

ON THE PERIODIC v_2 -SELF-MAP OF A_1 PRASIT BHATTACHARYA^{1,*}, PHILIP EGGER², AND MARK MAHOWALD³

This paper is dedicated to the memory of Mark Mahowald (1931-2013).

ABSTRACT. We prove that the minimal v_2 -self-map of the 2-local spectrum A_1 has periodicity 32.

Keywords: stable homotopy, v_2 -periodicity

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Convention. Throughout this paper we work in the stable homotopy category of spectra localized at the prime 2.

1. INTRODUCTION

Let $K(n)$ be the n^{th} Morava K -theory. Let \mathcal{C}_0 be the category of 2-local finite spectra, $\mathcal{C}_n \subset \mathcal{C}_0$ be the full subcategory of $K(n-1)$ -acyclics and \mathcal{C}_∞ be the full subcategory of contractible spectra. Hopkins and Smith [NilpII] showed that the \mathcal{C}_n are thick subcategories of \mathcal{C}_0 (in fact, they are the only thick subcategories of \mathcal{C}_0) and they fit into a sequence

$$\mathcal{C}_0 \supset \mathcal{C}_1 \supset \dots \supset \mathcal{C}_n \supset \dots \supset \mathcal{C}_\infty.$$

We say a finite spectrum X is of type n if $X \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$.

A self-map $v : \Sigma^k X \rightarrow X$ of a finite spectrum X is called a v_n -self-map if

$$K(n)_*(v) : K(n)_*(X) \longrightarrow K(n)_*(X)$$

is an isomorphism. For a finite spectrum X , a self-map $v : \Sigma^k X \rightarrow X$ can also be regarded as an element of $\pi_k(X \wedge DX)$, where DX is the Spanier-Whitehead dual of X .

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For any ring spectrum E , let H_E denote the E -Hurewicz natural transformation

$$H_E : \pi_*(_) \longrightarrow E_*(_).$$

Let $k(n)$ denote the connective cover of $K(n)$. If $v : S^k \rightarrow X \wedge DX$ is a v_n -self-map then $H_{k(n)}(v) \in k(n)_*(X \wedge DX)$ has to be the image of $v_n^m \in k(n)_* \cong \mathbb{F}_2[v_n]$, for some positive integer m , under the map

$$k(n)_*\iota : k(n)_* \longrightarrow k(n)_*(X \wedge DX),$$

where $\iota : S^0 \rightarrow X \wedge DX$ is the unit map. The value m is called the *periodicity* of the v_n -self-map v . We call v a *minimal v_n -self-map* for X , if v is a v_n -self-map with smallest periodicity. An easy consequence of [NilpII, Theorem 9] is that the periodicity of a minimal v_n -self-map is always a power of 2.

Hopkins and Smith showed, among other things, that every type n spectrum admits a v_n -self-map and the cofiber of a v_n -self-map is of type $n + 1$. However, not much is known about the minimal periodicity of such v_n -self-maps.

The sphere spectrum S^0 is a type 0 spectrum with a v_0 -self-map $2 : S^0 \rightarrow S^0$. The cofiber of this v_0 -self-map is the type 1 spectrum $M(1)$. The spectrum $M(1)$ is known to admit a unique minimal v_1 -self-map of periodicity 4. The cofiber of this v_1 -self-map is denoted by $M(1, 4)$. In 2008, Behrens, Hill, Hopkins and the third author [BHHM] showed that the minimal v_2 -self-map on $M(1, 4)$ is $v : \Sigma^{192}M(1, 4) \rightarrow M(1, 4)$, which has periodicity 32.

Instead of S^0 , we can start with the type 0 spectrum $C\eta$, the cofiber of $\eta : S^1 \rightarrow S^0$. The spectrum $C\eta$ admits a non-zero v_0 -self-map $2 \wedge 1_{C\eta} : C\eta \rightarrow C\eta$, with cofiber $M(1) \wedge C\eta := Y$. The type 1 spectrum Y admits eight minimal v_1 -self-maps of periodicity 1. These eight maps are constructed in [DM81] using stunted projective spaces. The cofiber of any of the v_1 -self-maps is referred to as A_1 . Though there are eight different v_1 -self-maps, there are only four different homotopy types of the cofibers A_1 (see [DM81, Proposition 2.1]).

Let $A(1)$ be the subalgebra of the Steenrod algebra A generated by Sq^1 and Sq^2 . It turns out that the cohomology of any homotopy type of A_1 is a free $A(1)$ -module on one generator. However, different homotopy types of A_1 have different A -module structures, which are distinguished by the action of Sq^4 . We depict the cohomologies of the four different spectra A_1 in Figure 1.1 where the red square brackets represent an action of Sq^4 , the blue curved lines represent an action of Sq^2 , and the black straight lines represent an action of Sq^1 . The subalgebra $A(1)$ has four different A -module structures each of which corresponds to a homotopy type of A_1 . Any A -module structure on $A(1)$ has a nontrivial Sq^4 action on the generator in degree 1 forced by the Adem relations. However, there are choices for Sq^4 actions to be trivial or nontrivial on generators in degree 0 and degree 2, thus giving us four different A -module structures. We denote different homotopy types of A_1 using the notation $A_1[ij]$ where i and j are the indicator functions for the action of Sq^4 on the generators in degree 0 and degree 2 respectively. The spectra $A_1[01]$ and $A_1[10]$ are self-dual, i.e. $A_1[01] = \Sigma^6 DA_1[01]$ and $A_1[10] = \Sigma^6 DA_1[10]$, whereas $A_1[00]$ and $A_1[11]$ are dual to each other, i.e. $A_1[00] = \Sigma^6 DA_1[11]$. This is a consequence of the fact that

$$\chi(Sq^4) = Sq^4 + Sq^3 Sq^1,$$

where $\chi : A \rightarrow A$ is the canonical antiautomorphism of the Steenrod algebra.

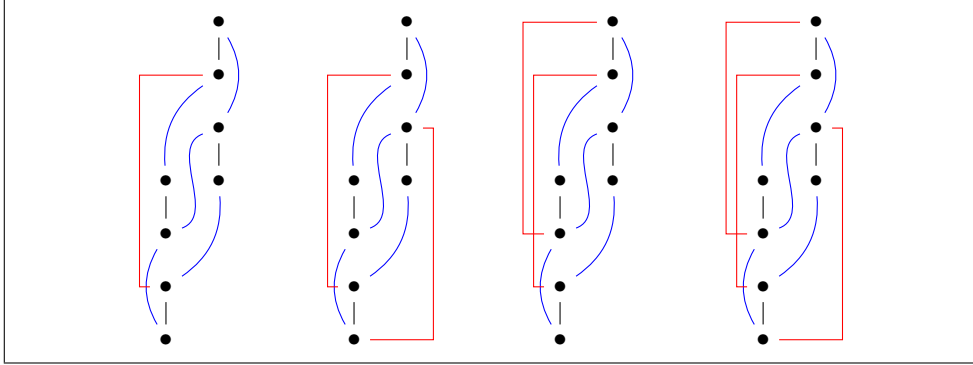


FIGURE 1.1. The A -module structures of $H^*(A_1[00])$, $H^*(A_1[10])$, $H^*(A_1[01])$ and $H^*(A_1[11])$.

It is worth noting that A_1 is created in a way similar to $M(1,4)$, where $C\eta$ is analogous to S^0 , and Y is analogous to $M(1)$. Therefore, it is reasonable to ask whether A_1 has the same v_2 -periodicity as $M(1,4)$. The minimal v_1 -self-map of Y has periodicity 1, which is less than the periodicity of the minimal v_1 -self-map on $M(1)$, which is 4. Hence, it is natural to ask if any of the four models of A_1 admit a v_2 -self-map of periodicity 2^k , where $k \leq 4$. In [BHHM], the third author conjectured that the minimal v_2 -self-map of A_1 should have periodicity 32. The goal of this paper is to prove the following ¹:

Main Theorem 1. *For all four models of A_1 , the minimal v_2 -self-map*

$$v : \Sigma^{192} A_1 \rightarrow A_1$$

has periodicity 32.

Notation 1.1. For any ring spectrum E , $\iota_E : S^0 \rightarrow E$ will denote the unit map. The unit map ι_E induces the the Hurewicz natural transformation

$$H_E : \pi_*(_) \longrightarrow E_*(_)$$

as introduced earlier. When $E = A_1 \wedge DA_1$, we simply use $\iota : S^0 \rightarrow A_1 \wedge DA_1$ to denote the unit map. Let $i : S^0 \hookrightarrow A_1$ be the map that represents the inclusion of the bottom cell. Let $j : A_1 \wedge DA_1 \rightarrow A_1$ denote the map $1_{A_1} \wedge Di$.

Notation 1.2. To lighten the notations, we use $Ext_T^{s,t}(X)$ to denote $Ext_T^{s,t}(H^*(X), \mathbb{F}_2)$, where T is a subalgebra of the Steenrod algebra A .

1.1. Outline. To prove Main Theorem 1, we use the fact that the spectrum tmf detects certain v_2 -periodic elements. More specifically, the unit map $\iota_{k(2)} : S^0 \rightarrow k(2)$ factors through tmf , i.e. we have

$$\iota_{k(2)} : S^0 \xrightarrow{\iota_{tmf}} tmf \xrightarrow{r} k(2).$$

The induced map in homotopy

$$r_* : tmf_* \longrightarrow k(2)_*$$

¹In [DM81], Davis and the third author claimed, incorrectly, that the periodicity of minimal v_2 -self-maps on $M(1,4)$ and the two self-dual models of A_1 , namely $A_1[01]$ and $A_1[10]$, as 8. After successfully correcting the v_2 -periodicity of $M(1,4)$ in [BHHM], the v_2 -periodicity of A_1 was called into question by the third author.

sends Δ^8 , the periodicity generator of tmf_* , to v_2^{32} . Since A_1 is a type 2 spectrum, we know that Δ^8 has a nonzero image under the composition

$$tmf_* \xrightarrow{r_*} k(2)_* \xrightarrow{k(2)_*\iota} k(2)_*(A_1 \wedge DA_1).$$

Therefore, from the commutative diagram

$$\begin{array}{ccc} tmf_* & \xrightarrow{tmf_*\iota} & tmf_*(A_1 \wedge DA_1) \\ r_* \downarrow & & \downarrow r_*(A_1 \wedge DA_1) \\ k(2)_* & \xrightarrow{k(2)_*\iota} & k(2)_*(A_1 \wedge DA_1) \end{array}$$

we see that $k(2)_*\iota(v_2^{32})$ lifts to $tmf_*(A_1 \wedge DA_1)$. We can choose the lift to be $tmf_*\iota(\Delta^8)$. This does not eliminate the possibility that smaller powers of $k(2)_*\iota(v_2)$ could lift to $tmf_*(A_1 \wedge DA_1)$. However, if $k(2)_*\iota(v_2^8)$ and $k(2)_*\iota(v_2^{16})$ do not lift to $tmf_*(A_1 \wedge DA_1)$, then they will not lift to $\pi_*(A_1 \wedge DA_1)$. So we analyse the map of Adams spectral sequences induced by $r : tmf \rightarrow k(2)$.

It is well-known that $H^*(tmf)$ as an A -module is isomorphic to $A//A(2)$, where $A(2)$ is the subalgebra of A generated by Sq^1, Sq^2 and Sq^4 . Therefore, applying a change of rings formula, we see that $Ext_{A(2)}^{s,t}(X)$ is the E_2 page of the Adams spectral sequence

$$E_2^{s,t} = Ext_{A(2)}^{s,t}(X) \Rightarrow tmf_{t-s}(X).$$

Similarly, we have an Adams spectral sequence

$$E_2^{s,t} = Ext_{E(Q_2)}^{s,t}(X) \Rightarrow k(2)_{t-s}(X),$$

which is a manifestation of the fact that $H^*(k(2)) = A//E(Q_2)$.

The map $\iota : S^0 \rightarrow A_1 \wedge DA_1$ induces the following commutative diagram of spectral sequences

$$(1.3) \quad \begin{array}{ccc} Ext_{A(2)}^{s,t}(S^0) & \xrightarrow{\iota_*^{tmf}} & Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) \\ \downarrow & & \downarrow \\ Ext_{E(Q_2)}^{s,t}(S^0) & \xrightarrow{\iota_*^{k(2)}} & Ext_{E(Q_2)}^{s,t}(A_1 \wedge DA_1). \end{array}$$

It is well known that

$$v_2^8 \in Ext_{E(Q_2)}^{8,48+8}(S^0)$$

is the image of the nonnilpotent element $b_{3,0}^4 \in Ext_{A(2)}^{8,48+8}(S^0)$ (see [Bau, Hen]).

Since A_1 is a type 2 spectrum, the element $\iota_*^{k(2)}(v_2^8) \in Ext_{E(Q_2)}^{8,48+8}(A_1 \wedge DA_1)$ is nonnilpotent. Consequently,

$$\iota_*^{tmf}(b_{3,0}^4) \in Ext_{A(2)}^{8,48+8}(A_1 \wedge DA_1)$$

is nonnilpotent. Thus, $\iota_*^{k(2)}(v_2^{8n})$ lifts to a nonzero element of $Ext_{A(2)}^{8n,48n+8n}(A_1 \wedge DA_1)$ for every $n \in \mathbb{N}$, which can be chosen to be $\iota_*^{tmf}(b_{3,0}^{4n})$.

In Section 2, we warm up by computing $Ext_{A(2)}^{s,t}(A_1)$ using the May spectral sequence and compute its vanishing line for later use. In Section 3 we show that

$\iota_*^{tmf}(b_{3,0}^A)$ admits a d_2 differential and $\iota_*^{tmf}(b_{3,0}^8)$ admits a d_3 differential in the Adams spectral sequence

$$E_2^{s,t} = Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) \implies tmf_{t-s}(A_1 \wedge DA_1).$$

This will imply the nonexistence of a 8-periodic or 16-periodic v_2 -self-map of A_1 . We will recall the algebraic tmf resolution of [BHHM] and use the resulting spectral sequence to show that for every $n \in \mathbb{N}$, the element $\iota_*^{tmf}(b_{3,0}^{4n})$ lifts to $Ext_A^{8n, 48n+8n}(A_1 \wedge DA_1)$ under the map induced by H_{tmf} . Furthermore, we show that the lifts of $\iota_*^{tmf}(b_{3,0}^A)$ and $\iota_*^{tmf}(b_{3,0}^8)$ support a d_2 and a d_3 differential respectively in the Adams spectral sequence

$$E_2^{s,t} = Ext_A^{s,t}(A_1 \wedge DA_1) \implies \pi_{t-s}(A_1 \wedge DA_1).$$

This extra effort enables us to identify some d_2 and d_3 differentials in the above spectral sequence, which will play a crucial role in the proof of the existence of a 32-periodic v_2 -self-map of A_1 . Thus, the existence of a 32-periodic v_2 -self-map of A_1 boils down to showing that the lift of $\iota_*^{tmf}(b_{3,0}^{16})$, which we'll call \bar{v} , is a permanent cycle in the Adams spectral sequence

$$E_2^{s,t} = Ext_A^{s,t}(A_1 \wedge DA_1) \implies \pi_{t-s}(A_1 \wedge DA_1).$$

Note that \bar{v} cannot be a target of a differential as its image in $Ext_{E(Q_2)}^{32, 192+32}(A_1 \wedge DA_1)$ is not a target of a differential. Further, \bar{v} cannot support a nontrivial d_2 or d_3 differential by the Leibniz rule. In Section 5 we use all prior knowledge of d_2 and d_3 differentials, including an important d_3 differential found in Section 4, to show that the potential targets of d_r differentials for $r \geq 4$ are either zero or not present in the Adams E_4 page. This will conclude the proof of Main Theorem 1.

Notation 1.4. For the rest of the paper, we will abusively denote any $x \in Ext_{A(2)}^{s,t}(S^0)$ and $\iota_*^{tmf}(x) \in Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$ and sometimes their lifts in $Ext_A^{s,t}(S^0)$ and $Ext_A^{s,t}(A_1 \wedge DA_1)$ respectively under H_{tmf*} , just by x . This will allow us to suppress cumbersome notations. We will make sure that the ambient group in which x belongs is clear from the context.

1.2. Use of Bruner's Ext software. We will use this software (see Appendix A or [Bru] for a description of the program) for two purposes. Given any $A(2)$ -module M , finitely generated as an \mathbb{F}_2 -vector space, the program can compute the groups $Ext_{A(2)}^{s,t}(M, \mathbb{F}_2)$ to the extent of identifying generators in each bidegree within a finite range, determined by the user. Since we are interested in $Ext_{A(2)}^{s,t}(X)$ for finite spectra X , such as $A_1 \wedge DA_1$, whose cohomology structures as $A(2)$ -modules are known, this suits our task perfectly. The second purpose is the following: As any finite spectrum X is an S^0 -module, $Ext_{A(2)}^{*,*}(X)$ is a module over $Ext_{A(2)}^{*,*}(S^0)$. Given an element $x \in Ext_{A(2)}^{s,t}(X)$, the action of $Ext_{A(2)}^{*,*}(S^0)$ can be computed using the `dolifts` functionality of the software. Summary of the output of the Bruner's program that is needed for some of the results in Section 4 and Section 5 are listed in Appendix B and Appendix C respectively.

One should also be aware that Main Theorem 1 is by no means a consequence of the programming output. However, parts of the proof are reduced to pure algebraic computation, which can be performed using Bruner's program.

2. COMPUTATION OF $Ext_{A(2)}^{s,t}(A_1)$ AND ITS VANISHING LINE

J.P. May in his thesis [May] introduced a filtration of the Steenrod algebra called the May filtration, which induces a filtration of the cobar complex $C(\mathbb{F}_2, A_*, \mathbb{F}_2)$. This filtration gives a trigraded spectral sequence

$$E_1^{s,t,u} = \mathbb{F}_2[h_{i,j} : i \geq 1, j \geq 0] \Rightarrow Ext_A^{s,t}(S^0), |h_{i,j}| = (1, 2^j(2^i - 1), 2i - 1),$$

with differentials d_r of tridegree $(1, 0, 1 - 2r)$, which converges to the E_2 page of the Adams spectral sequence

$$Ext_A^{s,t}(S^0) \Rightarrow \pi_{t-s}(S^0).$$

The element $h_{i,j}$ corresponds to the class $[\xi_i^{2^j}]$ in the cobar complex $C(\mathbb{F}_2, A_*, \mathbb{F}_2)$. We stick to the notation introduced by Tangora in his thesis [Tan]. For example, $h_{1,j}$ is abbreviated by h_j . Meanwhile, there are many elements $h_{i,j}$ that are not d_1 -cycles in the May spectral sequence, however, even in these cases, the Leibniz rule means that $h_{i,j}^2$ will be d_1 -cycles. To get around the awkwardness of talking about $h_{i,j}^2$ in later pages of the May spectral sequence, where $h_{i,j}$ may not even exist, Tangora uses $b_{i,j}$ to denote $h_{i,j}^2$ from the E_2 page onwards.

One can use the same May filtration on the subalgebra $A(2)$ of A , to obtain a filtration on the cobar complex $C(\mathbb{F}_2, A(2)_*, \mathbb{F}_2)$. Thus we get a May spectral sequence with finitely many differentials

$$\mathbb{F}_2[h_0, h_1, h_2, h_{2,0}, h_{2,1}, h_{3,0}] \Rightarrow Ext_{A(2)}^{s,t}(S^0)$$

all of which have been computed (see [DM]). The bigraded ring $Ext_{A(2)}^{s,t}(S^0)$ is the Adams E_2 page for the homotopy groups of tmf .

We have obtained A_1 by a series of cofibrations,

$$S^1 \xrightarrow{\eta} S^0 \longrightarrow C\eta$$

$$C\eta \xrightarrow{2} C\eta \longrightarrow Y$$

and

$$\Sigma^2 Y \xrightarrow{v_1} Y \longrightarrow A_1.$$

The maps 2 , η and v_1 are detected by h_0 , h_1 and $h_{2,0}$, respectively, in the May spectral sequence. Using the fact that cofiber sequences induce long exact sequences of E_1 pages of the May spectral sequence, we get that the E_1 page of the May spectral sequence converging to $Ext_{A(2)}^{s,t}(A_1)$ is

$$\mathbb{F}_2[h_2, h_{2,1}, h_{3,0}] \Longrightarrow Ext_{A(2)}^{s,t}(A_1).$$

Alternatively, using a change of rings formula, we see that the cobar complex (whose cohomology is $Ext_{A(2)}^{s,t}(A_1)$) is

$$C(\mathbb{F}_2, A(2)_*, A(1)_*) \cong C(\mathbb{F}_2, (A(2)//A(1))_*, \mathbb{F}_2),$$

hence a quotient of $C(\mathbb{F}_2, A(2)_*, \mathbb{F}_2)$. Thus, the filtration on $C(\mathbb{F}_2, A(2)_*, \mathbb{F}_2)$ induces a filtration on $C(\mathbb{F}_2, A(2)_*, A(1)_*)$ as a result of which $\mathbb{F}_2[h_2, h_{2,1}, h_{3,0}]$ is a module over $\mathbb{F}_2[h_0, h_1, h_2, h_{2,0}, h_{2,1}, h_{3,0}]$.

The d_1 differentials in the May spectral sequence

$$\mathbb{F}_2[h_0, h_1, h_2, h_{2,0}, h_{2,1}, h_{3,0}] \Rightarrow Ext_{A(2)}^{s,t}(S^0)$$

come from the coproduct on $A(2)_*$. It is well known that $d_1(h_2) = 0$, $d_1(h_{2,1}) = h_1h_2$ and $d_1(h_{3,0}) = h_0h_{2,1} + h_2h_{2,0}$. Under the quotient map

$$\mathbb{F}_2[h_0, h_1, h_2, h_{2,0}, h_{2,1}, h_{3,0}] \twoheadrightarrow \mathbb{F}_2[h_2, h_{2,1}, h_{3,0}]$$

all the images of the above differentials map to zero. Therefore, there are no d_1 differentials in the May spectral sequence

$$\mathbb{F}_2[h_2, h_{2,0}, h_{3,0}] \Rightarrow Ext_{A(2)}(A_1).$$

One can use Nakamura's formula to compute higher May differentials. The operations Sq_i on the cobar complex of $C(\mathbb{F}_2, A_*, \mathbb{F}_2)$, defined by $Sq_i(x) = x \cup_i x + \delta x \cup_{i+1} x$ (see [Nak]), satisfy

$$\begin{aligned} Sq_0(h_{i,j}) &= h_{i,j}^2 \\ Sq_0(b_{i,j}) &= b_{i,j}^2 \\ Sq_1(h_{i,j}) &= h_{i,j+1} \end{aligned}$$

as well as Cartan's formulas (see [Nak, Proposition 4.4 and Proposition 4.5])

$$\begin{aligned} Sq_0(xy) &= Sq_0(x)Sq_0(y) \\ Sq_1(xy) &= Sq_1(x)Sq_0(y) + Sq_0(x)Sq_1(y) \end{aligned}$$

whenever x and y are represented by elements in appropriate pages of the May spectral sequence. In particular we have

$$Sq_1(x^2) = 0$$

for every x . The differential δ in the cobar complex $C(\mathbb{F}_2, A_*, \mathbb{F}_2)$, satisfies the relation

$$(2.1) \quad \delta Sq_i = Sq_{i+1} \delta$$

for $i \geq 0$ (see [Nak, Lemma 4.1]) and is often called Nakamura's formula in the literature.

Since the May spectral sequence is obtained by filtering the cobar complex, the above formula helps in detecting differentials in the May spectral sequence. Since the cobar complex

$$C(\mathbb{F}_2, A(2)_*, A(1)_*) \cong C(\mathbb{F}_2, (A(2)//A(1))_*, \mathbb{F}_2),$$

is a quotient of $C(\mathbb{F}_2, A(2)_*, \mathbb{F}_2)$, we apply (2.1) to find differentials in the May spectral sequence for A_1 .

Lemma 2.2. *In the May spectral sequence*

$$\mathbb{F}_2[h_2, h_{2,1}, h_{3,0}] \Rightarrow Ext_{A(2)}^{s,t}(A_1),$$

we have

- $d_2(b_{2,1}) = h_2^3$
- $d_3(b_{3,0}) = h_2^2 h_{2,1}$
- $d_4(b_{3,0}^2) = h_2 b_{2,1}^2$

and the spectral sequence collapses at E_5 .

Proof. In the May spectral sequence

$$(2.3) \quad \mathbb{F}_2[h_0, h_1, h_2, h_{2,0}, h_{2,1}, h_{3,0}] \Rightarrow Ext_{A(2)}^{s,t}(S^0)$$

the differentials $d_2(b_{2,1}) = h_2^3$ and $d_4(b_{3,0}^2) = h_2 b_{2,1}^2$ translate into differentials in $Ext_{A(2)}(A_1)$. In the cobar complex, $b_{3,0}$ is represented by the element $[\xi_3|\xi_3]$. Since $b_{3,0} = Sq_0 h_{3,0}$, we apply (2.1), to obtain

$$\begin{aligned} d_3(Sq_0 h_{3,0}) &= Sq_1(d_1 h_{3,0}) \\ &= Sq_1(h_0 h_{2,1} + h_2 h_{2,0}) \\ &= h_0^2 h_{2,2} + h_1 h_{2,1}^2 + h_2^2 h_{2,1} + h_3 h_{2,0}^2 \\ &= h_2^2 h_{2,1} \quad \text{in the May spectral sequence for } A_1. \end{aligned}$$

Therefore, in the cobar complex $C(\mathbb{F}_2, A(2)_*, A(1)_*)$, it must be the case that,

$$\delta([\xi_3|\xi_3]) = [\xi_1^2|\xi_1^2|\xi_2^2] + \text{elements of higher May filtration.}$$

As a result we have

$$d_3(b_{3,0}) = h_2^2 h_{2,1}.$$

The May spectral sequence 2.3 does not have any differentials d_r for $r \geq 5$, consequently no differentials in the May spectral sequence

$$\mathbb{F}_2[h_2, h_{2,1}, h_{3,0}] \Rightarrow Ext_{A(2)}^{s,t}(A_1).$$

□

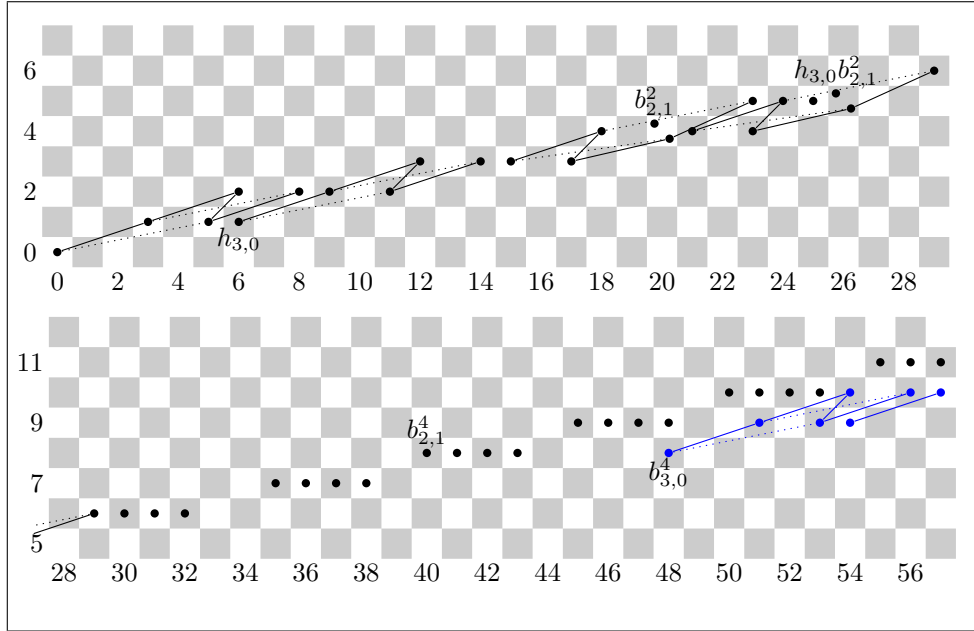


FIGURE 2.1. The E_∞ -page of the May spectral sequence for $Ext_{A(2)}(A_1)$.

In Figure 2.1, the solid line of slope 1 represents multiplication by h_1 , the solid line of slope $\frac{1}{3}$ represents multiplication by h_2 , while the dotted line of slope $\frac{1}{5}$ represents multiplication by $h_{2,1}$. The element $b_{3,0}^4$ is the periodicity generator of $Ext_{A(2)}^{*,*}(A_1)$ and the blue part is simply a repetition of the earlier pattern.

This matches the output of Bruner's program [Bru] for $Ext_{A(2)}^{s,t}(A_1)$, though different models of A_1 may have different extensions some of which might not be detected in the May spectral sequence.

Having computed the E_2 page $Ext_{A(2)}^{s,t}(A_1)$, we give a vanishing line of this spectral sequence, which will come in handy later on in the paper.

Lemma 2.4. *The group $Ext_{A(2)}^{s,t}(A_1)$ is zero if*

$$s > \frac{1}{5}(t - s) + 1,$$

and for $t - s \geq 29$, it is zero if

$$s > \frac{1}{5}(t - s).$$

In other words, there is a vanishing line

$$y = \frac{1}{5}x + 1.$$

Proof. Of the three generators of the E_1 page, h_2 has slope $\frac{1}{3}$, $h_{2,1}$ has slope $\frac{1}{5}$, and $h_{3,0}$ has slope $\frac{1}{6}$. However, while $Ext_{A(2)}^{s,t}(A_1)$ contains infinitely large powers of $h_{2,1}$ and $h_{3,0}$, it only contains powers up to 2 of h_2 . Hence, the vanishing line of $Ext_{A(2)}^{s,t}(A_1)$ must have slope $\frac{1}{5}$, determined by $b_{2,1}^2$. Now, since $h_2 b_{2,1}^2 = 0$, the vanishing line for stems greater than 29 is $y = \frac{1}{5}x$ and a glance at Figure 2.1 gives us the y -intercept of the overall vanishing line. \square

3. A d_2 AND A d_3 DIFFERENTIAL

In this section we first show that $b_{3,0}^4$ and $b_{3,0}^8$ in $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$ support a d_2 and a d_3 differential respectively. Then we show that these differentials lift to $Ext_A^{s,t}(A_1 \wedge DA_1)$ under the map of spectral sequences induced by H_{tmf} . Some of the proofs in this section as well as in the subsequent sections use Bruner's program [Bru]. We provide Appendix A to help readers familiarize themselves with this software.

In the Adams spectral sequence

$$E_2^{s,t} = Ext_{A(2)}^{s,t}(S^0) \implies tmf_{t-s}$$

it is well known that $d_2(b_{3,0}^4) = e_0r$ and $d_3(b_{3,0}^8) = wgr$ (see [Hen]). Using Bruner's program, we see that e_0r and wgr both have nonzero images in $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$.

Lemma 3.1. *In the Adams spectral sequence*

$$E_2^{s,t} = Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) \implies tmf_{t-s}(A_1 \wedge DA_1)$$

we have $d_2(b_{3,0}^4) = e_0r$ and $d_3(b_{3,0}^8) = wgr$.

Proof. In the map of Adams spectral sequences,

$$\begin{array}{ccc} E_2^{s,t} = Ext_{A(2)}^{s,t}(S^0) & \implies & tmf_{t-s} \\ \downarrow & & \downarrow \\ E_2^{s,t} = Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) & \implies & tmf_{t-s}(A_1 \wedge DA_1) \end{array}$$

we have established that (beware of our abusive notations as explained in Notation 1.1)

$$\begin{aligned} \text{Ext}_{A(2)}^{s,t}(S^0) &\xrightarrow{\iota_*^{tmf}} \text{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \\ b_{3,0}^4 &\mapsto b_{3,0}^4 \\ b_{3,0}^8 &\mapsto b_{3,0}^8 \\ e_0 r &\mapsto e_0 r \\ wgr &\mapsto wgr. \end{aligned}$$

Since $d_2(b_{3,0}^4) = e_0 r$ in the Adams spectral sequence for tmf_* , it follows that we have a d_2 -differential

$$d_2(b_{3,0}^4) = e_0 r.$$

As a consequence of the Leibniz rule, $d_2(b_{3,0}^8) = 0$ and hence $b_{3,0}^8$ and its image under ι_*^{tmf} are nonzero elements in the E_3 pages of Adams spectral sequences for tmf_* and $tmf_*(A_1 \wedge DA_1)$, respectively.

Since there is a d_3 differential $d_3(b_{3,0}^8) = wgr$ in the Adams spectral sequence for tmf_* , it will follow that $b_{3,0}^8$ supports a d_3 -differential in the Adams spectral sequence for $tmf_*(A_1 \wedge DA_1)$, provided the image of wgr is nonzero in the E_3 -page of the Adams spectral sequence for $tmf_*(A_1 \wedge DA_1)$. Thus we have to show that there does not exist a differential of the form $d_2(x) = wgr$.

Using Bruner's program [Bru], we check that $wgr \in \text{Ext}_{A(2)}^{19,95+19}(S^0)$ maps nontrivially to $\text{Ext}_{A(2)}^{19,95+19}(A_1)$. Thus, if there exists an x such that $d_2(x) = wgr$ in

$$\text{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \implies \text{tmf}_{t-s}(A_1 \wedge DA_1),$$

then the image of x , call it x' , must be nontrivial under the map

$$j_* : \text{Ext}_{A(2)}^{17,96+17}(A_1 \wedge DA_1) \longrightarrow \text{Ext}_{A(2)}^{17,96+17}(A_1)$$

and we will have $d_2(x') = wgr$ in

$$\text{Ext}_{A(2)}^{s,t}(A_1) \implies \text{tmf}_{t-s}(A_1).$$

There is exactly one generator of $\text{Ext}_{A(2)}^{17,96+17}(A_1)$, and that generator is $b_{3,0}^4 \cdot y$ under the pairing

$$\text{Ext}_{A(2)}^{8,48+8}(S^0) \otimes \text{Ext}_{A(2)}^{9,48+9}(A_1) \longrightarrow \text{Ext}_{A(2)}^{17,96+17}(A_1).$$

It is clear that $d_2(y) = 0$ as $\text{Ext}_{A(2)}^{11,47+11}(A_1) = 0$ (see Chart 2.1). Thus using the Leibniz rule, we see that

$$d_2(b_{3,0}^4 y) = e_0 r \cdot y.$$

Using [Bru], we check that $e_0 r \cdot y = 0$. Therefore, wgr is nonzero in the E_3 -page of the spectral sequence

$$\text{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \implies \text{tmf}_{t-s}(A_1 \wedge DA_1),$$

and therefore

$$d_3(b_{3,0}^8) = wgr$$

in this spectral sequence. □

As a consequence of Lemma 3.1, we see that v_2^8 and v_2^{16} in $k(2)_*(A_1 \wedge DA_1)$ do not lift to $tmf_*(A_1 \wedge DA_1)$ and hence cannot lift to $\pi_*(A_1 \wedge DA_1)$. Thus we have established:

Theorem 3.2. *The spectra A_1 do not admit an 8-periodic or 16-periodic v_2 -self-map.*

Next we describe an algebraic resolution which will allow us to lift the d_2 differential and the d_3 differential of Lemma 3.1 to the Adams spectral sequence

$$E_2^{s,t} = Ext_A^{s,t}(A_1 \wedge DA_1) \implies \pi_{t-s}(A_1 \wedge DA_1).$$

We will briefly recall the resolution described in [BHHM, Section 5], and how it is used to lift elements of Ext groups over $A(2)$ to Ext groups over A . Consider the A -module

$$A//A(2) := A \otimes_{A(2)} \mathbb{F}_2$$

and denote by $\overline{A//A(2)}$ the kernel of the augmentation map

$$A//A(2) \longrightarrow \mathbb{F}_2.$$

When we consider the triangulated structure of the derived category of A -modules, we get maps

$$A//A(2) \longrightarrow \mathbb{F}_2 \longrightarrow \overline{A//A(2)}[1],$$

and a resulting diagram

$$\begin{array}{ccccccc} \mathbb{F}_2 & \longrightarrow & \overline{A//A(2)}[1] & \longrightarrow & \overline{A//A(2)}^{\otimes 2}[2] & \longrightarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ A//A(2) & & A//A(2) \otimes \overline{A//A(2)}[1] & & A//A(2) \otimes \overline{A//A(2)}^{\otimes 2}[2] & & \end{array}$$

to which we shall apply the functor $Ext_A^{s,t}(H^*(X) \otimes -, \mathbb{F}_2)$ to get a spectral sequence, which we shall refer to as the algebraic tmf spectral sequence to reflect the fact that $A//A(2)$ is the cohomology of tmf . This spectral sequence will be trigraded, with E_1 page

$$\begin{aligned} E_1^{s,t,n} &= Ext_A^{s,t}(H^*(X) \otimes A//A(2) \otimes \overline{A//A(2)}^{\otimes n}[n], \mathbb{F}_2) \\ &\cong Ext_{A(2)}^{s-n,t}(H^*(X) \otimes \overline{A//A(2)}^{\otimes n}, \mathbb{F}_2) \end{aligned}$$

which converges to

$$Ext_A^{s,t}(H^*(X), \mathbb{F}_2).$$

For any element in the algebraic tmf spectral sequence in tridegree (s, t, n) , we will refer to s as its Adams filtration, t as the internal degree and n as the algebraic tmf filtration. The differential d_r has tridegree $(1, 0, r)$. It is shown in [DM] that

$$A//A(2) \cong \bigoplus_{i \geq 0} H^*(\Sigma^{8i} bo_i),$$

where bo_i denotes the i -th bo -Brown-Gitler spectrum of [GJM]. As a result the E_1 page of the algebraic tmf spectral sequence simplifies to

$$E_1^{s,t,n} = \bigoplus_{i_1, \dots, i_n \geq 1} Ext_{A(2)}^{s-n, t-8(i_1+\dots+i_n)}(X \wedge bo_{i_1} \wedge \dots \wedge bo_{i_n}) \implies Ext_A^{s,t}(X).$$

We will attempt to exploit the relative sparseness of the E_1 page, especially its vanishing line properties, in the case when $X = A_1 \wedge DA_1$.

Remark 3.3 (The cellular structure of bo -Brown-Gitler spectra). The spectrum bo_0 is the sphere spectrum. The cohomology of the spectrum bo_1 as a module over the Steenrod algebra can be described through the following picture, with the generators labelled by cohomological degree:



where the black, blue and red lines describe the actions of Sq^1 , Sq^2 and Sq^4 respectively. Note that the 4-skeleton of bo_1 is $C\nu$. Indeed, the bo_i 's fit together to form the following cofiber sequence

$$bo_{i-1} \longrightarrow bo_i \longrightarrow \Sigma^{4i}B(i)$$

where $B(i)$ is the i -th integral Brown-Gitler spectrum as described in [GJM]. Therefore for every $i \geq 1$, the 7-skeleton of bo_i is bo_1 and the 4-skeleton of bo_i is $C\nu$.

One can compute $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1 \wedge bo_i)$ from $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$ using the Atiyah-Hirzebruch spectral sequence or with Bruner's program [Bru].

Lemma 3.4. *The group*

$$Ext_{A(2)}^{s,t}(A_1 \wedge DA_1 \wedge bo_{i_1} \wedge \dots \wedge bo_{i_n})$$

is zero if $s > \frac{1}{5}((t-s) + 6)$.

Proof. We showed in Lemma 2.4 that $Ext_{A(2)}^{s,t}(A_1)$ has a vanishing line $s = \frac{1}{5}(t-s)$ for $t-s \geq 30$ and a vanishing line of $s = \frac{1}{5}(t-s) + 1$ overall. The only generator of $Ext_{A(2)}^{s,t}(A_1)$ with a slope greater than $\frac{1}{5}$ is h_2 , so if we kill off h_2 by considering $Ext_{A(2)}^{s,t}(A_1 \wedge C\nu)$ then the vanishing line is precisely $s = \frac{1}{5}(t-s)$.

As we mentioned in Remark 3.3, the 4-skeleton of any bo_i is $C\nu$ and the next cell is in dimension 6. So we can build bo_i by attaching finitely many cells to $C\nu$ of dimension ≥ 6 . Hence by using the Atiyah-Hirzebruch spectral sequence and the fact that $\frac{1}{5}(x-6) + 1 < \frac{1}{5}x$, one can see that the vanishing line of $A_1 \wedge bo_i$ is $s = \frac{1}{5}(t-s)$. One can build $A_1 \wedge bo_{i_1} \wedge \dots \wedge bo_{i_n}$ from $A_1 \wedge bo_{i_1}$, iteratively using cofiber sequences, which depend on the cell structure of $bo_{i_2} \wedge \dots \wedge bo_{i_n}$. Since we have already established that $Ext_{A(2)}^{s,t}(A_1 \wedge bo_{i_1})$ has vanishing line $s = \frac{1}{5}(t-s)$ and that $bo_{i_2} \wedge \dots \wedge bo_{i_n}$ is a connected spectrum, we conclude, using the Atiyah-Hirzebruch spectral sequence, that the vanishing line for $Ext_{A(2)}^{s,t}(A_1 \wedge bo_{i_1} \wedge \dots \wedge bo_{i_n})$ is $s = \frac{1}{5}(t-s)$.

However, DA_1 has cells in negative dimension, in fact the bottom cell is in dimension -6 . Again by using the Atiyah-Hirzebruch spectral sequence, one concludes that the vanishing line for $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1 \wedge bo_{i_1} \wedge \dots \wedge bo_{i_n})$ is

$$s = \frac{1}{5}(t-s+6)$$

for any $i_k \geq 1$, completing the proof. \square

Corollary 3.5. *The group $Ext_A^{s,t}(A_1 \wedge DA_1)$ is zero if*

$$s > \frac{1}{5}(t - s) + \frac{11}{5}$$

and for $t - s \leq 23$, it is zero if

$$s > \frac{1}{5}(t - s) + \frac{6}{5}.$$

The result is a straightforward consequence of Lemma 2.4, Lemma 3.4 and the algebraic tmf spectral sequence.

Lemma 3.6. *The element*

$$b_{3,0}^4 \in Ext_{A(2)}^{8,48+8}(A_1 \wedge DA_1)$$

is in the image of the map

$$Ext_A^{8,48+8}(A_1 \wedge DA_1) \longrightarrow Ext_{A(2)}^{8,48+8}(A_1 \wedge DA_1).$$

Proof. Clearly $b_{3,0}^4$ is in bidegree $(s, t) = (8, 48 + 8) = (8, 56)$ of the E_1 page of the algebraic tmf spectral sequence, so we must verify that it is a permanent cycle, which we will do by showing that the E_1 page is zero in bidegree $(s, t) = (9, 56)$ when $n \geq 1$. Namely, we must show that for every $n \geq 1$, the group

$$\bigoplus_{i_1, \dots, i_n \geq 1} Ext_{A(2)}^{9-n, 56-8(i_1+\dots+i_n)}(A_1 \wedge DA_1 \wedge bo_{i_1} \wedge \dots \wedge bo_{i_n})$$

is zero. Using the vanishing line in Lemma 3.4, the group is zero for all $i_1, \dots, i_n \geq 1$ such that

$$\frac{1}{5}(56 - 8(i_1 + \dots + i_n) - 9 + n + 6) < 9 - n$$

or

$$(3.7) \quad \frac{1}{5}(53 + n - 8(i_1 + \dots + i_n)) < 9 - n.$$

Of course, we have

$$\frac{1}{5}(53 + n - 8(i_1 + \dots + i_n)) \leq \frac{1}{5}(53 - 7n),$$

and if $n > 4$, we also have

$$\frac{1}{5}(53 - 7n) < 9 - n.$$

Assume $n = 1$, then (3.7) becomes

$$\frac{1}{5}(54 - 8i_1) < 8,$$

or

$$i_1 > 1,$$

so it suffices to verify that

$$Ext_{A(2)}^{8,48}(A_1 \wedge DA_1 \wedge bo_1) = 0.$$

Assume $n = 2$, then (3.7) becomes

$$\frac{1}{5}(55 - 8(i_1 + i_2)) < 7,$$

or

$$i_1 + i_2 > 2,$$

so it suffices to verify that

$$\text{Ext}_{A(2)}^{7,40}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1) = 0.$$

Assume $n = 3$, then (3.7) becomes

$$\frac{1}{5}(56 - 8(i_1 + i_2 + i_3)) < 6,$$

or

$$i_1 + i_2 + i_3 > 3,$$

so it suffices to verify that

$$\text{Ext}_{A(2)}^{6,32}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1 \wedge bo_1) = 0.$$

Assume $n = 4$, then (3.7) becomes

$$\frac{1}{5}(57 - 8(i_1 + i_2 + i_3 + i_4)) < 5,$$

or

$$i_1 + i_2 + i_3 + i_4 > 4,$$

so it suffices to verify that

$$\text{Ext}_{A(2)}^{5,24}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1 \wedge bo_1 \wedge bo_1) = 0.$$

For all four models of A_1 , Bruner's program [Bru] shows that all the groups we expected to be zero are in fact zero. \square

Corollary 3.8. *For all $n \in \mathbb{N}$, the elements $b_{3,0}^{4n} \in \text{Ext}_{A(2)}^{8n,48n+8n}(A_1 \wedge DA_1)$ lift to $\text{Ext}_A^{8n,48n+8n}(A_1 \wedge DA_1)$ under the map induced by H_{tmf} .*

Proof. Since $A_1 \wedge DA_1$ is a ring spectrum, it follows that the map

$$\text{Ext}_A^{s,t}(A_1 \wedge DA_1) \longrightarrow \text{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1)$$

induced by H_{tmf} is a map of algebras. By Lemma 3.6, $b_{3,0}^4$ lifts and thus $b_{3,0}^{4n}$ lifts for every $n \in \mathbb{N}$. \square

Remark 3.9. The lift of $b_{3,0}^{4n} \in \text{Ext}_{A(2)}^{8n,48n+8n}(A_1 \wedge DA_1)$ to $\text{Ext}_A^{8n,48n+8n}(A_1 \wedge DA_1)$ may not be unique. The conclusions of Lemma 3.10 will not depend on the choice of lift.

Lemma 3.10. *In the Adams spectral sequence*

$$E_2^{s,t} = \text{Ext}_A^{s,t}(A_1 \wedge DA_1) \implies \pi_{t-s}(A_1 \wedge DA_1)$$

there is a d_2 -differential

$$d_2(b_{3,0}^4) = \widetilde{e_0 r} = e_0 r + R$$

and a d_3 -differential

$$d_3(b_{3,0}^8) = \widetilde{wgr} = wgr + S$$

for some R and S in algebraic tmf filtration greater than zero.

Proof. Recall that the element $e_0r \in Ext_A^{10,47+10}(S^0)$ (see [Tan]) maps to a nonzero element in $Ext_{A(2)}^{10,47+10}(S^0)$ which is also called e_0r in the literature, and that $d_2(b_{3,0}^4) = e_0r$ in

$$Ext_{A(2)}^{s,t}(S^0) \implies tmf_{t-s}.$$

In Lemma 3.1, we argued that e_0r has a nonzero image under the map

$$Ext_{A(2)}^{10,47+10}(S^0) \longrightarrow Ext_{A(2)}^{10,47+10}(A_1 \wedge DA_1).$$

Therefore by inspecting the commutative diagram

$$(3.11) \quad \begin{array}{ccc} Ext_A^{10,47+10}(S^0) & \longrightarrow & Ext_A^{10,47+10}(A_1 \wedge DA_1) \\ \downarrow & & \downarrow \\ Ext_{A(2)}^{10,47+10}(S^0) & \longrightarrow & Ext_{A(2)}^{10,47+10}(A_1 \wedge DA_1), \end{array}$$

we see that $e_0r \in Ext_A^{10,47+10}(S^0)$ has a nonzero image in $Ext_A^{10,47+10}(A_1 \wedge DA_1)$. Since $d_2(b_{3,0}^4) = e_0r$ in $Ext_{A(2)}(A_1 \wedge DA_1)$, it follows that

$$d_2(b_{3,0}^4) = e_0r + R$$

in $Ext_A(A_1 \wedge DA_1)$ for some R in algebraic tmf filtration greater than zero.

Consequently, $d_2(b_{3,0}^8) = 0$ in

$$Ext_A^{s,t}(A_1 \wedge DA_1) \implies \pi_{t-s}(A_1 \wedge DA_1),$$

and clearly $b_{3,0}^8$ is not hit by a d_2 in this spectral sequence, otherwise it would be hit by a differential in

$$Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) \implies tmf_{t-s}(A_1 \wedge DA_1).$$

However, $b_{3,0}^8$ could support a nonzero d_3 . The element $wgr \in Ext_A^{19,95+19}(S^0)$ maps to a nonzero element of $Ext_{A(2)}^{19,95+19}(S^0)$ we will also call wgr . We showed, in Lemma 3.1, that the image of wgr is nonzero in $Ext_{A(2)}^{19,95+19}(A_1 \wedge DA_1)$. The diagram

$$(3.12) \quad \begin{array}{ccc} Ext_A^{19,95+19}(S^0) & \longrightarrow & Ext_A^{19,95+19}(A_1 \wedge DA_1) \\ \downarrow & & \downarrow \\ Ext_{A(2)}^{19,95+19}(S^0) & \longrightarrow & Ext_{A(2)}^{19,95+19}(A_1 \wedge DA_1), \end{array}$$

makes it clear that the image of wgr is nonzero in $Ext_A^{19,95+19}(A_1 \wedge DA_1)$.

Note that $wgr \in Ext_A^{19,95+19}(A_1 \wedge DA_1)$ cannot support a d_2 -differential as $d_2(wgr)$ would have bidegree $(21, 94 + 21)$ and

$$Ext_A^{21,94+21}(A_1 \wedge DA_1) = 0$$

by Corollary 3.5. Moreover, wgr cannot be target of a d_2 -differential as this will force a d_2 -differential in $Ext_{A(2)}(A_1 \wedge DA_1)$, which is not possible, as we argued in the proof of Lemma 3.1. Thus, wgr is in the E_3 -page.

From Lemma 3.1, we know that $d_3(b_{3,0}^8) = wgr$ in the Adams spectral sequence for $tmf_*(A_1 \wedge DA_1)$. It follows that

$$d_3(b_{3,0}^8) = wgr + S,$$

for some S in algebraic tmf filtration greater than zero, in the Adams spectral sequence for $\pi_*(A_1 \wedge DA_1)$. \square

4. ANOTHER d_3 DIFFERENTIAL

In the Adams spectral sequence

$$Ext_{A(2)}^{s,t}(S^0) \implies tmf_{t-s},$$

there is a well-known d_3 differential

$$d_3(v_2^{20}h_1) = g^6.$$

The element g is Tangora's name [Tan] for the element detected by $b_{2,1}^2$ in the E_∞ page of the May spectral sequence

$$\mathbb{F}_2[h_{i,j} : i > 0, j \geq 0] \implies Ext_A^{s,t}(S^0).$$

In the literature, the same name is adopted for its image in $Ext_{A(2)}^{24,120+24}(S^0)$. The goal of this section is to show that this differential induces a d_3 differential in

$$Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) \implies tmf_{t-s}(A_1 \wedge DA_1)$$

and it lifts to a d_3 differential under the map of spectral sequences

$$\begin{array}{ccc} Ext_A^{s,t}(A_1 \wedge DA_1) & \implies & \pi_{t-s}(A_1 \wedge DA_1) \\ \downarrow & & \downarrow \\ Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) & \implies & tmf_{t-s}(A_1 \wedge DA_1). \end{array}$$

Lemma 4.1. *In the Adams spectral sequence*

$$Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) \implies tmf_{t-s}(A_1 \wedge DA_1),$$

the element g^6 is killed by a d_3 differential

$$d_3(v_2^{20}h_1) = g^6.$$

Proof. From the calculation in Lemma 2.2, it is clear that $g^6 = b_{2,1}^{12}$ has a nonzero image in $Ext_{A(2)}^{24,120+24}(A_1)$. Since we have a factorization of maps

$$Ext_{A(2)}^{24,120+24}(S^0) \longrightarrow Ext_{A(2)}^{24,120+24}(A_1 \wedge DA_1) \longrightarrow Ext_{A(2)}^{24,120+24}(A_1),$$

g^6 must also be nonzero in $Ext_{A(2)}^{24,120+24}(A_1 \wedge DA_1)$. Furthermore, because it is hit by a d_3 differential in

$$Ext_{A(2)}^{s,t}(S^0) \implies tmf_{t-s},$$

it must also be hit by a d_3 differential in

$$Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) \implies tmf_{t-s}(A_1 \wedge DA_1).$$

However, this does not preclude the possibility that it might be hit by a d_2 differential in this spectral sequence. Indeed, there are elements $\tilde{x} \in Ext_{A(2)}^{22,121+22}(A_1 \wedge DA_1)$ that could support a d_2 differential

$$d_2(x) = g^6.$$

In such a case, x would have to map to a nonzero element $x \in Ext_{A(2)}^{22,121+22}(A_1)$ and there would exist a differential

$$d_2(x) = g^6$$

in

$$Ext_{A(2)}^{s,t}(A_1) \implies tmf_{t-s}(A_1).$$

From the calculations of Lemma 2.2, there is exactly one possible nonzero $x \in Ext_{A(2)}^{22,121+22}(A_1)$. Using Bruner's program [Bru] (see Equation (A.2)) we see that this x is a multiple of $gb_{3,0}^4$ under the pairing

$$\begin{aligned} Ext_{A(2)}^{12,68+12}(S^0) \otimes Ext_{A(2)}^{10,53+10}(A_1) &\longrightarrow Ext_{A(2)}^{22,121+22}(A_1) \\ gb_{3,0}^4 \otimes \bar{x} &\mapsto x. \end{aligned}$$

Clearly $d_2(\bar{x}) = 0$ as $Ext_{A(2)}^{9,55+9}(A_1) = 0$. We apply the Leibniz rule to see that

$$d_2(x) = ge_0r \cdot \bar{x}.$$

However, $ge_0r = 0$ in $Ext_{A(2)}^{14,67+14}(S^0)$, therefore $d_2(x) = 0$. Consequently, g^6 is present and nonzero in the E_3 page of the spectral sequence

$$Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) \implies tmf_{t-s}(A_1 \wedge DA_1).$$

Since we have a map of spectral sequences

$$\begin{array}{ccc} Ext_{A(2)}^{s,t}(S^0) & \implies & tmf_{t-s} \\ \downarrow & & \downarrow \\ Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) & \implies & tmf_{t-s}(A_1 \wedge DA_1). \end{array}$$

the result follows. \square

Our next goal is to lift this d_3 differential to the Adams spectral sequence

$$Ext_A^{s,t}(A_1 \wedge DA_1) \implies \pi_{t-s}(A_1 \wedge DA_1).$$

The main tool at our disposal is the algebraic tmf spectral sequence, described in Section 3.

Notation 4.2. The elements of $E_1^{s,t,n}$, the E_1 page of the algebraic tmf spectral sequence for $A_1 \wedge DA_1$, which are nonzero permanent cycles, will detect nonzero elements of $Ext_A^{s,t}(A_1 \wedge DA_1)$. Therefore we place an element $x \in E_1^{s,t,n}$ in bidegree $(t-s-n, s+n)$. Thus the elements that may contribute to the same bidegree of $Ext_A^{s,t}(A_1 \wedge DA_1)$ are placed together. With this arrangement any differential in the algebraic tmf spectral sequence will look like Adams d_1 differential. The generators of

$$E_1^{s,t,n} = \bigoplus_{i_1, \dots, i_n \geq 1} Ext_{A(2)}^{s-n, t-8(i_1+\dots+i_n)}(A_1 \wedge DA_1 \wedge bo_{i_1} \wedge \dots \wedge bo_{i_n})$$

will be denoted by dots in the following manner (recall that $bo_0 = S^0$):

- elements with $n = 0$ are denoted by a \bullet ,
- elements with $n = 1, i_1 = 1$ are denoted by a \circ^1 ,
- elements with $n = 1, i_1 = 2$ are denoted by a \circ^2 ,
- elements with $n = 2, i_1 = 1, i_2 = 1$ are denoted by a \odot ,

- and N/A stands for ‘not applicable,’ i.e. coordinates of the table which are irrelevant to our arguments.

Lemma 4.3. *The elements g^6 and $v_2^{20}h_1$ lift to $Ext_A^{s,t}(A_1 \wedge DA_1)$ under the map*

$$\iota_* : Ext_A^{s,t}(A_1 \wedge DA_1) \longrightarrow Ext_{A(2)}^{s,t}(A_1 \wedge DA_1).$$

Proof. We use the algebraic tmf spectral sequence to show that g^6 and $v_2^{20}h_1$ lift to $Ext_A(A_1 \wedge DA_1)$. A d_r differential in the algebraic tmf spectral sequence will increase the algebraic tmf filtration by r . Since g^6 and $v_2^{20}h_1$ are in algebraic tmf filtration 0, they cannot be a target of a differential. We will now show that both g^6 and $v_2^{20}h_1$ cannot support a nonzero differential. The argument varies for different models of A_1 .

Case 1. When $A_1 = A_1[01]$, Table 4.0.1 shows the relevant part of the E_1 page of the algebraic tmf spectral sequence.

TABLE 4.0.1. E_1 page of the algebraic tmf spectral sequence for $Ext_A^{s,t}(A_1 \wedge DA_1)$, where $A_1 = A_1[01]$

$s \setminus t - s$	119	120	121
25	0	N/A	N/A
24	N/A	$\{\bullet \bullet \bullet \bullet\} := X_{24}^0 \ni g^6$	N/A
23	N/A	N/A	N/A
22	N/A	0	N/A
21	N/A	N/A	$\{\bullet\} := X_{21}^0 \ni v_2^{20}h_1$ $\circ^1 \circ^1 \circ^1 \circ^1 \circ^1$ $\circ^2 \circ^2 \circ^2 \circ^2$ $\odot \odot$

Elements of X_{24}^0 or X_{21}^0 in Table 4.0.1 clearly do not support a differential, and hence g^6 and $v_2^{20}h_1$ lift to $Ext_A^{s,t}(A_1 \wedge DA_1)$.

Case 2. The case $A_1 = A_1[10]$ is very similar to the previous one.

TABLE 4.0.2. E_1 page of the algebraic tmf spectral sequence for $Ext_A^{s,t}(A_1 \wedge DA_1)$, where $A_1 = A_1[10]$

$s \setminus t - s$	119	120	121
25	0	N/A	N/A
24	N/A	$\{\bullet \bullet \bullet \bullet\} := X_{24}^0 \ni g^6$	N/A
23	N/A	N/A	N/A
22	N/A	0	N/A
21	N/A	N/A	$\{\bullet\} := X_{21}^0 \ni v_2^{20}h_1$ $\circ^1 \circ^1 \circ^1 \circ^1 \circ^1$ $\circ^2 \circ^2 \circ^2 \circ^2$ $\odot \odot$

Elements of X_{24}^0 or X_{21}^0 in Table 4.0.2 clearly do not support a differential, and hence g^6 and $v_2^{20}h_1$ lift to $Ext_A^{s,t}(A_1 \wedge DA_1)$.

Case 3. The analysis for $A_1 = A_1[00]$ or $A_1 = A_1[11]$ are the same as $A_1[00]$ and $A_1[11]$ are dual to each other. In either case the E_1 -page of the algebraic tmf spectral sequence around stem 120 looks like the following:

TABLE 4.0.3. E_1 page of the algebraic tmf spectral sequence for $Ext_A^{s,t}(A_1 \wedge DA_1)$, where $A_1 = A_1[00]$ or $A_1 = A_1[11]$

$s \setminus t - s$	119	120	121
25	•	N/A	N/A
24	N/A	{•••••} := $X_{24}^0 \ni g^6$	N/A
23	N/A	N/A	N/A
22	N/A	••• ⊙ ⊙ ⊙ ⊙	N/A
21	N/A	N/A	{••} := $X_{21}^0 \ni v_2^{20}h_1$ ⊙ ¹ ⊙ ¹ ⊙ ¹ ⊙ ¹ ⊙ ¹ ⊙ ² ⊙ ² ⊙ ² ⊙ ² ⊙ ² ⊙ ² ⊙ ⊙ ⊙ ⊙ ⊙ ⊙

Elements of X_{24}^0 in Table 4.0.3 clearly do not support a differential, and hence g^6 lifts to $Ext_A^{s,t}(A_1 \wedge DA_1)$. Unfortunately, it is possible that an element of X_{21}^0 might support a differential.

However, it is known that $v_2^{20}h_1$ is a multiple of $b_{3,0}^8$ under the pairing

$$\begin{aligned} Ext_{A(2)}^{16,96+16}(S^0) \otimes Ext_{A(2)}^{5,25+5}(S^0) &\longrightarrow Ext_{A(2)}^{21,121+21}(S^0) \\ b_{3,0}^8 \otimes v_2^4 h_1 &\mapsto v_2^{20} h_1. \end{aligned}$$

Therefore the same is true for $v_2^{20}h_1 \in Ext_{A(2)}^{21,121+21}(A_1 \wedge DA_1)$ as

$$\iota_* : Ext_{A(2)}^{s,t}(S^0) \longrightarrow Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$$

is a map of algebras. By Corollary 3.8, we know that $b_{3,0}^8$ lifts to $Ext_A^{16,96+16}(A_1 \wedge DA_1)$. If we show that $v_2^4 h_1$ lifts to $Ext_A^{5,25+5}(A_1 \wedge DA_1)$ as well, then the result will follow as

$$H_{tmf*} : Ext_A^{s,t}(A_1 \wedge DA_1) \longrightarrow Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$$

is a map of algebras. Looking at Table 4.0.4 makes it clear that every element of $b_{3,0}^{-8} X_{21}^0$, including $v_2^4 h_1$, lifts to $Ext_A^{5,25+5}(A_1 \wedge DA_1)$, and hence that every element of X_{21}^0 , including $v_2^{20}h_1$, lifts to $Ext_A^{21,121+21}(A_1 \wedge DA_1)$.

TABLE 4.0.4. E_1 page of the algebraic tmf spectral sequence for $Ext_A^{s,t}(A_1 \wedge DA_1)$, where $A_1 = A_1[00]$ or $A_1 = A_1[11]$

$s \setminus t - s$	24	25
6	•	•••
5	N/A	$\{\bullet\bullet\} = b_{3,0}^{-8} X_{21}^0$ \circ^1

□

The lift of $v_2^{20}h_1 \in Ext_{A(2)}^{21,121+21}(A_1 \wedge DA_1)$ to $Ext_A^{21,121+21}(A_1 \wedge DA_1)$ found in Lemma 4.3 is not unique. More precisely, every such lift is

$$v_2^{20}h_1 + S \in Ext_A^{21,121+21}(A_1 \wedge DA_1)$$

for some element S in the higher algebraic tmf filtration. Notice that the Adams differentials $d_i(S)$ are zero for $i \in \{2, 3\}$ as there are no element of algebraic tmf filtration greater than zero in $Ext_A^{10,10+47}(A_1 \wedge DA_1)$ and $Ext_A^{11,11+47}(A_1 \wedge DA_1)$. Therefore the following lemma holds for any choice of lift of $v_2^{20}h_1 \in Ext_{A(2)}^{21,121+21}(A_1 \wedge DA_1)$.

Lemma 4.4. *In the Adams spectral sequence*

$$Ext_A^{s,t}(A_1 \wedge DA_1) \implies \pi_{t-s}(A_1 \wedge DA_1),$$

there exists a d_3 differential

$$d_3(v_2^{20}h_1) = g^6.$$

Proof. Consider the map of Adams spectral sequence

$$\begin{array}{ccc} E_2^{s,t} = Ext_A^{s,t}(A_1 \wedge DA_1) & \implies & \pi_{t-s}(A_1 \wedge DA_1) \\ \downarrow & & \downarrow \\ E_2^{s,t} = Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) & \implies & tmf_{t-s}(A_1 \wedge DA_1) \end{array}$$

induced by H_{tmf} . The fact that g^6 and $v_2^{20}h_1$ are nonzero in the E_3 page of the Adams spectral sequence for $tmf_*(A_1 \wedge DA_1)$ (see Lemma 4.1), forces g^6 and $v_2^{20}h_1$ have nonzero lift in the E_3 page of the Adams spectral sequence for $\pi_*(A_1 \wedge DA_1)$. Moreover the map of E_3 pages of the spectral sequences commutes with differentials. Thus in the E_3 page of the Adams spectral sequence for $\pi_*(A_1 \wedge DA_1)$

$$d_3(v_2^{20}h_1) = g^6 + R,$$

where R is an element of algebraic tmf filtration greater than zero. Furthermore, Table 4.0.1, Table 4.0.2 and Table 4.0.3 make clear that in the bidegree of g^6 , there are no elements of higher algebraic tmf filtration, and therefore $R = 0$. □

5. A_1 ADMITS A 32 PERIODIC v_2 -SELF-MAP

In Section 3, we established that the potential candidates for 8-periodic and 16-periodic v_2 -self-maps on A_1 support a d_2 and a d_3 differentials respectively (see Lemma 3.10). So we know by the Leibniz formula that the candidates for 32-periodic v_2 -self-map is a nonzero d_3 -cycle. So the only way these candidates can fail to converge to an element of $\pi_*(A_1 \wedge DA_1)$ is by supporting a d_r differential for $r \geq 4$ in the Adams spectral sequence

$$E_2 = Ext_A(A_1 \wedge DA_1) \Rightarrow \pi_*(A_1 \wedge DA_1).$$

So we look for potential targets in $Ext_A^{s,t}(A_1 \wedge DA_1)$ when $t - s = 191$ with Adams filtration $s \geq 36$. In order to detect elements with $t - s = 191$ we use the *algebraic-tmf* spectral sequence

$$E_1^{s,t,n} = Ext_{A(2)}^{s-n,t}(\overline{A//A(2)})^{\otimes n} \otimes H^*(X, \mathbb{Z}/2).$$

As pointed out in Remark 3.9 the candidates for 32-periodic v_2 -self-map may not be unique. To show the existence it is enough to show that one of those candidates is a nonzero permanent cycle in the E_∞ page of the Adams spectral sequence. We conveniently choose $b_{3,0}^{4n} \in Ext_A^{8n,48n+8n}(A_1 \wedge DA_1)$ to be the lift of $b_{3,0}^{4n} \in Ext_{A(2)}^{8n,48n+8n}(A_1 \wedge DA_1)$ whose algebraic *tmf* filtration is precisely zero.

Recall that, as an $A(2)$ -module

$$A//A(2) = \bigoplus_{i \in \mathbb{N}} H^*(\Sigma^{8i} bo_i)$$

where the bo_i are the *bo* Brown-Gitler spectra defined by Goerss, Jones and the third author [GJM]. Because of this splitting we get

$$E_1^{s,t,n} = \bigoplus_{i_1, \dots, i_n \geq 1} Ext_{A(2)}^{s-n, t-8(i_1+\dots+i_n)}(bo_{i_1} \wedge \dots \wedge bo_{i_n} \wedge A_1 \wedge DA_1)$$

for the E_1 page of the algebraic *tmf* spectral sequence.

An easy consequence of the vanishing line established in Lemma 3.4 is the following.

Lemma 5.1. *The only potential contributors to $Ext_A^{s,t}(A_1 \wedge DA_1)$ for $t - s = 191$ and $s \geq 36$ come from the following summands of the algebraic *tmf* E_1 page:*

$$\begin{aligned} & Ext_{A(2)}^{s,t}(A_1 \wedge DA_1) \\ & \oplus \bigoplus_{1 \leq i \leq 3} Ext_{A(2)}^{s-1, t-8i}(A_1 \wedge DA_1 \wedge bo_i) \\ & \oplus \bigoplus_{1 \leq i \leq 2} Ext_{A(2)}^{s-2, t-8-8i}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_i) \\ & \oplus Ext_{A(2)}^{s-3, t-24}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1 \wedge bo_1). \end{aligned}$$

We know that, in the Adams spectral sequence for $A_1 \wedge DA_1$, $b_{3,0}^{16}$ can support d_r differential only if $r \geq 4$. The broad idea is to show that all potential targets for a d_r differential for $r \geq 4$ are either zero or do not lift to E_4 page. While the result holds for all models of A_1 , the computations will be slightly different for different models, and so we will treat these models separately. Since $A_1[00]$ and $A_1[11]$ are Spanier-Whitehead dual to each other, we can treat the cases of $A_1[00]$ and $A_1[11]$ as one case. We will then have to treat the cases of the selfdual spectra $A_1[01]$ and

$A_1[10]$ separately. The completeness of the tables in this section will be justified by the more detailed tables in Appendix C.

5.1. **The case $A_1 = A_1[00]$ or $A_1 = A_1[11]$.** We begin by laying out, in Table 5.1.1, the elements of the E_1 page of algebraic tmf spectral sequence, in Notation 4.2. The table makes it clear that all elements with $t - s = 191$, with the possible exception of those in X_{36}^0 , are permanent cycles in the algebraic tmf spectral sequence.

TABLE 5.1.1. E_1 page of the algebraic tmf spectral sequence for $Ext_A^{s,t}(A_1 \wedge DA_1)$, where $A_1 = A_1[00]$ or $A_1 = A_1[11]$, stem 189-191

$s \setminus t - s$	189	190	191
40	0	0	0
39	0	$\{\bullet\bullet\} := Y_{39}^0$	$\{\bullet\bullet\bullet\} := X_{39}^0$
38	N/A	$\{\bullet\bullet\bullet\bullet\} := Y_{38}^0$	$\{\bullet\bullet\bullet\} := X_{38}^0$
37	N/A	$\bullet\bullet\bullet\bullet$ $\circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1$	$\{\bullet\bullet\bullet\bullet\bullet\} := X_{37}^0$ $\{\circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1\} := X_{37}^1$
36	N/A	N/A	$\{\bullet\bullet\bullet\} := X_{36}^0$ $\{\circ^1 \circ^1\} := X_{36}^1$ $\{\circ \circ \circ \circ \circ \circ\} := X_{36}^{1,1}$

Our goal is to show that every linear combination of elements in $X_s^{i_1, \dots, i_n}$ were either absent or zero in the E_4 page of the Adams spectral sequence. Using Bruner's program (for details see Tables C.1.1, C.1.2, C.1.3, and C.1.4 of Appendix C), we observe that a lot of these elements are multiples of g^6 in the E_1 page of the algebraic tmf spectral sequence, which we record in Table 5.1.2.

TABLE 5.1.2. E_1 page of the algebraic tmf spectral sequence for $Ext_A^{s,t}(A_1 \wedge DA_1)$, where $A_1 = A_1[00]$ or $A_1 = A_1[11]$, stem 70-71

$s \setminus t - s$	70	71
15	$\{\bullet\bullet\} = g^{-6} Y_{39}^0$	$\{\bullet\bullet\bullet\} = g^{-6} X_{39}^0$
14	$\{\bullet\bullet\bullet\bullet\} = g^{-6} Y_{38}^0$	$\{\bullet\bullet\bullet\bullet\} = g^{-6} X_{38}^0$
13	$\bullet\bullet\bullet\bullet$ $\circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1$	$\{\bullet\bullet\bullet\bullet\bullet\} = g^{-6} X_{37}^0$ $\{\circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1\} = g^{-6} X_{37}^1$
12	N/A	$\{\circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1\} = g^{-6} X_{36}^1$ $\{\circ \circ \circ \circ \circ \circ\} = g^{-6} X_{36}^{1,1}$

Tables C.1.1, C.1.2, C.1.3, and C.1.4 make clear that

- multiplication by g^6 surjects onto $X_{39}^0 \oplus X_{38}^0 \oplus X_{37}^0 \oplus X_{37}^1 \oplus X_{36}^1 \oplus X_{36}^{1,1}$, and
- Elements in $g^{-1}(X_{39}^0 \oplus X_{38}^0 \oplus X_{37}^0 \oplus X_{37}^1 \oplus X_{36}^1 \oplus X_{36}^{1,1})$ have nonzero images under multiplication by $v_2^{20} h_1$ if and only if multiplication by g^6 is nonzero.

Lemma 5.2. *Every element of*

$$X_{39}^0 \oplus X_{38}^0 \oplus X_{37}^0 \oplus X_{37}^1 \oplus X_{36}^1 \oplus X_{36}^{1,1}$$

is present in the Adams E_2 page, but is either zero or absent in the Adams E_4 page.

Proof. Notice that for any $x = g^6 \cdot y \in X_{39}^0 \oplus X_{38}^0 \oplus X_{37}^0 \oplus X_{37}^1 \oplus X_{36}^1 \oplus X_{36}^{1,1}$, both x and y is a nonzero permanent cycle of the algebraic tmf spectral sequence. Indeed, the target of any differential supported by y , must have algebraic tmf filtration greater than y and from Table 5.1.2 it is clear no such element is present in appropriate bidegree. Hence y is present in the Adams E_2 page. Same argument holds for x .

Case 1. When $x = g^6 \cdot y \in X_{39}^0 \oplus X_{38}^0$, clearly y is then a permanent cycle in the Adams spectral sequence. Using Leibniz rule, we see that

$$d_2(x) = d_2(g^6 \cdot y) = 0$$

and

$$d_3(v_2^{20} h_1 \cdot y) = v_2^{20} h_1 \cdot d_3(y) + d_3(v_2^{20} h_1) \cdot y = g^6 \cdot y = x.$$

Therefore, if $x = g^6 \cdot y$ is nonzero in E_3 page, then x is zero in E_4 page.

Case 2. When $x = g^6 \cdot y \in X_{37}^1 \oplus X_{36}^1 \oplus X_{36}^{1,1}$, then $d_r(y)$ for $r \geq 2$, if nonzero, must have algebraic tmf filtration greater than zero, as

$$\begin{array}{ccc} Ext_A(A_1 \wedge DA_1) & \Longrightarrow & \pi_*(A_1 \wedge DA_1) \\ \downarrow & & \downarrow \\ Ext_{A(2)}(A_1 \wedge DA_1) & \Longrightarrow & tmf_*(A_1 \wedge DA_1) \end{array}$$

is a map of spectral sequence. Since there are no elements of algebraic tmf filtration greater than zero in bigree $(s, 71 + s)$ for $s \geq 14$, it follows that $d_r(y) = 0$ for $r \geq 2$ and y a permanent cycle in the Adams spectral sequence. If y is a target of a differential in algebraic tmf spectral sequence or a Adams d_2 differential, then $x = 0$ in E_3 page. Consequently, $g^6 x = 0$ in the E_3 page as well. If y is not a target of such differentials, then we have

$$d_3(v_2^{20} h_1 \cdot y) = v_2^{20} h_1 \cdot d_3(y) + d_3(v_2^{20} h_1) \cdot y = g^6 \cdot y = x.$$

In either case, $g^6 \cdot y = x = 0$ in E_4 page.

Case 3. When $x = g^6 \cdot y \in X_{37}^0$ and y is a permanent cycle, then we can argue $g^6 \cdot y = x = 0$ in the E_4 page as we did in the previous cases. If

$$d_2(y) = y'$$

then y' must belong to $g^{-1}Y_{39}^0$. Since multiplication by g^6 is a bijection between $g^{-1}Y_{39}^0$ and Y_{39}^0 , we get

$$d_2(x) = d_2(g^6 \cdot y) = g^6 \cdot d_2(y) + d_2(g^6) \cdot y = g^6 \cdot y' \neq 0.$$

Therefore, x is absent in the E_4 page. □

Thus we are left with the case when $x \in X_{36}^0$.

Lemma 5.3. *Every element of X_{36}^0 is either zero or absent in the Adams E_4 page.*

Proof. X_{36}^0 is spanned by three generators $\{s_1, t_1, t_2\}$. Using Bruner's program (see), we explore the following relations:

$$\begin{aligned} s_1 &= b_{3,0}^4 \cdot x_1 \\ t_1 &= b_{3,0}^4 \cdot y_1 = b_{3,0}^8 \cdot z_1 \\ t_2 &= b_{3,0}^4 \cdot y_2 = b_{3,0}^8 \cdot z_2 \\ Y_{38}^0 \ni e_0 r \cdot x_1 &\neq 0 \\ e_0 r \cdot y_1 &= 0 \\ e_0 r \cdot y_2 &= 0 \\ Y_{39}^0 \ni wgr \cdot z_1 &\neq 0 \\ Y_{39}^0 \ni wgr \cdot z_2 &\neq 0 \end{aligned}$$

and $wgr \cdot z_1$ and $wgr \cdot z_2$ are linearly independent. In Bruner's notation, $s_1 = 36_{64}$, $t_1 = 36_{65}$, $t_2 = 36_{66}$, $x_1 = 28_{32}$, $e_0 r \cdot x_1 = 38_{25}$, $y_1 = 28_{33}$, $y_2 = 28_{34}$, $z_1 = 20_1$, $wgr \cdot z_1 = 39_1$, $z_2 = 20_2$ and $wgr \cdot z_2 = 39_2$ (see Table C.1.5) .

TABLE 5.1.3. E_1 page of the algebraic tmf spectral sequence for $Ext_A^{s,t}(A_1 \wedge DA_1)$, where $A_1 = A_1[00]$ or $A_1 = A_1[11]$

$s \setminus t - s$	94	95
23	0	0
22	0	0
21	0	0
20	N/A	$\{\bullet = z_1, \bullet = z_2\} := Z_{20}$
$s \setminus t - s$	142	143
30	0	0
29	•••••	•••••
28	N/A	$\{\bullet = x_1, \bullet = y_1, \bullet = y_2\} := Z_{28}$ $\circ^1 \circ^1$

From the Table 5.1.3, it is clear that any element in Z_{20} and Z_{28} are permanent cycles.

Case 1. If $x = \epsilon_1 s_1 + \delta_1 t_1 + \delta_2 t_2 \neq 0$ in the Adams E_2 page with $\epsilon_1 \neq 0$, then

$$d_2(x) = \epsilon_1 (e_0 r \cdot x_1) \neq 0.$$

Thus x is not present in E_4 page.

Case 2. If $x = \delta_1 t_1 + \delta_2 t_2 \neq 0$, then

$$d_2(x) = 0.$$

If $x \neq 0$ in the Adams E_3 page then

$$d_3(x) = \delta_1 d_3(b_{3,0}^4 \cdot z_1) + \delta_2 d_3(b_{3,0}^4 \cdot z_2) = wgr \cdot (\delta_1 z_1 + \delta_2 z_2) \neq 0$$

Thus x is not present in E_4 page.

□

This proves Theorem 1 in the cases $A_1 = A_1[00]$ or $A_1 = A_1[11]$.

5.2. **The case $A_1 = A_1[01]$ or $A_1 = A_1[01]$.** Even though, in principle, we should treat $A_1[01]$ and $A_1[10]$ as two different cases, but it turns out that Tables 5.2.1, and 5.2.2 are identical in both the case and the arguments remain exactly the same for both of them. For $A_1[01]$, refer to Tables C.2.1, C.2.2, C.2.3, and C.2.4 of Appendix C, and for $A_1[10]$, refer to Tables C.3.1, C.3.2, C.3.3, and C.3.4 of Appendix C, to observe that most of the elements in Table 5.2.1 are multiples by g^6 of elements in Table 5.2.2.

TABLE 5.2.1. E_1 page of the algebraic tmf spectral sequence for $Ext_A^{s,t}(A_1 \wedge DA_1)$, where $A_1 = A_1[01]$

$s \setminus t - s$	190	191
39	0	$\{\bullet\} := X_{39}^0$
38	$\{\bullet\bullet\bullet\} := Y_{38}^0$	$\{\bullet\} := X_{38}^0$
37	$\bullet\bullet\bullet\bullet$ $\circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1$	$\{\bullet\bullet\bullet\bullet\} := X_{37}^0$ $\{\circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1\} := X_{37}^1$
36	N/A	$\{\circ\circ\} := X_{36}^{1,1}$

TABLE 5.2.2. E_1 page of the algebraic tmf spectral sequence for $Ext_A^{s,t}(A_1 \wedge DA_1)$, where $A_1 = A_1[01]$

$s \setminus t - s$	70	71
15	0	$\{\bullet\} = g^{-6} X_{39}^0$
14	$\{\bullet\bullet\bullet\} = g^{-6} Y_{38}^0$	$\{\bullet\bullet\} = g^{-6} X_{38}^0$
13	$\bullet\bullet\bullet\bullet$ $\circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1$	$\{\bullet\bullet\bullet\bullet\bullet\bullet\} = g^{-6} X_{37}^0$ $\{\circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1\} = g^{-6} X_{37}^1$
12	N/A	$\circ^1 \circ^1 \circ^1 \circ^1 \circ^1 \circ^1$ $\{\circ\circ\} = g^{-6} X_{36}^{1,1}$

Tables C.2.1, C.2.2, C.2.3, and C.2.4 and Tables C.3.1, C.3.2, C.3.3, and C.3.4 make clear that

- multiplication by g^6 is surjective onto $X_{39}^0 \oplus X_{38}^0 \oplus X_{37}^0 \oplus X_{37}^1 \oplus X_{36}^{1,1}$, and
- elements in $g^{-6}(X_{39}^0 \oplus X_{38}^0 \oplus X_{37}^0 \oplus X_{37}^1 \oplus X_{36}^{1,1})$ have nonzero images under multiplication by $v_2^{20}h_1$ if and only if multiplication by g^6 is nonzero.

Lemma 5.4. *All elements of*

$$X_{39}^0 \oplus X_{38}^0 \oplus X_{37}^0 \oplus X_{37}^1 \oplus X_{36}^{1,1}$$

are present in the Adams E_2 page, but are zero in the Adams E_4 page.

Proof. Notice that for any $x = g^6 \cdot y \in X_{39}^0 \oplus X_{38}^0 \oplus X_{37}^0 \oplus X_{37}^1 \oplus X_{36}^{1,1}$, both x and y is a nonzero permanent cycle of the algebraic tmf spectral sequence. Indeed, the target of any differential supported by y , must have algebraic tmf filtration greater than y and from Table 5.2.2 it is clear no such elements are present in appropriate bidegrees. Hence y is present in the Adams E_2 page. Same argument holds for x .

Any $y \in g^{-6}(X_{39}^0 \oplus X_{38}^0 \oplus X_{37}^0 \oplus X_{37}^1)$ is a permanent cycle in Adams spectral sequence, as it is clear from Table 5.2.2 that $Ext_A^{s,70+s}(A_1 \wedge DA_1) = 0$ for $s \geq 15$. If

$y \in g^{-6}X_{36}^{1,1}$, then y has algebraic tmf filtration greater than zero, therefore $d_r(y)$ must have algebraic tmf filtration greater than zero. From Table 5.2.2, we observe that $Ext_A^{s,70+s}(A_1 \wedge DA_1)$ when $s \geq 14$, does not contain any element of algebraic tmf filtration greater than zero. Therefore, any $y \in g^{-6}X_{36}^{1,1}$ is a permanent cycle as well.

Since $d_2(g^6) = 0$, for any $x = g^6 \cdot y \in X_{39}^0 \oplus X_{38}^0 \oplus X_{37}^0 \oplus X_{37}^1 \oplus X_{36}^{1,1}$

$$d_2(x) = d_2(g^6 \cdot y) = d_2(g^6) \cdot y + g^6 \cdot d_2(y) = 0.$$

Hence x is present in E_3 page.

If $x = g^6 \cdot y$ is nonzero in E_3 page, Bruner's program shows that $v_2^{20}h_1 \cdot y$ is nonzero as well. Thus, using Leibniz rule

$$d_3(v_2^{20}h_1 \cdot y) = d_3(v_2^{20}h_1) \cdot y + v_2^{20}h_1 \cdot d_3(y) = g^6 \cdot y = x.$$

Thus, x is zero in E_4 page. □

APPENDIX A. GENERAL REMARKS ON THE USE OF BRUNER'S PROGRAM

Since many of our proofs relied on the output of Bruner's program, we append some facts about the program to justify our claims.

The program takes as input a graded module M over A (or $A(2)$) that is a finite dimensional \mathbb{F}_2 -vector space and computes $Ext_A^{s,t}(M, \mathbb{F}_2)$ (or $Ext_{A(2)}^{s,t}(M, \mathbb{F}_2)$) for t in a user-defined range, and $0 \leq s \leq \text{MAXFILT}$, where one has $\text{MAXFILT} = 40$ by default. The structure of M as an A -module is encoded in a text file named M , placed in the directory $A/\text{samples}$ in the way we will now describe.

The first line of the text file M consists of a positive integer n , the dimension of M as an \mathbb{F}_2 -vector space, whose basis elements we will call g_0, \dots, g_{n-1} . The second line consists of an ordered list of integers d_0, \dots, d_{n-1} , which are the respective degrees of the g_i . Every subsequent line in the text file describes a nontrivial action of some Sq^k on some generator g_i . For instance, if we have

$$Sq^k(g_i) = g_{j1} + \dots + g_{jl},$$

we would encode this fact by writing the line

i k l j1 ...jl

followed by a new line. Every action not encoded by such a line is assumed to be trivial. To ensure that such a text file in fact represents an honest A -module, we must run the `newconsistency` script, which will alert us if:

- the text file contains a line

i k l j1 ...jl

and it turns out that one of the d_j 's is not equal to $d_i + k$, or

- the module taken as a whole fails to satisfy a particular Adem relation.

Example A.1. Consider the A -module given by Figure A.1, where generators are depicted by dots and actions of Sq^1 , Sq^2 , and Sq^4 are depicted by black, blue and red lines respectively:

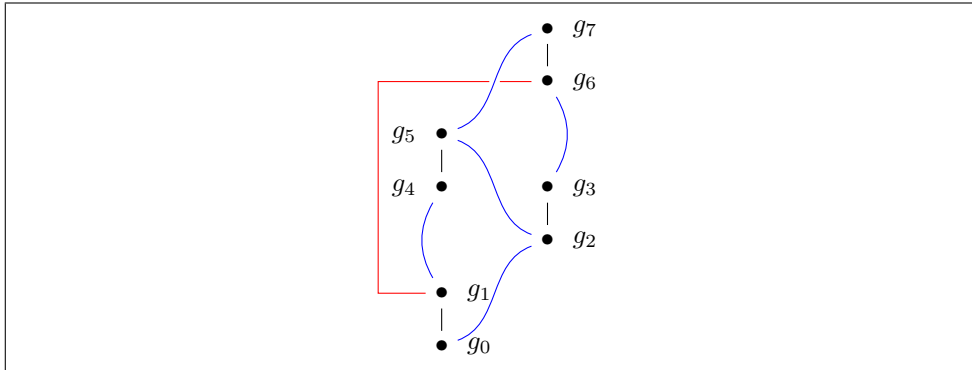


FIGURE A.1. $H^*A_1[00]$ as an A -module

Based on this picture, we get the text file in figure A.2, which we call `A1-00_def`: We go to the directory `A2` and run

```

8
0 1 2 3 3 4 5 6
0 1 1 1
0 2 1 2
1 2 1 4
1 4 1 6
2 1 1 3
2 2 1 5
3 2 1 6
4 1 1 5
5 2 1 7
6 1 1 7

```

FIGURE A.2. The text file `A/samples/A1-00_def`

```

./newmodule A1-00 ../A/samples/A1-00_def
cd A1-00.

```

Now we are ready to compute. Running the script

```
./dims 0 250&
```

will compute $Ext_{A(2)}^{s,t}(A_1[00])$ for $0 \leq s \leq \text{MAXFILT} = 40$ and $0 \leq t \leq 250$. The `&` is not strictly necessary, but may be a good idea if running a computation expected to take a long time and if one would like to do other things in the meantime. Then, to see the Ext group, one runs

```

./report summary
./vsumm A1-00 > A1-00.tex
pdflatex A1-00.tex

```

to produce a pdf document `A1-00.pdf` resembling Figure A.3. As the figure makes apparent, the generators of the Ext group (as an \mathbb{F}_2 vector space) are stored in the computer with names such as s_g , where s is the Adams filtration of the generator, and g is some way of ordering all generators of filtration s . It should be emphasized that g is not the stem of the generator. In figure A.3 for instance, the generator 1_2 is the second generator of filtration 1, but it is in stem 6. The figure also tells us the action of the Hopf elements h_0 through h_3 , so that in our example, h_2 multiplied by the generator 1_2 equals the generator 2_2 .

By running

```
./display 0 A1-00_
```

to produce single-page pdf documents `A1-00_1.pdf`, `A1-00_2.pdf`..., it is also possible to see the Ext group in the visually more appealing form of a chart, as shown in figure A.4. What is gained in esthetics is however lost in completeness, as these charts only display the action of h_0 (via a vertical solid line), h_1 (via a solid line of slope 1), and h_2 (via a dotted line of slope $\frac{1}{3}$).

A1-00

July 11, 2015

Notes:

1. Stem refers to the geometric total degree $n = t - s$, where t is the internal degree and s is the homological degree (or 'filtration').
2. If a stem is not printed, there are no elements in that stem.
3. The notation s_g refers to generator number g in filtration s .
4. Dashes (-) are used to indicate that an h_i multiplen is beyond the range which has been calculated.

Table 1: Stem 0

n	s	g	h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7
0	0	0_0				1_0				

Table 2: Stem 3

n	s	g	h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7
3	1	1_0				2_0				

Table 3: Stem 5

n	s	g	h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7
5	1	1_1		2_0	2_1					

Table 4: Stem 6

n	s	g	h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7
6	1	1_2			2_2					
6	2	2_0								

1

FIGURE A.3. First page of A1-00.pdf

The program is also capable of computing dual modules via the `dualizeDef` script, and tensor products via the `tensorDef` script. Both executables are conveniently located in the `A/samples` directory where we put our module definition text files. Thus, running

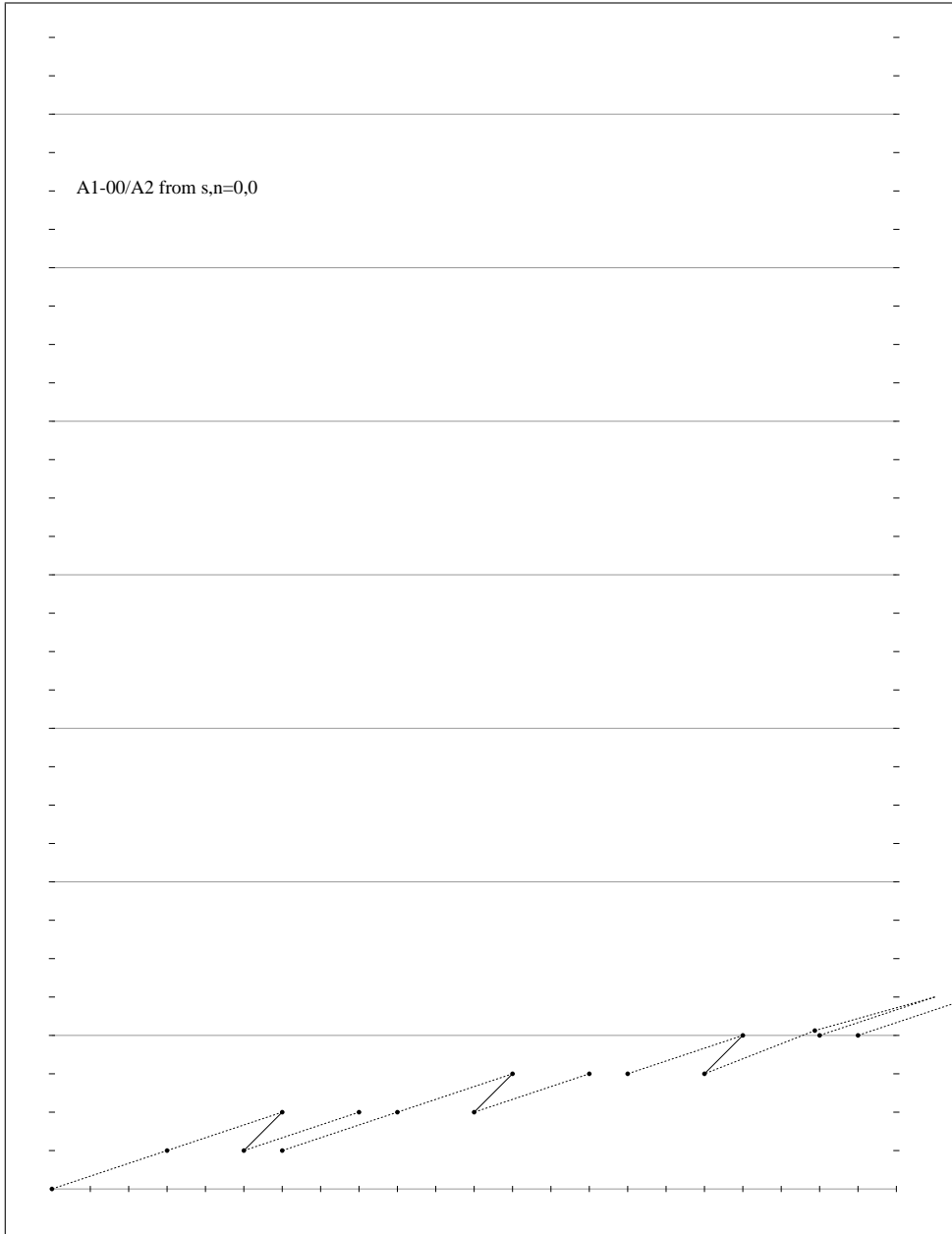


FIGURE A.4. The file A1-00.1.pdf

```
./dualizeDef A1-00_def DA1-00_def
./tensorDef A1-00_def DA1-00_def ADA1-00_def,
```

produces the text file ADA1-00_def, with which we proceed in the same way as earlier with A1-00_def.

$$\begin{aligned}
g &= 4_8 \in Ext_{A(2)}^{4,20+4}(S^0) \\
b_{3,0}^4 &= 8_{19} \in Ext_{A(2)}^{8,48+8}(S^0) \\
e_{0r} &= 10_{18} \in Ext_{A(2)}^{10,47+10}(S^0) \\
b_{3,0}^8 &= 16_{54} \in Ext_{A(2)}^{16,96+16}(S^0) \\
wgr &= 19_{56} \in Ext_{A(2)}^{19,95+19}(S^0) \\
v_2^{20}h_1 &= 21_{85} \in Ext_{A(2)}^{21,121+21}(S^0) \\
g^6 &= 24_{90} \in Ext_{A(2)}^{24,120+24}(S^0)
\end{aligned}$$

FIGURE A.5. s_g representation of important elements of $Ext_{A(2)}^{s,t}(S^0)$

While `ADA1-00.pdf` only shows the action of the Hopf elements h_0 through h_3 , the scripts `cocycle` and `dolifts` enable the user to input a specific generator and find the action of much of $Ext_{A(2)}^{s,t}(S^0)$ on that specific generator. Let us do this with the generator $0_6 \in Ext_{A(2)}^{0,0}(A_1[00] \wedge DA_1[00])$ by going to directory `A2` and running

```
./cocycle ADA1-00 0 6
```

which will create a subdirectory `A2/ADA1-00/0.6`. To find the action of all elements of $Ext_{A(2)}^{s,t}(S^0)$ with $0 \leq s \leq 20$ on 0_6 , we go back to directory `A2/ADA1-00` and run

```
./dolifts 0 20 maps
```

Now `ADA1-00/0.6` will contain several text files, among them `brackets.sym` (which contains information about Massey products) and `Map.aug` (which contains information about the action of $Ext_{A(2)}^{s,t}(S^0)$ on 0_6).

The generators of $Ext_{A(2)}^{s,t}(S^0)$ are stored in the computer in the format s_g . In figure A.5, we include a list of important elements of $Ext_{A(2)}^{s,t}(S^0)$ and their s_g representation.

We'd like to know what $s_g(0_6) \in Ext_{A(2)}(A_1[00] \wedge DA_1[00])$ is in the notation of `ADA1-00.pdf`. Of course, $s_g(0_6)$ is in filtration s , so we only need to specify which of the generators in filtration s make up $s_g(0_6)$. If, for instance, we have

$$s_g(0_6) = s_{g1} + \dots + s_{gn},$$

then `ADA1-00/0.6/Map.aug` will contain the lines

```

s g1 g
s g2 g
...
s gn g.

```

Now, in the Adams spectral sequence

$$Ext_{A(2)}^{s,t}(S^0) \Rightarrow tmf_{t-s},$$

we have

$$d_2(b_{3,0}^4) = e_{0r} = 10_{18} \in Ext_{A(2)}^{10,47+10}(S^0), d_3(b_{3,0}^8) = 19_{56} \in Ext_{A(2)}^{19,95+19}(S^0).$$

It follows that if $10_{18}(0_6) = 10_x \in Ext_{A(2)}^{8,8+47}(A_1 \wedge DA_1)$ and $19_{56}(0_6) = 19_y \in Ext_{A(2)}^{19,19+95}(A_1 \wedge DA_1)$, then $b_{3,0}^4 \in Ext_{A(2)}^{8,48+8}(A_1 \wedge DA_1)$ supports a d_2 differential,

and $b_{3,0}^8 \in Ext_{A(2)}^{16,96+16}(A_1 \wedge DA_1)$ supports a d_3 differential. By doing the above steps for all four versions of A_1 , and checking the respective `Map.aug` files, each contain lines

```
10 x 18
19 y 56,
```

justifying the claim in Lemma 3.1.

Using the tools we have so far described, it is easy to verify the claim from the proof of Lemma 4.1, that for all four models of A_1 we have

$$(A.2) \quad gb_{3,0}^4 \cdot 10_3 = 22_7.$$

It is similarly easy to verify that if $A_1 = A_1[00]$ or $A_1 = A_1[11]$, we have

$$ge_0r \cdot 10_3 = 0,$$

while if $A_1 = A_1[01]$ or $A_1 = A_1[10]$, we have

$$ge_0r \cdot 10_3 = 24_0 = g^6.$$

Finally, in order to run the algebraic tmf spectral sequence, we will also need to do computations involving the bo -Brown-Gitler spectra. We give the A -module definitions for the cohomologies of bo_1 and bo_2 here:

```
4
0 4 6 7
0 4 1 1
0 6 1 2
0 7 1 3
1 2 1 2
1 3 1 3
2 1 1 3
```

(A) The text file `bo1.def`


```
11
0 4 6 7 8 10 11 12 13 14 15
0 4 1 1
0 6 1 2
0 7 1 3
1 2 1 2
1 3 1 3
2 1 1 3
2 4 1 5
2 5 1 6
3 4 1 6
3 6 1 8
4 2 1 5
4 3 1 6
4 4 1 7
4 5 1 8
4 6 1 9
4 7 1 10
5 1 1 6
6 2 1 8
7 1 1 8
7 2 1 9
7 3 1 10
9 1 1 10
```

(B) The text file `bo2_def`

APPENDIX B. TABLES FROM SECTION 4

B.1. The cases $A_1 = A_1[00]$ or $A_1 = A_1[11]$.TABLE B.1.1. $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25	25_0	120	25	25_2	121	25	25_5
			120	25	25_1	121	25	25_4
						121	25	25_3
119	24		120	24	24_{25}	121	24	24_{28}
			120	24	24_{24}	121	24	24_{27}
			120	24	24_{23}	121	24	24_{26}
			120	24	24_{22}			
			120	24	24_{21}			
119	23		120	23	23_{42}	121	23	23_{47}
			120	23	23_{41}	121	23	23_{46}
			120	23	23_{40}	121	23	23_{45}
			120	23	23_{39}	121	23	23_{44}
			120	23	23_{38}	121	23	23_{43}
119	22		120	22	22_{64}	121	22	22_{68}
			120	22	22_{63}	121	22	22_{67}
			120	22	22_{62}	121	22	22_{66}
						121	22	22_{65}
119	21		120	21		121	21	21_{91}
						121	21	21_{90}

$t-s$	s	s_g	$t-s$	s	s_g
24	6	6_1	25	6	6_4
			25	6	6_3
			25	6	6_2
24	5	5_{25}	25	5	5_{27}
24	5	5_{24}	25	5	5_{26}
24	5	5_{23}			
24	5	5_{22}			

x	$b_{3,0}^8 x$
5_{27}	21_{91}
5_{26}	21_{90}

TABLE B.1.2. $Ext_{A(2)}^{s-1,t-8}(A_1 \wedge DA_1 \wedge bo_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25		120	25		121	25	
119	24		120	24		121	24	
119	23	22_5	120	23	22_{11}	121	23	22_{19}
119	23	22_4	120	23	22_{10}	121	23	22_{18}
119	23	22_3	120	23	22_9	121	23	22_{17}
119	23	22_2	120	23	22_8	121	23	22_{16}
			120	23	22_7	121	23	22_{15}
			120	23	22_6	121	23	22_{14}
						121	23	22_{13}
						121	23	22_{12}
119	22	21_{31}	120	22		121	22	21_{33}
119	22	21_{30}	120	22		121	22	21_{32}
119	21	20_{51}	120	21	20_{57}	121	21	20_{62}
119	21	20_{50}	120	21	20_{56}	121	21	20_{61}
119	21	20_{49}	120	21	20_{55}	121	21	20_{60}
119	21	20_{48}	120	21	20_{54}	121	21	20_{59}
119	21	20_{47}	120	21	20_{53}	121	21	20_{58}
119	21	20_{46}	120	21	20_{52}			
119	21	20_{45}						
119	21	20_{44}						

$t-s$	s	s_g	$t-s$	s	s_g
24	6		25	6	
24	5		25	5	4_0

TABLE B.1.3. $Ext_{A(2)}^{s-1,t-16}(A_1 \wedge DA_1 \wedge bo_2)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25		120	25		121	25	
119	24		120	24		121	24	
119	23		120	23		121	23	
119	22		120	22		121	22	
119	21		120	21	20_0	121	21	20_2
						121	21	20_1

$t-s$	s	s_g	$t-s$	s	s_g
24	6		25	6	
24	5		25	5	

TABLE B.1.4. $Ext_{A(2)}^{s-2,t-16}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25		120	25		121	25	
119	24		120	24		121	24	
119	23		120	23		121	23	
119	22	20_1	120	22	20_5	121	22	20_{11}
119	22	20_0	120	22	20_4	121	22	20_{10}
			120	22	20_3	121	22	20_9
			120	22	20_2	121	22	20_8
						121	22	20_7
						121	22	20_6
119	21	19_{41}	120	21	19_{51}	121	21	19_{57}
119	21	19_{40}	120	21	19_{50}	121	21	19_{56}
119	21	19_{39}	120	21	19_{49}	121	21	19_{55}
119	21	19_{38}	120	21	19_{48}	121	21	19_{54}
119	21	19_{37}	120	21	19_{47}	121	21	19_{53}
119	21	19_{36}	120	21	19_{46}	121	21	19_{52}
119	21	19_{35}	120	21	19_{45}			
119	21	19_{34}	120	21	19_{44}			
119	21	19_{33}	120	21	19_{43}			
119	21	19_{32}	120	21	19_{42}			

$t-s$	s	s_g	$t-s$	s	s_g
24	6		25	6	
24	5		25	5	

B.2. The case $A_1 = A_1[01]$.

TABLE B.2.1. $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25		120	25		121	25	25 ₀
119	24	24 ₁₄	120	24	24 ₁₈	121	24	24 ₁₉
119	24	24 ₁₃	120	24	24 ₁₇			
119	24	24 ₁₂	120	24	24 ₁₆			
119	24	24 ₁₁	120	24	24 ₁₅			
119	24	24 ₁₀						
119	23	23 ₂₀	120	23	23 ₂₄	121	23	23 ₂₉
			120	23	23 ₂₃	121	23	23 ₂₈
			120	23	23 ₂₂	121	23	23 ₂₇
			120	23	23 ₂₁	121	23	23 ₂₆
						121	23	23 ₂₅
119	22	22 ₄₁	120	22		121	22	22 ₄₁
119	22	22 ₄₀						
119	21	21 ₆₀	120	21	21 ₆₄	121	21	21 ₆₅
119	21	21 ₅₉	120	21	21 ₆₃			
119	21	21 ₅₈	120	21	21 ₆₂			
119	21	21 ₅₇	120	21	21 ₆₁			
119	21	21 ₅₆						
119	21	21 ₅₅						
119	21	21 ₅₄						

$t-s$	s	s_g	$t-s$	s	s_g
24	6		25	6	6 ₁
24	5	5 ₂₄	25	5	5 ₂₅
24	5	5 ₂₃			
24	5	5 ₂₂			
24	5	5 ₂₁			

x	$b_{3,0}^8 x$
5 ₂₅	21 ₆₅

TABLE B.2.2. $Ext_{A(2)}^{s-1, t-8}(A_1 \wedge DA_1 \wedge bo_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25		120	25		121	25	
119	24		120	24		121	24	
119	23	22_3	120	23	22_9	121	23	22_{17}
119	23	22_2	120	23	22_8	121	23	22_{16}
119	23	22_1	120	23	22_7	121	23	22_{15}
119	23	22_0	120	23	22_6	121	23	22_{14}
			120	23	22_5	121	23	22_{13}
			120	23	22_4	121	23	22_{12}
						121	23	22_{11}
						121	23	22_{10}
119	22		120	22		121	22	
119	21	20_{45}	120	21	20_{51}	121	21	20_{56}
119	21	20_{44}	120	21	20_{50}	121	21	20_{55}
119	21	20_{43}	120	21	20_{49}	121	21	20_{54}
119	21	20_{42}	120	21	20_{48}	121	21	20_{53}
119	21	20_{41}	120	21	20_{47}	121	21	20_{52}
119	21	20_{40}	120	21	20_{46}			
119	21	20_{39}						
119	21	20_{38}						

$t-s$	s	s_g	$t-s$	s	s_g
24	6		25	6	
24	5		25	5	4_0

TABLE B.2.3. $Ext_{A(2)}^{s-1, t-16}(A_1 \wedge DA_1 \wedge bo_2)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25		120	25		121	25	
119	24		120	24		121	24	
119	23		120	23		121	23	
119	22		120	22		121	22	
119	21		120	21		121	21	

$t-s$	s	s_g	$t-s$	s	s_g
24	6		25	6	
24	5		25	5	

TABLE B.2.4. $Ext_{A(2)}^{s-2,t-16}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25		120	25		121	25	
119	24		120	24		121	24	
119	23		120F	23		121	23	
119	22		120	22		121	22	20 ₁
						121	22	20 ₀
119	21	19 ₂₉	120	21	19 ₃₇	121	21	19 ₃₉
119	21	19 ₂₈	120	21	19 ₃₆	121	21	19 ₃₈
119	21	19 ₂₇	120	21	19 ₃₅			
119	21	19 ₂₆	120	21	19 ₃₄			
119	21	19 ₂₅	120	21	19 ₃₃			
119	21	19 ₂₄	120	21	19 ₃₂			
119	21	19 ₂₃	120	21	19 ₃₁			
119	21	19 ₂₂	120	21	19 ₃₀			
119	21	19 ₂₁						
119	21	19 ₂₀						

$t-s$	s	s_g	$t-s$	s	s_g
24	6		25	6	
24	5		25	5	

B.3. The case $A_1 = A_1[10]$.TABLE B.3.1. $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25		120	25		121	25	25 ₀
119	24	24 ₁₄	120	24	24 ₁₈	121	24	24 ₁₉
119	24	24 ₁₃	120	24	24 ₁₇			
119	24	24 ₁₂	120	24	24 ₁₆			
119	24	24 ₁₁	120	24	24 ₁₅			
119	24	24 ₁₀						
119	23	23 ₂₀	120	23	23 ₂₄	121	23	23 ₂₉
			120	23	23 ₂₃	121	23	23 ₂₈
			120	23	23 ₂₂	121	23	23 ₂₇
			120	23	23 ₂₁	121	23	23 ₂₆
						121	23	23 ₂₅
119	22	22 ₄₀	120	22		121	22	22 ₄₁
119	22	22 ₃₉						
119	21	21 ₆₁	120	21	21 ₆₅	121	21	21 ₆₆
119	21	21 ₆₀	120	21	21 ₆₄			
119	21	21 ₅₉	120	21	21 ₆₃			
119	21	21 ₅₈	120	21	21 ₆₂			
119	21	21 ₅₇						
119	21	21 ₅₆						
119	21	21 ₅₅						

$t-s$	s	s_g	$t-s$	s	s_g
24	6		25	6	6 ₁
24	5	5 ₂₅	25	5	5 ₂₆
24	5	5 ₂₄			
24	5	5 ₂₃			
24	5	5 ₂₂			

x	$b_{3,0}^8 x$
5 ₂₆	21 ₆₆

TABLE B.3.2. $Ext_{A(2)}^{s-1,t-8}(A_1 \wedge DA_1 \wedge bo_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25		120	25		121	25	
119	24		120	24		121	24	
119	23	22_3	120	23	22_9	121	23	22_{17}
119	23	22_2	120	23	22_8	121	23	22_{16}
119	23	22_1	120	23	22_7	121	23	22_{15}
119	23	22_0	120	23	22_6	121	23	22_{14}
			120	23	22_5	121	23	22_{13}
			120	23	22_4	121	23	22_{12}
						121	23	22_{11}
						121	23	22_{10}
119	22		120	22		121	22	
119	21	20_{45}	120	21	20_{51}	121	21	20_{56}
119	21	20_{44}	120	21	20_{50}	121	21	20_{55}
119	21	20_{43}	120	21	20_{49}	121	21	20_{54}
119	21	20_{42}	120	21	20_{48}	121	21	20_{53}
119	21	20_{41}	120	21	20_{47}	121	21	20_{52}
119	21	20_{40}	120	21	20_{46}			
119	21	20_{39}						
119	21	20_{38}						

$t-s$	s	s_g	$t-s$	s	s_g
24	6		25	6	
24	5		25	5	4_0

TABLE B.3.3. $Ext_{A(2)}^{s-1,t-16}(A_1 \wedge DA_1 \wedge bo_2)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25		120	25		121	25	
119	24		120	24		121	24	
119	23		120	23		121	23	
119	22		120	22		121	22	
119	21		120	21		121	21	

$t-s$	s	s_g	$t-s$	s	s_g
24	6		25	6	
24	5		25	5	

TABLE B.3.4. $Ext_{A(2)}^{s-2,t-16}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
119	25		120	25		121	25	
119	24		120	24		121	24	
119	23		120	23		121	23	
119	22		120	22		121	22	20_1
						121	22	20_0
119	21	19_{29}	120	21	19_{37}	121	21	19_{39}
119	21	19_{28}	120	21	19_{36}	121	21	19_{38}
119	21	19_{27}	120	21	19_{35}			
119	21	19_{26}	120	21	19_{34}			
119	21	19_{25}	120	21	19_{33}			
119	21	19_{24}	120	21	19_{32}			
119	21	19_{23}	120	21	19_{31}			
119	21	19_{22}	120	21	19_{30}			
119	21	19_{21}						
119	21	19_{20}						

$t-s$	s	s_g	$t-s$	s	s_g
24	6		25	6	
24	5		25	5	

APPENDIX C. TABLES FROM SECTION 5

C.1. The case $A_1 = A_1[00]$ or $A_1 = A_1[11]$.

TABLE C.1.1. $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
70	15	15 ₂	71	15	15 ₅	190	39	39 ₂	191	39	39 ₅
70	15	15 ₁	71	15	15 ₄	190	39	39 ₁	191	39	39 ₄
			71	15	15 ₃				191	39	39 ₃
70	14	14 ₂₅	71	14	14 ₂₉	190	38	38 ₂₅	191	38	38 ₂₈
70	14	14 ₂₄	71	14	14 ₂₈	190	38	38 ₂₄	191	38	38 ₂₇
70	14	14 ₂₃	71	14	14 ₂₇	190	38	38 ₂₃	191	38	38 ₂₆
70	14	14 ₂₂	71	14	14 ₂₆	190	38	38 ₂₂			
70	14	14 ₂₁				190	38	38 ₂₁			
70	13	13 ₄₆	71	13	13 ₅₃	190	37	37 ₄₂	191	37	37 ₄₇
70	13	13 ₄₅	71	13	13 ₅₂	190	37	37 ₄₁	191	37	37 ₄₆
70	13	13 ₄₄	71	13	13 ₅₁	190	37	37 ₄₀	191	37	37 ₄₅
70	13	13 ₄₃	71	13	13 ₅₀	190	37	37 ₃₉	191	37	37 ₄₄
70	13	13 ₄₂	71	13	13 ₄₉	190	37	37 ₃₈	191	37	37 ₄₃
			71	13	13 ₄₈						
			71	13	13 ₄₇						
						190	36	36 ₆₃	191	36	36 ₆₆
						190	36	36 ₆₂	191	36	36 ₆₅
						190	36	36 ₆₁	191	36	36 ₆₄

n	i_1, \dots, i_n	x	$g^6 x$	$v_2^{20} h_1 x$
0	0	15 ₅	39 ₅	36 ₆₉
0	0	15 ₄	39 ₄	36 ₆₈
0	0	15 ₃	39 ₃	36 ₆₇
0	0	15 ₂	39 ₂	36 ₆₆
0	0	15 ₁	39 ₁	36 ₆₅
0	0	14 ₂₉	0	0
0	0	14 ₂₈	38 ₂₈	35 ₉₂
0	0	14 ₂₇	38 ₂₇	35 ₉₁
0	0	14 ₂₆	38 ₂₆	35 ₉₀
0	0	14 ₂₅	38 ₂₅	35 ₈₉
0	0	14 ₂₄	38 ₂₄	35 ₈₈
0	0	14 ₂₃	38 ₂₃	35 ₈₇
0	0	14 ₂₂	38 ₂₂	35 ₈₆
0	0	14 ₂₁	38 ₂₁	35 ₈₅
0	0	13 ₅₃	0	0
0	0	13 ₅₂	0	0
0	0	13 ₅₁	37 ₄₄	34 ₁₀₈
0	0	13 ₅₀	37 ₄₃	34 ₁₀₇
0	0	13 ₄₉	37 ₄₃ + 37 ₄₅	34 ₁₀₇ + 34 ₁₀₉
0	0	13 ₄₈	37 ₄₅ + 37 ₄₆ + 37 ₄₇	34 ₁₀₉ + 34 ₁₁₀ + 34 ₁₁₁
0	0	13 ₄₇	37 ₄₃ + 37 ₄₅ + 37 ₄₆	34 ₁₀₇ + 34 ₁₀₉ + 34 ₁₁₀

TABLE C.1.2. $Ext_{A(2)}^{s-1, t-8}(A_1 \wedge DA_1 \wedge bo_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
70	15		71	15		190	39		191	39	
70	14		71	14		190	38		191	38	
70	13	12_{11}	71	13	12_{19}	190	37	36_{11}	191	37	36_{19}
70	13	12_{10}	71	13	12_{18}	190	37	36_{10}	191	37	36_{18}
70	13	12_9	71	13	12_{17}	190	37	36_9	191	37	36_{17}
70	13	12_8	71	13	12_{16}	190	37	36_8	191	37	36_{16}
70	13	12_7	71	13	12_{15}	190	37	36_7	191	37	36_{15}
70	13	12_6	71	13	12_{14}	190	37	36_6	191	37	36_{14}
			71	13	12_{13}				191	37	36_{13}
			71	13	12_{12}				191	37	36_{12}
70	12	11_{40}	71	12	11_{46}	190	36		191	36	35_{33}
70	12	11_{39}	71	12	11_{45}				191	36	35_{32}
70	12	11_{38}	71	12	11_{44}						
70	12	11_{37}	71	12	11_{43}						
70	12	11_{36}	71	12	11_{42}						
70	12	11_{35}	71	12	11_{41}						
70	12	11_{34}									

n	i_1, \dots, i_n	x	$g^6 x$	$v_2^{20} h_1 x$
1	1	12_{19}	36_{19}	33_{83}
1	1	12_{18}	36_{18}	33_{82}
1	1	12_{17}	36_{17}	$33_{79} + 33_{83}$
1	1	12_{16}	36_{16}	$33_{79} + 33_{81}$
1	1	12_{15}	36_{15}	33_{80}
1	1	12_{14}	36_{14}	$33_{78} + 33_{79} + 33_{81} + 33_{83}$
1	1	12_{13}	36_{13}	33_{77}
1	1	12_{12}	36_{12}	$33_{76} + 33_{83}$
1	1	11_{46}	0	0
1	1	11_{45}	0	0
1	1	11_{44}	0	0
1	1	11_{43}	35_{33}	32_{97}
1	1	11_{42}	35_{32}	32_{96}
1	1	11_{41}	0	0

TABLE C.1.3. $Ext_{A(2)}^{s-1, t-16}(A_1 \wedge DA_1 \wedge bo_2)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
70	15		71	15		190	39		191	39	
70	14		71	14		190	38		191	38	
70	13		71	13		190	37		191	37	
70	12		71	12		190	36		191	36	

TABLE C.1.4. $Ext_{A(2)}^{s-2,t-16}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
70	15		71	15		190	39		191	39	
70	14		71	14		190	38		191	38	
70	13		71	13		190	37		191	37	
70	12	10_5	71	12	10_{11}	190	36	34_5	191	36	34_{11}
70	12	10_4	71	12	10_{10}	190	36	34_4	191	36	34_{10}
70	12	10_3	71	12	10_9	190	36	34_3	191	36	34_9
70	12	10_2	71	12	10_8	190	36	34_2	191	36	34_8
			71	12	10_7				191	36	34_7
			71	12	10_6				191	36	34_6

n	i_1, \dots, i_n	x	$g^6 x$	$v_2^{20} h_1 x$
2	1, 1	10_{11}	34_{11}	31_{139}
2	1, 1	10_{10}	34_{10}	31_{138}
2	1, 1	10_9	34_9	31_{137}
2	1, 1	10_8	34_8	31_{136}
2	1, 1	10_7	34_7	31_{135}
2	1, 1	10_6	34_6	$31_{134} + 31_{137} + 31_{138}$

TABLE C.1.5. $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
95	20	20_2	143	28	28_{34}	191	36	36_{66}
95	20	20_1	143	28	28_{33}	191	36	36_{65}
			143	28	28_{32}	191	36	36_{64}

x	$b_{3,0}^4 \cdot x$	$e_0 r \cdot x$	$b_{3,0}^8 \cdot x$	$wgr \cdot x$
28_{34}	36_{66}	0	N/A	N/A
28_{33}	36_{65}	0	N/A	N/A
28_{32}	36_{64}	38_{25}	N/A	N/A
20_2	28_{34}	N/A	36_{66}	39_2
20_1	28_{33}	N/A	36_{65}	39_1

C.2. The case $A_1 = A_1[01]$.TABLE C.2.1. $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
70	15		71	15	15 ₀	190	39		191	39	39 ₀
70	14	14 ₁₈	71	14	14 ₂₀	190	38	38 ₁₈	191	38	38 ₁₉
70	14	14 ₁₇	71	14	14 ₁₉	190	38	38 ₁₇			
70	14	14 ₁₆				190	38	38 ₁₆			
70	14	14 ₁₅				190	38	38 ₁₅			
70	13	13 ₃₃	71	13	13 ₄₀	190	37	37 ₂₄	191	37	37 ₂₉
70	13	13 ₃₂	71	13	13 ₃₉	190	37	37 ₂₃	191	37	37 ₂₈
70	13	13 ₃₁	71	13	13 ₃₈	190	37	37 ₂₂	191	37	37 ₂₇
70	13	13 ₃₀	71	13	13 ₃₇	190	37	37 ₂₁	191	37	37 ₂₆
70	13	13 ₂₉	71	13	13 ₃₆				191	37	37 ₂₅
			71	13	13 ₃₅						
			71	13	13 ₃₄						
70	12		71	12		190	36		191	36	

x	g^6x	$v_2^{20}h_1x$
15 ₀	39 ₀	36 ₄₀
14 ₂₀	0	0
14 ₁₉	38 ₁₉	35 ₅₉
14 ₁₈	38 ₁₈	35 ₅₈
14 ₁₇	38 ₁₇	35 ₅₇ + 35 ₅₈
14 ₁₆	38 ₁₆	35 ₅₆
14 ₁₅	38 ₁₅	35 ₅₅ + 35 ₅₈
13 ₄₀	0	0
13 ₃₉	0	0
13 ₃₈	37 ₂₉	34 ₆₉
13 ₃₇	37 ₂₈	34 ₆₈
13 ₃₆	37 ₂₇	34 ₆₇ + 34 ₆₈
13 ₃₅	37 ₂₆	34 ₆₆ + 34 ₆₇ + 34 ₆₉
13 ₃₄	37 ₂₅	34 ₆₅ + 34 ₆₆

TABLE C.2.2. $Ext_{A(2)}^{s-1,t-8}(A_1 \wedge DA_1 \wedge bo_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
70	15		71	15		190	39		191	39	
70	14		71	14		190	38		191	38	
70	13	12_9	71	13	12_{17}	190	37	36_9	191	37	36_{17}
70	13	12_8	71	13	12_{16}	190	37	36_8	191	37	36_{16}
70	13	12_7	71	13	12_{15}	190	37	36_7	191	37	36_{15}
70	13	12_6	71	13	12_{14}	190	37	36_6	191	37	36_{14}
70	13	12_5	71	13	12_{13}	190	37	36_5	191	37	36_{13}
70	13	12_4	71	13	12_{12}	190	37	36_4	191	37	36_{12}
			71	13	12_{11}				191	37	36_{11}
			71	13	12_{10}				191	37	36_{10}
70	12	11_{36}	71	12	11_{42}	190	36		191	36	
70	12	11_{35}	71	12	11_{41}						
70	12	11_{34}	71	12	11_{40}						
70	12	11_{33}	71	12	11_{39}						
70	12	11_{32}	71	12	11_{38}						
70	12	11_{31}	71	12	11_{37}						
70	12	11_{30}									

x	g^6x	$v_2^{20}h_1x$
12_{17}	36_{17}	33_{73}
12_{16}	36_{16}	$33_{72} + 33_{73}$
12_{15}	36_{15}	33_{71}
12_{14}	36_{14}	$33_{70} + 33_{71}$
12_{13}	36_{13}	$33_{69} + 33_{71} + 33_{72} + 33_{73}$
12_{12}	36_{12}	$33_{68} + 33_{69} + 33_{71} + 33_{72} + 33_{73}$
12_{11}	36_{11}	$33_{67} + 33_{68} + 33_{69} + 33_{72}$
12_{10}	36_{10}	$33_{66} + 33_{72}$

TABLE C.2.3. $Ext_{A(2)}^{s-1,t-16}(A_1 \wedge DA_1 \wedge bo_2)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
70	15		71	15		190	39		191	39	
70	14		71	14		190	38		191	38	
70	13		71	13		190	37		191	37	
70	12		71	12		190	36		191	36	

TABLE C.2.4. $Ext_{A(2)}^{s-2,t-16}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
70	15		71	15		190	39		191	39	
70	14		71	14		190	38		191	38	
70	13		71	13		190	37		191	37	
70	12		71	12	10_1	190	36		191	36	34_1
			71	12	10_0				191	36	34_0

x	g^6x	$v_2^{20}h_1x$
10_1	34_1	31_{81}
10_0	34_0	31_{80}

C.3. The case $A_1 = A_1[10]$.TABLE C.3.1. $Ext_{A(2)}^{s,t}(A_1 \wedge DA_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
			71	15	15_0				191	39	39_0
70	14	14_{18}	71	14	14_{20}	190	38	38_{18}	191	38	38_{19}
70	14	14_{17}	71	14	14_{19}	190	38	38_{17}			
70	14	14_{16}				190	38	38_{16}			
70	14	14_{15}				190	38	38_{15}			
70	13	13_{34}	71	13	13_{41}	190	37	37_{24}	191	37	37_{29}
70	13	13_{33}	71	13	13_{40}	190	37	37_{23}	191	37	37_{28}
70	13	13_{32}	71	13	13_{39}	190	37	37_{22}	191	37	37_{27}
70	13	13_{31}	71	13	13_{38}	190	37	37_{21}	191	37	37_{26}
70	13	13_{30}	71	13	13_{37}				191	37	37_{25}
			71	13	13_{36}						
			71	13	13_{35}						

x	g^6x	$v_2^{20}h_1x$
15_0	39_0	36_{40}
14_{20}	0	0
14_{19}	38_{19}	35_{59}
13_{41}	0	0
13_{40}	0	0
13_{39}	37_{29}	34_{69}
13_{38}	37_{28}	34_{68}
13_{37}	37_{27}	34_{67}
13_{36}	37_{26}	34_{66}
13_{35}	37_{25}	34_{65}

TABLE C.3.2. $Ext_{A(2)}^{s-1,t-8}(A_1 \wedge DA_1 \wedge b_{01})$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
70	15		71	15		190	39		191	39	
70	14		71	14		190	38		191	38	
70	13	12_9	71	13	12_{17}	190	37	36_9	191	37	36_{17}
70	13	12_8	71	13	12_{16}	190	37	36_8	191	37	36_{16}
70	13	12_7	71	13	12_{15}	190	37	36_7	191	37	36_{15}
70	13	12_6	71	13	12_{14}	190	37	36_6	191	37	36_{14}
70	13	12_5	71	13	12_{13}	190	37	36_5	191	37	36_{13}
70	13	12_4	71	13	12_{12}	190	37	36_4	191	37	36_{12}
			71	13	12_{11}				191	37	36_{11}
			71	13	12_{10}				191	37	36_{10}
70	12	11_{39}	71	12	11_{45}	190	36		191	36	
70	12	11_{38}	71	12	11_{44}						
70	12	11_{37}	71	12	11_{43}						
70	12	11_{36}	71	12	11_{42}						
70	12	11_{35}	71	12	11_{41}						
70	12	11_{34}	71	12	11_{40}						
70	12	11_{33}									

x	g^6x	$v_2^{20}h_1x$
12_{17}	36_{17}	33_{73}
12_{16}	36_{16}	33_{72}
12_{15}	36_{15}	33_{71}
12_{14}	36_{14}	33_{70}
12_{13}	36_{13}	33_{69}
12_{12}	36_{12}	$33_{68} + 33_{73}$
12_{11}	36_{11}	$33_{67} + 33_{73}$
12_{10}	36_{10}	$33_{66} + 33_{71} + 33_{72}$

TABLE C.3.3. $Ext_{A(2)}^{s-1,t-16}(A_1 \wedge DA_1 \wedge bo_2)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
70	15		71	15		190	39		191	39	
70	14		71	14		190	38		191	38	
70	13		71	13		190	37		191	37	
70	12		71	12		190	36		191	36	

TABLE C.3.4. $Ext_{A(2)}^{s-2,t-16}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1)$

$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g	$t-s$	s	s_g
70	15		71	15		190	39		191	39	
70	14		71	14		190	38		191	38	
70	13		71	13		190	37		191	37	
70	12		71	12	10_1	190	36		191	36	34_1
			71	12	10_0				191	36	34_0

x	$g^6 x$	$v_2^{20} h_1 x$
10_1	34_1	31_{81}
10_0	34_0	31_{80}

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