

AN ALGEBRAIC MODEL FOR RATIONAL
NAÏVE-COMMUTATIVE G -EQUIVARIANT RING SPECTRA
FOR FINITE G

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(communicated by Brooke Shipley)

Abstract

Equipping a non-equivariant topological E_∞ -operad with the trivial G -action gives an operad in G -spaces. The algebra structure encoded by this operad in G -spectra is characterised homotopically by having no non-trivial multiplicative norms. Algebras over this operad are called naïve-commutative ring G -spectra. In this paper we let G be a finite group and we show that commutative algebras in the algebraic model for rational G -spectra model the rational naïve-commutative ring G -spectra. In other words, a rational naïve-commutative ring G -spectrum is given in the algebraic model by specifying a $\mathbb{Q}[W_G(H)]$ -differential graded algebra for each conjugacy class of subgroups H of G . Here $W_G(H) = N_G(H)/H$ is the Weyl group of H in G .

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1. Introduction

We are interested in the category of G -spectra, where G is a finite group and the indexing universe (unless specifically stated otherwise) is a complete G -universe U . This is a model for the equivariant stable homotopy category where all G -representation spheres are invertible. Building on work of Greenlees and May [GM95, Appendix A] and Barnes [Bar09a], Kędziołek [Kęd] gave a symmetric monoidal algebraic model for the category of rational G -spectra, when G is a finite group. As a consequence, one obtains a model for rational ring G -spectra in terms of monoids in the algebraic model. However, this does not imply results about strict commutative rational ring G -spectra (Comm-algebras). This is because of the well-known but surprising result that the derived functor of a *symmetric* monoidal Quillen functor can fail to preserve *commutative* monoids (algebras for an operad \mathbf{Comm}) in the equivariant setting.

Recent work of Blumberg and Hill [BH15] describes a class of commutative multiplicative structures on the equivariant stable homotopy category. These multiplicative structures are governed by G -operads called N_∞ -operads. Roughly speaking, a multiplicative structure is characterised by the set of Hill-Hopkins-Ravenel norms which exist on the commutative algebras corresponding to it.

This work of Blumberg and Hill raises a question: which level of commutative ring G -spectra is modelled by commutative algebras in the algebraic model of [Kęd]?

Contribution of this paper

In this paper we show that commutative algebras in the algebraic model for rational G -spectra are Quillen equivalent to rational naïve-commutative ring G -spectra, that is, the algebras over the *non-equivariant* E_∞ -operad equipped with the trivial G -action, denoted E_∞^1 . In the view of [BH15] this operad has the least commutative structure, in particular, it encodes the algebra structure without multiplicative norms.

Furthermore, the Quillen equivalences between the algebraic model and rational G -spectra do not lift to Quillen equivalences of any more structured commutative objects, such as \mathbf{Comm} -algebras or algebras for a different N_∞ -operad. Thus we have an interesting example of where symmetric monoidal Quillen equivalences do not induce Quillen equivalences of categories of \mathbf{Comm} -algebras.

There are clear reasons why it should be expected that the algebraic model only captures the lowest level of commutativity in the equivariant setting. The methods used to obtain the monoidal algebraic model for rational G -spectra rely on the complete idempotent splitting using methods of Barnes [Bar09b] and left Bousfield localisation techniques. The splitting operation cannot preserve norms, and hence will not preserve more structured commutative objects (see for example McClure [McC96] or Section 7). Equally, the operad \mathbf{Comm} in the algebraic model does not encode any additional structure beyond that of commutative algebras in the usual sense, and thus the category of its algebras can only model the rational naïve-commutative ring G -spectra. In the non-rational setting, Böhme [Boe18] gives a thorough study of which idempotents of the non-rational Burnside ring preserve these commutative structures.

The new results of the paper are in the analysis of the relation between a left Bousfield localisation at an object E of G -spectra (denoted by $L_E G\text{-Sp}$) and an

N_∞ -operad \mathcal{O} . In Section 4 we identify conditions on \mathcal{O} and E which ensure that $\mathcal{O}\text{-alg-in-}L_E G\text{-Sp}$ has a right-lifted model structure from $L_E G\text{-Sp}$. In particular, the fibrant replacement functor of this lifted model structure on $\mathcal{O}\text{-alg-in-}L_E G\text{-Sp}$ is a functorial E -localisation that preserves \mathcal{O} -algebras.

In the appendix we construct a model structure on $\mathbf{Comm}\text{-alg-in-}G\text{-Sp}_\mathbb{Q}$, where G is any compact Lie group. This is a model structure on commutative ring G -spectra where weak equivalences are rational equivalences. This procedure does not follow from the previous analysis, since \mathbf{Comm} is not an N_∞ -operad, but instead uses the existence of a commutative multiplication on the rational sphere spectrum. The authors included this appendix as they were unable to find a sufficiently detailed reference for this result, which is needed in Section 5.

This paper is also the first step in understanding what level of commutativity is modelled by the algebraic model for rational $SO(2)$ -spectra of [BGKS17]. In this case various left Bousfield localisations and idempotent splittings are also used to obtain the algebraic model. Understanding the interplay between left Bousfield localisations, idempotents and operads \mathcal{O} is crucial in this more complicated setting. This in turn will be used in [BGK] to show that rational $SO(2)$ -equivariant elliptic cohomology constructed in [Gre99] is modelled by a rational E_∞^1 -ring- G -spectrum and hence there is a description of the category of modules over this ring spectrum in terms of algebraic geometry.

Relation to other work

There are recent results by Hill and Hopkins [HH16], Gutiérrez and White [GW17], and Hill [Hil17] on the interaction of Bousfield localisations (understood as a fibrant replacement in the left Bousfield localised model structure) with the structure of \mathcal{O} -algebras, for various N_∞ -operads \mathcal{O} . Our approach differs from the references mentioned above as we are primarily interested in creating a version of E -local model structure on \mathcal{O} -algebras in G -spectra. This gives a stronger result than simply looking at images of \mathcal{O} -algebras under localisation functors. The existence of such a model structure implies that one can preserve the commutative structure modelled by \mathcal{O} while doing a fibrant replacement relative to E . In particular, the conditions of Section 4 are much more direct than those of [HH16, Theorems 6.2 and 6.3].

An algebraic model

Our primary aim is Theorem 6.8 which states that for a finite group G , there is a zig-zag of Quillen equivalences

$$E_\infty^1\text{-alg-in-}(G\text{-Sp})_\mathbb{Q} \simeq \mathbf{Comm}\text{-alg-in-} \prod_{(H)\leq G} \mathbf{Ch}(\mathbb{Q}[W_G H]),$$

where $W_G H = N_G H/H$ is the Weyl group of H in G . This is based on the chain of symmetric monoidal Quillen equivalences of [Kęd, Corollary 5.10]. To illustrate the whole path of that result, we present a diagram which shows every step of this comparison. The reader may wish to refer to this diagram now, but the notation will be introduced as we proceed. Left Quillen functors are placed on the left and we use the notation $N = N_G H$ and $W = W_G H = N_G H/H$. Also, $e_{(H)_G}$ denotes the idempotent supported on the conjugacy class of H in the rational Burnside ring

of G . We present on the right the series of Quillen equivalence at the level of (naïve) commutative ring objects.

$$\begin{array}{ccc}
L_{e_{(H)}_G S_{\mathbb{Q}}}(G\text{-Sp}) & & E_{\infty}^1\text{-alg-in-}L_{e_{(H)}_G S_{\mathbb{Q}}}(G\text{-Sp}) \\
\downarrow i^* \uparrow F_N(G_+, -) & & \downarrow i^* \uparrow F_N(G_+, -) \\
L_{e_{(H)}_N S_{\mathbb{Q}}}(N\text{-Sp}) & & E_{\infty}^1\text{-alg-in-}L_{e_{(H)}_N S_{\mathbb{Q}}}(N\text{-Sp}) \\
\downarrow \epsilon^* \uparrow (-)^H & & \downarrow \epsilon^* \uparrow (-)^H \\
L_{e_1 S_{\mathbb{Q}}}(W\text{-Sp}) & & E_{\infty}^1\text{-alg-in-}L_{e_1 S_{\mathbb{Q}}}(W\text{-Sp}) \\
\downarrow I_t^c \uparrow \text{res} & & \downarrow I_t^c \uparrow \text{res} \\
\text{Sp}_{\mathbb{Q}}[W] & & E_{\infty}^1\text{-alg-in-Sp}_{\mathbb{Q}}[W] \\
\downarrow \mathbb{P}o|-| \uparrow \text{Sing} \circ U & & \downarrow \text{change of} \uparrow \text{operads} \\
\text{Sp}_{\mathbb{Q}}^{\Sigma}[W] & & \text{Comm-alg-in-Sp}_{\mathbb{Q}}[W] \\
\downarrow H\mathbb{Q}\wedge- \uparrow U & & \downarrow \mathbb{P}o|-| \uparrow \text{Sing} \circ U \\
(H\mathbb{Q}\text{-mod})[W] & & \text{Comm-alg-in-Sp}_{\mathbb{Q}}^{\Sigma}[W] \\
\downarrow \text{zig-zag of Quillen} & & \downarrow H\mathbb{Q}\wedge- \uparrow U \\
\text{Ch}(\mathbb{Q}[W]) & & \text{Comm-alg-in-}(H\mathbb{Q}\text{-mod})[W] \\
& & \downarrow \text{zig-zag of Quillen} \\
& & \text{Comm-alg-in-Ch}(\mathbb{Q}[W])
\end{array}$$

Outline of the paper

The aim is to show that the right hand column of the above diagram is a series of Quillen equivalences. We start by recalling basic definitions and results from Blumberg and Hill [BH15] on N_{∞} -operads in Section 2. Section 3 describes a general strategy of lifting adjunctions and model structures to the level of algebras over operads. It also provides reasonable conditions which imply that a Quillen equivalence at the level of underlying categories lifts to a Quillen equivalence at the level of algebras over an operad.

In Section 4, we give a general result with directly checkable assumptions that ensures there is a right-lifted model structure on \mathcal{O} -algebras in $L_E G\text{-Sp}$ (and related categories), see Theorem 4.5. The technical heart of the paper is Section 5 where the details of Theorem 4.5 are proven.

In Section 6 we verify that the assumptions of the previous results hold in the cases of interest: the zig-zag of Quillen equivalences from [Kęd]. This section gives the main result, of the paper: Theorem 6.8.

Finally, the appendix shows that there is a model structure on *rational* commutative ring G -spectra right-lifted from the rational positive stable model structure on

G -spectra. This uses a different method to the results on \mathcal{O} -algebras for N_∞ -operads \mathcal{O} and works for any compact Lie group G .

Notation

We stick to the convention of writing a left adjoint in any adjoint pair on top or on the left of a right one.

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2. Recollections on N_∞ -operads

Definition 2.1. A G -operad \mathcal{O} consists of a sequence of $G \times \Sigma_n$ -spaces \mathcal{O}_n , $n \geq 1$, such that:

1. There is a G -fixed identity element $1 \in \mathcal{O}_1$.
2. There are G -equivariant composition maps

$$\mathcal{O}_k \times \mathcal{O}_{n_1} \times \mathcal{O}_{n_2} \times \cdots \times \mathcal{O}_{n_k} \longrightarrow \mathcal{O}_{n_1 + \cdots + n_k}$$

satisfying the usual compatibility conditions with each other and with the symmetric group actions, see [CW91, 2.1].

Recall that for a group G a *family* is a collection of subgroups of G closed under conjugation and taking subgroups. If \mathcal{F} is a family, then a universal space $E\mathcal{F}$ for the family \mathcal{F} is a G -space with the property that

$$(E\mathcal{F})^H \simeq \begin{cases} * & H \in \mathcal{F}, \\ \emptyset & H \notin \mathcal{F}. \end{cases}$$

Definition 2.2 ([BH15, Definition 1.1]). A G -operad \mathcal{O} is an N_∞ -operad if:

- $\mathcal{O}(n)$ is universal space for a family $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ that contains all subgroups of the form $H \times \{1\}$.
- Σ_n acts freely on $\mathcal{O}(n)$.
- $\mathcal{O}(0)$ is G -contractible.

If we forget the G action on an N_∞ -operad we obtain an E_∞ -operad in the classical sense. The condition that Σ_n acts freely on $\mathcal{O}(n)$ is precisely the statement that for any subgroup of the form $\{e\} \times K$ of $G \times \Sigma_n$, $\mathcal{O}(n)^{\{e\} \times K} = \emptyset$. It follows that $\mathcal{F}_n(\mathcal{O})$ cannot contain any subgroup of the form $\{e\} \times K$. Thus any $\Gamma \in \mathcal{F}_n(\mathcal{O})$ must have trivial intersection with $\{e\} \times \Sigma_n$ (or Γ would share a non-trivial subgroup with $\{e\} \times \Sigma_n$, which in turn would be an element of the family $\mathcal{F}_n(\mathcal{O})$). We say that a subgroup Γ of $G \times \Sigma_n$ is **admissible** if $\Gamma \cap \{e\} \times \Sigma_n = \{e \times e\}$.

Since N_∞ -operads are characterised by the families of subgroups from the definition above, let us fix some notation. We will use the notation E_∞^1 for an N_∞ -operad, where for every n we take the family $\mathcal{F}_1(\Sigma_n) = \{H \times \{e\} \mid H \leq G\}$. Notice that those are the **minimal** families for an N_∞ -operad with respect to G . The superscript 1 refers to the trivial group in the sense that E_∞^1 is given by inflating a non-equivariant E_∞ -operad to a G -operad. On the other hand, we can take the **maximal** families in the definition above, that is, for $n \geq 1$ take the family $\mathcal{F}_G(\Sigma_n)$ of all admissible subgroups of $G \times \Sigma_n$. We will use the notation E_∞^G for any operad with $\mathcal{F}_n(\mathcal{O}) = \mathcal{F}_G(\Sigma_n)$ for all $n \geq 0$.

Example 2.3. Let U denote a G -universe, i.e. a countably infinite-dimensional real G -inner product space which contains each finite dimensional sub-representation infinitely often. The *linear isometries operad* $\mathcal{L}(U)$ is a G -operad such that

$$\mathcal{L}(U)(n) := \mathcal{L}(U^n, U),$$

where $\mathcal{L}(U^n, U)$ denotes non-equivariant linear isometries from U^n to U . It is a $G \times \Sigma_n$ space by conjugation and diagonal action. The identity map $U \rightarrow U$ is the distinguished element of $\mathcal{L}(U)(1)$ and the structure maps are given by composition.

If U is a **complete** G -universe then $\mathcal{L}(U)$ is an example of E_∞^G -operad.

If U is a **trivial** G -universe then $\mathcal{L}(U)$ is an example of E_∞^1 -operad.

Both are examples of N_∞ -operads.

Since the category of G -spectra is tensored over G -spaces we can consider \mathcal{O} -algebras in G -spectra, where \mathcal{O} is a G -operad (an operad in G -spaces). In particular, this applies when \mathcal{O} is an N_∞ -operad.

In the next proposition we consider the category of orthogonal G -spectra with the positive stable model structure of [MM02, Section III.5].

Proposition 2.4 ([BH15, Proposition A.1]). *For any N_∞ -operad \mathcal{O} such that each $\mathcal{O}(n)$ has a homotopy type of a $G \times \Sigma_n$ -CW complex there exists a right-lifted model structure on \mathcal{O} -algebras in orthogonal G -spectra (i.e. the weak equivalences and fibrations are created in the category of orthogonal G -spectra with the positive stable model structure).*

By [BH15, Definition 3.9] a map $\mathcal{O} \rightarrow \mathcal{O}'$ of G -operads is a weak equivalence if for any $n \geq 1$ the map $\mathcal{O}(n)^\Gamma \rightarrow \mathcal{O}'(n)^\Gamma$ is an equivalence for all subgroups $\Gamma \subseteq G \times \Sigma_n$.

With this definition we recall the comparison result for algebras over weakly equivalent N_∞ -operads.

Lemma 2.5 ([BH15, Theorem A.3]). *Suppose $f: \mathcal{O} \rightarrow \mathcal{O}'$ is a weak equivalence of N_∞ -operads, such that $\mathcal{O}(n)$ and $\mathcal{O}'(n)$ have homotopy types of $G \times \Sigma_n$ -CW complexes and both $\mathcal{O}(1)$ and $\mathcal{O}'(1)$ have nondegenerate G -fixed basepoints. Then the adjunction (f_*, f^*) is a Quillen equivalence between categories of their algebras.*

3. Strategy for lifting model structures

We are interested in considering \mathcal{O} -algebras in various symmetric monoidal categories \mathbb{C} for an operad \mathcal{O} . We want to put model structures on these categories of

algebras, and to show that Quillen equivalences on the underlying categories lift to Quillen equivalences on categories of algebras. The purpose of this section is to record the formal part of this process, showing the basic conditions that need to be satisfied in order to perform a right lifting of model structures and Quillen equivalences. We will apply this strategy numerous times in the rest of the paper.

Suppose \mathcal{O} is an operad in a category V and \mathbb{C} is tensored over V . In that situation there is an adjunction

$$F_{\mathcal{O}} : \mathbb{C} \rightleftarrows \mathcal{O}\text{-alg-in-}\mathbb{C} : U ,$$

where U is the forgetful functor and $F_{\mathcal{O}}$ is the free \mathcal{O} -algebra functor

$$F_{\mathcal{O}}X = \coprod_n \mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n}.$$

If \mathbb{C} is a model category we want to right lift the model structure from \mathbb{C} to the category $\mathcal{O}\text{-alg-in-}\mathbb{C}$ of \mathcal{O} -algebras in \mathbb{C} using the right adjoint U , i.e. we want the weak equivalences and fibrations to be created by U .

To do that we will use Kan’s result for right lifting model structures.

Lemma 3.1 ([Hir03, Theorem 11.3.2]). *Suppose \mathbb{C} is a cofibrantly generated model category with a set of generating acyclic cofibrations J . If*

- $F_{\mathcal{O}}$ preserves small objects (or U preserves filtered colimits) and
- every transfinite composition of pushouts (cobase extensions) of elements of $F_{\mathcal{O}}(J)$ is sent to a weak equivalence of \mathbb{C} by U ,

then the model structure on \mathbb{C} may be lifted using the right adjoint to give a cofibrantly generated model structure on $\mathcal{O}\text{-alg-in-}\mathbb{C}$. The functor U then creates fibrations and weak equivalences.

Since later on we will be interested in lifting Quillen equivalences we consider the following situation. Suppose there are two symmetric monoidal categories \mathbb{C} and \mathbb{D} , both tensored over a closed symmetric monoidal category \mathbb{V} . Let \mathcal{O} be an operad in \mathbb{V} . Suppose further that we have an adjunction

$$\mathbb{C} \rightleftarrows^L_R \mathbb{D}.$$

If both categories \mathbb{C}, \mathbb{D} have model structures which can be lifted to \mathcal{O} algebras we obtain the following diagram

$$\begin{array}{ccc} \mathcal{O}\text{-alg-in-}\mathbb{C} & & \mathcal{O}\text{-alg-in-}\mathbb{D} \\ F_{\mathcal{O}} \uparrow \downarrow U & & F_{\mathcal{O}} \uparrow \downarrow U \\ \mathbb{C} & \rightleftarrows^L_R & \mathbb{D} \end{array}$$

It is natural to ask under what assumptions on L and R the adjunction lifts to the adjunction at the level of algebras and, moreover, when is it a Quillen pair and a Quillen equivalence. We will answer these questions below.

Remark 3.2. Notice that we can consider a generalisation of a situation above, namely algebras over different operads in \mathbb{C} and in \mathbb{D} . Since we will not use that generalisation

in this paper, we will stick to the simpler case of algebras over the same operad both in \mathbb{C} and in \mathbb{D} .

For the rest of this section, we assume that the hypotheses of Lemma 3.1 have been verified for both $(\mathbb{C}, \mathcal{O})$ and $(\mathbb{D}, \mathcal{O})$, so that $\mathcal{O}\text{-alg-in-}\mathbb{C}$ and $\mathcal{O}\text{-alg-in-}\mathbb{D}$ have both been given the lifted model structures.

Now suppose that the left adjoint L is tensored over a closed symmetric monoidal category \mathbb{V} and that L is a strong symmetric monoidal functor. This implies that the right adjoint R is lax symmetric monoidal and both functors lift to the level of \mathcal{O} -algebras. Note that a \mathbb{V} -enriched adjunction lifts to algebras over compatible \mathbb{V} -operads.

The more general case of lifting adjunctions in case of non-commutative monoids was dealt with in [SS03, Subsection 3.3]. These are all special cases of the adjoint lifting theorem, see for example [Bor94, Theorem 4.5.6].

Lemma 3.3. *Let (L, R) be a \mathbb{V} -enriched strong symmetric monoidal adjunction between \mathbb{V} -tensored symmetric monoidal categories \mathbb{C} and \mathbb{D} .*

Let \mathcal{O} be an operad in \mathbb{V} , then L and R extend to functors of \mathcal{O} -operads in \mathbb{C} and \mathbb{D} , and we obtain a commutative square of adjunctions

$$\begin{array}{ccc} \mathcal{O}\text{-alg-in-}\mathbb{C} & \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} & \mathcal{O}\text{-alg-in-}\mathbb{D} \\ F_{\mathcal{O}} \uparrow \downarrow U & & F_{\mathcal{O}} \uparrow \downarrow U \\ \mathbb{C} & \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} & \mathbb{D}. \end{array}$$

Remark 3.4. Notice, that we can consider a more general situation of lifting adjoint pairs to the level of algebras, when the left adjoint is *not* strong symmetric monoidal. However, in our applications all left adjoints are strong symmetric monoidal, thus we restrict attention to this particular situation.

It is automatic that the lifted adjunction of Lemma 3.3 forms a Quillen pair.

Lemma 3.5. *Suppose that L and R are a Quillen pair and that the hypotheses of Lemma 3.1 hold so that the categories of \mathcal{O} -algebras have lifted cofibrantly generated model structures and the functors lift to an adjoint pair L, R . Then L, R is a Quillen pair at the level of \mathcal{O} -algebras.*

Proof. The result follows from the fact that the right adjoint at the level of algebras is the same as at the level of underlying categories and fibrations and weak equivalences are created in the underlying categories. \square

In the situation where Lemma 3.3 holds, we can look for criteria which will guarantee that a Quillen equivalence on the original category lifts to a Quillen *equivalence* on algebras. With \mathcal{O} -algebras replaced by non-commutative monoids, this is the content of [SS03, Theorem 3.12]. This lemma will be used to obtain most of the Quillen equivalences needed for the main result, Theorem 6.8.

Lemma 3.6. *Suppose that L is a strong monoidal functor, L and R form a Quillen equivalence (at the level of categories \mathbb{C} and \mathbb{D}) and that the hypotheses of Lemma 3.1*

hold so that the categories of \mathcal{O} -algebras have lifted cofibrantly generated model structures and the functors lift to an adjoint pair L, R at the level of \mathcal{O} -algebras.

If U preserves cofibrant objects then the lifted Quillen pair L and R at the level of \mathcal{O} -algebras is a Quillen equivalence.

Proof. To show that (L, R) is a Quillen equivalence at the level of \mathcal{O} -algebras we need to show that for a cofibrant X in $\mathcal{O}\text{-alg-in-}\mathbb{C}$ and a fibrant Y in $\mathcal{O}\text{-alg-in-}\mathbb{D}$, $X \rightarrow RY$ is an equivalence if and only if $LX \rightarrow Y$ is an equivalence.

Since weak equivalences are created in the original categories, the first condition is that $UX \rightarrow URY = RUY$ is an equivalence in \mathbb{C} .

Now UY is fibrant because the fibrations are created by U and UX is cofibrant by assumption. It follows from the fact that (L, R) is a Quillen equivalence at the level of underlying categories that $UX \rightarrow RUY$ being a weak equivalence is equivalent to $LUX \rightarrow UY$ being a weak equivalence.

Finally, since $LU = UL$ the latter condition is equivalent to $ULX \rightarrow UY$ being a weak equivalence, which finishes the proof. \square

4. Lifting model structures for G -spectra

In this section we lift various model structures on the category $G\text{-Sp}$ of orthogonal G -spectra to the level of algebras over an operad \mathcal{O} in G -spaces. The main example is the left Bousfield localisation of a positive stable model structure on $G\text{-Sp}$ at a G -spectrum E , denoted by $L_E(G\text{-Sp})$. We establish compatibility conditions between E and \mathcal{O} , which ensure that the right-lifted model structure on \mathcal{O} -algebras from $L_E(G\text{-Sp})$ exists.

Recall that a left Bousfield localisation of a positive stable model structure on $G\text{-Sp}$ at a cofibrant object E is a new model structure, where cofibrations are the positive stable cofibrations and a map f is an E -equivalence if $E \wedge f$ is a stable weak equivalence.

We assume in this section that \mathcal{O} is an N_∞ -operad, thus by Definition 2.2 $\mathcal{O}(n)$ is universal space for a family of subgroups of $G \times \Sigma_n$ that contains all subgroups of the form $H \times \{e\}$. We call this family $\mathcal{F}_n(\mathcal{O})$. Two primary examples of such an operad are E_∞^G and E_∞^1 , see Example 2.3.

First, we consider compatibility conditions between a G -spectrum E and a G -operad \mathcal{O} .

Definition 4.1. We define the **geometric isotropy** of a G -spectrum E as

$$\text{Ig}(E) = \{H \leq G \mid \Phi^H E \not\cong *\}.$$

Let $p_G: G \times \Sigma_n \rightarrow G$ be the projection. We say that a G -spectrum E is \mathcal{O} -compatible if for all $n \geq 1$, whenever $\Gamma \in \mathcal{F}_n(\mathcal{O})$ and $p_G \Gamma \in \text{Ig}(E)$, then $\Gamma = p_G \Gamma \times \{e\}$.

Remark 4.2. To relate the above definition to admissible sets (see [BH15, Definition 4.3]), notice that E is \mathcal{O} -compatible if whenever H is in the geometric isotropy of E , then the admissible H -sets for \mathcal{O} are all trivial. By [BH15, Theorem 7.12] this means that an \mathcal{O} -algebra will have no norms to H for any H in the geometric isotropy of E .

Example 4.3. Notice that all G -spectra are compatible with an E_∞^1 -operad. Any free G -spectrum E is compatible with any N_∞ -operad, in particular, with E_∞^G .

Example 4.4. Let $G = C_6$ and take $\mathcal{O} := \mathcal{D}(U)$, the little disc operad for the G -universe U generated by $\mathbb{R}[C_6/C_2]$ (see [BH15, Definition 3.11 (ii)] for the definition of the little disc operad). We observe that C_6 -spectrum $E := \Sigma^\infty C_6/C_{2+}$ is \mathcal{O} -compatible.

First notice, that for $H \leq G$ by [BH15, Theorem 4.19] the admissible H -sets T for \mathcal{O} are the ones such that there exists an H -equivariant embedding $T \rightarrow U$. Thus the only C_2 -admissible sets are the trivial ones.

Note that if $\Gamma \subset G \times \Sigma_n$ is such that $\Gamma \cap (\{e\} \times \Sigma_n) = \{e\}$, then there is a subgroup H in G and a homomorphism $f: H \rightarrow \Sigma_n$ such that Γ is the graph of f . By [BH15, Definition 4.3] an H -set T is admissible for an operad \mathcal{O} if the graph of the homomorphism $\Gamma_T: H \rightarrow \Sigma_{|T|}$ defining the H -structure on T is in $\mathcal{F}_{|T|}(\mathcal{O})$.

Since the C_2 -admissible sets T are the trivial ones, their graphs $\Gamma_T: C_2 \rightarrow \Sigma_{|T|}$ give subgroups of the form $C_2 \times \{e\}$.

This implies the condition from the definition of \mathcal{O} -compatibility if $\text{Ig}(E) = \{1, C_2\}$. Thus the C_6 -spectrum $E := \Sigma^\infty C_6/C_{2+}$ is \mathcal{O} -compatible.

Below we consider G -spectra with the model structure which is the left Bousfield localisation of the positive stable model structure at E , denoted by $L_E(G\text{-Sp})$.

Theorem 4.5. *For E an \mathcal{O} -compatible G -spectrum, there is a cofibrantly generated model structure on \mathcal{O} -algebras in $G\text{-Sp}$ where the weak equivalences are those maps which forget to E -equivalences of G -spectra. We call this the **E -local model structure on \mathcal{O} -algebras in G -spectra**. Furthermore, the forgetful functor sends cofibrant algebras to cofibrant spectra in $L_E G\text{-Sp}$.*

Proof. The argument is similar to that of [BH15, Proposition A.1] which is based on the general result of [MMSS01, Proposition 5.13]. This general result has two parts, the first part is about the interaction of pushouts, sequential colimits and h -cofibrations (also known as the Cofibration Hypothesis). The second part is that every map built using pushouts (cobase extensions) and sequential colimits from $F_{\mathcal{O}}J_E$ (maps of the form $F_{\mathcal{O}}j$, for j a generating acyclic cofibration of $L_E(G\text{-Sp})$) should be E -equivalences.

The first part is well known and uses a standard technique of describing pushouts of \mathcal{O} -algebras as sequential colimits in underlying category. This argument appears in several places, one of the earlier occurrences is Elmendorf et al. [EKMM97, Chapters VII and VIII] where they construct model structure of commutative algebras and localisations of commutative algebras. A more recent description is given by Harper and Hess in [HH13, Proposition 5.10].

The second part can be broken into a number of steps:

Step 1 If $f: X \rightarrow Y$ is a cofibration of cofibrant G -spectra that is an E -equivalence, then $F_{\mathcal{O}}f$ is an E -equivalence and an h -cofibration of underlying G -spectra.

Step 2 Any pushout (cobase extension) of $F_{\mathcal{O}}f$ is an E -equivalence and an h -cofibration of underlying G -spectra.

Step 3 Any sequential colimit of such maps is an E -equivalence.

Step 1 follows from Lemma 5.6. Step 2 requires us to rewrite the pushout in the previously-mentioned manner of Harper and Hess and check that each stage of their construction preserves E -equivalences. This is standard and uses Lemma 5.2 to show that certain constructions are E -equivalences.

Step 3 is routine. A sequential colimit of a collection of maps of \mathcal{O} -algebras (which forget to h -cofibrations) is given by the sequential colimit in G -spectra (this is part of the Cofibration Hypothesis). The result holds since smashing with E commutes with sequential colimits of G -spectra. It follows that the model structure exists.

The statement about cofibrant objects follows from [HH13, Theorem 5.18]. \square

By Example 4.3 we have two classes of model structures where the E -local model structure on \mathcal{O} -algebras in G -spectra always exists.

Corollary 4.6. *For E a G -spectrum, there is a cofibrantly generated model structure on E_∞^1 -algebras in $G\text{-Sp}$ where the weak equivalences are those maps which forget to E -equivalences of G -spectra. Furthermore, the forgetful functor preserves cofibrant objects.*

Remark 4.7. Notice that Corollary 4.6 holds for any compact Lie group G .

Corollary 4.8. *For E a free G -spectrum, there is a cofibrantly generated model structure on \mathcal{O} -algebras in $G\text{-Sp}$ where the weak equivalences are those maps which forget to E -equivalences of G -spectra. Furthermore, the forgetful functor preserves cofibrant objects.*

Remark 4.9. For specific E , such as $S_{\mathbb{Q}}$ in Appendix A, much more can be said. In particular, while compatibility is sufficient, the appendix shows that it is not necessary.

We may apply these results to some cases of interest to our main result. For H a subgroup of G , let $e_{(H)G}$ be the idempotent of the rational Burnside ring of G supported by the G -conjugacy class of H . Let N be the normaliser of H , and $e_{(H)N}$ the analogous idempotent to $e_{(H)G}$ in the rational Burnside ring of N . We let e_1 be the idempotent corresponding to the trivial subgroup of $W = N/H$. The above results show that we have lifted model structures on each of the following categories:

$$\begin{array}{ll} E_\infty^1\text{-alg-in-}L_{e_{(H)G}S_{\mathbb{Q}}}(G\text{-Sp}), & E_\infty^1\text{-alg-in-}L_{e_{(H)N}S_{\mathbb{Q}}}(N\text{-Sp}), \\ E_\infty^1\text{-alg-in-}L_{e_1S_{\mathbb{Q}}}(W\text{-Sp}), & E_\infty^1\text{-alg-in-Sp}_{\mathbb{Q}}[W]. \end{array}$$

Furthermore, in each case forgetting the algebra structure preserves cofibrant objects (even though cofibrations will not usually be preserved).

5. Isotropic combinatorics

In this section we complete the proof of Theorem 4.5 by proving Lemmas 5.2 and 5.6. Throughout this section, our model structures will be based on the positive stable model structure on $G\text{-Sp}$, so $L_E G\text{-Sp}$ will be the left Bousfield localisation of the positive stable model structure at E .

Assume that $f: X \rightarrow Y$ is a positive cofibration of positive cofibrant G -spectra that is an E -equivalence. We want to show that

$$F_{\mathcal{O}}^n f: \mathcal{O}(n)_+ \wedge_{\Sigma_n} X^{\wedge n} \rightarrow \mathcal{O}(n)_+ \wedge_{\Sigma_n} Y^{\wedge n}$$

is an E -equivalence of G -spectra for each $n \in \mathbb{N}$, provided that E is \mathcal{O} -compatible (see Definition 4.1).

Before we can prove Lemma 5.6 we need to study the Σ_n -orbits functor in some detail. We start by relating smashing with E to coequalisation over Σ_n and show that

$$((\mathcal{O}(n)_+ \wedge X^{\wedge n}) \wedge E)/\Sigma_n \rightarrow ((\mathcal{O}(n)_+ \wedge X^{\wedge n})/\Sigma_n) \wedge E = (\mathcal{O}(n)_+ \wedge_{\Sigma_n} X^{\wedge n}) \wedge E$$

is an isomorphism of G -spectra. While obvious at the space level, the spectrum level construction depends upon the choice of universe for $(G \times \Sigma_n)\text{-Sp}$.

Remark 5.1. Whenever we consider $G \times \Sigma_n$ -spectra we index on a Σ_n fixed universe, such as the inflation of a complete G -universe to a $G \times \Sigma_n$ -universe. The indexing spaces of such a universe are G -representations considered as $G \times \Sigma_n$ -representations via the projection $p_G: G \times \Sigma_n \rightarrow G$. It follows that a G -spectrum B , inflated to a $G \times \Sigma_n$ -spectrum is given by simply applying the space level inflation functor $(p_G^* B)(V) = p_G^*(B(V))$ for V a G -representation.

Lemma 5.2. *Let A be a $G \times \Sigma_n$ -spectrum and B a G -spectrum. Then there is a natural isomorphism of G -spectra*

$$(A \wedge B)/\Sigma_n \rightarrow (A/\Sigma_n) \wedge B.$$

Proof. Let U be an indexing space for G -spectra, then we can consider the smash product before coequalising over \mathbb{S} , denoted $\bar{\wedge}$:

$$\begin{aligned} ((A \bar{\wedge} B)/\Sigma_n)(U) &= (A \wedge B)(U)/\Sigma_n \\ &= \left(\int^{V,W} A(V) \wedge B(W) \wedge \mathcal{L}(V \oplus W, U) \right) / \Sigma_n \\ &= \left(\int^{V,W} A(V)/\Sigma_n \wedge B(W) \wedge \mathcal{L}(V \oplus W, U) \right) \\ &= (A/\Sigma_n \bar{\wedge} B)(U). \end{aligned}$$

This depends on the representations V , W and U all being Σ_n -fixed, so that we can move the Σ_n -orbits functor past the coend and the smash products of spaces. It follows that this isomorphism passes to the actual smash product of orthogonal spectra. \square

Whilst this lemma helps with the categorical nature of Σ_n , we still need to consider the homotopical properties of $(-)/\Sigma_n$ and understand when $(-)/\Sigma_n$ preserve weak equivalences. Recall the \mathcal{F} -model structure on $(G \times \Sigma_n)\text{-Sp}$ for \mathcal{F} a family of subgroups of $G \times \Sigma_n$ from [MM02, Section IV.6]. Here the weak equivalences are given by taking Γ -fixed points for each $\Gamma \in \mathcal{F}$ and are called \mathcal{F} -equivalences. The cofibrations are the positive cofibrations constructed from space level cofibrations but where one uses only the homogeneous spaces $(G \times \Sigma_n)/\Gamma_+$ for $\Gamma \in \mathcal{F}$. This model structure is denoted $\mathcal{F}(G \times \Sigma_n)\text{-Sp}$.

We use these definitions in the following lemma, which is given so that we can later clarify why we need some relation between the spectrum E and the operad \mathcal{O} in order to get a model structure $\mathcal{O}\text{-alg-in-}L_E G\text{-Sp}$, right-lifted from $L_E G\text{-Sp}$. We state the lemma for the positive model structures, but it also holds in the non-positive case.

Lemma 5.3. *Let \mathcal{F} be a family of subgroups of $(G \times \Sigma_n)\text{-Sp}$, such that for any $H \leq G$, $H \times \{e\} \in \mathcal{F}$. Then there is a Quillen pair*

$$(-)/\Sigma_n : \mathcal{F}(G \times \Sigma_n)\text{-Sp} \rightleftarrows G\text{-Sp} : p_G^*.$$

The right adjoint p_G^ preserves positive cofibrations and the left adjoint $(-)/\Sigma_n$ sends \mathcal{F} -equivalences between positive \mathcal{F} -cofibrant objects to weak equivalences of G -spectra.*

Proof. This is a Quillen pair as p_G^* preserves fibrations and acyclic fibrations since (at the level of spaces) $(p_G^*A)^\Gamma = A^{p_G^\Gamma}$. The statement about $(-)/\Sigma_n$ is then immediate from Ken Brown's Lemma. It follows from the isomorphism

$$p_G^*G/H_+ \cong (G \times \Sigma_n)/(H \times \{e\})_+$$

and the assumption $H \times \{e\} \in \mathcal{F}$, that p_G^* preserves positive cofibrations. \square

This lemma has some simple consequences. Consider a map $f: X \rightarrow Y$ of $G \times \Sigma_n$ -spectra which are (positive) cofibrant in the \mathcal{F} -model structure for some family \mathcal{F} . Then to see that f/Σ_n is a weak equivalence in $G\text{-Sp}$, one only has to check the Γ -fixed points of f are weak equivalences of spectra for Γ in \mathcal{F} . Hence for a fixed f , the more subgroups that are needed in \mathcal{F} to make X and Y cofibrant the more fixed points we need to check. In particular, if $\mathcal{F} = \{H \times \{e\} \mid H \leq G\}$ then one only has to check the $H \times \{e\}$ -fixed points for varying H . That is, one only has to check that f is a weak equivalence of G -spectra after forgetting the Σ_n -action.

We can now reduce our problem substantially.

We follow the same plan as related results in [EKMM97, Chapter VIII], but we find it easier to give references to more recent sources. Let $f: X \rightarrow Y$ be a positive cofibration of positive cofibrant G -spectra and an E -equivalence. We want to show that

$$F_0^n f : \mathcal{O}(n)_+ \wedge_{\Sigma_n} X^{\wedge n} \longrightarrow \mathcal{O}(n)_+ \wedge_{\Sigma_n} Y^{\wedge n}$$

is a weak equivalence of G -spectra. Lemmas 5.2 and 5.3 show that it suffices to prove that

$$\mathcal{O}(n)_+ \wedge f^{\wedge n} \wedge E : \mathcal{O}(n)_+ \wedge X^{\wedge n} \wedge E \longrightarrow \mathcal{O}(n)_+ \wedge Y^{\wedge n} \wedge E$$

is an $\mathcal{F}_n(\mathcal{O})$ -equivalence of $\mathcal{F}_n(\mathcal{O})$ -cofibrant $G \times \Sigma_n$ spectra. Work of Hill, Hopkins, and Ravenel [HHR11, Remark B.112 and Proposition B.89] shows that $X^{\wedge n}$ is a positive cofibrant $G \times \Sigma_n$ -spectrum. Lemma 5.3 shows that E , inflated to a $G \times \Sigma_n$ -spectrum is also cofibrant. Hence [MM02, Lemma IV.6.6] tells us that $\mathcal{O}(n)_+ \wedge X^{\wedge n} \wedge E$ is an $\mathcal{F}_n(\mathcal{O})$ -cofibrant $G \times \Sigma_n$ spectrum. To see that $F_0^n f$ is an E -equivalence, we must check that $\mathcal{O}(n)_+ \wedge f^{\wedge n} \wedge E$ is an $\mathcal{F}_n(\mathcal{O})$ -equivalence. In fact, this is equivalent to checking that $\mathcal{O}(n)_+ \wedge f^{\wedge n} \wedge E$ is a weak equivalence of $(G \times \Sigma_n)\text{-Sp}$ by [MM02, Proposition IV.6.7]. To show this last statement holds, we use geometric fixed points and see how the geometric isotropy subgroups of E interact with $\mathcal{F}_n(\mathcal{O})$.

Definition 5.4. Let us say that a subgroup $\Gamma \leq G \times \Sigma_n$ is **irrelevant** if the Φ^Γ -fixed points of $\mathcal{O}(n)_+$ or p_G^*E are trivial and **relevant** otherwise.

If E is \mathcal{O} -compatible, then the only relevant subgroups are of the form $H \times \{e\}$. Let $p_G: G \times \Sigma_n \rightarrow G$ be the projection, then for $A \leq G \times \Sigma_n$, the A -geometric fixed

points of E inflated to a $G \times \Sigma_n$ spectrum are given by $\Phi^{p_G(A)}E$. The next lemma is then immediate.

Lemma 5.5. *The relevant subgroups for $\mathcal{O}(n)_+$ and E lie in*

$$\mathcal{F}_n(\mathcal{O}) \cap \mathrm{Ig}_{G \times \Sigma_n}(p_G^*E) = \{A \mid A \in \mathcal{F}_n(\mathcal{O}), p_G(A) \in \mathrm{Ig}_G(E)\}.$$

We now prove that if E is \mathcal{O} -compatible, then $F_{\mathcal{O}}$ preserves E -equivalences.

Lemma 5.6. *Let $f: X \rightarrow Y$ be a positive cofibration between positive cofibrant G -spectra and an E -equivalence. Assume that E is \mathcal{O} -compatible, then*

$$\mathcal{O}(n)_+ \wedge f^{\wedge n}: \mathcal{O}(n)_+ \wedge X^{\wedge n} \longrightarrow \mathcal{O}(n)_+ \wedge Y^{\wedge n}$$

is an E -equivalence.

Proof. We only need to consider the relevant subgroups, as both sides are trivial at an irrelevant subgroup. By the compatibility condition, any relevant subgroup is of the form $H \times \{e\}$. If we view $X^{\wedge n}$ just as an $H \times \{e\}$ -spectrum then $X^{\wedge n}$ is just a smash product of copies of X and we have $\Phi^{H \times \{e\}}(X^{\wedge n}) = (\Phi^H X)^{\wedge n}$. Hence

$$\Phi^{H \times \{e\}}(\mathcal{O}(n)_+ \wedge X^{\wedge n} \wedge E) \simeq \Phi^{H \times \{e\}}(\mathcal{O}(n)_+) \wedge (\Phi^H X)^{\wedge n} \wedge \Phi^H E.$$

If $f: X \rightarrow Y$ is an E -equivalence, $\Phi^H X \rightarrow \Phi^H Y$ is an $\Phi^H E$ -equivalence. Moreover, Φ^H preserves cofibrations and the smash product of E' -equivalences of cofibrant objects is an E' -equivalence for any suitable spectrum E' , such as $\Phi^H E$. \square

This result is equivalent to showing that if Z is E -acyclic then the unit $S^0 \rightarrow F_{\mathcal{O}}Z$ is an E -equivalence. This is the condition needed by Hill and Hopkins in [HH16, Theorem 6.2] to conclude that localisation at E preserves \mathcal{O} -algebras. The original results along these lines can be found in Elmendorf et al. [EKMM97, Theorem VIII 2.2 and Lemma VIII 2.7].

6. Lifting the Quillen equivalences

In this section we construct our sequence of Quillen equivalences between E_{∞}^1 -algebras in rational G -spectra and commutative algebras in the algebraic model. The strategy is to replace E_{∞}^1 by \mathbf{Comm} as soon as we have moved from $L_{e_{(H)}G}S_{\mathbb{Q}}(G\text{-Sp})$ to $W_G H$ -objects in spectra. We then apply the results of [RS17] to compare commutative algebras in rational spectra to commutative algebras in rational chain complexes.

Before we check the compatibility of the zig-zag of Quillen equivalences from [Kęd] with algebras over E_{∞}^1 -algebras we note that this zig-zag of Quillen equivalences works for positive stable model structures.

Remark 6.1. The model structures and Quillen equivalences of [Kęd] relating rational G -spectra to the algebraic model can all be obtained using the positive stable model structure on the various categories of spectra. Moreover, the adjunction

$$HQ \wedge - : Sp_{\mathbb{Q}}^{\Sigma}[W] \rightleftarrows (HQ\text{-mod})[W] : U$$

is a Quillen equivalence when the left side is considered with the positive stable model structure and the right hand side is considered with the positive flat stable model

structure of [Shi04]. This model structure on $(H\mathbb{Q}\text{-mod})[W]$ is the starting point for the work in [RS17].

The results of Section 4 allow us to apply Lemmas 3.3 and 3.6 to obtain Quillen equivalences

$$\begin{array}{ccc}
 E_{\infty}^1\text{-alg-in-}L_{e_{(H)}G}S_{\mathbb{Q}}(G\text{-Sp}) & & \\
 \downarrow i^* \quad \uparrow F_N(G_+, -) & & \\
 E_{\infty}^1\text{-alg-in-}L_{e_{(H)}N}S_{\mathbb{Q}}(N\text{-Sp}) & & \\
 \uparrow \epsilon^* \quad \downarrow (-)^H & & \\
 E_{\infty}^1\text{-alg-in-}L_{e_1}S_{\mathbb{Q}}(W_G H\text{-Sp}) & & \\
 \uparrow I_t^c \quad \downarrow \text{res} & & \\
 E_{\infty}^1\text{-alg-in-Sp}_{\mathbb{Q}}[W_G H]. & &
 \end{array}$$

Now we change the operad from the E_{∞}^1 to Comm obtaining a Quillen equivalence.

Lemma 6.2. *There is an adjunction*

$$\eta_* : E_{\infty}^1\text{-alg-in-Sp}_{\mathbb{Q}}[W] \rightleftarrows \text{Comm-alg-in-Sp}_{\mathbb{Q}}[W] : \eta^*$$

induced by the map of operads in Top_* , $\eta : E_{\infty}^1 \rightarrow \text{Comm}$. This adjunction is a Quillen equivalence with respect to right-induced model structures from the right induced model structure on W -objects from the rational positive stable model structure on $\text{Sp}_{\mathbb{Q}}$.

Proof. We first prove the result in the case W is the trivial group. The model structure on the right exists by Theorem A.2 and the model structure on the left exists by Corollary 4.6 (here G is the trivial group). Since both model structures are lifted and η^* forgets to the identity functor of $\text{Sp}_{\mathbb{Q}}$, it follows that the right adjoint is a right Quillen functor that preserves all weak equivalences. The derived unit is the same as the derived unit of the non-rational adjunction and hence is a π_* -isomorphism.

Now we consider non-trivial W . Let \mathbb{C} be a symmetric monoidal category, W a finite group and \mathcal{O} an operad such that the category of \mathcal{O} -algebras in \mathbb{C} exists. It is easy to see that the category of \mathcal{O} -algebras in \mathbb{C} with W -action, $(\mathcal{O}\text{-alg-in-}\mathbb{C})[W]$ is isomorphic to the category of \mathcal{O} -algebras in $\mathbb{C}[W]$, $\mathcal{O}\text{-alg-in-}\mathbb{C}[W]$. Extending to categories of W -objects preserves Quillen equivalences by [Ked, Proposition 5.4], so the general result follows immediately. \square

Lemma 6.3. *For W a finite group, there is a Quillen equivalence*

$$\mathbb{P} \circ | - | : \text{Comm-alg-in-Sp}_{\mathbb{Q}}^{\Sigma}[W] \rightleftarrows \text{Comm-alg-in-Sp}_{\mathbb{Q}}[W] : \text{Sing} \circ \mathbb{U},$$

where both model structures are right induced from the rational positive stable model structures.

Proof. This adjunction satisfies the conditions of Lemma 3.6, hence we can lift the equivalence between $Sp_{\mathbb{Q}}^{\Sigma}[W]$ and $Sp_{\mathbb{Q}}[W]$ to the level of **Comm**-algebras. \square

Proposition 6.4. *For W a finite group, there is a Quillen equivalence*

$$H\mathbb{Q} \wedge - : \mathbf{Comm}\text{-alg-in-}Sp_{\mathbb{Q}}^{\Sigma}[W] \rightleftarrows \mathbf{Comm}\text{-alg-in-}(H\mathbb{Q}\text{-mod})[W] : U.$$

Proof. The right hand model structure exists by [HH13, Theorem 7.16]. This adjunction exists by Lemma 3.3. It satisfies the conditions of Lemma 3.6, hence we can lift the Quillen equivalence between $Sp_{\mathbb{Q}}^{\Sigma}[W]$ and $(H\mathbb{Q}\text{-mod})[W]$ to the level of **Comm**-algebras. \square

The paper [RS17] of Richter and Shipley proves that the model category of commutative HR -algebras (in symmetric spectra) is Quillen equivalent to E_{∞} -algebras in unbounded chain complexes of R -modules. When $R = \mathbb{Q}$ this gives the following

Theorem 6.5 ([RS17, Corollary 8.4]). *There is a zig-zag of Quillen equivalences between the model category of commutative $H\mathbb{Q}$ -algebras (in symmetric spectra) and differential graded commutative \mathbb{Q} -algebras.*

The model structure on commutative $H\mathbb{Q}$ -algebras is lifted from the positive stable model structure on symmetric spectra, see for example [Shi04] or [HSS00]. The model structure on differential graded commutative \mathbb{Q} -algebras has fibrations the surjections and weak equivalences the homology isomorphisms.

Corollary 6.6. *Let W be a finite group. There is a zig-zag of Quillen equivalences between the model category of commutative $H\mathbb{Q}$ -algebras with W -action and differential graded commutative \mathbb{Q} -algebras with a W -action.*

Theorem 6.7. *There is a zig-zag of Quillen equivalences*

$$E_{\infty}^1\text{-alg-in-}L_{e_{(H)}G}S_{\mathbb{Q}}(G\text{-Sp}) \simeq \mathbf{Comm}\text{-alg-in-}Ch(\mathbb{Q}[W_G H]).$$

Proof. The above results give a zig-zag of Quillen equivalences

$$E_{\infty}^1\text{-alg-in-}L_{e_{(H)}G}S_{\mathbb{Q}}(G\text{-Sp}) \simeq \mathbf{Comm}\text{-alg-in-}(H\mathbb{Q}\text{-mod})[W_G H].$$

Corollary 6.6 shows that $\mathbf{Comm}\text{-alg-in-}(H\mathbb{Q}\text{-mod})[W_G H]$ is Quillen equivalent to the model category $\mathbf{Comm}\text{-alg-in-}Ch(\mathbb{Q}[W_G H])$ so the result holds. \square

We collect all these Quillen equivalences into one result.

Theorem 6.8. *There is a zig-zag of Quillen equivalences*

$$E_{\infty}^1\text{-alg-in-}(G\text{-Sp})_{\mathbb{Q}} \simeq \mathbf{Comm}\text{-alg-in-} \prod_{(H) \leq G} Ch(\mathbb{Q}[W_G H]).$$

Proof. It is clear that

$$\mathbf{Comm}\text{-alg-in-} \prod_{(H) \leq G} Ch(\mathbb{Q}[W_G H]) = \prod_{(H) \leq G} \mathbf{Comm}\text{-alg-in-}Ch(\mathbb{Q}[W_G H]).$$

Whereas on the topological side, the symmetric monoidal Quillen equivalence

$$\Delta : (G\text{-Sp})_{\mathbb{Q}} \rightleftarrows \prod_{(H) \leq G} L_{e_{(H)}G}S_{\mathbb{Q}}(G\text{-Sp}) : \prod$$

of [Bar09a, Corollary 6.4] lifts to the level of E_{∞}^1 -algebras by Lemma 3.6. Since taking

E_∞^1 -algebras commutes with the product of categories, we may then use Theorem 6.7 on each factor to obtain the desired zig-zag of Quillen equivalences. \square

Remark 6.9. Each of the following categories has a model structure lifted from the corresponding category of spectra:

$$\begin{array}{ll} E_\infty^1\text{-alg-in-}L_{e_1 S_\mathbb{Q}}(W\text{-Sp}), & \text{Comm-alg-in-}L_{e_1 S_\mathbb{Q}}(W\text{-Sp}), \\ E_\infty^1\text{-alg-in-}Sp_\mathbb{Q}^\Sigma[W], & \text{Comm-alg-in-}Sp_\mathbb{Q}^\Sigma[W], \\ E_\infty^1\text{-alg-in-}Sp_\mathbb{Q}[W], & \text{Comm-alg-in-}Sp_\mathbb{Q}[W], \\ E_\infty^1\text{-alg-in-}(H\mathbb{Q}\text{-mod})[W], & \text{Comm-alg-in-}(H\mathbb{Q}\text{-mod})[W]. \end{array}$$

Since each of the model structures are lifted, the model categories of the left hand column are all Quillen equivalent and similarly so for the right hand column. Lemma 6.2 gives a Quillen equivalence between the two columns (at the level of $Sp_\mathbb{Q}[W]$), hence all of the above model categories are Quillen equivalent. In particular, we have several routes to obtain the Quillen equivalence of Theorem 6.8, but they all give the same derived equivalence.

We cannot, however, put a lifted model structure on the two model categories

$$\text{Comm-alg-in-}L_{e_{(H)}_G S_\mathbb{Q}}(G\text{-Sp}), \quad \text{Comm-alg-in-}L_{e_{(H)}_N S_\mathbb{Q}}(N\text{-Sp}).$$

The difficulty is on the existence of multiplicative norms, as we discuss in Section 7.

7. Norm type obstructions

The first step in the zig-zag of Quillen equivalences used to provide an algebraic model for rational G -spectra, when G is a finite group is the splitting result of Barnes [Bar09b]. It uses the idempotents of the rational Burnside ring corresponding to conjugacy classes of all subgroups of G . This splitting is precisely the step which fails to preserve any more (G -commutative) structure on algebras than the one for E_∞^1 -algebras. This is closely related to the fact that at the homotopy level any algebra with more structure will have non-trivial norm maps and the complete idempotent splitting in the first step of the zig-zag kills any non-trivial norm maps. This shows, that a Quillen equivalence of E_∞^1 -algebras and commutative algebras in the algebraic model is the most one could hope for using this approach. Let us analyse the situation in more detail below.

First recall that if X is a G -spectrum which is non-equivariantly contractible, such that $\pi_0^H(X)$ is a unital ring for all $H \leq G$ and X has norms then $X \simeq *$. By having norms we mean that there is a map $N_1^H: \pi_0^1(X) \rightarrow \pi_0^H(X)$ which is compatible with the product in the sense that $N_1^H(0) = 0$, $N_1^H(1) = 1$ and $N_1^H(xy) = N_1^H(x)N_1^H(y)$. Now if $\pi_0^1(X) = 0$ then $1 = 0$, and it follows that $1 = 0$ in $\pi_0^H(X)$ so that $\pi_0^H(X) = 0$. Hence $\pi_*^H(X) = 0$ for all H and $X \simeq *$.

Remark 7.1. The argument above implies that there is no right-lifted model structure on $E_\infty^\mathcal{F}$ -algebras from $L_{e_{(H)}_G S_\mathbb{Q}}(G\text{-Sp})$, when $H \neq 1$ and $\mathcal{F} \neq 1$. Suppose there is one. Then for any $E_\infty^\mathcal{F}$ -algebra A there is a fibrant replacement (in $E_\infty^\mathcal{F}$ -algebras) A^f , which is underlying $e_{(H)}_G$ -local and thus by the argument above it is equivariantly contractible. Thus any A is weakly equivalent to a point in the model structure lifted from the $e_{(H)}_G$ -local model structure on G -spectra. However, not every $E_\infty^\mathcal{F}$ -algebra

in G -spectra is $e_{(H)_G}$ -locally equivalent to a point (for example the sphere spectrum). Since the forgetful functor was supposed to create weak equivalences at the level of $E_\infty^{\mathcal{F}}$ -algebras, we get a contradiction. A similar argument shows that there is no right-lifted model structure on \mathbf{Comm} -algebras from $L_{e_{(H)_G} S_{\mathbb{Q}}}^G(G\text{-Sp})$.

Recent work of Böhme discusses similar concerns in the non-rational setting. For example, [Boe18, Corollary G] describes which localisations of the sphere spectrum at an idempotent of the (non-rational) Burnside ring are E_∞^G -algebras.

Appendix A. Rationalisation of commutative rings

In this section we show that there is a right-lifted model structure on commutative algebras in rational G -spectra, where G is a compact Lie group. In particular, this argument shows that there is a right-lifted model structure on the category of commutative algebras in rational *non-equivariant* spectra, used in several lemmas in Section 6. This seems to be a folklore result, however, the authors do not know of a reference for it in the literature.

Fix $E = S_{\mathbb{Q}}$, the non-equivariant rational sphere spectrum that we inflate to the level of G -spectra whenever needed. The spectrum $S_{\mathbb{Q}}$ is constructed as the homotopy colimit of the sequence

$$S^0 \xrightarrow{2} S^0 \xrightarrow{3} S^0 \xrightarrow{4} S^0 \xrightarrow{5} \dots$$

in $\mathbf{Ho}(\mathbf{Sp})$, hence it is a cofibrant spectrum (and can be made positive cofibrant by replacing each S^0 by its positive cofibrant replacement before taking homotopy colimits). It is weakly equivalent to the non-equivariant Eilenberg-MacLane spectrum $H\mathbb{Q}$, which is a commutative ring spectrum.

We want to show that free algebra functor for \mathbf{Comm} takes rational equivalences (that are cofibrations between cofibrant objects in the rational positive stable model structure) to rational equivalences. We have to use a different approach to the one in Section 5, as \mathbf{Comm} uses the family of all subgroups of $G \times \Sigma_n$ and is not an N_∞ -operad. The key fact that we will use is that $S_{\mathbb{Q}}$ has a Σ_n -equivariant ‘multiplication’ map $S_{\mathbb{Q}}^{\wedge n} \rightarrow S_{\mathbb{Q}}$ that is a weak equivalence of Σ_n -spectra. This map is modelling the multiplication of $H\mathbb{Q}$.

Lemma A.1. *There is a Σ_n -equivariant map $S_{\mathbb{Q}}^{\wedge n} \rightarrow S_{\mathbb{Q}}$ of Σ_n -spectra that is a π_* -isomorphism on Σ_n -spectra (indexed on a trivial universe).*

Furthermore, if we inflate this to $(G \times \Sigma_n)\text{-Sp}$ then it becomes a weak equivalence of $(G \times \Sigma_n)\text{-Sp}$.

Proof. Consider the commutative diagram in $\mathbf{Ho}(\Sigma_n\text{-Sp})$:

$$\begin{array}{ccccccc} (S^0)^{\wedge n} & \xrightarrow{2^{\wedge n}} & (S^0)^{\wedge n} & \xrightarrow{3^{\wedge n}} & (S^0)^{\wedge n} & \xrightarrow{4^{\wedge n}} & (S^0)^{\wedge n} \xrightarrow{5^{\wedge n}} \dots \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ S^0 & \xrightarrow{2^n} & S^0 & \xrightarrow{3^n} & S^0 & \xrightarrow{4^n} & S^0 \xrightarrow{5^n} \dots \end{array}$$

the vertical maps are the commutative multiplication operation of S^0 and hence are Σ_n -equivariant. Taking homotopy colimits we obtain the desired weak equivalence of $\Sigma_n\text{-Sp}$: $S_{\mathbb{Q}}^{\wedge n} \rightarrow S_{\mathbb{Q}}$.

The second statement follows since for any Σ_n -spectrum A $(p_{\Sigma_n}^* A)^\Gamma = A^{p_{\Sigma_n} \Gamma}$, where $p_{\Sigma_n} \Gamma$ is the image of Γ under the projection to Σ_n . \square

Theorem A.2. *There is a cofibrantly generated model structure on Comm -algebras in $G\text{-Sp}$ where the weak equivalences are those maps which forget to rational equivalences of G -spectra.*

Proof. The proof is very similar to the case of a general operad. The main changes are in the three steps used to show that every map built using pushouts (cobase extensions) and sequential colimits from $F_\circ J_\mathbb{Q}$ (maps of the form $F_\circ j$, for j a generating cofibration of the rational positive stable model structure on $G\text{-Sp}$) is a rational equivalence.

Step 1 If $f: X \rightarrow Y$ is a cofibration of cofibrant G -spectra that is a rational equivalence, then $F_{\text{Comm}} f$ is a rational equivalence and an h -cofibration of underlying G -spectra.

Step 2 Any pushout (cobase extension) of $F_{\text{Comm}} f$ is a rational equivalence and an h -cofibration of underlying G -spectra.

Step 3 Any sequential colimit of such maps is a rational equivalence.

The first step is a difficult one, we start by comparing Comm to the E_∞^G (this is the same method as is used to show that there is a model structure on Comm -algebras in $G\text{-Sp}$). If X is a cofibrant G -spectrum then the map

$$(E_G \Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n} \longrightarrow X^{\wedge n} / \Sigma_n$$

is a (natural) weak equivalence of $G \times \Sigma_n$ -spectra by [HHR11, Proposition B.117] or [MM02, Lemma 8.4]. Here $(E_G \Sigma_n)_+$ is the universal space for the family \mathcal{F}_n^G . Hence it suffices to show that if $f: X \rightarrow Y$ is a cofibration of cofibrant G -spectra that is a rational equivalence, then

$$(E_G \Sigma_n)_+ \wedge_{\Sigma_n} f^{\wedge n} : (E_G \Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n} \longrightarrow (E_G \Sigma_n)_+ \wedge_{\Sigma_n} Y^{\wedge n}$$

is a rational equivalence.

By Lemma A.1 there is a $G \times \Sigma_n$ -equivariant map $S_{\mathbb{Q}}^{\wedge n} \rightarrow S_{\mathbb{Q}}$ that is a weak equivalence of $G \times \Sigma_n$ -spectra. We use this to obtain the following series of maps of G -spectra

$$\begin{aligned} ((E_G \Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n}) \wedge S_{\mathbb{Q}} &\xrightarrow{\cong} ((E_G \Sigma_n)_+ \wedge X^{\wedge n} \wedge S_{\mathbb{Q}}) / \Sigma_n \\ &\longleftarrow ((E_G \Sigma_n)_+ \wedge X^{\wedge n} \wedge S_{\mathbb{Q}}^{\wedge n}) / \Sigma_n \\ &\xrightarrow{\cong} ((E_G \Sigma_n)_+ \wedge (X \wedge S_{\mathbb{Q}})^{\wedge n}) / \Sigma_n. \end{aligned}$$

The second map is a weak equivalence by Lemma 5.3, hence we have a zig-zag of weak equivalences of G -spectra. Since $X \rightarrow Y$ is a rational equivalence, it follows that

$$((E_G \Sigma_n)_+ \wedge (X \wedge S_{\mathbb{Q}})^{\wedge n}) / \Sigma_n \longrightarrow ((E_G \Sigma_n)_+ \wedge (Y \wedge S_{\mathbb{Q}})^{\wedge n}) / \Sigma_n$$

is a weak equivalence of G -spectra. Hence the map $X^{\wedge n} / \Sigma_n \rightarrow Y^{\wedge n} / \Sigma_n$ is a rational equivalence of G -spectra.

The remaining steps hold just as before. \square

The same strategy gives a related result for N_∞ -operads in place of Comm .

Corollary A.3. *For an N_∞ -operad \mathcal{O} , there is a cofibrantly generated model structure on \mathcal{O} -algebras in $G\text{-Sp}$ where the weak equivalences are those maps which forget to rational equivalences of G -spectra.*

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