AN $\mathbb{R}$-MOTIVIC v_1–SELF-MAP OF PERIODICITY 1

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Abstract. We consider a nontrivial action of C_2 on the type 1 spectrum $\mathcal{Y} := \mathrm{M}_2(1) \wedge \mathrm{C}(\eta)$, which is well-known for admitting a 1-periodic v_1–self-map. The resultant finite C_2-equivariant spectrum $\mathcal{Y}^{\mathrm{C}_2}$ can also be viewed as the complex points of a finite $\mathbb{R}$-motivic spectrum $\mathcal{Y}^{\mathbb{R}}$. In this paper, we show that one of the 1-periodic v_1–self-maps of $\mathcal{Y}$ can be lifted to a self-map of $\mathcal{Y}^{\mathrm{C}_2}$ as well as $\mathcal{Y}^{\mathbb{R}}$. Further, the cofiber of the self-map of $\mathcal{Y}^{\mathbb{R}}$ is a realization of the subalgebra $\mathcal{A}^{\mathbb{R}}(1)$ of the $\mathbb{R}$-motivic Steenrod algebra. We also show that the C_2-equivariant self-map is nilpotent on the geometric fixed-points of $\mathcal{Y}^{\mathrm{C}_2}$.

1. Introduction

In classical stable homotopy theory, the interest in periodic v_n–self-maps of finite spectra lies in the fact that one can associate to each v_n–self-map an infinite family in the chromatic layer n stable homotopy groups of spheres. Therefore, interest lies in constructing type n spectra and finding v_n–self-maps of lowest possible periodicity on a given type n spectrum. This, in general, is a difficult problem, though progress has been made sporadically throughout the history of the subject [T, DM, BP, BHHM, N, BEM, BE]. With the modern development of motivic stable homotopy theory, one may ask if there are similar periodic self-maps of finite motivic spectra.

Classically any non-contractible finite p-local spectrum admits a periodic v_n–self-map for some $n \geq 0$. This is a consequence of the thick-subcategory theorem [HS, Theorem 7], aided by a vanishing line argument [HS, §4.2]. In the classical case all the thick tensor ideals of $\mathbf{Sp}_{p,\mathrm{fin}}$ (the homotopy category of finite p-local spectra) are also prime (in the sense of [B]). The thick tensor-ideals of the homotopy category of cellular motivic spectra over $\mathbb{C}$ or $\mathbb{R}$ are not completely known (but see [HO]). However, one can gather some knowledge about the prime thick tensor-ideals in $\mathrm{Ho}(\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}})$ (the homotopy category of 2-local cellular $\mathbb{R}$-motivic spectra) through the Betti realization functor

$$\beta : \mathrm{Ho}(\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}) \longrightarrow \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$$

using the complete knowledge of prime thick-subcategories of $\mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ [BS].

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The prime thick tensor-ideals of $\mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}^{\mathrm{C}_2})$ are essentially the pull-back of the classical thick subcategories along the two functors, the geometric fix point functor

$$\Phi^{\mathrm{C}_2} : \mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}^{\mathrm{C}_2}) \longrightarrow \mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}})$$

and the forgetful functor

$$\Phi^{e} : \mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}^{\mathrm{C}_2}) \longrightarrow \mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}).$$

Let $\mathcal{C}_n$ denote the thick subcategory of $\mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}})$ consisting of spectra of type at least n. The prime thick-subcategories,

$$\mathcal{C}(e,n) = (\Phi^e)^{-1}(\mathcal{C}_n) \text{ and } \mathcal{C}(\mathrm{C}_2,n) = (\Phi^{\mathrm{C}_2})^{-1}(\mathcal{C}_n),$$

are the only prime thick subcategories of $\mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}^{\mathrm{C}_2})$.

Definition 1.1. We say a spectrum $X \in \mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}^{\mathrm{C}_2})$ is of *type* (n,m) iff $\Phi^e(X)$ is of type n and $\Phi^{\mathrm{C}_2}(X)$ is of type m.

For a type (n,m) spectrum X, a self-map $f : X \to X$ is periodic if and only if at least one of $\{\Phi^e(f), \Phi^{\mathrm{C}_2}(f)\}$ are periodic (see [BGH, Proposition 3.17]).

Definition 1.2. Let $X \in \mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}^{\mathrm{C}_2})$ be of type (n,m). We say a self-map $f : X \to X$ is

(i) a $v_{(n,m)}$*–self-map* of mixed periodicity (i,j) if $\Phi^e(f)$ is a v_n–self-map of periodicity i and $\Phi^{\mathrm{C}_2}(f)$ is a v_m–self-map of periodicity j,

(ii) a $v_{(n,\mathrm{nil})}$*–self-map* of periodicity i if $\Phi^e(f)$ is a v_n–self-map of periodicity i and $\Phi^{\mathrm{C}_2}(f)$ is nilpotent, and,

(iii) a $v_{(\mathrm{nil},m)}$*–self-map* of periodicity j if $\Phi^e(f)$ is a nilpotent self-map and $\Phi^{\mathrm{C}_2}(f)$ is a v_m–self-map of periodicity j.

Example 1.3. The sphere spectrum $\mathbb{S}_{\mathrm{C}_2}$ is of type $(0,0)$. The degree 2 map is a $v_{(0,0)}$–self-map. In general, if we consider the v_n–self-map of a type n spectrum with trivial action of C_2, then the resultant equivariant self-map is a $v_{(n,n)}$–self-map.

Example 1.4. Let $\mathrm{S}_{\mathrm{C}_2}^{1,1}$ denote the C_2-equivariant sphere which is the one-point compactification of the real sign representation. The unstable twist-map

$$\epsilon_u : \mathrm{S}_{\mathrm{C}_2}^{1,1} \wedge \mathrm{S}_{\mathrm{C}_2}^{1,1} \longrightarrow \mathrm{S}_{\mathrm{C}_2}^{1,1} \wedge \mathrm{S}_{\mathrm{C}_2}^{1,1}$$

stabilizes to a nonzero element $\epsilon \in \pi_{0,0}(\mathbb{S}_{\mathrm{C}_2})$. Let $\mathsf{h} = 1 - \epsilon \in \pi_{0,0}(\mathbb{S}_{\mathrm{C}_2})$ be the stabilization of the map

$$\mathsf{h}_u = 1 - \epsilon_u : \mathrm{S}_{\mathrm{C}_2}^{3,2} \longrightarrow \mathrm{S}_{\mathrm{C}_2}^{3,2}.$$

Note that on the underlying space ϵ is of degree -1, while on the fixed points it is the identity. Therefore $\Phi^e(\mathsf{h})$ is multiplication by 2, whereas $\Phi^{\mathrm{C}_2}(\mathsf{h})$ is trivial. Thus h is a $v_{(0,\mathrm{nil})}$–self-map. Thus $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ is of type $(1,0)$.

Example 1.5. The equivariant Hopf-map $\eta_{1,1} \in \pi_{1,1}(\mathbb{S}_{\mathrm{C}_2})$ is the Betti realization of the $\mathbb{R}$-motivic Hopf-map η [M2, DI3]. Up to a unit, it is the stabilization of the projection map

$$\pi : \mathrm{S}^{3,2}_{\mathrm{C}_2} \simeq \mathbb{C}^2 \setminus \{\mathbf{0}\} \longrightarrow \mathbb{CP}^1 \cong \mathrm{S}^{2,1}_{\mathrm{C}_2},$$

where the domain and the codomain are given the C_2-structure using complex conjugation. On fixed-points, the map π is the projection map

$$\pi : \mathbb{R}^2 \setminus \{\mathbf{0}\} \longrightarrow \mathbb{RP}^1,$$

which is a degree 2 map. From this we learn that while $\Phi^e(\eta_{1,1})$ is nilpotent, $\Phi^{\mathrm{C}_2}(\eta_{1,1})$ is the periodic v_0–self-map. Hence, $\eta_{1,1}$ is a $v_{(\mathrm{nil},0)}$–self-map and the cofiber $\mathrm{C}(\eta_{1,1})$ is of type $(0,1)$.

Remark 1.6. In the C_2-equivariant stable homotopy groups, the usual Hopf-map (sometimes referred to as the 'topological Hopf-map') is different from $\eta_{1,1}$ of Example 1.5. The 'topological Hopf-map' $\eta_{1,0} \in \pi_{1,0}(\mathbb{S}_{\mathrm{C}_2})$ should be thought of as the stabilization of the unstable Hopf-map

$$(\eta_{1,0})_u \colon \mathrm{S}^{3,0}_{\mathrm{C}_2} \longrightarrow \mathrm{S}^{2,0}_{\mathrm{C}_2}$$

where both domain and codomain are given the trivial C_2-action.

Definition 1.7. We say a spectrum $X \in \mathrm{Ho}(\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}})$ is of type (n,m) if $\beta(X)$ is of type (n,m). We call an $\mathbb{R}$-motivic self-map

$$f : X \to X$$

a $v_{(n,m)}$–self-map, where m and n are in $\mathbb{N} \cup \{\mathrm{nil}\}$ (but not both nil), if $\beta(f)$ is a C_2-equivariant $v_{(n,m)}$–self-map.

Remark 1.8. The maps 'multiplication by 2' (of Example 1.3), h (of Example 1.4), and $\eta_{1,1}$ (of Example 1.5) admit $\mathbb{R}$-motivic lifts along β and provide us with examples of a $v_{(0,0)}$–self-map, $v_{(0,\mathrm{nil})}$–self-map and $v_{(\mathrm{nil},0)}$–self-map of the $\mathbb{R}$-motivic sphere spectrum $\mathbb{S}_{\mathbb{R}}$, respectively.

A theorem of Balmer and Sanders [BS] asserts that $\mathcal{C}(e,n) \subset \mathcal{C}(\mathrm{C}_2,m)$ if and only if $n \geq m+1$. In particular, $\mathcal{C}(e,n)$ is contained in $\mathcal{C}(\mathrm{C}_2,n-1)$. Consequently, there are no type (n,m) (C_2-equivariant or $\mathbb{R}$-motivic) spectra if $n \geq m+2$. Their result also implies the following:

Proposition 1.9. *Let $X \in \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ be of type $(n+1,n)$ for some n. Then X cannot support a $v_{(n+1,\mathrm{nil})}$–self-map.*

The proposition holds since the cofiber of such a self-map would be of type $(n+2,n)$, contradicting the results of Balmer-Sanders. In particular, neither $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ nor $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ supports a $v_{(1,\mathrm{nil})}$–self-map. However, it is possible that $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ as well as $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ can admit a $v_{(1,0)}$–self-map or a $v_{(\mathrm{nil},0)}$–self-map. In fact, $\eta_{1,1} \in \pi_{1,1}(\mathbb{S}_{\mathbb{R}})$ and $\eta_{1,1} \in \pi_{1,1}(\mathbb{S}_{\mathrm{C}_2})$ induce $v_{(\mathrm{nil},0)}$–self-maps of $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ respectively. In Section 5, we show that:

Theorem 1.10. *The spectrum $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ does not admit a $v_{(1,0)}$–self-map.*

However, it is possible that $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ admits a $v_{(1,0)}$–self-map (for details see Remark 5.6). In contrast to the classical case, there is no guarantee that a finite C_2-equivariant or $\mathbb{R}$-motivic spectrum will admit *any* periodic self-map, or at least nothing concrete is known yet. This question must be studied!

The goal of this paper is rather modest. We consider the classical spectrum

$$\mathcal{Y} := \mathrm{M}_2(1) \wedge \mathrm{C}(\eta)$$

that admits, up to homotopy, 8 different v_1–self-maps of periodicity 1 [DM, Section 2] (see also [BEM]). We ask ourselves if the v_1–self-maps can preserve symmetries upon providing $\mathcal{Y}$ with interesting C_2-equivariant structures. We will consider four C_2-equivariant lifts of the spectrum $\mathcal{Y}$,

(i) $\mathcal{Y}^{\mathrm{C}_2}_{\mathrm{triv}}$, where the action of C_2 is trivial,

(ii) $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)} := \mathrm{C}^{\mathrm{C}_2}(2) \wedge \mathrm{C}^{\mathrm{C}_2}(\eta_{1,1})$, with $\Phi^{\mathrm{C}_2}(\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}) = \mathrm{M}_2(1) \wedge \mathrm{M}_2(1)$,

(iii) $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)} := \mathrm{C}^{\mathrm{C}_2}(\mathsf{h}) \wedge \mathrm{C}^{\mathrm{C}_2}(\eta_{1,0})$, with $\Phi^{\mathrm{C}_2}(\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}) = \Sigma\mathrm{C}(\eta) \vee \mathrm{C}(\eta)$, and,

(iv) $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)} := \mathrm{C}^{\mathrm{C}_2}(\mathsf{h}) \wedge \mathrm{C}^{\mathrm{C}_2}(\eta_{1,1})$, with $\Phi^{\mathrm{C}_2}(\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}) = \Sigma\mathrm{M}_2(1) \vee \mathrm{M}_2(1)$.

The C_2-spectra $\mathcal{Y}^{\mathrm{C}_2}_{\mathrm{triv}}$, $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}$ and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$ are of type $(1,1)$, and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}$ is of type $(1,0)$. There are unique $\mathbb{R}$-motivic lifts of the classes 2, h, $\eta_{1,0}$, and $\eta_{1,1}$, and therefore we have unique $\mathbb{R}$-motivic lifts of $\mathcal{Y}^{\mathrm{C}_2}_{\mathrm{triv}}$, $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}$, $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}$, and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$ which we will simply denote by $\mathcal{Y}^{\mathbb{R}}_{\mathrm{triv}}$, $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$, $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$, and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$, respectively. In this paper we prove:

Theorem 1.11. *The $\mathbb{R}$-motivic spectrum $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ admits a $v_{1,\mathrm{nil}}$–self-map*

$$v : \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \longrightarrow \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$$

of periodicity 1*.*

By applying the Betti realization functor we get:

Corollary 1.12. *The C_2-equivariant spectrum $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$ admits a 1-periodic $v_{1,\mathrm{nil}}$–self-map*

$$\beta(v) : \Sigma^{2,1}\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)} \longrightarrow \mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}.$$

Corollary 1.13. *The geometric fixed-point spectrum of the telescope*

$$\beta(v)^{-1}\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$$

is contractible.

Classically, the cofiber of a v_1–self-map on $\mathcal{Y}$ is a realization of the finite subalgebra $\mathcal{A}(1)$ of the Steenrod algebra $\mathcal{A}$. We see a very similar phenomenon in the $\mathbb{R}$-motivic as well as in the C_2-equivariant cases. The C_2-equivariant Steenrod algebra $\mathcal{A}^{\mathrm{C}_2}$ as well as the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ admit subalgebras analogous to $\mathcal{A}(1)$ (generated by Sq^1 and Sq^2) [H, R2], which we denote by $\mathcal{A}^{\mathrm{C}_2}(1)$ and $\mathcal{A}^{\mathbb{R}}(1)$, respectively. We observe that:

Theorem 1.14. *The spectrum* $\mathrm{C}^{\mathbb{R}}(v) := \mathrm{Cof}(v : \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \to \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ *is a type* $(2,1)$ *complex whose bigraded cohomology is a free* $\mathcal{A}^{\mathbb{R}}(1)$*-module on one generator.*

Corollary 1.15. *The bigraded cohomology of the* C_2*-equivariant spectrum*

$$\mathrm{C}^{\mathrm{C}_2}(\beta(v)) \cong \beta(\mathrm{C}^{\mathbb{R}}(v))$$

is a free $\mathcal{A}^{\mathrm{C}_2}(1)$*-module on one generator.*

Our last main result in this paper is the following.

Theorem 1.16. *The spectrum* $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$ *does not admit a* $v_{(1,0)}$*–self-map.*

The above results immediately raise some obvious questions. Pertaining to self-maps one may ask: Does $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ admit a $v_{1,\mathrm{nil}}$–self-map? Does $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ or $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ admit a $v_{(1,1)}$–self-map? Does $\mathcal{Y}^{\mathbb{R}}_{\mathrm{triv}}$, $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ or $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ admit $v_{(\mathrm{nil},1)}$–self-map? Or more generally, how many different homotopy types of each kind of periodic self-maps exist? Related to $\mathcal{A}^{\mathbb{R}}(1)$, one may inquire: How many different $\mathcal{A}^{\mathbb{R}}$-module structures can be given to $\mathcal{A}^{\mathbb{R}}(1)$? Can those $\mathcal{A}^{\mathbb{R}}$-modules be realized as a spectrum? Are the realizations of $\mathcal{A}^{\mathbb{R}}(1)$ equivalent to cofibers of periodic self-maps of $\mathcal{Y}^{\mathbb{R}}_{(i,j)}$? We hope to address most, if not all, of the above questions in our upcoming work (see Remark 3.21, Remark 4.18 and Remark 5.6).

1.1. **Outline of our method.** We first construct a spectrum $\mathcal{A}^{\mathbb{R}}_1$ which realizes the algebra $\mathcal{A}^{\mathbb{R}}(1)$ using a method of Smith (outlined in [R1, Appendix C]) which constructs new finite spectra (potentially with larger number of cells) from known ones. The idea is as follows. If X is a p-local finite spectrum then the permutation group Σ_n acts on $X^{\wedge n}$. One may then use an idempotent $e \in \mathbb{Z}_{(p)}[\Sigma_n]$ to obtain a split summand of the spectrum $X^{\wedge n}$. As explained in [R1, Appendix C], Young tableaux provide a rich source of such idempotents. For a judicious choice of e and X, the spectrum $e(X^{\wedge n})$ can be interesting.

We exploit the relation that $\mathsf{h} \cdot \eta_{1,1} = 0$ in $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ [M2] to construct an $\mathbb{R}$-motivic analogue of the question mark complex. The cell-diagram of the question mark complex is as described in the picture below. For a choice of idempotent element e

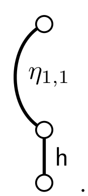
$\mathcal{Q}_{\mathbb{R}} =$ 

FIGURE 1.17. Cell-diagram of the $\mathbb{R}$-motivic question mark complex

in the group ring $\mathbb{Z}_{(2)}[\Sigma_3]$, we observe that $e(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})^{\otimes 3})$ is a free $\mathcal{A}^{\mathbb{R}}(1)$-module. This is the cohomology of an $\mathbb{R}$-motivic spectrum $\tilde{e}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3})$, which we call $\Sigma^{1,0}\mathcal{A}^{\mathbb{R}}_1$ (see (3.4) for details). The observation requires us to develop a criterion that will detect freeness for modules over certain subalgebras of $\mathcal{A}^{\mathbb{R}}$. Writing $\mathbb{M}^{\mathbb{R}}_2$ for the $\mathbb{R}$-motivic cohomology of a point, we prove:

Theorem 1.18. *A finitely generated $\mathcal{A}^{\mathbb{R}}(1)$-module* M *is free if and only if*

(1) M *is free as an $\mathbb{M}_2^{\mathbb{R}}$-module, and*

(2) $\mathbb{F}_2 \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathrm{M}$ *is a free* $\mathbb{F}_2 \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(1)$*-module.*

The cohomology of $\mathcal{A}_1^{\mathbb{R}}$ provides an $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$, which immediately gives us a short exact sequence

$$0 \to \mathrm{H}^{*,*}(\Sigma^{3,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \to \mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}}) \to \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \to 0$$

of $\mathcal{A}^{\mathbb{R}}$-modules. Thus, we get a candidate for a $v_{1,\mathrm{nil}}$–self-map in the $\mathbb{R}$-motivic Adams spectral sequence

$$\overline{v} \in \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) \Rightarrow [\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}, \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}]_{*,*}$$

which survives as there is no potential target for a differential supported by $\overline{v}$.

Organization of the paper. In Section 2, we review the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$, discuss the structure of its subalgebra $\mathcal{A}^{\mathbb{R}}(n)$, and prove Theorem 1.18. In Section 3, we construct the spectrum $\mathcal{A}_1^{\mathbb{R}}$ that realizes the subalgebra $\mathcal{A}^{\mathbb{R}}(1)$ and prove that it is of type $(2,1)$. In Section 4, we prove Theorem 1.11 and Theorem 1.14; i.e., we show that $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ admits a $v_{1,\mathrm{nil}}$–self-map and that its cofiber has the same $\mathcal{A}^{\mathbb{R}}$-module structure as that of $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$. In Section 5, we show the non-existence of a $v_{(1,0)}$–self-map on $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$; i.e., we prove Theorem 1.10 and Theorem 1.16.

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2. The $\mathbb{R}$-motivic Steenrod algebra and a freeness criterion

We begin by reviewing the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ following Voevodsky [V]. The algebra $\mathcal{A}^{\mathbb{R}}$ is the bigraded homotopy classes of self-maps of the $\mathbb{R}$-motivic Eilenberg-Mac Lane spectrum $\mathrm{H}\mathbb{F}_2^{\mathbb{R}}$:

$$\mathcal{A}^{\mathbb{R}} = [\mathrm{H}\mathbb{F}_2^{\mathbb{R}}, \mathrm{H}\mathbb{F}_2^{\mathbb{R}}]_{*,*}.$$

The unit map $\mathbb{S}_{\mathbb{R}} \to \mathrm{H}\mathbb{F}_2^{\mathbb{R}}$ induces a canonical projection map

$$\epsilon : \mathcal{A}^{\mathbb{R}} \longrightarrow \mathbb{M}_2^{\mathbb{R}} := [\mathbb{S}_{\mathbb{R}}, \mathrm{H}\mathbb{F}_2^{\mathbb{R}}]_{*,*} \cong \mathbb{F}_2[\tau, \rho],$$

where $|\tau| = (0, -1)$ and $|\rho| = (-1, -1)$. Further, using the multiplication map $\mathrm{H}\mathbb{F}_2^{\mathbb{R}} \wedge \mathrm{H}\mathbb{F}_2^{\mathbb{R}} \to \mathrm{H}\mathbb{F}_2^{\mathbb{R}}$ one can give $\mathcal{A}^{\mathbb{R}}$ a left $\mathbb{M}_2^{\mathbb{R}}$-module structure as well as a right $\mathbb{M}_2^{\mathbb{R}}$-module structure. Voevodsky shows that $\mathcal{A}^{\mathbb{R}}$ is a free left $\mathbb{M}_2^{\mathbb{R}}$-module. There is an analogue of the classical Adem basis in the motivic setting, and Voevodsy established motivic Adem relations, thereby completely describing the multiplicative structure of $\mathcal{A}^{\mathbb{R}}$.

The motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ also admits a diagonal map, so that its left $\mathbb{M}_2^{\mathbb{R}}$-linear dual is an algebra over $\mathbb{F}_2$. Note that $\mathcal{A}^{\mathbb{R}}$ is an $\mathbb{F}_2$-algebra but not an $\mathbb{M}_2^{\mathbb{R}}$-algebra as τ is not a central element since

$$\mathrm{Sq}^1(\tau) = \rho \neq \tau \mathrm{Sq}^1. \tag{2.1}$$

This complication is also reflected in the fact that the pair $(\mathbb{M}_2^{\mathbb{R}}, \hom_{\mathbb{M}_2^{\mathbb{R}}}(\mathcal{A}^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}))$ is a Hopf-algebroid, and not a Hopf-algebra like its complex counterpart. The underlying algebra of the dual $\mathbb{R}$-motivic Steenrod algebra is given by

$$\mathcal{A}_*^{\mathbb{R}} \cong \mathbb{M}_2^{\mathbb{R}}[\xi_{i+1}, \tau_i : i \geq 0]/(\tau_i^2 = \tau\xi_{i+1} + \rho\tau_{i+1} + \rho\tau_0\xi_{i+1})$$

where ξ_i and τ_i live in bidegree $(2^{i+1}-2, 2^i-1)$ and $(2^{i+1}-1, 2^i-1)$, respectively. The complete description of the Hopf-algebroid structure can be found in [V].

Ricka[1] [R2] identified the quotient Hopf-algebroids of $\mathcal{A}_*^{\mathbb{R}}$ (see also [H]). In particular, there are quotient Hopf-algebroids

$$\mathcal{A}^{\mathbb{R}}(n)_* = \mathcal{A}_*^{\mathbb{R}}/(\xi_1^{2^n}, \dots, \xi_n^2, \xi_{n+1}, \dots, \tau_0^{2^{n+1}}, \dots, \tau_n^2, \tau_{n+1}, \dots)$$

which can be thought of as analogues of the quotient Hopf-algebra

$$\mathcal{A}(n)_* = \mathcal{A}_*/(\xi_1^{2^{n+1}}, \dots, \xi_{n+1}^2, \xi_{n+2}, \dots)$$

of the classical dual Steenrod algebra $\mathcal{A}_*$. It is an algebraic fact that

$$\tau^{-1}(\mathcal{A}^{\mathbb{R}}(n)_*/(\rho)) \cong \mathbb{F}_2[\tau^{\pm 1}] \otimes \mathcal{A}(n)_*$$

as Hopf algebras. The quotient Hopf-algebroid $\mathcal{A}^{\mathbb{R}}(n)_*$ is the $\mathbb{M}_2^{\mathbb{R}}$-linear dual of the subalgebra $\mathcal{A}^{\mathbb{R}}(n)$ of $\mathcal{A}^{\mathbb{R}}$, which is generated by $\{\tau, \rho, \mathrm{Sq}^1, \mathrm{Sq}^2, \dots, \mathrm{Sq}^{2^n}\}$.

Even though τ is not in the center (2.1), ρ is in the center. We make use of this fact to prove the following result.

Lemma 2.2. *A finitely-generated $\mathcal{A}^{\mathbb{R}}(n)$-module M is free if and only if*

(1) M is free as an $\mathbb{F}_2[\rho]$-module, and,

(2) $\mathrm{M}/(\rho)$ is a free $\mathcal{A}^{\mathbb{R}}(n)/(\rho)$-module.

Proof. The 'only if' part is trivial. For the 'if' part, choose a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of $\mathrm{M}/(\rho)$ and let $\tilde{b}_i \in M$ be any lift of b_i. Let F denote the free $\mathcal{A}^{\mathbb{R}}(n)$-module generated by $\mathcal{B}$ and consider the map

$$f : \mathrm{F} \to \mathrm{M}$$

which sends $b_i \mapsto \tilde{b}_i$. We show that f is an isomorphism by inductively proving that f induces an isomorphism $\mathrm{F}/(\rho^n) \cong \mathrm{M}/(\rho^n)$ for all $n \geq 1$. The case of $n = 1$ is true by assumption.

[1] Ricka actually identified the quotient Hopf-algebroids of the C_2-equivariant dual Steenrod algebra. However, the difference between the $\mathbb{R}$-motivic Steenrod algebra and the C_2-equivariant Steenrod algebra lies only in the coefficient rings and results of Ricka easily identifies the quotient Hopf-algebroids of the $\mathbb{R}$-motivic Steenrod algebra.

For the inductive argument, first note that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{F}/(\rho^{n-1}) & \xrightarrow{\cdot\rho} & \mathrm{F}/(\rho^{n}) & \longrightarrow & \mathrm{F}/(\rho) & \longrightarrow & 0 \\ \| & & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_0 & & \| \\ 0 & \longrightarrow & \mathrm{M}/(\rho^{n-1}) & \xrightarrow{\cdot\rho} & \mathrm{M}/(\rho^{n}) & \longrightarrow & \mathrm{M}/(\rho) & \longrightarrow & 0 \end{array}$$

is a diagram of $\mathcal{A}^{\mathbb{R}}(n)$-modules (since ρ is in the center) where the horizontal rows are exact. The map f_0 is an isomorphism by assumption *(2)*, and f_{n-1} is an isomorphism by the inductive hypothesis; hence, f_n is an isomorphism by the five lemma. □

Proof of Theorem 1.18. The result follows immediately from Lemma 2.2 combined with [HK, Theorem B] and the fact that

$$\mathcal{A}^{\mathbb{C}}(n) = \mathcal{A}^{\mathbb{R}}(n)/(\rho).$$ □

The work of Adams and Margolis [AM] provides a freeness criterion for modules over finite-dimensional subalgebras of the classical Steenrod algebra. For an $\mathcal{A}(n)$-module M and element $x \in \mathcal{A}(n)$ such that $x^2 = 0$, one can define the Margolis homology of M with respect to x as

$$\mathcal{M}(\mathrm{M}, x) = \frac{\ker(x : \mathrm{M} \to \mathrm{M})}{\mathrm{img}(x : \mathrm{M} \to \mathrm{M})}.$$

Theorem 2.3. [AM, Theorem 4.4] *A finitely generated $\mathcal{A}(n)$-module M is free if and only if $\mathcal{M}(\mathrm{M}, \mathrm{P}^s_t) = 0$ for $0 < s < t$ with $s + t \leq n$.*

Remark 2.4. In the classical Steenrod algebra, P^s_t is the element dual to $\xi_t^{2^s}$. In terms of the Milnor basis, this is $\mathrm{Sq}(\underbrace{0, \ldots, 0}_{t-1}, 2^s)$. The element P^0_t is often denoted by Q_{t-1}.

Note that

$$\mathcal{A}^{\mathbb{R}}(n)_*/(\rho, \tau) = \frac{\mathbb{F}_2[\xi_1, \ldots, \xi_n]}{(\xi_1^{2^n}, \ldots, \xi_n^2)} \otimes \Lambda(\tau_0, \ldots, \tau_n)$$

as a Hopf-algebra. Further, if we forget the motivic grading, we have an isomorphism

$$\mathcal{A}^{\mathbb{R}}(n)/(\rho, \tau) \cong \varphi\mathcal{A}(n-1) \otimes \Lambda(\mathrm{P}^0_1, \ldots, \mathrm{P}^0_n), \tag{2.5}$$

where $\varphi\mathcal{A}(n-1)$ denotes the 'double' (see [M1, Chapter 15, Proposition 11]) of $\mathcal{A}(n-1)$. Let

$$\overline{\mathrm{P}}^s_t = (\xi_t^{2^s})^* \in \mathcal{A}^{\mathbb{R}}(n).$$

It can be shown that

$$(\overline{\mathrm{P}}^s_t)^2 \equiv 0 \mod (\rho, \tau)$$

for $s \leq t$. Combining (2.5), Theorem 2.3 and a similar result for primitively generated exterior Hopf-algebras [AM, Theorem 2.2], we deduce:

Lemma 2.6. *A finitely generated $\mathcal{A}^{\mathbb{R}}(n)/(\rho, \tau)$-module $\overline{\mathrm{M}}$ is free if and only if $\mathcal{M}(\overline{\mathrm{M}}, \overline{\mathrm{P}}^s_t) = 0$ whenever $0 \leq s \leq t$ and $1 \leq s + t \leq n + 1$.*

We end this section by recording the following corollary, which is immediate from Theorem 1.18 and Lemma 2.6.

Corollary 2.7. *A finitely generated $\mathcal{A}^{\mathbb{R}}(n)$ module* M *is free if and only if*

(1) M *is free as an $\mathbb{M}_2^{\mathbb{R}}$-module, and,*

(2) $\mathcal{M}(\mathrm{M} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{F}_2, \overline{\mathrm{P}}_s^t) = 0$ *for* $0 \le t \le s$ *and* $s + t = n + 1$.

3. A realization of $\mathcal{A}^{\mathbb{R}}(1)$

Consider the $\mathbb{R}$-motivic question mark complex $\mathcal{Q}_{\mathbb{R}}$, as introduced in Subsection 1.1. Let Σ_n act on $\mathcal{Q}_{\mathbb{R}}^{\wedge n}$ by permutation. Any element $e \in \mathbb{Z}_{(2)}[\Sigma_n]$ produces a canonical map

$$\tilde{e} : \mathcal{Q}_{\mathbb{R}}^{\wedge n} \longrightarrow \mathcal{Q}_{\mathbb{R}}^{\wedge n}.$$

Now let e be the idempotent

$$e = \tfrac{1+(1\ 2)-(1\ 3)-(1\ 3\ 2)}{3}$$

in $\mathbb{Z}_{(2)}[\Sigma_3]$, and denote by $\overline{e}$ the resulting idempotent of $\mathbb{F}_2[\Sigma_3]$. We record the following important property of $\overline{e}$ which is a special case of [R1, Theorem C.1.5].

Lemma 3.1. *If V is a finite-dimensional $\mathbb{F}_2$-vector space, then $\overline{e}(V^{\otimes 3}) = 0$ if and only if $\dim V \le 1$.*

The following result, which gives the values of $\overline{e}$ on induced representations, is also straightforward to verify:

Lemma 3.2. *Suppose that $W = \mathrm{Ind}_{C_2}^{\Sigma_3} \mathbb{F}_2$ is induced up from the trivial representation of a cyclic 2-subgroup. Then $\overline{e}(W) \cong \mathbb{F}_2$. Moreover, for the regular representation $\mathbb{F}_2[\Sigma_3] = \mathrm{Ind}_e^{\Sigma_3} \mathbb{F}_2$, we have $\dim \overline{e}(\mathbb{F}_2[\Sigma_3]) = 2$.*

We also record the fact that when $\dim_{\mathbb{F}_2} V = 2$ and $\dim_{\mathbb{F}_2} W = 3$ then

$$\dim_{\mathbb{F}_2} \overline{e}(V^{\otimes 3}) = 2 \quad \text{and} \quad \dim_{\mathbb{F}_2} \overline{e}(W^{\otimes 3}) = 8, \tag{3.3}$$

as we will often use this.

The bottom cell of $\tilde{e}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3})$ is in degree $(1, 0)$, and we define

$$\mathcal{A}_1^{\mathbb{R}} := \Sigma^{-1,0} \tilde{e}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3}) = \Sigma^{-1,0} \underset{\rightarrow}{\mathrm{hocolim}}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3} \xrightarrow{\tilde{e}} \mathcal{Q}_{\mathbb{R}}^{\wedge 3} \xrightarrow{\tilde{e}} \dots). \tag{3.4}$$

The purpose of this section is to prove the following theorem.

Theorem 3.5. *The spectrum $\mathcal{A}_1^{\mathbb{R}}$ is a type $(2, 1)$ complex whose bi-graded cohomology $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$ is a free $\mathcal{A}^{\mathbb{R}}(1)$-module on one generator.*

3.1. **$\mathcal{A}_1^{\mathbb{R}}$ is of type $(2,1)$.** Let $\mathcal{A}_1^{\mathrm{C}_2} := \beta(\mathcal{A}_1^{\mathbb{R}})$ and $\mathcal{Q}_{\mathrm{C}_2} := \beta(\mathcal{Q}_{\mathbb{R}})$. Note that we have a C_2-equivariant splitting

$$\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3} \simeq \tilde{e}(\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3}) \vee (1-\tilde{e})(\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3})$$

which splits the underlying spectra as well as the geometric fixed-points, as both Φ^e and Φ^{C_2} are additive functors.

We will identify the underlying spectrum $\Phi^e(\mathcal{A}_1^{\mathrm{C}_2})$ by studying the $\mathcal{A}$-module structure of its cohomology with $\mathbb{F}_2$-coefficients. Firstly, note that

$$\Phi^e(\mathcal{A}_1^{\mathrm{C}_2}) \simeq \Sigma^{-1}\tilde{e}(\Phi^e(\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3})) \simeq \Sigma^{-1}\tilde{e}(\mathcal{Q}^{\wedge 3}),$$

where $\mathcal{Q}$ is the classical question mark complex, whose $\mathrm{H}\mathbb{F}_2$-cohomology as an $\mathcal{A}$-module is well understood. It consists of three $\mathbb{F}_2$-generators a, b, and c in internal degrees 0, 1, and 3, such that $\mathrm{Sq}^1(a) = b$ and $\mathrm{Sq}^2(b) = c$ are the only nontrivial relations, as displayed in Figure 3.6.

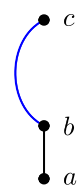
$\mathrm{H}^*(\mathcal{Q};\mathbb{F}_2) =$ 

FIGURE 3.6. We depict the $\mathcal{A}$-structure of $\mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)$ by marking Sq^1-action by black straight lines and Sq^2-action by blue curved lines between the $\mathbb{F}_2$-generators.

Because of the Kunneth isomorphism and the fact that the Steenrod algebra is cocommutative, we have an isomorphism of $\mathcal{A}$-modules

$$\mathrm{H}^{*+1}(\Phi^e(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2) \cong \mathrm{H}^*(\tilde{e}(\mathcal{Q}^{\wedge 3});\mathbb{F}_2) \cong \overline{e}(\mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)^{\otimes 3}).$$

Lemma 3.7. *The underlying $\mathcal{A}(1)$-module structure of $\mathrm{H}^*(\Phi^e(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2)$ is free on a single generator.*

Proof. Let us denote the $\mathcal{A}$-module $\mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)$ by V. Since $\dim \mathcal{M}(\mathrm{V}, Q_i) = 1$ for $i \in \{0,1\}$, it follows from the Kunneth isomorphism of Q_i-Margolis homology groups, cocommutativity of the Steenrod algebra, and Lemma 3.1 that

$$\mathcal{M}(\overline{e}(\mathrm{V}^{\otimes 3}), Q_i) = \overline{e}(\mathcal{M}(\mathrm{V}, Q_i)^{\otimes 3}) = 0$$

for $i = \{1,2\}$. It follows from [AM, Theorem 3.1] that $\mathrm{H}^*(\Phi^e(\mathcal{A}_1^{\mathbb{R}});\mathbb{F}_2)$ is free as an $\mathcal{A}(1)$-module. It is singly generated because of (3.3). □

We explicitly identify the image of $\overline{e} : \mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)^{\otimes 3} \longrightarrow \mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)^{\otimes 3}$ in Figure 3.8.

Remark 3.9. Using the Cartan formula, we can identify the action of Sq^4 on $\Phi^e(\mathcal{A}_1^{\mathrm{C}_2})$. We notice that its $\mathcal{A}$-module structure is isomorphic to $A_1[00]$ of [BEM]. Since such an $\mathcal{A}$-module is realized by a unique 2-local finite spectrum, we conclude

$$\Phi^e(\mathcal{A}_1^{\mathrm{C}_2}) \simeq A_1[00]$$

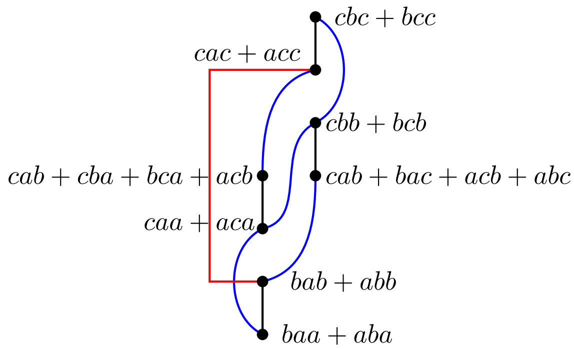


FIGURE 3.8. The $\mathcal{A}$-module structure of $\mathrm{H}^*(\Phi^e(\mathcal{A}_1^{\mathrm{C}_2};\mathbb{F}_2)$: Black straight lines, blue curved lines, and red boxed lines represent the Sq^1-action, Sq^2-action, and Sq^4-action, respectively.

and is of type 2.

Our next goal is to understand the homotopy type of the geometric fixed-point spectrum $\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2})$. First observe that the geometric fixed-points of the C_2-equivariant question mark complex $\mathcal{Q}_{\mathrm{C}_2}$ is the *exclamation mark* complex

$$\mathcal{E} :=$$

$$\simeq \quad \mathbb{S}^0 \vee \Sigma\mathrm{M}_2(1)!$$

This is because $\Phi^{\mathrm{C}_2}(\mathsf{h}) = 0$ and $\Phi^{\mathrm{C}_2}(\eta_{1,1}) = 2$. Secondly,

$$\mathrm{H}^{*+1}(\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2) \cong \mathrm{H}^*(\tilde{e}(\mathcal{E}^{\wedge 3});\mathbb{F}_2) \cong \overline{e}(\mathrm{H}^*(\mathcal{E};\mathbb{F}_2)^{\otimes 3})$$

is an isomorphism of $\mathcal{A}$-modules. We explicitly calculate the $\mathcal{A}$-module structure of $\mathrm{H}^*(\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2)$ from the above isomorphism and record it in Figure 3.10.

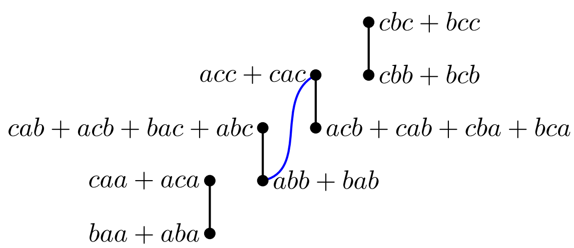


FIGURE 3.10. The $\mathcal{A}$-module structure of $\mathrm{H}^*(\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2)$.

Lemma 3.11. *The finite spectrum $\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2})$ is a type 1 spectrum and equivalent to*

$$\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2}) \simeq \mathrm{M}_2(1) \vee \Sigma\Big(\mathrm{M}_2(1) \wedge \mathrm{M}_2(1)\Big) \vee \Sigma^3\mathrm{M}_2(1).$$

Proof. From Figure 3.10, it is clear that we have an isomorphism of $\mathcal{A}$-modules

$$\mathrm{H}^*(\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2) \cong \mathrm{H}^*\Big(\mathrm{M}_2(1)\vee\Sigma\big(\mathrm{M}_2(1)\wedge\mathrm{M}_2(1)\big)\vee\Sigma^3\mathrm{M}_2(1);\mathbb{F}_2\Big).$$

It is possible that the $\mathcal{A}$-module $\mathrm{H}^*(\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2)$ may not realize to a unique finite spectrum (up to weak equivalence). However, other possibilities can be eliminated from the fact that $\mathcal{E}^{\wedge 3}$ splits Σ_3-equivariantly into four components:

$$\mathcal{E}^{\wedge 3}\simeq \mathbb{S}\vee\left(\bigvee_{i=1}^{3}\Sigma\mathrm{M}_2(1)\right)\vee\left(\bigvee_{i=1}^{3}\Sigma^2\mathrm{M}_2(1)^{\wedge 2}\right)\vee\Sigma^3\mathrm{M}_2(1)^{\wedge 3}.$$

The idempotent $\tilde{e}$ annihilates $\mathbb{S}\cong\mathbb{S}^{\wedge 3}$, and Lemma 3.2 implies that

$$\tilde{e}\left(\bigvee_{i=1}^{3}\Sigma\mathrm{M}_2(1)\right)\simeq\Sigma\mathrm{M}_2(1)\qquad\text{and}$$

$$\tilde{e}\left(\bigvee_{i=1}^{3}\Sigma^2\mathrm{M}_2(1)\wedge\mathrm{M}_2(1)\right)\simeq\Sigma^2\mathrm{M}_2(1)\wedge\mathrm{M}_2(1).$$

Similarly, we see using (3.3) that

$$\mathrm{H}^*\left(\tilde{e}\left(\Sigma^3\mathrm{M}_2(1)^{\wedge 3}\right)\right)\cong\overline{e}\left(\mathrm{H}^*\left(\Sigma\mathrm{M}_2(1)\right)^{\otimes 3}\right)\cong\mathrm{H}^*(\Sigma^3\mathrm{M}_2(1)).$$

Hence, the result. □

3.2. **The cohomology of $\mathcal{A}_1^{\mathbb{R}}$ is free over $\mathcal{A}^{\mathbb{R}}(1)$.** Next, we analyze the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$. We begin by recalling some general properties of the cohomology of motivic spectra.

If $X, Y\in\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ such that $\mathrm{H}^{*,*}(X)$ is free as a left $\mathbb{M}_2^{\mathbb{R}}$-module, then we have a Kunneth isomorphism [DI2, Proposition 7.7]

$$\mathrm{H}^{*,*}(X\wedge Y)\cong\mathrm{H}^{*,*}(X)\otimes_{\mathbb{M}_2^{\mathbb{R}}}\mathrm{H}^{*,*}(Y)\tag{3.12}$$

as the relevant Kunneth spectral sequence collapses. Further, if $\mathrm{H}^{*,*}(X)$ is free as a left $\mathbb{M}_2^{\mathbb{R}}$-module, then so is $\mathrm{H}^{*,*}(X\wedge Y)$. The $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(X\wedge Y)$ can then be computed using the Cartan formula. The comultiplication map of $\mathcal{A}^{\mathbb{R}}$ is left $\mathbb{M}_2^{\mathbb{R}}$-linear, coassociative and cocommutative [V, Lemma 11.9], which is also reflected in the fact that its $\mathbb{M}_2^{\mathbb{R}}$-linear dual is a commutative and associative algebra. Thus, when $\mathrm{H}^{*,*}(X)$ is a free left $\mathbb{M}_2^{\mathbb{R}}$-module, the elements of $\mathbb{F}_2[\Sigma_n]$ acts on

$$\mathrm{H}^{*,*}(X^{\wedge n})\cong\mathrm{H}^{*,*}(X)\otimes_{\mathbb{M}_2^{\mathbb{R}}}\cdots\otimes_{\mathbb{M}_2^{\mathbb{R}}}\mathrm{H}^{*,*}(X)$$

via permutation and commutes with the action of $\mathcal{A}^{\mathbb{R}}$. This also implies that $\mathbb{F}_2[\Sigma_n]$ also acts on

$$\mathrm{H}^{*,*}(X^{\wedge n})/(\rho,\tau)\cong(\mathrm{H}^{*,*}(X)/(\rho,\tau))\otimes\cdots\otimes\mathrm{H}^{*,*}(X)/(\rho,\tau)$$

and commutes with the action of $\mathcal{A}^{\mathbb{R}}/\!\!/\mathbb{M}_2^{\mathbb{R}}$. From the above discussion we may conclude that

$$\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})\cong\Sigma^{-1}\overline{e}(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})^{\otimes 3})\tag{3.13}$$

is an isomorphism of $\mathcal{A}^{\mathbb{R}}$-module.

We will also rely upon the following important property of the action of the motivic Steenrod algebra on the cohomology of a motivic space (as opposed to a motivic spectrum):

Remark 3.14 (Instability condition for $\mathbb{R}$-motivic cohomology)**.** If X is an $\mathbb{R}$-motivic space then $\mathrm{H}^{*,*}(X)$ admits a ring structure, and, for any $u \in \mathrm{H}^{n,i}(X)$, the $\mathbb{R}$-motivic squaring operations obey the rule

$$\mathrm{Sq}^{2i}(u) = \begin{cases} 0 & \text{if } n < 2i \\ u^2 & \text{if } n = 2i. \end{cases}$$

This is often referred to as the *instability condition*.

To understand the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$, we first make the following observation regarding $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ (as $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ is a sub-complex of $\mathcal{Q}_{\mathbb{R}}$) using an argument very similar to [DI1, Lemma 7.4].

Proposition 3.15. *There are two extensions of $\mathcal{A}^{\mathbb{R}}(0)$ to an $\mathcal{A}^{\mathbb{R}}$-module, and these $\mathcal{A}^{\mathbb{R}}$-modules are realized as the cohomology of $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathrm{C}^{\mathbb{R}}(2)$.*

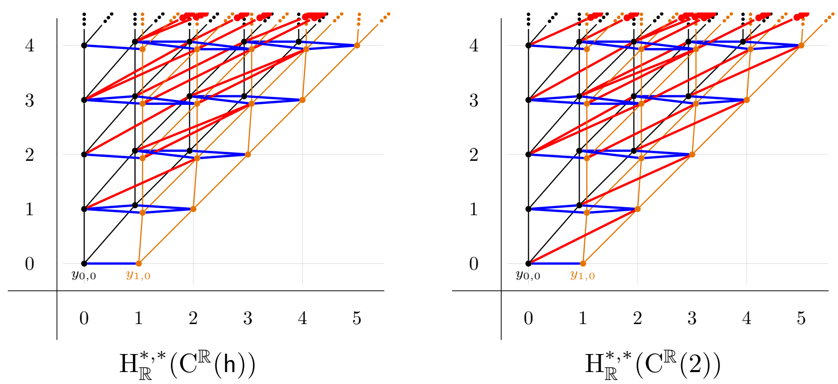


FIGURE 3.16. The x-axis represents the negative of topological dimension, y-axis represents the negative of motivic weight, vertical line of length $(0,1)$ represents τ-multiplication, diagonal line of length $(1,1)$ represents ρ-multiplication, blue line represents Sq^1-action and red line represents Sq^2-action.

Proof. For degree reasons, the only choice in extending $\mathcal{A}^{\mathbb{R}}(0)$ to an $\mathcal{A}^{\mathbb{R}}$-module is the action of Sq^2 on the generator in bidegree $(0,0)$. Writing $y_{0,0}$ for the generator in degree (0, 0) and $y_{1,0}$ for $\mathrm{Sq}^1(y_{0,0})$ in (cohomological) bidegree $(1,0)$. The two possible choices are

- $\mathrm{Sq}^2(y_{0,0}) = 0$ and
- $\mathrm{Sq}^2(y_{0,0}) = \rho \cdot y_{1,0}$.

We can realize the degree 2 map as an unstable map $\mathrm{S}^{1,0} \longrightarrow \mathrm{S}^{1,0}$, and we will write $\mathrm{C}^{\mathbb{R}}(2)_u$ for the cofiber. We deduce information about the $\mathcal{A}^{\mathbb{R}}$-module structure of

$\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$ by analyzing the cohomology ring of $\mathrm{S}^{1,1} \wedge \mathrm{C}^{\mathbb{R}}(2)_u$ using the instability condition of Remark 3.14. First, note that in

$$\mathrm{H}^{*,*}(\mathrm{S}^{1,1}) \cong \mathbb{M}_2^{\mathbb{R}} \cdot \iota_{1,1}$$

we have the relation $\iota_{1,1}^2 = \rho \cdot \iota_{1,1}$ [V, Lemma 6.8]. Also note that

$$\mathrm{H}^{*,*}((\mathrm{C}^{\mathbb{R}}(2)_u)_+) \cong \mathbb{M}_2^{\mathbb{R}}[x]/(x^3)$$

where x is in cohomological degrees $(1,0)$. Therefore, in

$$\mathrm{H}^{*,*}(\mathrm{S}^{1,1} \wedge \mathrm{C}^{\mathbb{R}}(2)_u) = \mathbb{M}_2^{\mathbb{R}} \cdot \iota_{1,1} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{M}_2^{\mathbb{R}}\{x, x^2\}$$

the instability condition implies

$$\mathrm{Sq}^2(\iota_{1,1} \otimes x) = \iota_{1,1}^2 \otimes x^2 = \rho \cdot \iota_{1,1} \otimes x^2.$$

Here the space-level cohomology class x^2 corresponds to the spectrum-level class $y_{1,0}$. Therefore, $\mathrm{Sq}^2(y_{0,0}) = \rho \cdot y_{1,0}$ in $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$. This is also reflected in the fact that multiplication by 2 is detected by $h_0 + \rho h_1$ in the $\mathbb{R}$-motivic Adams spectral sequence [DI1, §8].

On the other hand h is the 'zeroth $\mathbb{R}$-motivic Hopf-map' detected by the element h_0 in the motivic Adams spectral sequence . It follows that $\mathrm{Sq}^2(y_{0,0}) = 0$. □

In order to express the $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathrm{H}^{*,*}(X)$ for a finite spectrum X, it is enough to specify the action of $\mathcal{A}^{\mathbb{R}}$ on its left $\mathbb{M}_2^{\mathbb{R}}$-generators as the action of τ and ρ multiples are determined by the Cartan formula.

Example 3.17. Let $\{y_{0,0}, y_{1,0}\} \subset \mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ denote a left $\mathbb{M}_2^{\mathbb{R}}$-basis of $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$. The data that

- $\mathrm{Sq}^1(y_{0,0}) = y_{1,0}$
- $\mathrm{Sq}^2(y_{0,0}) = 0$

completely determines the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$.

Proposition 3.18. $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$ *is a free* $\mathbb{M}_2^{\mathbb{R}}$*-module generated by* a, b *and* c *in cohomological bidegrees* $(0,0), (1,0)$ *and* $(3,1)$*, and the relations*

(1) $\mathrm{Sq}^1(a) = b$*,*

(2) $\mathrm{Sq}^2(b) = c$*,*

(3) $\mathrm{Sq}^4(a) = 0$*.*

completely determine the $\mathcal{A}^{\mathbb{R}}$*-module structure of* $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$*.*

Proof. $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$ is a free $\mathbb{M}_2^{\mathbb{R}}$-module because the attaching maps of $\mathcal{Q}_{\mathbb{R}}$ induce trivial maps in $\mathrm{H}^{*,*}(-)$. The first two relations can be deduced from the obvious maps

(1) $\mathrm{C}^{\mathbb{R}}(\mathsf{h}) \to \mathcal{Q}_{\mathbb{R}}$

(2) $\mathcal{Q}_{\mathbb{R}} \to \Sigma^{1,0}\,\mathrm{C}^{\mathbb{R}}(\eta_{1,1})$

which are respectively surjective and injective in cohomology.

Let $\mathsf{h}^u : \mathrm{S}^{3,2} \to \mathrm{S}^{3,2}$ and $\eta^u_{1,1} : \mathrm{S}^{3,2} \to \mathrm{S}^{2,1}$ denote the unstable maps that stabilize to h and $\eta_{1,1}$, respectively. The unstable $\mathbb{R}$-motivic space $\mathcal{Q}^u_{\mathbb{R}}$ (which stabilizes to $\mathcal{Q}_{\mathbb{R}}$) can be constructed using the fact that the composite of the unstable maps

$$\mathrm{S}^{4,3} \xrightarrow{\Sigma^{1,1}\eta^u_{1,1}} \mathrm{S}^{3,2} \xrightarrow{\mathsf{h}^u} \mathrm{S}^{3,2}$$

is null. Thus $\mathrm{H}^{*,*}(\mathcal{Q}^u_{\mathbb{R}})$ consists of three generators a_u, b_u and c_u in bidegrees $(3,2)$, $(4,2)$ and $(6,3)$. It follows from the instability condition that $\mathrm{Sq}^4(a_u) = 0$. □

Proof of Theorem 3.5. From Remark 3.9 and Lemma 3.11, we deduce that $\mathcal{A}^{\mathbb{R}}_1$ is a type $(2,1)$ complex. To show that the bi-graded $\mathbb{R}$-motivic cohomology of $\mathcal{A}^{\mathbb{R}}_1$ is free as an $\mathcal{A}^{\mathbb{R}}(1)$, we make use of Corollary 2.7.

Since $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$ is a summand of a free $\mathbb{M}^{\mathbb{R}}_2$-module, it is projective as an $\mathbb{M}^{\mathbb{R}}_2$-module. In fact, $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$ is free, as projective modules over (graded) local rings are free. Also note that the elements

$$\overline{\mathrm{P}}^0_1, \overline{\mathrm{P}}^1_1, \overline{\mathrm{P}}^0_2 \in \mathcal{A}^{\mathbb{R}}(1)/(\rho,\tau) \cong \Lambda(\overline{\mathrm{P}}^0_1, \overline{\mathrm{P}}^1_1, \overline{\mathrm{P}}^0_2)$$

are primitive. Hence we have a Kunneth isomorphism in the respective Margolis homologies, in particular we have,

$$\mathcal{M}(\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)/(\rho,\tau), \overline{\mathrm{P}}^s_t) = \overline{e}(\mathcal{M}(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})/(\rho,\tau), \overline{\mathrm{P}}^s_t)^{\otimes 3})$$

for $(s,t) \in \{(0,1),(1,1),(0,2)\}$. Since $\dim_{\mathbb{F}_2} \mathcal{M}(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})/(\rho,\tau), \overline{\mathrm{P}}^s_t) = 1$, by Lemma 3.1

$$\mathcal{M}(\mathcal{A}^{\mathbb{R}}_1/(\rho,\tau), \overline{\mathrm{P}}^s_t) = 0$$

for $(s,t) \in \{(0,1),(1,1),(0,2)\}$. Thus, by Corollary 2.7 we conclude that $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$ is a free $\mathcal{A}^{\mathbb{R}}(1)$-module. A direct computation shows that

$$\dim_{\mathbb{F}_2} \mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)/(\rho,\tau) = 8,$$

hence $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$ is $\mathcal{A}^{\mathbb{R}}(1)$-free of rank one. □

3.3. **The $\mathcal{A}^{\mathbb{R}}$-module structure.** Using the description (3.13) and Cartan formula we make a complete calculation of the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$. Let $a,b,c \in \mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$ as in Proposition 3.18. In Figure 3.20 we provide a pictorial representation with the names of the generators that are in the image of the idempotent $\overline{e}$. For convenience we relabel the generators in Figure 3.20, where the indexing on a new label records the cohomological bidegrees of the corresponding generator. The following result is straightforward, and we leave it to the reader to verify.

Lemma 3.19. *In $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$, the underlying $\mathcal{A}^{\mathbb{R}}(1)$-module structure, along with the relations*

(1) $\mathrm{Sq}^4(v_{0,0}) = 0$,

(2) $\mathrm{Sq}^4(v_{1,0}) = \tau \cdot w_{5,2}$,

(3) $\mathrm{Sq}^4(v_{2,1}) = 0$,

(4) $\mathrm{Sq}^4(v_{3,1}) = 0 = \mathrm{Sq}^4(w_{3,1})$,

(5) $\mathrm{Sq}^8(v_{0,0}) = 0$,

completely determine the $\mathcal{A}^{\mathbb{R}}$*-module structure.*

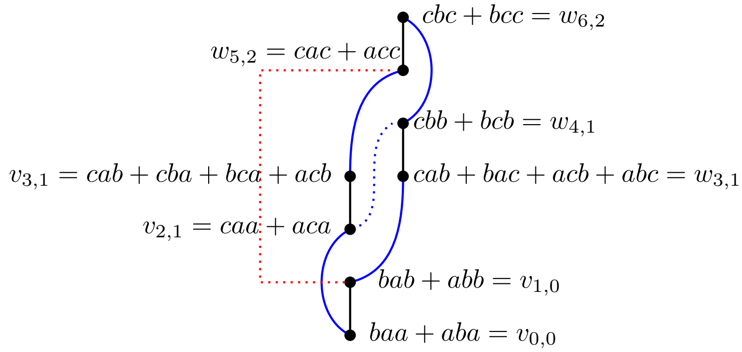


FIGURE 3.20. We depict the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{A}_1)$. The black, blue, and red lines represent the action of motivic Sq^1, Sq^2, and Sq^4, respectively. Black dots represent $\mathbb{M}_2^{\mathbb{R}}$-generators, and a dotted line represents that the action hits the τ-multiple of the given $\mathbb{M}_2^{\mathbb{R}}$-generator.

Remark 3.21. In upcoming work, we show that $\mathcal{A}^{\mathbb{R}}(1)$ admits 128 different $\mathcal{A}^{\mathbb{R}}$-module structures. Whether all of the 128 $\mathcal{A}^{\mathbb{R}}$-module structures can be realized by $\mathbb{R}$-motivic spectra, or not, is currently under investigation.

4. An $\mathbb{R}$-motivic v_1–self-map

With the construction of $\mathcal{A}_1^{\mathbb{R}}$, we hope that any one of $\mathcal{Y}_{(i,j)}^{\mathbb{R}}$ fits into an exact triangle

$$\Sigma^{2,1}\mathcal{Y}_{(i,j)}^{\mathbb{R}} \xrightarrow{v} \mathcal{Y}_{(i,j)}^{\mathbb{R}} \longrightarrow \mathcal{A}_1^{\mathbb{R}} \longrightarrow \Sigma^{3,1}\mathcal{Y}_{(i,j)}^{\mathbb{R}} \xrightarrow{\Sigma v} \dots \tag{4.1}$$

in $\mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}})$. The motivic weights prohibit $\mathcal{A}_1^{\mathbb{R}}$ from being the cofiber of a self-map on $\mathcal{Y}_{\mathrm{triv}}$ or $\mathcal{Y}_{(\mathsf{h},0)}$. We will also see that the spectrum $\mathcal{Y}_{(2,1)}^{\mathbb{R}}$ cannot be a part of (4.1) because of its $\mathcal{A}^{\mathbb{R}}$-module structure (see Lemma 4.5). If $\mathcal{Y}_{(i,j)} = \mathcal{Y}_{(\mathsf{h},1)}^{\mathbb{R}}$ in (4.1), then the map v will necessarily be a $v_{1,\mathrm{nil}}$–self-map because $\mathcal{Y}_{(\mathsf{h},1)}^{\mathbb{R}}$ is of type $(1,1)$ and $\mathcal{A}_1^{\mathbb{R}}$ is of type $(2,1)$. The main purpose of this section is to prove Theorem 1.11 and Theorem 1.14 by showing that $\mathcal{Y}_{(\mathsf{h},1)}^{\mathbb{R}}$ does fit into an exact triangle very similar to (4.1)

$$\Sigma^{2,1}\mathcal{Y}_{(i,j)}^{\mathbb{R}} \xrightarrow{v} \mathcal{Y}_{(i,j)}^{\mathbb{R}} \longrightarrow \mathrm{C}^{\mathbb{R}}(v) \longrightarrow \Sigma^{3,1}\mathcal{Y}_{(i,j)}^{\mathbb{R}} \xrightarrow{\Sigma v} \dots$$

where $\mathrm{C}^{\mathbb{R}}(v)$ is of type $(2,1)$ and $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(v)) \cong \mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$ as $\mathcal{A}^{\mathbb{R}}$-modules but potentially may have a homotopy type different than that of $\mathcal{A}_1^{\mathbb{R}}$. We begin by discussing the $\mathcal{A}^{\mathbb{R}}$-module structures of $\mathrm{H}^{*,*}(\mathcal{Y}_{(\mathsf{h},1)}^{\mathbb{R}})$.

Using Adem relations, one can show that the element

$$Q_1 := \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1 \in \mathcal{A}^{\mathbb{R}}(1)$$

squares to zero. Let $\Lambda(Q_1)$ denote the exterior subalgebra $\mathbb{M}_2^{\mathbb{R}}[Q_1]/(Q_1^2)$ of $\mathcal{A}^{\mathbb{R}}(1)$. Let $\mathcal{B}^{\mathbb{R}}(1)$ denote the $\mathcal{A}^{\mathbb{R}}(1)$-module

$$\mathcal{B}^{\mathbb{R}}(1) := \mathcal{A}^{\mathbb{R}}(1) \otimes_{\Lambda(Q_1)} \mathbb{M}_2^{\mathbb{R}}.$$

Both $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ are realizations of $\mathcal{B}^{\mathbb{R}}(1)$. In other words:

Proposition 4.2. *There is an isomorphism of $\mathcal{A}^{\mathbb{R}}(1)$-modules*

$$\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)}) \cong \mathcal{B}^{\mathbb{R}}(1)$$

for $(i,j) \in \{(2,1),(\mathsf{h},1)\}$.

Proof. Note that $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)})$ is cyclic as an $\mathcal{A}^{\mathbb{R}}(1)$-module for $(i,j) \in \{(2,1),(\mathsf{h},1)\}$. Thus we have a map

$$f_i : \mathcal{A}^{\mathbb{R}}(1) \to \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)}). \tag{4.3}$$

The result follows from the fact that Q_1 acts trivially on $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)})$ and a dimension counting argument. □

Remark 4.4. Let $\{y_{0,0}, y_{1,0}\}$ be the $\mathbb{M}_2^{\mathbb{R}}$-basis of $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ or $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$, so that $\mathrm{Sq}^1(y_{0,0}) = y_{1,0}$, and let $\{x_{0,0}, x_{2,1}\}$ a basis of $\mathrm{C}^{\mathbb{R}}(\eta_{1,1})$, so that $\mathrm{Sq}^2(x_{0,0}) = x_{2,1}$. If we consider the $\mathbb{M}_2^{\mathbb{R}}$-basis $\{v_{0,0}, v_{1,0}, v_{2,1}, v_{3,1}, w_{3,1}, w_{3,2}, w_{4,2}, w_{5,3}, w_{6,3}\}$ of $\mathcal{A}^{\mathbb{R}}(1)$ from Subsection 3.3, then the maps f_i of (4.3) are given as in Table 1.

TABLE 1. The maps f_2 and f_{h}

x	$f_2(x)$	$f_{\mathsf{h}}(x)$
$v_{0,0}$	$y_{0,0}x_{0,0}$	$y_{0,0}x_{0,0}$
$v_{1,0}$	$y_{1,0}x_{0,0}$	$y_{1,0}x_{0,0}$
$v_{2,1}$	$y_{0,0}x_{2,0} + \rho \cdot y_{1,0}x_{0,0}$	$y_{0,0}x_{2,0}$
$v_{3,1}$	$y_{1,0}x_{2,0}$	$y_{1,0}x_{2,0}$
$w_{3,1}$	$y_{1,0}x_{2,0}$	$y_{1,0}x_{2,0}$
$w_{4,2}$	0	0
$w_{5,3}$	0	0
$w_{6,3}$	0	0

Lemma 4.5. *The $\mathcal{A}^{\mathbb{R}}$-module structures on $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(2,1)})$ and $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ are given as in Figure 4.6.*

Proof. The result is an easy consequence of a calculation using the Cartan formula

$$\mathrm{Sq}^4(xy) = \mathrm{Sq}^4(x)y + \tau\mathrm{Sq}^3(x)\mathrm{Sq}^1(y) + \mathrm{Sq}^2(x)\mathrm{Sq}^2(y) + \tau\mathrm{Sq}^1(x)\mathrm{Sq}^3(y) + x\mathrm{Sq}^4(y)$$

and the fact that $\mathrm{Sq}^2(y_{0,0}) = \rho y_{1,0}$ in $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$, whereas $\mathrm{Sq}^2(y_{0,0}) = 0$ in $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ (see Proposition 3.15). □

Remark 4.7. Comparing Lemma 4.5 and Lemma 3.19, we see that the $\mathcal{A}^{\mathbb{R}}(1)$-module map f_2, as in Remark 4.4, cannot be extended to a map of $\mathcal{A}^{\mathbb{R}}$-modules.

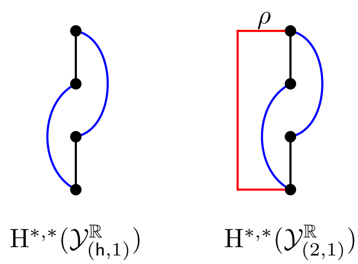


FIGURE 4.6. Black, blue, and red lines represent the action of Sq^1, Sq^2, and Sq^4, respectively. Black dots represent $\mathbb{M}_2^{\mathbb{R}}$-generators, and in the case of $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$, Sq^4 on the bottom cell is ρ times the top cell.

Corollary 4.8. *There is an exact sequence of $\mathcal{A}^{\mathbb{R}}$-modules*

$$0 \longrightarrow \mathrm{H}^{*,*}(\Sigma^{3,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \xrightarrow{\pi^*} \mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}}) \xrightarrow{\iota^*} \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \longrightarrow 0. \tag{4.9}$$

Proof. From the description of the map f_{h} in Remark 4.4, along with Lemma 3.19 and Lemma 4.5, it is easy to check that f_{h} extends to an $\mathcal{A}^{\mathbb{R}}$-module map and that

$$\ker f_{\mathsf{h}} \cong \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$$

as $\mathcal{A}^{\mathbb{R}}$-modules. □

The exact sequence (4.9) corresponds to a nonzero element in the E_2-page of the $\mathbb{R}$-motivic Adams spectral sequence (also see Remark 4.12 and Remark 4.14)

$$\overline{v} \in \mathrm{Ext}^{2,1,1}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \wedge D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathbb{M}_2^{\mathbb{R}}) \Rightarrow [\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}, \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}]_{2,1}, \tag{4.10}$$

where $D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} := \mathrm{F}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}, \mathbb{S}_{\mathbb{R}})$ is the Spanier-Whitehead dual of $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$. If

Notation 4.11. Note that we follow [DI1, BI] in grading $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}$ as $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}}$, where s is the stem, f is the Adams filtration, and w is the weight. We will also follow [GI1] in referring to the difference $s - w$ as the *coweight*.

Remark 4.12. Since $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ is $\mathbb{M}_2^{\mathbb{R}}$-free, an appropriate universal-coefficient spectral sequence collapses and we get $\mathrm{H}^{*,*}(D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \cong \hom_{\mathbb{M}_2^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathbb{M}_2^{\mathbb{R}})$. Further, the Kunneth isomorphism of (3.12) gives us

$$\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \wedge D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \cong \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathrm{H}^{*,*}(D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}),$$

and therefore,

$$\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \wedge D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) \cong \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})).$$

Theorem 1.11 follows immediately if we show that the element $\overline{v}$ is a nonzero permanent cycle. The following lemma implies that a d_r-differential (for $r \geq 2$) supported by $\overline{v}$ has no potential nonzero target.

Proposition 4.13. *For $f \geq 3$, $\mathrm{Ext}^{1,f,1}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) = 0$.*

In order to calculate $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$, we filter the spectrum $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ via the evident maps

$$\begin{array}{ccccccc} Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3. \\ \| & & \| & & \| & & \| \\ \mathbb{S}_{\mathbb{R}} & & \mathrm{C}^{\mathbb{R}}(\mathsf{h}) & & \mathrm{C}^{\mathbb{R}}(\mathsf{h}) \cup_{\mathbb{S}_{\mathbb{R}}} \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) & & \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \end{array}$$

Note that $\mathrm{H}^{*,*}(Y_j)$ are free $\mathbb{M}^{\mathbb{R}}_2$-modules. The above filtration results in cofiber sequences

$$Y_0 \longrightarrow Y_1 \longrightarrow \Sigma^{1,0}\mathbb{S}_{\mathbb{R}},$$

$$Y_1 \longrightarrow Y_2 \longrightarrow \Sigma^{2,1}\mathbb{S}_{\mathbb{R}}, \qquad \text{and}$$

$$Y_2 \longrightarrow Y_3 \longrightarrow \Sigma^{3,1}\mathbb{S}_{\mathbb{R}},$$

which induce short exact sequences of $\mathcal{A}^{\mathbb{R}}$-modules as the connecting map

$$\mathrm{C}^{\mathbb{R}}(Y_j \to Y_{j+1}) \longrightarrow \Sigma Y_j$$

induces the zero map in $\mathrm{H}^{*,*}(-)$. Thus, applying the functor $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}_{(\mathsf{h},1)}), -)$ to these short-exact sequences, we get long exact sequences, which can be spliced together to obtain an Atiyah-Hirzebruch like spectral sequence

$$\begin{array}{c} \mathrm{E}_1^{*,*,*,*} = \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}_{(\mathsf{h},1)}), \mathbb{M}^{\mathbb{R}}_2)\{g_{0,0}, g_{1,0}, g_{2,1}, g_{3,1}\} \\ \Downarrow \\ \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})). \end{array}$$

An element $x \cdot g_{i,j}$ in the E_2-page contributes to the degree $|x| - (i, 0, j)$ of the abutment. Thus, Proposition 4.13 is a straightforward consequence of the following Proposition 4.15.

Remark 4.14. Because, $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ is $\mathbb{M}^{\mathbb{R}}_2$-free and finite, we have

$$\mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \cong \mathrm{hom}_{\mathbb{M}^{\mathbb{R}}_2}(\mathrm{H}^{*,*}(\mathcal{Y}_{(\mathsf{h},1)}), \mathbb{M}^{\mathbb{R}}_2),$$

and therefore, $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathbb{M}^{\mathbb{R}}_2) \cong \mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$.

Proposition 4.15. *For $f \geq 3$ and $(i,j) \in \{(0,0),(1,0),(2,1),(3,1)\}$, we have that*

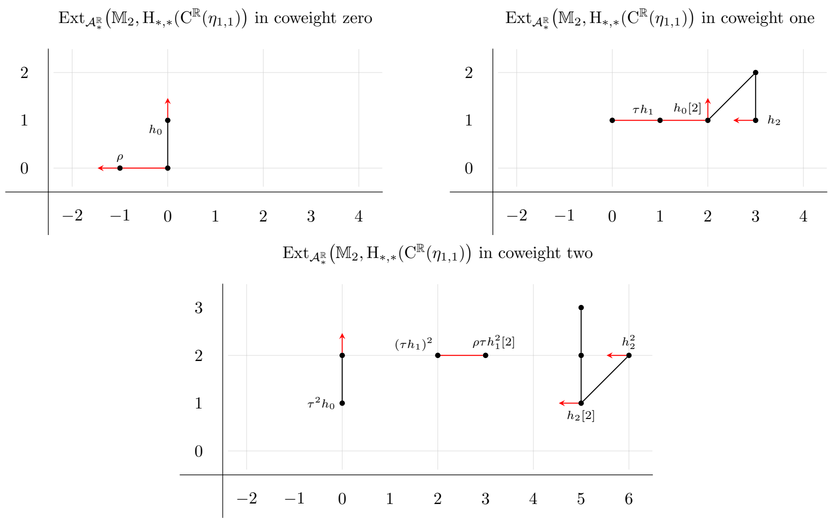
$$\mathrm{Ext}^{1+i,f,1+j}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) = 0.$$

Proof. Our desired vanishing concerns only the groups $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$ in coweights 0, 1 and 2. These groups can be easily calculated starting from the computations of $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathbb{M}^{\mathbb{R}}_2)$ in [DI1] and [BI] and using the short exact sequences in $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}$ arising from the cofiber sequences

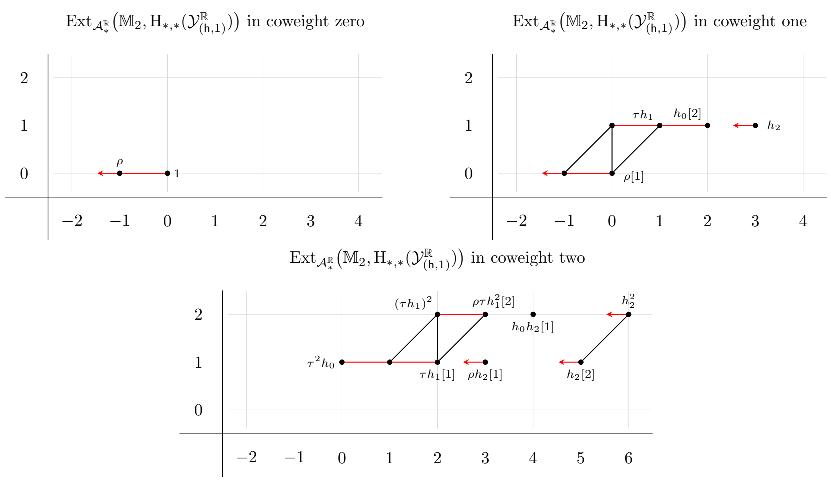
$$\Sigma^{1,1}S_{\mathbb{R}} \xrightarrow{\eta_{1,1}} S_{\mathbb{R}} \longrightarrow \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) \quad \text{and}$$

$$\mathrm{C}^{\mathbb{R}}(\eta_{1,1}) \xrightarrow{\mathsf{h}} \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) \longrightarrow \mathrm{C}^{\mathbb{R}}(\mathsf{h}) \wedge \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) = \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}.$$

We display $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathrm{C}^{\mathbb{R}}(\eta_{1,1})))$ in coweights 0, 1 and 2 in the charts below.



We find that $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}_2^{\mathbb{R}}, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$ is, in coweights zero, one and two, also given by the charts below.



The result follows from the above charts. □

Remark 4.16. One can also resolve Proposition 4.15 directly using the ρ-Bockstein spectral sequence

$$\begin{array}{c} \mathrm{E}_1 := \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}_*}(\mathbb{F}_2[\tau], \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{C}}_{(\mathsf{h},1)})) \otimes \mathbb{F}_2[\rho] \\ \Downarrow \\ \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}_2^{\mathbb{R}}, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) \end{array} \tag{4.17}$$

and identifying a vanishing region for $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{C}}_*}(\mathbb{F}_2[\tau], \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{C}}_{(\mathsf{h},1)}))$. Even a rough estimate of the vanishing region using the E_1-page of the $\mathbb{C}$-motivic May spectral sequence leads to Proposition 4.15. Such an approach would avoid explicit calculations of $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}$ as in [DI1] and [BI].

Proof of Theorem 1.11. Since Proposition 4.15 $\implies$ Proposition 4.13, every map

$$v : \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \longrightarrow \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$$

detected by $\overline{v}$ of (4.10) is a nonzero permanent cycle. In order to finish the proof of Theorem 1.11 we must show that v is necessarily $v_{(1,\mathrm{nil})}$–self-map of periodicity 1. It is easy to see that the underlying map

$$\Phi^e(\beta(v)) : \Sigma^2\mathcal{Y} \longrightarrow \mathcal{Y}$$

is a v_1–self-map of periodicity 1 as

$$\mathrm{C}(\Phi^e(\beta(v))) \simeq \Phi^e(\beta(\mathrm{C}^{\mathbb{R}}(v))) \simeq \mathcal{A}_1[00]$$

is of type 1 (see Remark 3.9). On the other hand,

$$\Phi^{\mathrm{C}_2}(\beta(v)) : \Sigma^2(\Sigma\mathrm{M}_2(1) \vee \mathrm{M}_2(1)) \longrightarrow \Sigma\mathrm{M}_2(1) \vee \mathrm{M}_2(1)$$

is necessarily a nilpotent map because of [HS, Theorem 3(ii)] and the fact that a v_1–self-map of $\mathrm{M}_2(1)$ has periodicity at least 4 (see [DM] for details) which lives in $[\mathrm{M}_2(1), \mathrm{M}_2(1)]_{8k}$ for $k \geq 1$. □

Proof of Theorem 1.14. Since v is a $v_{(1,\mathrm{nil})}$–self-map and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ is of type $(1,1)$, it follows that $\mathrm{C}^{\mathbb{R}}(v)$ is of type $(2,1)$. Moreover,

$$\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(v)) \cong \mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$$

as v is detected by $\overline{v}$ of (4.10) in the E_2-page of the Adams spectral sequence. Thus, $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(v))$ is a free $\mathcal{A}^{\mathbb{R}}(1)$-module on single generator. □

Remark 4.18. It is likely that realizing a different $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$ as a spectrum (see also Remark 3.21) may lead to a 1-periodic v_1–self-map on $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ as well as on $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}$. We explore such possibilities in upcoming work.

5. Nonexistence of $v_{1,0}$–self-map on $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$

Let X be a finite $\mathbb{R}$-motivic spectrum and let $f\colon \Sigma^{i,j}X \to X$ be a map such that

$$\Phi^{\mathrm{C}_2}(\beta(f))\colon \Sigma^{i-j}\Phi^{\mathrm{C}_2}(\beta(X)) \longrightarrow \Phi^{\mathrm{C}_2}(\beta(X))$$

is a v_0–self-map. Then it must be the case that $i = j$, as v_0–self-maps preserve dimension. Note that both $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$ are of type $(1,0)$.

Proposition 5.1. *The v_1–self-maps of $\mathrm{M}_2(1)$ are not in the image of the underlying homomorphism*

$$\Phi^e \circ \beta\colon [\Sigma^{8k,8k}\mathrm{C}^{\mathbb{R}}(\mathsf{h}), \mathrm{C}^{\mathbb{R}}(\mathsf{h})]^{\mathbb{R}} \longrightarrow [\Sigma^{8k}\mathrm{M}_2(1), \mathrm{M}_2(1)].$$

Proof. The minimal periodicity of a v_1–self-map of $\mathrm{M}_2(1)$ is 4. Let $v : \Sigma^{8k}\mathrm{M}_2(1) \to \mathrm{M}_2(1)$ be a $4k$-periodic v_1–self-map. It is well-known that the composite

$$\Sigma^{8k}\mathbb{S} \hookrightarrow \Sigma^{8k}\mathrm{M}_2(1) \xrightarrow{\ v\ } \mathrm{M}_2(1) \longrightarrow \Sigma^1\mathbb{S} \tag{5.2}$$

is not null (and equals $P^{k-1}(8\sigma)$ where P is a periodic operator given by the Toda bracket $\langle \sigma, 16, - \rangle$.)

Suppose there exists $f : \Sigma^{8k,8k}\mathrm{C}^{\mathbb{R}}(\mathsf{h}) \to \mathrm{C}^{\mathbb{R}}(\mathsf{h})$ such that $\Phi^e \circ \beta(f) = v$. Then (5.2) implies that the composition

$$\Sigma^{8k,8k}\mathbb{S}_{\mathbb{R}} \hookrightarrow \Sigma^{8k,8k}\mathrm{C}^{\mathbb{R}}(\mathsf{h}) \xrightarrow{v} \mathrm{C}^{\mathbb{R}}(\mathsf{h}) \longrightarrow \Sigma^{1,0}\mathbb{S} \tag{5.3}$$

is nonzero as the functor $\Phi^e \circ \beta$ is additive. The composite of the maps in (5.3) is a nonzero element of $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ in negative co-weight. This contradicts the fact that $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ is trivial in negative co-weights [DI1]. □

Proposition 5.4. *The v_1–self-maps of $\mathcal{Y}$ are not in the image of the underlying homomorphism*

$$\Phi^e \circ \beta \colon [\Sigma^{2k,2k}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}, \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}]^{\mathbb{R}} \longrightarrow [\Sigma^{8k}\mathcal{Y}, \mathcal{Y}].$$

Proof. Let $v : \Sigma^{2k}\mathcal{Y} \to \mathcal{Y}$ denote a v_1–self-map of periodicity k. Notice that the composite

$$\mathrm{S}^{2k} \hookrightarrow \Sigma^{2k}\mathcal{Y} \xrightarrow{v} \mathcal{Y} \longrightarrow \mathcal{Y}_{\geq 1} \tag{5.5}$$

where $\mathcal{Y}_{\geq 1}$ is the first coskeleton, must be nonzero. If not, then v factors through the bottom cell resulting in a map $\mathrm{S}^{2k} \to \Sigma^{2k}\mathcal{Y} \to \mathbb{S}$ which induces an isomorphism in $\mathrm{K}(1)$-homology, contradicting the fact that $\mathbb{S}$ is of type 0.

If $f : \Sigma^{2k,2k}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)} \to \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$ were a map such that $\Phi^e \circ \beta(f) = v$, then (5.5) would force one among the hypothetical composites (A), (B) or (C) in the diagram

$$\begin{array}{ccccccccc}
\Sigma^{2k,2k}\mathbb{S}_{\mathbb{R}} & \hookrightarrow & \Sigma^{2k,2k}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)} & \longrightarrow & \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)} & \overset{p_3}{\dashrightarrow} & \Sigma^{3,0}\mathbb{S}_{\mathbb{R}} & & (A) \\
& & & \searrow & \uparrow & & & & \\
& & & & \mathrm{Fib}(p_3) & \overset{p_2}{\dashrightarrow} & \Sigma^{2,0}\mathbb{S}_{\mathbb{R}} & & (B) \\
& & & \searrow & \uparrow & & & & \\
& & & & \mathrm{Fib}(p_2) & \overset{p_1}{\dashrightarrow} & \Sigma^{1,0}\mathbb{S}_{\mathbb{R}} & & (C)
\end{array}$$

to exist as a nonzero map, thereby contradicting the fact that $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ is trivial in negative co-weights. □

Remark 5.6. The above results do not preclude the existence of a $v_{1,0}$–self-map on $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}$. Forthcoming work [GI2] of the second author and Isaksen shows that 8σ is in the image of $\Phi^e \colon \pi_{7,8}(\mathbb{S}_{\mathrm{C}_2}) \longrightarrow \pi_7(\mathbb{S})$ and suggests that $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ supports a $v_{1,0}$–self-map.

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