K-THEORY AND POLYNOMIAL FUNCTORS

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Abstract. We show that the algebraic K-theory space of stable ∞-categories is canonically functorial in polynomial functors.

1. Introduction

The purpose of this note is to provide an additional structure on the higher algebraic K-theory of stable ∞-categories, arising from polynomial rather than exact functors.

In the case of the Grothendieck group K_0, the construction is due to Dold [Dol72] and Joukhovitski [Jou00]. Let $\mathcal{A}$ be an additive category. The group $K_0(\mathcal{A})$ is defined to be the group completion of the additive monoid of isomorphism classes of objects of $\mathcal{A}$. By construction, an additive functor $F : \mathcal{A} \to \mathcal{B}$ induces a map of abelian groups $K_0(\mathcal{A}) \to K_0(\mathcal{B})$.

The results of *loc. cit.* provide additional functoriality on the construction K_0, and show that if $F : \mathcal{A} \to \mathcal{B}$ is merely a *polynomial* functor in the sense of [EML54], then F nevertheless induces a canonical map of *sets* $F_* : K_0(\mathcal{A}) \to K_0(\mathcal{B})$, such that F_* carries the class of an object $x \in \mathcal{A}$ to the class of $F(x) \in \mathcal{B}$. This polynomial functoriality yields, for example, the λ-operations on $K_0(R)$ for a commutative ring R, which arise from the exterior power operations on R-modules: the ith exterior power functor $\bigwedge^i$ induces a polynomial endofunctor on finitely generated projective R-modules, and hence a map of sets $\lambda^i : K_0(R) \to K_0(R)$. Here we will extend this polynomial functoriality to higher algebraic K-theory. To do this, it is convenient to use the setup of the K-theory of stable ∞-categories.

Let $\mathcal{C}$ be a stable ∞-category. As in [BGT13, Bar16], one constructs an algebraic K-theory space $K(\mathcal{C})$ via the Waldhausen $S_\bullet$-construction applied to $\mathcal{C}$; an exact functor $\mathcal{C} \to \mathcal{D}$ of stable ∞-categories induces a map of spaces $K(\mathcal{C}) \to K(\mathcal{D})$. For example, when $\mathcal{C} = \mathrm{Perf}(X)$ is the stable ∞-category of perfect complexes over a quasi-compact and quasi-separated scheme X, this is the K-theory space of X (introduced in [TT90], or [Qui72] if X is affine). Moreover, one characterizes [BGT13, Bar16] the construction $\mathcal{C} \mapsto K(\mathcal{C})$, when considered as an invariant of all stable ∞-categories and exact functors between them, via a universal property.

In this paper, we provide additional structure on the construction of algebraic K-theory in analogy with the results on K_0 from [Dol72, Jou00], and characterize it by the same universal property.

To formulate the result, let $\mathrm{Cat}_\infty^{\mathrm{perf}}$ denote the ∞-category of small, idempotent-complete stable ∞-categories and exact functors between them, and let $\mathcal{S}$ be the

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∞-category of spaces. Algebraic K-theory defines a functor

$$K : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}.$$

It receives a natural transformation from the functor $\iota : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}$ which carries $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ to the space of objects in $\mathcal{C}$, i.e., we have a map $\iota \to K$ of functors $\mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}$. The universal property of K-theory [BGT13, Bar16] states that K is the initial functor $\mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}$ receiving a map from ι such that K preserves finite products, splits semiorthogonal decompositions, and is grouplike.

Let $\mathcal{C}, \mathcal{D}$ be small, stable idempotent-complete ∞-categories. A functor $f : \mathcal{C} \to \mathcal{D}$ is said to be *polynomial* if it is n-excisive for some n in the sense of [Goo92]. Let $\mathrm{Cat}_\infty^{\mathrm{poly}}$ denote the ∞-category of small, idempotent-complete stable ∞-categories and polynomial functors between them. Thus, we have an inclusion $\mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Cat}_\infty^{\mathrm{poly}}$; note that $\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathrm{Cat}_\infty^{\mathrm{poly}}$ have the same objects, but $\mathrm{Cat}_\infty^{\mathrm{poly}}$ has many more morphisms. Our main result states that K-theory can be defined on $\mathrm{Cat}_\infty^{\mathrm{poly}}$.

Theorem 1.1. *There is a canonical extension of the functor $K : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}$ to a functor $\widetilde{K} : \mathrm{Cat}_\infty^{\mathrm{poly}} \to \mathcal{S}$.*

In fact, the construction $\widetilde{K}$ is characterized by a similar universal property. Namely, one has a canonical extension of the functor $\iota : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}$ to a functor $\iota : \mathrm{Cat}_\infty^{\mathrm{poly}} \to \mathcal{S}$ since ι can be defined on all ∞-categories and functors between them. One defines the functor $\widetilde{K}$ (with a natural map $\iota \to \widetilde{K}$) by enforcing the same universal property on $\mathrm{Cat}_\infty^{\mathrm{poly}}$. The main computation one then carries out is that $\widetilde{K}$ restricts to K on $\mathrm{Cat}_\infty^{\mathrm{perf}}$, i.e., one recovers the original K-theory functor.

Remark 1.2. Theorem 1.1, together with the theory of the Bousfield–Kuhn functor [Kuh89, Bou01], implies that for $n \geq 1$, the (telescopic) $T(n)$-localization of the algebraic K-theory *spectrum* of a stable ∞-category is functorial in polynomial functors, i.e., extends to $\mathrm{Cat}_\infty^{\mathrm{poly}}$.

Motivation and related work. Many previous authors have considered various types of non-additive operations on algebraic K-theory spaces, which provided substantial motivation for this work.

An important example is given by operations in the K-theory space (and on the K-groups) of a ring R arising from exterior and symmetric power functors on R-modules. Constructions of such maps appear in many sources, including [Hil81, Kra80, Sou85, Gra89, Nen91, Lev97, HKT17]. We expect, but have not checked, that these maps agree with the maps provided by Theorem 1.1 using the derived exterior and symmetric power functors on $\mathrm{Perf}(R)$. Another example in this vein is given by the multiplicative norm maps along finite étale maps constructed in [BH17].

A different instance of non-additive operations in K-theory arises in Segal's approach to the Kahn–Priddy theorem [Seg74]. These maps arise from the K-theory of non-additive categories (such as the category of finite sets) and cannot be obtained from Theorem 1.1.

Notation and conventions. We freely use the language of ∞-categories and higher algebra as in [Lur09, Lur14]. Throughout, we let $\mathcal{S}$ denote the ∞-category of spaces, and Sp the ∞-category of spectra.

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2. Polynomial functors

2.1. **Simplicial and filtered objects.** In this subsection, we review basic facts about simplicial objects in a stable ∞-category. In particular, we review the stable version of the Dold-Kan correspondence, following Lurie [Lur14], which connects simplicial and filtered objects.

To begin with, we review the classical Dold-Kan correspondence. A general reference for this is [GJ99, III.2] for the category of abelian groups or [Wei94, 8.4] for an abelian category. We refer to [Lur14, 1.2.3] for a treatment for arbitrary additive categories.

Theorem 2.1 (Dold-Kan correspondence)**.** *Let $\mathcal{A}$ be an additive category which is idempotent-complete. Then we have an equivalence of categories*

$$\mathrm{Fun}(\Delta^{op}, \mathcal{A}) \simeq \mathrm{Ch}_{\geq 0}(\mathcal{A}),$$

between the category $\mathrm{Fun}(\Delta^{op}, \mathcal{A})$ *of simplicial objects in $\mathcal{A}$ and the category* $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ *of nonnegatively graded chain complexes in $\mathcal{A}$.*

The Dold-Kan equivalence arises as follows. Given a simplicial object $X_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathcal{A})$, we form an associated chain complex C_* such that:

(1) C_n is a direct summand of X_n, and is given by the intersection of the kernels $\bigcap_{i\geq 1} \ker(d_i)$ where the d_i's give the face maps $X_n \to X_{n-1}$. If $\mathcal{A}$ is only assumed additive, the existence of this kernel is not a priori obvious (cf. [Lur14, Rmk. 1.2.3.15]). However, we emphasize that the object C_n depends only on the face maps $d_i, i \geq 1$.
(2) The differential $C_n \to C_{n-1}$ comes from the face map d_0 in the simplicial structure.

The Dold-Kan correspondence has an analog for stable ∞-categories, formulated in [Lur14, Sec. 1.2.4], yielding a correspondence between simplicial and *filtered* objects.

Theorem 2.2 (Lurie [Lur14, Th. 1.2.4.1])**.** *Let $\mathcal{C}$ be a stable ∞-category. Then we have an equivalence of stable ∞-categories*

$$\mathrm{Fun}(\Delta^{op}, \mathcal{C}) \simeq \mathrm{Fun}(N\mathbb{Z}_{\geq 0}, \mathcal{C}),$$

which sends a simplicial object $X_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ to the filtered object $|\mathrm{sk}_0 X_\bullet| \to |\mathrm{sk}_1 X_\bullet| \to |\mathrm{sk}_2 X_\bullet| \to \dots$.

Remark 2.3 (Making Dold-Kan explicit)**.** We will need to unwind the correspondence as follows. Given a filtered object $Y_0 \to Y_1 \to Y_2 \to \dots$ in $\mathcal{C}$, we form the sequence of cofibers $Y_0, Y_1/Y_0, Y_2/Y_1, \dots$, i.e., the associated graded. We have boundary maps

$$Y_1/Y_0 \to \Sigma Y_0, \quad Y_2/Y_1 \to \Sigma Y_1/Y_0, \dots,$$

and the sequence

$$\cdots \to \Sigma^{-2} Y_2/Y_1 \to \Sigma^{-1} Y_1/Y_0 \to Y_0 \tag{1}$$

forms a chain complex in the homotopy category $\mathrm{Ho}(\mathcal{C})$: the composite of any two successive maps is nullhomotopic. If $\mathcal{C}$ is an idempotent-complete stable ∞-category, then $\mathrm{Ho}(\mathcal{C})$ is an idempotent-complete additive category. Given a simplicial object $X_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathcal{C})$, we can also extract a simplicial object in $\mathrm{Ho}(\mathcal{C})$ and thus a chain complex in $\mathrm{Ho}(\mathcal{C})$ by the classical Dold-Kan correspondence. A basic compatibility states that this produces the sequence (1), i.e., the additive and stable Dold-Kan correspondences are compatible [Lur14, Rem. 1.2.4.3].

2.2. **Polynomial functors of stable ∞-categories.** In this subsection, we review the notion of polynomial functor between stable ∞-categories. We first discuss the analogous notion for additive ∞-categories.

Definition 2.4 (Eilenberg-MacLane [EML54])**.** Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive ∞-categories and assume first that $\mathcal{B}$ is idempotent complete. Then:

- F is called *polynomial* of degree ≤ -1 if it is the trivial functor which sends everything to 0.
- F is called *polynomial* of degree ≤ 0 if it is a constant functor.
- Inductively, F is called polynomial of degree $\leq n$ if for each $Y \in \mathcal{A}$ the functor
$$(D_Y F)(X) := \ker\Big(F(Y \oplus X) \xrightarrow{F(\mathrm{pr}_X)} F(X)\Big)$$
is polynomial of degree at most $n-1$. Note that this kernel exists as we have assumed $\mathcal{B}$ to be idempotent complete and it is the complementary summand to $F(X)$.

We let $\mathrm{Fun}_{\leq n}(\mathcal{A}, \mathcal{B}) \subset \mathrm{Fun}(\mathcal{A}, \mathcal{B})$ be the subcategory spanned by functors of degree $\leq n$.[1] Finally, a functor $F : \mathcal{A} \to \mathcal{B}$ is *polynomial* if it is polynomial of degree $\leq n$ for some n.

The notion of a polynomial functor behaves in a very intuitive fashion. For example, the composite of a functor of degree $\leq m$ with one of degree $\leq n$ is of degree $\leq mn$. As another example, we have:

Example 2.5. Let R be a commutative ring. The symmetric power functors Sym^i and exterior power functors $\bigwedge^i$ on the category Proj_R^ω of finitely generated projective R-modules are polynomial of degree $\leq i$.

Let $\mathcal{C}$ be a stable ∞-category.

Definition 2.6 (n-skeletal simplicial objects)**.** We say that a simplicial object $X_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ is *n-skeletal* if it is left Kan extended from its restriction to $\Delta_{\leq n} \subset \Delta$.

[1] This notion easily generalizes when $\mathcal{B}$ is not idempotent complete. If $\mathcal{B}$ is not idempotent complete then $F : \mathcal{A} \to \mathcal{B}$ is called polynomial of degree $\leq n$ if the composition $\mathcal{A} \to \mathcal{B} \to \overline{\mathcal{B}}$ is polynomial of degree $\leq n$, where $\mathcal{B} \to \overline{\mathcal{B}}$ is an idempotent completion.

Remark 2.7 (n-skeletal geometric realizations exist)**.** Note that if $X_\bullet$ is n-skeletal for some n, then the geometric realization $|X_\bullet|$ exists in $\mathcal{C}$.

Remark 2.8 (n-skeletal objects via Dold-Kan)**.** The condition that $X_\bullet$ should be n-skeletal depends only on the underlying homotopy category of $\mathcal{C}$, considered as an additive category. Namely, using the Dold-Kan correspondence, we can form a chain complex C_* in the homotopy category of $\mathcal{C}$ from $X_\bullet$, and then we claim that $X_\bullet$ is n-skeletal if and only if C_* vanishes for $* > n$.

Indeed, if $X_\bullet$ is n-skeletal, then clearly the maps $|\mathrm{sk}_i X_\bullet| \to |\mathrm{sk}_{i+1} X_\bullet|$ are equivalences for $i \geq n$, which implies that $C_* = 0$ for $* > n$. Conversely, if $C_* = 0$ for $* > n$, then the map of simplicial objects $\mathrm{sk}_n X_\bullet \to X_\bullet$ has the property that it induces an equivalence on i-truncated geometric realizations for all $i \in \mathbb{Z}_{\geq 0}$, which implies by Theorem 2.2 that it is an equivalence of simplicial objects.

Definition 2.9. We say that a functor $f : \mathcal{C} \to \mathcal{D}$ of stable ∞-categories preserves *finite geometric realizations* if for every simplicial object $X_\bullet$ in $\mathcal{C}$ which is n-skeletal for some n, the colimit $|F(X_\bullet)|$ exists and the natural map $|F(X_\bullet)| \to F(|X_\bullet|)$ is an equivalence in $\mathcal{C}$.

Proposition 2.10. *Let $\mathcal{C}, \mathcal{D}$ be stable ∞-categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that the underlying functor on additive categories $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ is polynomial of degree $\leq d$. Then if $X_\bullet$ is an n-skeletal simplicial object in $\mathcal{C}$, $F(X_\bullet)$ is an nd-skeletal simplicial object in $\mathcal{D}$.*

Proof. As above in Remark 2.8, this is a statement purely at the level of homotopy categories. That is, the simplicial object $X_\bullet$ defines a simplicial object of the additive category $\mathrm{Ho}(\mathcal{C})$, and thus a chain complex C_* in $\mathrm{Ho}(\mathcal{C})$. Similarly $F(X_\bullet)$ defines a simplicial object of $\mathrm{Ho}(\mathcal{D})$ and thus a chain complex D_* of $\mathrm{Ho}(\mathcal{D})$. The claim is that if $C_* = 0$ for $* > n$ then $D_* = 0$ for $* > nd$. This is purely a statement about additive categories. For proofs, see [GS87, Lem. 3.3] or [DP61, 4.23]. □

Definition 2.11 (Polynomial functors)**.** Let $\mathcal{C}, \mathcal{D}$ be stable ∞-categories. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *polynomial of degree* $\leq d$ if the underlying functor $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ of additive categories is polynomial of degree $\leq d$ and if F preserves finite geometric realizations. We let $\mathrm{Fun}_{\leq d}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the full subcategory spanned by functors of degree $\leq d$.

This notion of a polynomial functor will be fundamental to this paper. The definition is equivalent to the more classical definition of a polynomial functor, due to Goodwillie [Goo92], via n-excisivity. Of course, the primary applications of the theory treat functors where either the domain or codomain is not stable. For convenience, we describe the comparison below.

Definition 2.12. Let $[n] = \{0, 1, \ldots, n\}$, and let $\mathcal{P}([n])$ denote the nerve of the poset of subsets of $[n]$, so that $\mathcal{P}([n]) \simeq (\Delta^1)^{n+1}$. An *$n$-cube* in $\mathcal{C}$ is a functor $f : \mathcal{P}([n]) \to \mathcal{C}$. The n-cube is said to be *strongly coCartesian* if it is left Kan extended from the subset $\mathcal{P}_{\leq 1}([n]) \subset \mathcal{P}([n])$ spanned by subsets of cardinality ≤ 1, and *coCartesian* if it is a colimit diagram. Note that a diagram is coCartesian if and only if it is a limit or *Cartesian* diagram by [Lur14, Prop. 1.2.4.13].

Definition 2.13 (Goodwillie [Goo92, Def. 3.1])**.** Let $\mathcal{C}, \mathcal{D}$ be small stable ∞-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. For $n \geq 0$, we say that F is *n-excisive* if F carries strongly coCartesian n-cubes to coCartesian n-cubes.

Example 2.14. For $n = 0$, the condition is that F should be a constant functor. For $n = -1$, we say that F is n-excisive if F is identically zero. For $n = 1$, F is 1-excisive (or simply excisive) if it carries pushouts to pullbacks.

Proposition 2.15. *The functor $F : \mathcal{C} \to \mathcal{D}$ is polynomial of degree $\leq n$ (in the sense of Definition 2.11) if and only if it is n-excisive.*

Proof. Suppose first that F is polynomial of degree $\leq n$, and fix a strongly coCartesian $(n+1)$-cube in $\mathcal{C}$, which is determined by a collection of maps $X \to Y_i$, $i = 0, 1, \ldots, n$, in $\mathcal{C}$. We would like to show that F carries this cube to a coCartesian one. Since every object in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ is a finite geometric realization of arrows of the form $A \to A \oplus B$, and F preserves finite geometric realizations, we may assume that we have $Y_i \simeq X \oplus Z_i$ for objects $Z_i, i = 0, 1, \ldots, n$.

Denote the above strongly coCartesian cube by $f : \mathcal{P}([n+1]) \to \mathcal{C}$. The cofiber $\varinjlim_{\mathcal{P}'([n+1])} F \circ f \to F(f([n]))$ is given precisely by the iterated derivative $D_{Z_0} D_{Z_1} \ldots D_{Z_n} F(X)$, as follows easily by induction on n. Thus, if F is polynomial of degree $\leq n$, we see that F is n-excisive.

Conversely, if F is n-excisive, the previous paragraph shows that $\mathrm{Ho}(F)$ is polynomial of degree $\leq n$. It remains to show that F preserves finite geometric realizations. This follows from the general theory and classification of n-excisive functors, cf. also [BM19, Prop. 3.36] for an account. We can form the embedding $\mathcal{D} \hookrightarrow \mathrm{Ind}(\mathcal{D})$ to replace $\mathcal{D}$ by a presentable stable ∞-category; this inclusion preserves finite geometric realizations. In this case, the general theory shows that F can be built up via a finite filtration from its *homogeneous layers*, and each homogeneous layer of degree i is of the form $X \mapsto B(X, X, \ldots, X)_{h\Sigma_i}$ for $B : \mathcal{C}^i \to \mathrm{Ind}(\mathcal{D})$ a functor which is exact in each variable and symmetric in its variables (see [Lur14, Sec. 6.1] for a reference in this setting). Therefore, it suffices to show that if $B : \mathcal{C}^i \to \mathrm{Ind}(\mathcal{D})$ is a functor which is exact in each variable, then $X \mapsto B(X, X, \ldots, X)$ preserves finite geometric realizations. This now follows from the cofinality of the diagonal $\Delta^{op} \to (\Delta^{op})^i$ and the fact that B preserves finite geometric realizations in each variable separately, since it is exact. □

2.3. **Construction of polynomial functors.** We now discuss some examples of polynomial functors on stable ∞-categories; these will arise from the derived functors of polynomial functors of additive ∞-categories. We first review the relationship between additive, prestable, and stable ∞-categories. Compare [Lur, Sec. C.1.5].

Construction 2.16 (The stable envelope)**.** Given any small additive ∞-category $\mathcal{A}$, there is a universal stable ∞-category $\mathrm{Stab}(\mathcal{A})$ equipped with an additive, fully faithful functor $\mathcal{A} \to \mathrm{Stab}(\mathcal{A})$. Given any small stable ∞-category $\mathcal{B}$, any additive functor $\mathcal{A} \to \mathcal{B}$ canonically extends to an exact functor $\mathrm{Stab}(\mathcal{A}) \to \mathcal{B}$. In other words, Stab is a left adjoint from the natural forgetful functor from the ∞-category of small stable ∞-categories to the ∞-category of small additive ∞-categories. We refer to $\mathrm{Stab}(\mathcal{A})$ as the *stable envelope* of $\mathcal{A}$.

Explicitly, $\mathrm{Stab}(\mathcal{A})$ is the stable subcategory of the ∞-category $\mathrm{Fun}^{\times}(\mathcal{A}^{op}, \mathrm{Sp})$ of finitely product-preserving presheaves of spectra on $\mathcal{A}$ generated by the image of the Yoneda embedding.

Construction 2.17 (The prestable envelope $\mathrm{Stab}(\mathcal{A})_{\geq 0}$)**.** Let $\mathcal{A}$ be a small additive ∞-category as above, and let $\mathrm{Stab}(\mathcal{A})$ be its stable envelope. We let $\mathrm{Stab}(\mathcal{A})_{\geq 0}$

denote the subcategory of the nonabelian derived ∞-category [Lur09, Sec. 5.5.8] $\mathcal{P}_\Sigma(\mathcal{A})$ generated under finite colimits by $\mathcal{A}$. Then $\mathrm{Stab}(\mathcal{A})_{\geq 0}$ is a prestable ∞-category [Lur, Appendix C] and is the universal prestable ∞-category receiving an additive functor $\mathcal{A} \to \mathcal{P}_\Sigma(\mathcal{A})$, which we call the *prestable envelope* of $\mathcal{A}$. There are fully faithful embeddings

$$\mathcal{A} \subset \mathrm{Stab}(\mathcal{A})_{\geq 0} \subset \mathrm{Stab}(\mathcal{A}),$$

and $\mathrm{Stab}(\mathcal{A})$ is also the stabilization of the prestable ∞-category $\mathrm{Stab}(\mathcal{A})_{\geq 0}$.

Example 2.18. Let R be a ring, or more generally a connective $\mathbb{E}_1$-ring spectrum. Then one has a natural additive ∞-category Proj_R^ω of finitely generated, projective R-modules. The stable envelope is given by the ∞-category $\mathrm{Perf}(R)$ of perfect R-modules, and the prestable envelope is given by the ∞-category $\mathrm{Perf}(R)_{\geq 0}$ of connective perfect R-modules.

Our main result is the following extension principle, which allows one to extend polynomial functors from an additive ∞-category to the stable envelope.

Theorem 2.19 (Extension of polynomial functors)**.** *Let $\mathcal{D}$ be a stable, idempotent complete ∞-category and $\mathcal{A}$ be an additive ∞-category. Pullback along the functor $\mathcal{A} \to \mathrm{Stab}(\mathcal{A})$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}_{\leq n}(\mathcal{A}, \mathcal{D}) \simeq \mathrm{Fun}_{\leq n}(\mathrm{Stab}(\mathcal{A}), \mathcal{D}),$$

between degree $\leq n$ functors $\mathcal{A} \to \mathcal{D}$ (in the sense of additive ∞-categories) and degree $\leq n$ functors (in the sense of stable ∞-categories) $\mathrm{Stab}(\mathcal{A}) \to \mathcal{D}$.

We use the following crucial observation due to Brantner.

Lemma 2.20 (Extending from the connective objects, cf. [BM19, Th. 3.35])**.** *Let $\mathcal{D}$ be a presentable, stable ∞-category. Restriction induces an equivalence of ∞-categories*

$$\mathrm{Fun}_{\leq n}(\mathrm{Stab}(\mathcal{A})_{\geq 0}, \mathcal{D}) \simeq \mathrm{Fun}_{\leq n}(\mathrm{Stab}(\mathcal{A}), \mathcal{D}),$$

between n-excisive functors $\mathrm{Stab}(\mathcal{A})_{\geq 0} \to \mathcal{D}$ and n-excisive functors $\mathrm{Stab}(\mathcal{A}) \to \mathcal{D}$.

Lemma 2.21. *Let $\mathcal{C}, \mathcal{D}$ be stable ∞-categories. Let $\mathcal{A} \subset \mathcal{C}$ be an additive subcategory which generates $\mathcal{C}$ as a stable subcategory. Let $F : \mathcal{C} \to \mathcal{D}$ be a degree $\leq n$ functor. Suppose $F|_{\mathcal{A}}$ has image contained in a stable subcategory $\mathcal{D}' \subset \mathcal{D}$. Then F has image contained in $\mathcal{D}'$.*

Proof. We use essentially the notions of *levels* in $\mathcal{C}$, cf. [ABIM10, Sec. 2], although we are not assuming idempotent completeness.

We define an increasing and exhaustive filtration of subcategories $\mathcal{C}_{\leq 1} \subset \mathcal{C}_{\leq 2} \subset \cdots \subset \mathcal{C}$ as follows. The subcategory $\mathcal{C}_{\leq 1}$ is the additive closure of $\mathcal{A}$ under shifts, so any object of $\mathcal{C}_{\leq 1}$ can be written in the form $\Sigma^{i_1} A_1 \oplus \cdots \oplus \Sigma^{i_n} A_n$ for some $A_1, \ldots, A_n \in \mathcal{A}$ and $i_1, \ldots, i_n \in \mathbb{Z}$. Inductively, we let $\mathcal{C}_{\leq n}$ denote the subcategory of objects that are extensions of objects in $\mathcal{C}_{\leq a}$ and $\mathcal{C}_{\leq b}$ for $0 < a, b < n$ with $a + b \leq n$. Each $\mathcal{C}_{\leq n}$ is closed under translates, and it is easy to see that $\bigcup_n \mathcal{C}_{\leq n}$ is stable and contains $\mathcal{A}$ and hence equals $\mathcal{C}$.

We claim first that $F(\mathcal{C}_{\leq 1}) \subset \mathcal{D}$. That is, we need to show that for $A_1, \ldots, A_n \in \mathcal{A}$ and $i_1, \ldots, i_n \in \mathbb{Z}$, we have $F(\Sigma^{i_1} A_1 \oplus \cdots \oplus \Sigma^{i_n} A_n) \in \mathcal{D}$. If $i_1, \ldots, i_n \geq 0$, then we can choose a simplicial object in $\mathcal{A}$ which is d-truncated for some d and whose geometric realization is $\Sigma^{i_1} A_1 \oplus \cdots \oplus \Sigma^{i_n} A_n$. Since F preserves finite geometric realizations, the claim follows. In general, for any object $X \in \mathcal{C}$, we can recover

$F(X)$ as a finite homotopy limit of $F(0), F(\Sigma X), F(\Sigma X \oplus \Sigma X), \dots$ via the T_n-construction, since F is n-excisive. Using this, we can remove the hypotheses that $i_1, \dots, i_n \geq 0$ and conclude that $F(\mathcal{C}_{\leq 1}) \subset \mathcal{D}$.

Given an object of $\mathcal{C}_{\leq n}$ with $n > 1$, it is an extension of objects in $\mathcal{C}_{\leq a}$ and $\mathcal{C}_{\leq b}$ for some $a, b < n$. In view of Construction 3.8 below, it follows that any such object can be written as a finite geometric realization of objects of level $< n$. Since F preserves finite geometric realizations, it follows by induction that $F(\mathcal{C}_{\leq n}) \subset \mathcal{D}$. □

Proof of Theorem 2.19. Embedding $\mathcal{D}$ inside $\mathrm{Ind}(\mathcal{D})$, we may assume that $\mathcal{D}$ is actually presentable. By Lemma 2.21, we do not lose any generality by doing so.

By the universal property of $\mathcal{P}_\Sigma$, we have an equivalence

$$\mathrm{Fun}_{\leq n}(\mathcal{A}, \mathrm{Ind}(\mathcal{D})) \simeq \mathrm{Fun}^{\Sigma}_{\leq n}(\mathcal{P}_\Sigma(\mathcal{A}), \mathrm{Ind}(\mathcal{D})), \tag{2}$$

where Σ denotes functors which preserve sifted colimits; the universal property gives this without the $\leq n$ condition, which we can then impose. Now the inclusion $\mathrm{Stab}(\mathcal{A})_{\geq 0} \subset \mathcal{P}_\Sigma(\mathcal{A})$ exhibits the target as the Ind-completion of the source, which yields an equivalence

$$\mathrm{Fun}_{\leq n}(\mathrm{Stab}(\mathcal{A})_{\geq 0}, \mathrm{Ind}(\mathcal{D})) \simeq \mathrm{Fun}^{\omega}_{\leq n}(\mathcal{P}_\Sigma(\mathcal{A}), \mathrm{Ind}(\mathcal{D})), \tag{3}$$

where ω denotes functors which preserve filtered colimits. By definition, any functor in $\mathrm{Fun}^{\omega}_{\leq n}(\mathcal{P}_\Sigma(\mathcal{A}), \mathrm{Ind}(\mathcal{D})$ preserves finite geometric realizations, and hence all geometric realizations; thus it also preserves all sifted colimits. This shows that the categories in (2) and (3) are identified. Now the result follows by combining this identification with Lemma 2.20. □

Corollary 2.22. *Let $\mathcal{A} \to \mathcal{B}$ be additive ∞-categories. Then a degree $\leq n$ functor $\mathcal{A} \to \mathcal{B}$ canonically prolongs to a degree $\leq n$ functor of stable ∞-categories,* $\mathrm{Stab}(\mathcal{A}) \to \mathrm{Stab}(\mathcal{B})$. □

Example 2.23. Let R be a commutative ring. Then we have a functor $\mathrm{Sym}^i : \mathrm{Perf}(R) \to \mathrm{Perf}(R)$ which is i-excisive and which extends the usual symmetric powers of finitely generated projective R-modules. We can regard this as a derived functor of the usual symmetric power, although we are allowing nonconnective objects as well.

The above construction of extending functors, for $\mathrm{Stab}(\mathcal{A})_{\geq 0}$, is the classical one of Dold-Puppe [DP61] of "nonabelian derived functors" constructed using simplicial resolutions. Compare also [JM99] for the connection between polynomial functors on additive categories and n-excisive functors. The extension to $\mathrm{Stab}(\mathcal{A})$, at least in certain cases, goes back to Illusie [Ill71, Sec. I-4] in work on the cotangent complex, using simplicial cosimplicial objects.

3. K_0 and polynomial functors

3.1. **Additive ∞-categories.** In this section, we review the result of [Dol72, Jou00] that K_0 of additive ∞-categories is naturally functorial in polynomial functors; this special case of Theorem 1.1 will play an essential role in its proof.

Definition 3.1 (Passi [Pas74])**.** Let M be an abelian monoid and A be an abelian group. We will define inductively when a map $f : M \to A$ (of sets) is called polynomial of degree $\leq n$.

- A map f is called *polynomial of degree* ≤ -1 if it is identically zero.

- A map f if called *polynomial of degree* $\leq n$ if for each $y \in M$ the map $D_y f : M \to A$ defined by
$$(D_y f)(x) := f(x+y) - f(x)$$
is polynomial of degree $\leq n-1$.

We say that f is *polynomial* if it is polynomial of degree n for some n.

We denote the set of polynomial maps $M \to A$ of degree $\leq n$ by $\mathrm{Hom}_{\leq n}(M, A)$. It is straightforward to check that composing polynomial maps whenever this is defined is again polynomial and changes the degree in the obvious way.

Example 3.2. A map $f : \mathbb{Z} \to \mathbb{Z}$ is polynomial of degree $\leq n$ precisely if it can be represented by a polynomial of degree n with rational coefficients. In this case it has to be of the form
$$f(x) = \sum_{i=0}^{n} \alpha_i \binom{x}{i}$$
with $\alpha_i \in \mathbb{Z}$.

Now let $i : M \to M^+$ be the group completion of the abelian monoid M. Then the following result states that we can always extend polynomial maps uniquely over the group completion. This is surprising if one thinks about how to extend to a formal difference.

Theorem 3.3 ([Jou00, Prop. 1.6])**.** *For any abelian monoid M and abelian group A, the map*
$$i^* : \mathrm{Hom}_{\leq n}(M^+, A) \to \mathrm{Hom}_{\leq n}(M, A)$$
is a bijection.

The proof in *loc. cit.* gives an explicit argument. For the convenience of the reader, we include an abstract argument via monoid and group rings.

Proof. The first step is to reformulate the condition for a map $f : M \to A$ to be polynomial of degree $\leq n$. To do this we will temporarily for this proof write M multiplicatively, in particular $1 \in M$ is the neutral element.

A map of sets $f : M \to A$ is polynomial of degree at most n precisely if the induced map $\overline{f} : \mathbb{Z}[M] \to A$ defined by
$$\sum_{m \in M} \alpha_m \cdot m \mapsto \sum_{m \in M} \alpha_m f(m)$$
(with $\alpha_m \in \mathbb{Z}$) factors over the $(n+1)$'st power I^{n+1} of the augmentation ideal $I \subseteq \mathbb{Z}[M]$. In other words, there is a canonical bijection
$$\mathrm{Hom}_{\leq n}(M, A) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}[M]/I^{n+1}, A). \tag{4}$$
This fact is proven in [Pas74], but let us give an argument. The augmentation ideal I is generated additively by elements of the form $(m-1)$ with $m \in M$. Therefore I^{n+1} is generated additively by elements of the form
$$(m_0 - 1) \cdot \ldots \cdot (m_n - 1)$$
with $m_i \in M$. A slightly bigger additive generating set for I^{n+1} is then given by
$$x \cdot (m_0 - 1) \cdot \ldots \cdot (m_n - 1)$$

with $m_i, x \in M$. Using this fact we have to show that $f : M \to A$ is polynomial of degree at most n precisely if $\overline{f} : \mathbb{Z}[M] \to A$ vanishes on these products for all $x, m_i \in M$. This follows from the following pair of observations:

- A map $f : M \to A$ is polynomial of degree at most n precisely if for each sequence $m_0, ..., m_n, x$ of elements in M we have
$$(D_{m_0} D_{m_1} ... D_{m_n} f)(x) = 0.$$
- There is an equality
$$(D_{m_0} D_{m_1} ... D_{m_n} f)(x) = \overline{f}\Big(x \cdot (m_0 - 1) \cdot (m_1 - 1) \cdot \ldots \cdot (m_n - 1)\Big).$$

The first of these observations is the definition. The second observation follows inductively from the case $n = 0$ which is obvious.

Now we can proceed to the proof of the theorem. By virtue of the natural bijection (4) it suffices to show that the map
$$\mathbb{Z}[M]/I^{n+1} \to \mathbb{Z}[M^+]/I^{n+1}$$
is an isomorphism of abelian groups. Both sides are actually rings and the map in question is a map of rings. In order to construct an inverse ring map, it suffices to check that all elements $m \in M$ represent multiplicative units in $\mathbb{Z}[M]/I^{n+1}$; however, this follows because $m - 1$ is nilpotent. □

The last result shows that group completion is universal with respect to polynomial maps and not only for additive maps. From this, the extended functoriality of K_0 readily follows, as in [Jou00]; we review the details below.

Definition 3.4 (K_0 of additive ∞-categories)**.** For an additive ∞-category $\mathcal{A}$, the group $K_0(\mathcal{A})$ is the group completion of the abelian monoid $\pi_0(\mathcal{A})$ of isomorphism classes of objects with $\oplus$ as addition. Concretely $K_0(\mathcal{A})$ is the abelian group generated from isomorphism classes of objects subject to the relation $[A] + [B] = [A \oplus B]$.

Let $\mathrm{Cat}^{\mathrm{add}}$ be the ∞-category of additive ∞-categories and additive functors. Let Ab denote the ordinary category of abelian groups. Then K_0 defines a functor
$$K_0 : \mathrm{Cat}^{\mathrm{add}} \to \mathrm{Ab}$$

Let $\mathrm{Cat}^{\mathrm{add,poly}}$ be the ∞-category of additive ∞-categories and polynomial functors between them. The next result appears in the present form in [Jou00] and (for modules over a ring) in [Dol72].

Proposition 3.5 ([Jou00, Prop. 1.8])**.** *There is a functor* $\widetilde{K}_0 : \mathrm{Cat}^{\mathrm{add}}_{\mathrm{poly}} \to \mathrm{Set}$ *with a transformation* $\pi_0 \to \widetilde{K}_0$ *such that the diagram*

$$\begin{tikzcd} \mathrm{Cat}^{\mathrm{add}} \arrow[r, "K_0"] \arrow[d] & \mathrm{Ab} \arrow[d] \\ \mathrm{Cat}^{\mathrm{add}}_{\mathrm{poly}} \arrow[r, "\widetilde{K}_0"] & \mathrm{Set} \end{tikzcd}$$

commutes up to natural isomorphism.

Proof. For a polynomial functor $F : \mathcal{A} \to \mathcal{B}$ the map $\pi_0(\mathcal{A}) \to \pi_0(\mathcal{B}) \to K_0(\mathcal{B})$ is polynomial. Thus by Theorem 3.3 it can be uniquely extended to a polynomial map $K_0(\mathcal{A}) \to K_0(\mathcal{B})$. This gives the desired maps, and it is easy to see that they define a functor $\widetilde{K}_0 : \mathrm{Cat}^{\mathrm{add,poly}} \to \mathrm{Set}$. □

3.2. **Stable ∞-categories.** In this section, we extend the results of the previous section to show that K_0 is functorial in polynomial functors of stable ∞-categories; the strategy of proof is similar to that of [Dol72]. Recall that for stable ∞-categories, one has the following definition of K_0, which only depends on the underlying triangulated homotopy category.

Definition 3.6. Given a stable ∞-category $\mathcal{C}$, we define the group $K_0(\mathcal{C})$ as the quotient of $K_0^{\mathrm{add}}(\mathcal{C})$ (i.e., K_0 of the underlying additive ∞-category) by the relations $[X]+[Z]=[Y]$ for cofiber sequences $X \to Y \to Z$ in $\mathcal{C}$.

Proposition 3.7. *Let $F : \mathcal{C} \to \mathcal{D}$ be a polynomial functor between the stable ∞-categories $\mathcal{C}, \mathcal{D}$. Then there is a unique polynomial map $F_* : K_0(\mathcal{C}) \to K_0(\mathcal{D})$ such that $F_*([X]) = [F(X)]$ for $X \in \mathcal{C}$.*

We have already seen an analog of this result for K_0^{add}, the K_0 of the underlying additive ∞-category (Proposition 3.5). The obstruction is to understand the interaction with cofiber sequences. For this, we will need the following construction, and a general lemma about simplicial resolutions.

Construction 3.8. Let $\mathcal{C}$ be a stable ∞-category. Suppose given a cofiber sequence $X' \to X \to X''$ in $\mathcal{C}$. Then we form the Čech nerve of the map $X \to X''$. This constructs a 1-skeletal simplicial object $A_\bullet$ in $\mathcal{C}$ of the form

$$\ldots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} X' \oplus X' \oplus X \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X' \oplus X \longrightarrow X \ .$$

Alternatively, we can consider this simplicial object as the two-sided bar construction of the abelian group object $X' \in \mathcal{C}$ acting on X (via $X' \to X$). We observe that each of the terms in the simplicial object, and each of the face maps $d_i, i \geq 1$ depend only on the objects X', X (and not on the map $X' \to X$). Also, $A_\bullet$ is augmented over X'' and is a resolution of X''.

Lemma 3.9. *Suppose $\mathcal{C}$ is a stable ∞-category and $X^1_\bullet, X^2_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ are two simplicial objects such that:*

1. *Both $X^1_\bullet, X^2_\bullet$ are d-skeletal for some d.*
2. *We have an identification $X^1_n \simeq X^2_n$ for each n.*
3. *Under the above identification, the face maps $d_i, i \geq 1$ for both simplicial objects are homotopic.*

Then $|X^1_\bullet|, |X^2_\bullet|$ define the same class in $K_0(\mathcal{C})$.

Proof. This follows from the fact that $X^1_\bullet, X^2_\bullet$ have finite filtrations whose associated gradeds are identified in view of the stable Dold-Kan correspondence. □

Proposition 3.10. *Let $f : A \to A'$ be a polynomial map between abelian groups. Let $M \subset A$ be an abelian submonoid. Suppose that for $m \in M$ and $x \in A$, we have $f(x+m) = f(x)$. Then for any m' belonging to the subgroup M' generated by M and $x \in A$, we have $f(x+m') = f(x)$ and f factors over A/M'.*

Proof. Fix $x \in A$. Consider the polynomial map $A \to A'$ sending $y \mapsto f(x+y) - f(x)$. Since this vanishes for $y \in M$, it vanishes on the image of $M^+ \to A$ and the result follows. □

Proposition 3.11. *Let $f : M \to A$ be a polynomial map from an abelian monoid M to an abelian group A. Let $N \subset M \times M$ be a submonoid which contains the diagonal. Suppose that for each $(m_1, m_2) \in N$, we have $f(m_1) = f(m_2)$. Then the*

unique polynomial extension $f^+ : M^+ \to A$ factors over the quotient of M^+ by the subgroup generated by $\{m_1 - m_2\}_{(m_1,m_2)\in N}$.

Proof. Note first that the collection $C = \{m_1 - m_2\}_{(m_1,m_2)\in N} \subset M^+$ is a submonoid. We claim that for any $x \in M^+$ and $c \in C$, we have $f^+(x) = f^+(x + c)$. Equivalently, for any $y \in M^+$, $f^+(y + m_1) = f^+(y + m_2)$. Since both are polynomial maps, it suffices to check this for $y \in M$, in which case it follows from our assumptions. Thus the function $f^+ : M^+ \to A$ is invariant under translations by elements of C. Since C is a monoid, it follows by Proposition 3.10 that f^+ is invariant under translations by elements of the subgroup generated by C. □

Proof of Proposition 3.7. By Theorem 3.5, we have a natural polynomial map on additive K-theory

$$K_0^{\mathrm{add}}(\mathcal{C}) \xrightarrow{F_*^{\mathrm{add}}} K_0^{\mathrm{add}}(\mathcal{D}),$$

such that $F_*^{\mathrm{add}}([X]) = [F(X)]$ for $X \in \mathcal{C}$. It suffices to show the composite $K_0^{\mathrm{add}}(\mathcal{C}) \xrightarrow{F_*^{\mathrm{add}}} K_0^{\mathrm{add}}(\mathcal{D}) \twoheadrightarrow K_0(\mathcal{D})$ factors through $K_0(\mathcal{C})$. To see this, recall that $K_0(\mathcal{C})$ is the quotient of $K_0^{\mathrm{add}}(\mathcal{C})$ (in turn the group completion of $\pi_0(\mathcal{C})$) by the relations $[X] = [X' \oplus X'']$ for each cofiber sequence

$$X' \to X \to X''. \tag{5}$$

The collection of such defines a submonoid of $\pi_0(\mathcal{C}) \times \pi_0(\mathcal{C})$ containing the diagonal. To prove the assertion, we need to show[2] that if (5) is a cofiber sequence in $\mathcal{C}$, then

$$[F(X)] = [F(X' \oplus X'')].$$

To see this, we construct two simplicial objects $C^1_\bullet$ and $C^2_\bullet$ as in Construction 3.8 such that:

(1) $C^1_\bullet, C^2_\bullet$ are identified in each degree n with $X' \oplus X''[-1]^{\oplus n}$ and the face maps $d_i, i \geq 1$ are homotopic.
(2) We have $|C^1_\bullet| \simeq X' \oplus X''$ and $|C^2_\bullet| \simeq X$.
(3) Both $C^1_\bullet, C^2_\bullet$ are 1-skeletal.

Namely, $C^1_\bullet$ is the Čech nerve of $X' \xrightarrow{(\mathrm{id},0)} X' \oplus X''$ while $C^2_\bullet$ is the Čech nerve of $X' \to X$. We then find that the simplicial objects $F(C^1_\bullet), F(C^2_\bullet)$ are n-skeletal (if F has degree $\leq n$) and the geometric realizations are given by $F(X' \oplus X''), F(X)$ respectively. Moreover, $F(C^1_\bullet), F(C^2_\bullet)$ agree in each degree and the face maps $d_i, i \geq 1$ are identified. By Lemma 3.9, it follows that their geometric realizations have the same class in K_0, as desired. □

4. The main result

4.1. **The universal property of higher K-theory.** Our first goal is to review the axiomatic approach to higher K-theory, and its characterization. We will use the K-theory of stable ∞-categories, as developed by [BGT13] and [Bar16], following ideas that go back to Waldhausen [Wal85] and ultimately Quillen [Qui73].

Throughout, we fix (for set-theoretic reasons) a regular cardinal κ. Recall that $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is compactly generated [BGT13, Cor. 4.25]. Let $\mathrm{Cat}_{\infty,\kappa}^{\mathrm{perf}}$ denote the subcategory of κ-compact objects.

[2]Compare also [Jou00, Th. A] for a related type of statement.

Definition 4.1 (Additive invariants)**.** (1) Let $\mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$ denote the ∞-category of finitely product-preserving functors $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa} \to \mathcal{S}$.

(2) We say that $f \in \mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$ is *additive* if f is grouplike and f carries semiorthogonal decompositions in $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}$ to products.

We let $\mathrm{Fun}^\pi_{\mathrm{add}}(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S}) \subset \mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$ be the subcategory of additive invariants. This inclusion admits a left adjoint $(-)_{\mathrm{add}}$, called *additivization.*

The construction ι which carries $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}$ to its underlying space (i.e., the nerve of the maximal sub ∞-groupoid) yields an object of $\mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$. The construction of the algebraic K-theory space $K(-)$ yields an additive invariant, by Waldhausen's additivity theorem.

Theorem 4.2 (Compare [BGT13, Bar16])**.** *The K-theory functor $K : \mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa} \to \mathcal{S}$ is the additivization of $\iota \in \mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$.*

Remark 4.3. As the results in *loc. cit.* are stated slightly differently (in particular, $\kappa = \aleph_0$ is assumed), we briefly indicate how to deduce the present form of Theorem 4.2.

To begin with, we reduce to the case $\kappa = \aleph_0$. Let $F = (\iota)_{\mathrm{add}}$ denote the additivization of ι considered as an object of $\mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$. We can also consider the additivization of ι considered as an object of $\mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\omega}, \mathcal{S})$ and left Kan extend from $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\omega}$ to $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}$; we denote this by $F' : \mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa} \to \mathcal{S}$. By left Kan extension, we also have a map $\iota \to F'$ in $\mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$.

Now F' is also an additive invariant, thanks to [HSS17, Prop. 5.5]. It follows that we have maps in $\mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$ under ι from $F' \to F$ and $F \to F'$, using the universal properties. It follows easily (from the universal properties in $\mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$ and $\mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\omega}, \mathcal{S})$) that the composites in both directions are the identity, whence $F \simeq F'$.

Thus, we may assume $\kappa = \aleph_0$ for the statement of Theorem 4.2. For $\kappa = \aleph_0$, we have that $\mathrm{Fun}^\pi_{\mathrm{add}}(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\omega}, \mathcal{S})$ is the ∞-category of $\mathrm{Sp}_{\geq 0}$-valued additive invariants in the sense of [BGT13], whence the result.

We will also need a slight reformulation of the universal property, using a variant of the definition of an additive invariant, which turns out to be equivalent. In the following, we write $\mathrm{Fun}_{\mathrm{ex}}(-,-)$ denote the ∞-category of exact functors between two stable ∞-categories.

Definition 4.4 (Universal K-equivalences)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}^{\mathrm{perf}}_\infty$ is said to be a *universal K-equivalence* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that

$$[G \circ F] = [\mathrm{id}_{\mathcal{C}}] \in K_0(\mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathcal{C})), \quad [F \circ G] = [\mathrm{id}_{\mathcal{D}}] \in K_0(\mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}, \mathcal{D})). \tag{6}$$

Equivalently, this holds if and only if for every $\mathcal{E} \in \mathrm{Cat}^{\mathrm{perf}}_\infty$, the natural map $\mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathcal{E})$ induces an isomorphism on K_0.

Example 4.5. The shear map $\mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, i.e., the functor $(X, Y) \mapsto (X \oplus Y, Y)$, is a universal K-equivalence. If $\mathcal{C}$ admits a semiorthogonal decomposition into subcategories $\mathcal{C}_1, \mathcal{C}_2$, then the projection $\mathcal{C} \to \mathcal{C}_1 \times \mathcal{C}_2$ is a universal K-equivalence.

Proposition 4.6. *A functor in $\mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$ is additive if and only if it carries universal K-equivalences to equivalences.*

Proof. By the above examples, any object in $\mathrm{Fun}^{\pi}(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$ which preserves universal K-equivalences is necessarily additive. Thus, it remains only to show that an additive invariant carries universal K-equivalences to equivalences. Note that an additive invariant $f : \mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa} \to \mathcal{S}$ naturally lifts to $\mathrm{Sp}_{\geq 0}$, since it is grouplike. Moreover, given $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}$, the map obtained by applying f,

$$\pi_0(\mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathcal{D})^{\simeq}) \xrightarrow{f} \pi_0 \mathrm{Hom}_{\mathrm{Sp}_{\geq 0}}(f(\mathcal{C}), f(\mathcal{D}))$$

has the property that it factors through $K_0(\mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathcal{D}))$: indeed, this follows using additivity for $\mathcal{D}^{\Delta^1}$. This easily shows that f sends universal K-equivalences to equivalences. □

We thus obtain the following result, showing that additivization is the Bousfield localization at the universal K-equivalences.

Corollary 4.7. $\mathrm{Fun}^{\pi}_{\mathrm{add}}(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$ *is the Bousfield localization of* $\mathrm{Fun}^{\pi}(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$ *at the class of maps in* $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}$ *(via the Yoneda embedding) which are universal K-equivalences.* □

4.2. **The universal property with polynomial functors.** In this subsection, we formulate the main technical result (Theorem 4.9) of the paper, which controls the additivization of a theory functorial in polynomial functors. Throughout, we fix a regular *uncountable* cardinal κ.

Definition 4.8. We let $\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}$ denote the ∞-category whose objects are κ-compact idempotent-complete, stable ∞-categories and whose morphisms are polynomial functors between them.

We consider the ∞-category $\mathrm{Fun}^{\pi}(\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}, \mathcal{S})$ of functors $\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa} \to \mathcal{S}$ which preserve finite products. We say that an object $T \in \mathrm{Fun}^{\pi}(\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}, \mathcal{S})$ is *additive* if its restriction to $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}$ is additive. We let $\mathrm{Fun}^{\pi}_{\mathrm{add}}(\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}, \mathcal{S}) \subset \mathrm{Fun}^{\pi}(\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}, \mathcal{S})$ denote the subcategory of additive objects. This inclusion admits a left adjoint $(-)_{\mathrm{addp}}$, called *polynomial additivization.*

As an example, the underlying ∞-groupoid functor still defines a functor $\iota : \mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa} \to \mathcal{S}$ which preserves finite products, and hence an object of $\mathrm{Fun}^{\pi}(\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}, \mathcal{S})$. We now state the main technical result, which states that the polynomial additivization recovers the additivization when restricted to $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}$. This will be proved below in section 4.4.

Theorem 4.9. *Let* $T \in \mathrm{Fun}^{\pi}(\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}, \mathcal{S})$ *and let* T_{addp} *denote its polynomial additivization. Then the map* $T \to T_{\mathrm{addp}}$*, when restricted to* $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}$*, exhibits the restriction* $T_{\mathrm{addp}}|_{\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}}$ *as the additivization of* $T|_{\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}}$.

A direct consequence of the theorem is a sort of converse: given any map $T \to T'$ such that the restricted transformation exhibits $T'|_{\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}}$ as the additivization of $T|_{\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}}$, then T' is already the polynomial additivization. To see this simply note that T' is additive since this is only a condition on the restricted functor. Thus we get a map $T_{\mathrm{addp}} \to T'$ which is, by the theorem, an equivalence when restricted to $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}$ and therefore an equivalence.

Taking $T = \iota$ and using the universal property of K-theory (as in Theorem 4.2), we obtain the polynomial functoriality of K-theory (Theorem 1.1 from the introduction).

Corollary 4.10. *There is a (unique) functor* $\widetilde{K} : \mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa} \to \mathcal{S}$ *together with a map* $\iota \to \widetilde{K}$ *in* $\mathrm{Fun}^{\pi}(\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}, \mathcal{S})$*, such that the underlying map* $\iota|_{\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}}$ $\widetilde{K}|_{\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}}$ *identifies* $\widetilde{K}|_{\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}}$ *with* K*. Moreover,* $\widetilde{K}$ *is the polynomial additivization of* ι*.* □

Remark 4.11. Proving such a result directly (e.g., by examining the $S_\bullet$-construction) seems to be difficult. In fact, since the maps on K-theory spaces induced by polynomial functors are in general not loop maps they cannot be induced by maps of the respective $S_\bullet$-constructions.

4.3. **Generalities on Bousfield localizations.** We need some preliminaries about strongly saturated collections, cf. [Lur09, Sec. 5.5.4].

Definition 4.12. Let $\mathcal{E}$ be a presentable ∞-category. A *strongly saturated* class of maps is a full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{E})$ which is closed under colimits, base-changes, and compositions.

Construction 4.13 (Strongly saturated classes correspond to Bousfield localizations)**.** Given a set of maps in $\mathcal{E}$, they generate a smallest strongly saturated class. A strongly saturated class arising in this way is said to be of *small generation.*

The class of maps in $\mathcal{E}$ that map to equivalences under a Bousfield localization $\mathcal{E} \to \mathcal{E}'$ of presentable ∞-categories is strongly saturated and of small generation, and this in fact establishes a correspondence between accessible localizations and strongly saturated classes of small generation [Lur09, Props. 5.5.4.15-16]. Specifically, given a set S of maps, the Bousfield localization corresponding to the strongly saturated class generating is the Bousfield localization whose image consists of the S-local objects. To summarize, given a presentable ∞-category $\mathcal{E}$, we have a correspondence between the following collections:

- Presentable ∞-categories $\mathcal{E}'$, equipped with fully faithful right adjoints $\mathcal{E}' \to \mathcal{E}$ (so the left adjoint is a localization functor).
- Strongly saturated classes of maps in $\mathcal{E}$ which are of small generation.
- Accessible localization functors $L : \mathcal{E} \to \mathcal{E}$.

Proposition 4.14. *Let* $\mathcal{C}$ *be a presentable* ∞*-category which is given as the non-abelian derived* ∞*-category of a subcategory* $\mathcal{C}_0 \subset \mathcal{C}$ *closed under finite coproducts. Let* S *be the strongly saturated collection of maps in* $\mathcal{C}$ *generated by a subset* $S_0 \subset \mathrm{Fun}(\Delta^1, \mathcal{C}_0)$*. Suppose that* S_0 *is closed under finite coproducts and contains the identity maps. Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor which preserves sifted colimits and let* V *be a strongly saturated class in* $\mathcal{D}$*. If* $F(S_0) \subset V$*, then* $F(S) \subset V$*.*

Proof. Consider the collection $\mathscr{M}$ of maps $x \to y$ in $\mathcal{C}$ such that for every map $x \to x'$, the map $F(x' \to y \cup_x x')) \in V$. This collection $\mathscr{M}$ (in $\mathrm{Fun}(\Delta^1, \mathcal{C})$) is clearly closed under base-change, composition, and sifted colimits. Therefore, $\mathscr{M}$ is closed under all colimits and is in particular a strongly saturated class.

We claim that this collection $\mathscr{M}$ contains all of S; it suffices to see that $S_0 \subset \mathscr{M}$. To see this, let $x_0 \to y_0$ be a map in S_0. We need to see that the base-change of this map along a map $x_0 \to x_0'$ is carried by F into V. Any map $x_0 \to x_0'$ can be written as a sifted colimit of maps $x_0 \to x_0 \sqcup z$ for $z \in \mathcal{C}$, so one reduces to this case. Writing z as a sifted colimit of objects in $\mathcal{C}_0$, we reduce to the case where $z = z_0 \in \mathcal{C}_0$. Then the assertion is part of the hypotheses, so we obtain $(x_0 \to y_0) \in \mathscr{M}$ as desired. □

Corollary 4.15. *Let $\mathcal{A}, \mathcal{B}$ be ∞-categories admitting finite coproducts. Let $F_0 : \mathcal{A} \to \mathcal{B}$ be a functor preserving finite coproducts, inducing a cocontinuous functor $F : \mathcal{P}_\Sigma(\mathcal{A}) \to \mathcal{P}_\Sigma(\mathcal{B})$ with a right adjoint $G : \mathcal{P}_\Sigma(\mathcal{B}) \to \mathcal{P}_\Sigma(\mathcal{A})$ which preserves sifted colimits.*

Let S_0 be a class of maps in $\mathcal{A}$ and let $T_0 = F(S_0)$ denote the induced class of maps in $\mathcal{B}$; let S, T be the induced strongly saturated classes of maps in $\mathcal{P}_\Sigma(\mathcal{A}) \to \mathcal{P}_\Sigma(\mathcal{B})$, and let L_S, L_T be the associated Bousfield localization functors. Suppose that the class of maps $G(T_0) = GF(S_0)$ in $\mathcal{P}_\Sigma(\mathcal{A})$ belongs to the strongly saturated class generated by S_0.

Then the functor $G : \mathcal{P}_\Sigma(\mathcal{B}) \to \mathcal{P}_\Sigma(\mathcal{A})$ commutes with the respective localization functors. More precisely:

1. *For any $Y \in \mathcal{P}_\Sigma(\mathcal{B})$ which is T-local, $G(Y)$ is S-local.*
2. *For any $Y' \in \mathcal{P}_\Sigma(\mathcal{B})$, the natural map $Y' \to L_T(Y')$ induces (by the property (1)) a map $L_S G(Y') \to G(L_T(Y'))$; this map is an equivalence.*
3. *G induces a functor $L_T\mathcal{P}_\Sigma(\mathcal{B}) \to L_S\mathcal{P}_\Sigma(\mathcal{A})$ which commutes with limits and sifted colimits, which is right adjoint to the functor $L_T F : L_S\mathcal{P}_\Sigma(\mathcal{A}) \to L_T\mathcal{P}_\Sigma(\mathcal{B})$.*

Proof. Part (1) follows because F (which preserves colimits) carries S into T, so the right adjoint necessarily carries T-local objects into S-local objects.

By Proposition 4.14, G carries the strongly saturated class T in $\mathcal{P}_\Sigma(\mathcal{B})$ into the strongly saturated class S in $\mathcal{P}_\Sigma(\mathcal{A})$. Now in (2), the map $Y' \to L_T(Y')$ belongs to the strongly saturated class T, whence $G(Y') \to G(L_T(Y'))$ belongs to the strongly saturated class S. Since the target of this map is S-local, it follows that $L_S G(Y') \xrightarrow{\sim} G(L_T(Y'))$. This proves part (2).

For (3), we already saw in (1) that G induces a functor $L_T\mathcal{P}_\Sigma(\mathcal{B}) \to L_S\mathcal{P}_\Sigma(\mathcal{A})$, and clearly this commutes with limits. It also commutes with sifted colimits since the functor $G : \mathcal{P}_\Sigma(\mathcal{B}) \to \mathcal{P}_\Sigma(\mathcal{A})$ commutes with sifted colimits and since G carries the L_T-localization into the L_S-localization. From this (3) follows. □

4.4. **Proof of Theorem 4.9.** The proof of Theorem 4.9 will require some more preliminaries. To begin with, we will need the construction of a universal target for a degree $\leq n$ functor.

Construction 4.16. Given $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$, we define the object $\Gamma_n\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ such that we have a natural equivalence for any $\mathcal{D} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$,

$$\mathrm{Fun}_{\mathrm{ex}}(\Gamma_n\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}_{\leq n}(\mathcal{C}, \mathcal{D}).$$

In particular, Γ_n receives a degree $\leq n$ functor $\mathcal{C} \to \Gamma_n\mathcal{C}$ and $\Gamma_n\mathcal{C}$ is universal for this structure. Explicitly, $\Gamma_n\mathcal{C}$ is obtained by starting with the free idempotent-complete stable ∞-category on $\mathcal{C}$, i.e., compact objects in Sp-valued presheaves on $\mathcal{C}$, and then forming the minimal exact localization such that the Yoneda functor becomes n-excisive.

Remark 4.17 (Some cardinality estimation)**.** Recall again that κ is assumed to be uncountable. If $\mathcal{C} \in \mathrm{Cat}_{\infty,\kappa}^{\mathrm{perf}}$, then we claim that $\Gamma_n\mathcal{C} \in \mathrm{Cat}_{\infty,\kappa}^{\mathrm{perf}}$ for all $n \geq 0$. In fact, we observe that $\mathcal{C} \in \mathrm{Cat}_{\infty,\kappa}^{\mathrm{perf}}$ if and only if $\mathcal{C}$ is κ-compact as an object of Cat_∞; moreover, this holds if and only if $\mathcal{C}$ has $< \kappa$ isomorphism classes of objects and the mapping spaces in $\mathcal{C}$ are κ-small.

The crucial observation, for our purposes, is simply that Γ_n behaves relatively well with respect to semiorthogonal decompositions: it transforms them into something that, while slightly more complicated, is very controllable on K-theory.

Proposition 4.18. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a universal K-equivalence. Then the map $\Gamma_n\mathcal{C} \to \Gamma_n\mathcal{C}'$ is a universal K-equivalence. That is, for every $\mathcal{D} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$, the functor*

$$F^* : \mathrm{Fun}_{\leq n}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}_{\leq n}(\mathcal{C}, \mathcal{D}),$$

induces an isomorphism on K_0.

Proof. Let $G : \mathcal{C}' \to \mathcal{C}$ be a functor such that $F \circ G, G \circ F$ satisfy (6). It suffices to show that the composite $\mathrm{Fun}_{\leq n}(\mathcal{C}, \mathcal{D}) \xrightarrow{G^*} \mathrm{Fun}_{\leq n}(\mathcal{C}', \mathcal{D}) \xrightarrow{F^*} \mathrm{Fun}_{\leq n}(\mathcal{C}, \mathcal{D})$ is the identity on K_0 (and the converse direction follows by symmetry).

To see this, fix a functor $f \in \mathrm{Fun}_{\leq n}(\mathcal{C}, \mathcal{D})$. We then have a degree n functor

$$f \circ \cdot : \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}_{\leq n}(\mathcal{C}, \mathcal{D}), \quad \phi \mapsto f \circ \phi.$$

By Proposition 3.7, this induces a unique map on K_0. It follows that since $G \circ F, \mathrm{id}$ define the same class in $K_0(\mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}', \mathcal{C}))$, the functors $f \circ G \circ F, f$ define the same class in $K_0(\mathrm{Fun}_{\leq n}(\mathcal{C}, \mathcal{D}))$. This shows precisely that $F^* \circ G^*$ induces the identity on K_0. Similarly, $G^* \circ F^*$ induces the identity on K_0. This completes the proof. □

Proof of Theorem 4.9. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}^\pi_{\mathrm{add}}(\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}, \mathcal{S}) & \longrightarrow & \mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}, \mathcal{S}) \\ \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} \\ \mathrm{Fun}^\pi_{\mathrm{add}}(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S}) & \longrightarrow & \mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S}) \end{array},$$

where the horizontal rows are the inclusions and the vertical arrows are given by restriction along $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa} \subset \mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}$. Our goal is to show that when we reverse the horizontal arrows by replacing the inclusion functors by additivizations, the diagram still commutes.

This statement fits into the setup of Corollary 4.15. Here we take $\mathcal{A} = (\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa})^{op}$ and $\mathcal{B} = (\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa})^{op}$, and F_0 to be the opposite of the inclusion $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa} \to \mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}$. Moreover, S_0 can be taken to be the class of universal K-equivalences in $\mathcal{A} = (\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa})^{op}$. The local objects then correspond to the additive invariants (Corollary 4.7).

Unwinding the definitions, we find that in order to apply Corollary 4.15, we now need to verify that if $\mathcal{C} \to \mathcal{D}$ is a universal K-equivalence in $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}$, then the map in $\mathrm{Fun}^\pi(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$ given by

$$\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}}(\mathcal{D}, -) \to \mathrm{Hom}_{\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}}(\mathcal{C}, -)$$

induces an equivalence upon additivizations. Now by definition we have

$$\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}}(\mathcal{D}, -) = \varinjlim_n \mathrm{Hom}_{\mathrm{Cat}^{\mathrm{perf}}_\infty}(\Gamma_n\mathcal{D}, -),$$

and similarly for $\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{poly}}_{\infty,\kappa}}(\mathcal{C}, -)$. It therefore suffices to show that

$$\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{perf}}_\infty}(\Gamma_n\mathcal{D}, -) \to \mathrm{Hom}_{\mathrm{Cat}^{\mathrm{perf}}_\infty}(\Gamma_n\mathcal{C}, -)$$

(as a map in $\mathrm{Fun}^{\pi}(\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}, \mathcal{S})$) induces an equivalence on additivizations. This follows from Proposition 4.18 and Corollary 4.7, noting that $\Gamma_n\mathcal{C}, \Gamma_n\mathcal{D}$ belong to $\mathrm{Cat}^{\mathrm{perf}}_{\infty,\kappa}$ since $\mathcal{C}, \mathcal{D}$ do and κ is uncountable. □

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