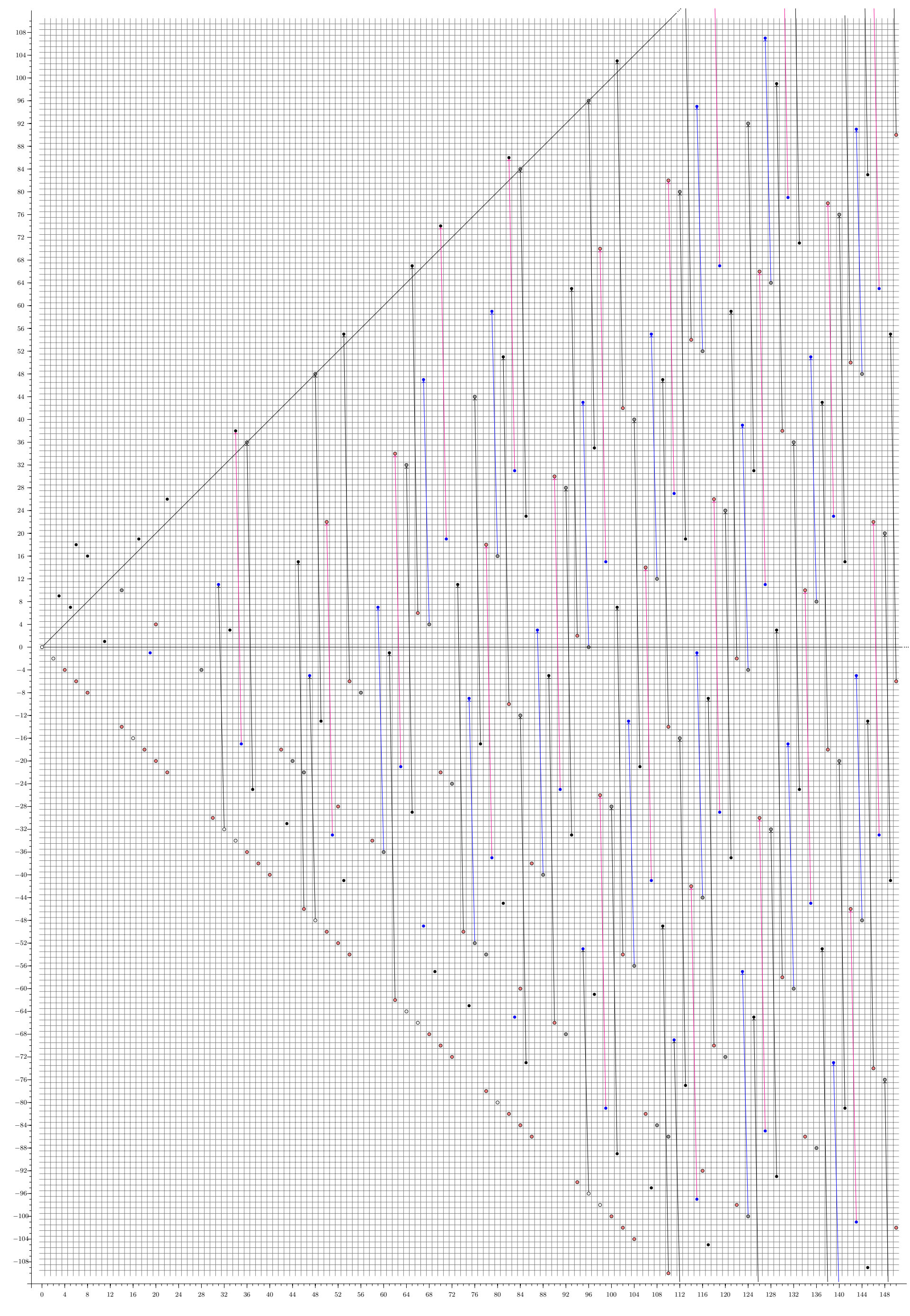

FIGURE 14. The d_{43} (blue), d_{55} (magenta), and d_{61}-differentials (black) in the a_λ-localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}\langle 2,2\rangle$.

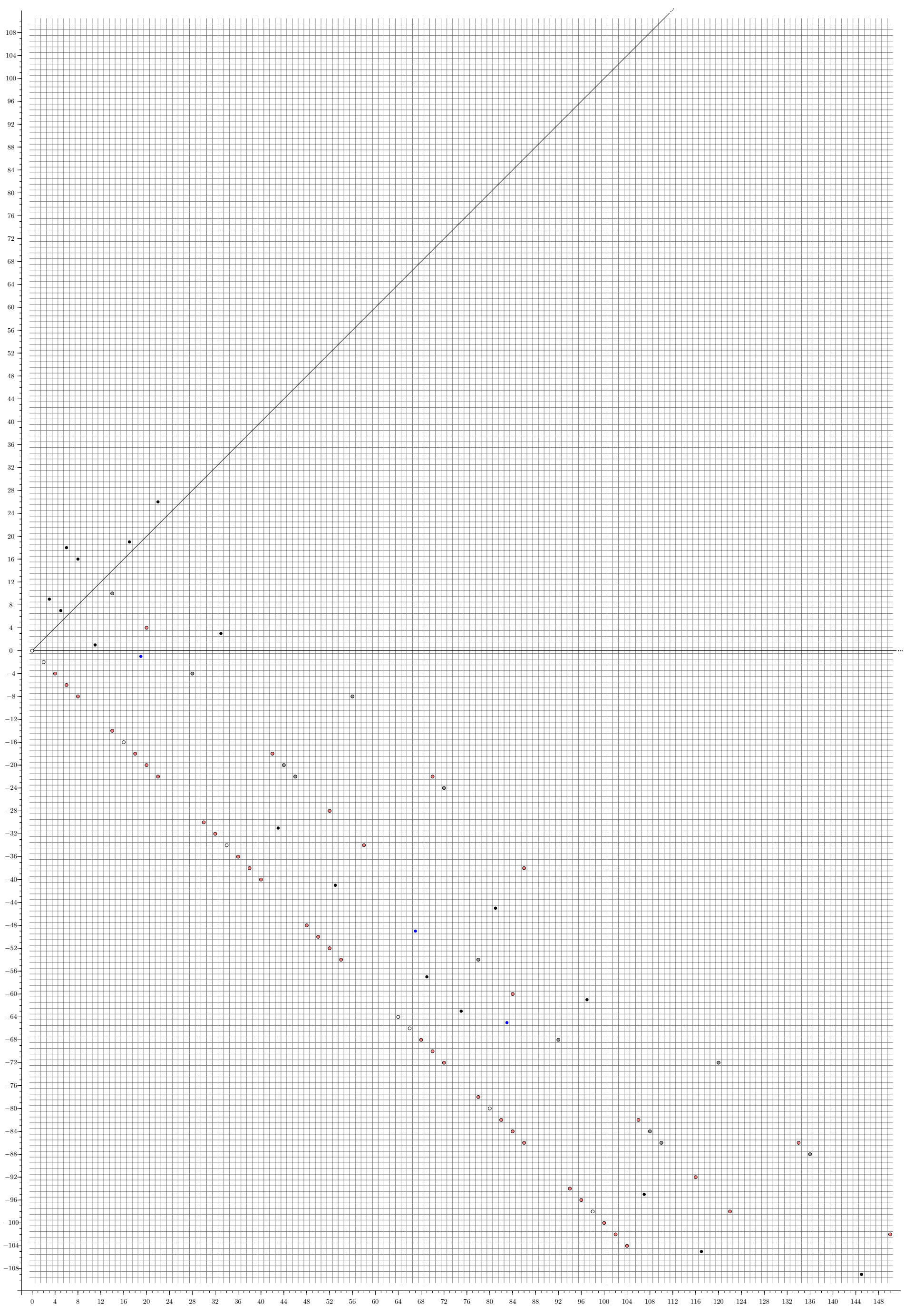

FIGURE 15. The E_∞-page of the a_λ-localized slice spectral sequence of $a_\lambda^{-1}BP^{((C_4))}\langle 2,2\rangle$.

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University of Colorado Boulder, Boulder, CO 80309
Email address: agnes.beaudry@colorado.edu

University of California Los Angeles, Los Angeles, CA 90095
Email address: mikehill@math.ucla.edu

University of Minnesota, Minneapolis, MN 55455
Email address: tlawson@umn.edu

University of Chicago, Chicago, IL 60637
Email address: dannyshixl@gmail.com

Max Planck Institute for Mathematics, 53111 Bonn, Germany
Email address: mingcongzeng@gmail.com